Classification of Systems:-

→ The Systems may be classified as

1. Continuous Time & discrete-time Systems
2. Lumped parameters and distributed-parameter Systems.
3. Static & dynamic Systems
4. Causal & Non-Causal Systems.
5. Linear & Non-linear Systems
6. Time-invariant & time-Variant Systems.
7. Stable & unstable Systems.

1. Continuous Time & discrete-time Systems:-

a) Continuous Time System:-

→ It is one which operates on a Continuous Time input Signal and produces a continuous Time Output Signal.

→ Let i/p of CTS is $x(t)$ and o/p of CT system is $y(t)$ then we can say that $x(t)$ is transformed to $y(t)$. That is

$$\boxed{y(t) = T[x(t)]}$$

$$x(t) \longrightarrow \boxed{\text{CT System}} \longrightarrow y(t)$$

Ex:- Amplifiers, filters, motors etc,

b) Discrete-Time Systems:-

→ It is one which operates on a Discrete Time input Signal and produces a Discrete Time Output Signal.

→ Let i/p of DT System is $x(n)$ & o/p of DTS is $y(n)$ then we can say that $x(n)$ is transformed to $y(n)$. That is

$$\boxed{y(n) = T[x(n)]}$$

$$\underset{x(n)}{\text{input}} \longrightarrow \boxed{\text{DT System}} \longrightarrow y(n)$$

2. Lumped-parameter and distributed-parameter Systems:-

→ In lumped parameter System each and every Component is lumped at one-point in Space. These Systems are described by Ordinary differential equations.

→ In distributed parameter Systems, the Signals are functions of Space as well as time. These Systems are described by partial differential equations.

2. Static and dynamic Systems:-

a) Static System:-

→ A System is called Static (or) memoryless if its output at any instant depends on the input at that instant but not on past (or) future values of input

pa P fu
× ✓ ×

Ex:- $y(t) = x^2(t)$; $y(n) = n\,x(n)$

b) Dynamic System:-

→ A System is called Dynamic (or) Memory if its output at any instant depends on the input at that instant and on past (or) future values of input is a dynamic System.

pa P fu.
✓ — (or) — ✓

Ex:- $y(t) = \frac{dx(t)}{dt}$; $y(n) = x(n-1)$;

4. Causal and non-Causal Systems:-

→ a) A causal System is one for which the output of the System at any time 't' depends on the present & past inputs but not future inputs. These are also known as anticipative Systems.

pa P futu
✓ ✓ ×

$$\boxed{\begin{aligned} Ex:-\ & y(t) = x(t) + x(t-1) \\ & y(n) = n\,x(n) + x(n-3) \end{aligned}}$$

→ b) Non-Causal System:-

Non-Causal System is one in which the output depends on the future values only.

pa P fu.
× × ✓

$$\boxed{\begin{aligned} Ex:-\ & y(t) = x(t+3) + x^2(t) \\ & y(n) = x(2n) \end{aligned}}$$

5. Linear and non-linear Systems:-

a) Linear Systems:-

→ Superposition principle States that the response to a weighted Sum of input Signals be equal to the weighted Sum of the outputs Corresponding to each of the individual input Signals.

→ A System which Satisfies both Scaling & Superposition are called Linear System.

Let $x_1(t)$ and $x_2(t)$ are i/ps

$y_1(t)$ and $y_2(t)$ are o/ps

then if $y(t) = a x_1(t) + b x_2(t) = a y_1(t) + b y_2(t)$

Where a & b are constants then ie.,

$$\boxed{T[a x_1(t) + b x_2(t)] = a T[x_1(t)] + b T[x_2(t)]}$$ is called linear system.

for DTS $$\boxed{T[a x_1(n) + b x_2(n)] = a T[x_1(n)] + b T[x_2(n)]}$$

From the Superposition & Scaling property we can find that a Zero input results a Zero output.

Non-linear Systems:-

If any System that do not obeys Scaling & Superposition then those Systems are called Non-linear Systems.

Means

$$T[ax_1(t)+bx_2(t)] \neq aT[x_1(t)] + bT[x_2(t)] \text{ — for CTS}$$

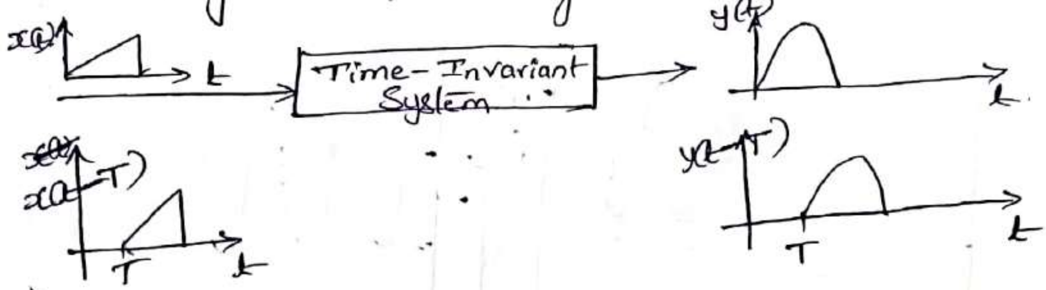
$$T[ax_1(n)+bx_2(n)] \neq aT[x_1(n)] + bT[x_2(n)] \text{ — for DTS.}$$

6. Time-Invariant and Time-Variant Systems:-

a) Time-invariant:-

A System is said to be time-invariant if its input-output Characteristics do not change with time.

Let a signal $x(t)$ to a System and obtain an output $y(t)$



→ The output due to the delayed input $x(t-T)$ for a time invariant System is given by

$$\boxed{y(t-T) = T[x(t-T)]}$$

b) Time-Variant:-

A System is said to be time Variant if the output due to the input $x(t-T)$ is not equal to $y(t-T)$ then the System is time-Variant.

→ For a Linear Time-Invariant (LTI) the co-efficients of differential equation describing the System are Constants.

→ If the co-efficients are function of time then the System is a linear-time Variant System.

7) Stable and Unstable Systems:-

a) Stable:-

→ An input Signal $x(t)$ is Said to be bounded, if it Satisfies the condition

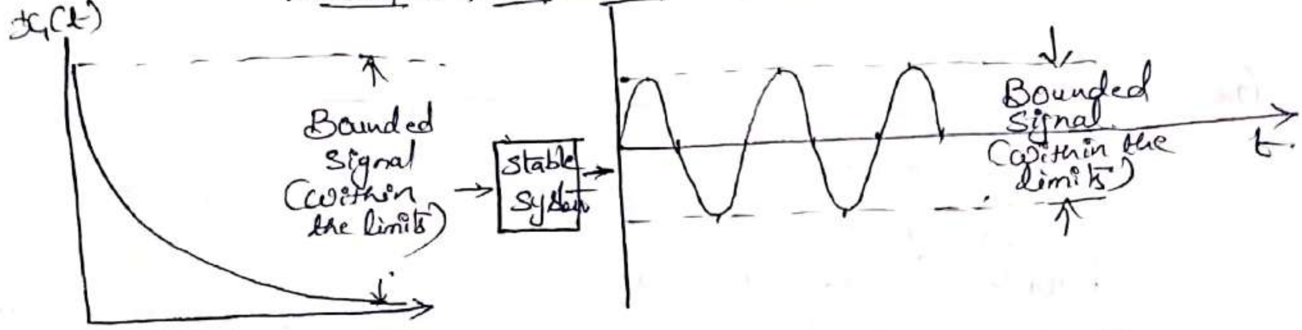
$$\boxed{|x(t)| \le M_x < \infty \quad \forall\ t}$$

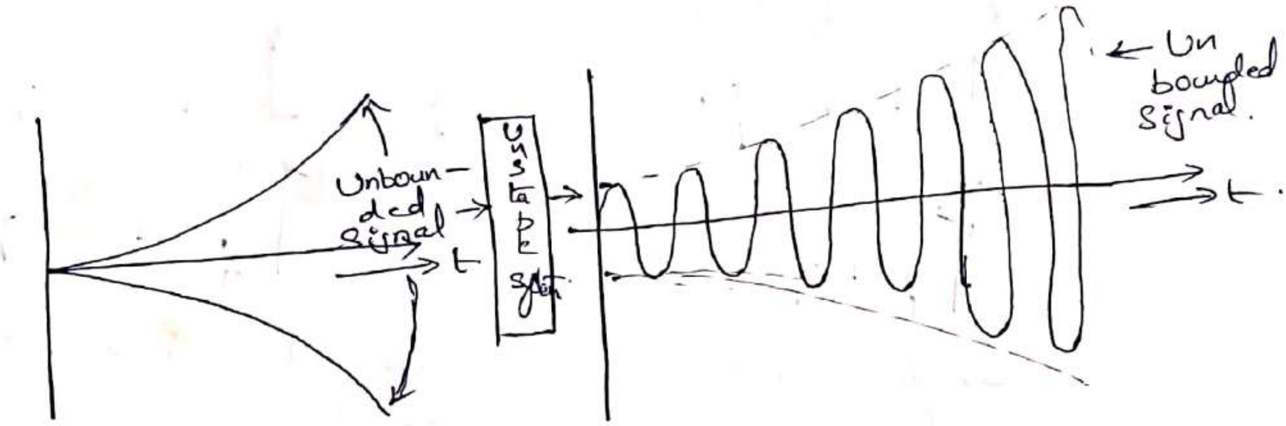
→ Similarly the output Signal is bounded if it Satisfies the condition

$$\boxed{|y(t)| \le M_y < \infty} \quad \forall\ t$$

where M_x and M_y are

Examples for Bounded and unbounded Signals





b) Unstable:-

If above conditions are not Satisfied i.e., an i/p Signal $x(t)$ is Said to be bounded output is not bounded then the System is unstable.

(or)

If $x(t)$ is not bounded Output is also not bounded then the System is unstable.

Definition:-

A System is Said to be bounded input and bounded Output Stable if and only if every bounded input produces a bounded Output.

Ex:- Find whether the following Systems are dynamic or not.

i) $y(t) = x(t-2)$ ii) $y(n) = x(n+2)$ iii) $y(t) = x^2(t)$

	Pa	P	future	
	×	✓	×	Memoryless
	✓		✓	Memory

Let i) $y(t) = x(t-2)$

$\left.\begin{array}{ll} @\ t=0 & y(0)=x(-2) \\ @\ t=1 & y(1)=x(-1) \\ @\ t=2 & y(2)=x(0) \end{array}\right\}$ so o/p depends on past input values

∴ System is dynamic

ii) $y(n) = x(n+2)$

$\left.\begin{array}{ll} @\ n=0 & y(0)=x(2) \\ @\ n=1 & y(1)=x(3) \\ @\ n=3 & y(3)=x(5) \\ @\ n=-1 & y(-1)=x(1) \end{array}\right\}$ o/p depends on future values of i/p.

∴ System is dynamic

iii) $y(t) = x^2(t)$

$\left.\begin{array}{ll} @\ t=0 & y(0)=x^2(0) \\ @\ t=1 & y(1)=x^2(1) \\ @\ t=2 & y(2)=x^2(2) \end{array}\right\}$ Present o/p at any time depends on present input

∴ System is Static.

Ex:- Check whether the following System are Causal or not

	pa	p	f
Caus	✓	✓	×
non-Caus	×	×	✓

i) $y(n) = x(n) + -\dfrac{1}{x(n-1)}$

$\left.\begin{array}{ll} @\ n=0 & y(0)=x(0)+\dfrac{-1}{x(-1)} \\ @\ n=2 & y(1)=x(1)+\dfrac{-1}{x(0)} \\ @\ n=-2 & y(-2)=x(-2)+\dfrac{-1}{x(-3)} \end{array}\right\}$ present & past

∴ System is Causal

ii) $y(t) = x^2(t) + x(t-2)$

$\left.\begin{array}{ll} @\ t=0 & y(0)=x^2(0)+x(-2) \\ @\ t=1 & y(1)=x^2(1)+x(-1) \\ @\ t=-1 & y(-1)=x^2(-1)+x(-3) \end{array}\right\}$ present & past

∴ System is Causal.

iii) $y(t) = x(t-2) + x(2-t)$

$\left.\begin{array}{ll} @\ t=0 & y(0)=x(-2)+x(2) \\ @\ t=1 & y(1)=x(-1)+x(1) \\ @\ t=-1 & y(-1)=x(-3)+x(3) \end{array}\right\}$ past & future

∴ System is non-causal.

iv) $y(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau$ Let $\int x(\tau)d\tau = z(t)$

$y(t) = \int z(t) \Big|_{-\infty}^{2t}$

at $t=0$ $z(t)\Big|_{-\infty}^{0} = z(0) - z(-\infty) = y(0)$

@ $t=1$ $\quad Z(t)$ is $\int_{-\infty}^{2}$ i.e ; $Z(t)\Big|_{-\infty}^{2} = Z(2) - Z(-\infty) = y(1)$

at $t=2$ $\quad Z(t)\Big|_{-\infty}^{4}$ i.e ; $Z(t)\Big|_{-\infty}^{4} = Z(4) - Z(-\infty) = y(4)$

in this o/p is depends on present i/p & future values of i/p

So the System is non-causal.

v) $y(n) = x(-n)$ Non-causal

vi) $y(n) = x(n^2)$ non-causal.

Ex:- $y(t) = x(t)x(t-1)$ (C) ; $y(t) = x(t/2)$ (NC)

$y(t) = x(t^{1/2})$ (NC) ; $y(n) = \sum_{k=-\infty}^{n+1} x(k)$ (NC)

$y(n) = \cos[x(n)]$ (C)

Ex:- Check whether the following System are linear & not.

1) $\frac{dy}{dt} + 3t\,y(t) = t^2 x(t)$

2) $\frac{dy(t)}{dt} + 2y(t) = x^2(t)$

3) $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

4) $\frac{dy}{dt} + 2y(t) = x(t)\frac{dx(t)}{dt}$

5) $y(n) = A\,x(n) + B$

6) $y(n) = 2x(n) + \frac{1}{x(n-1)}$

7) $y(n) = n\,x(n)$

1) $\frac{dy(t)}{dt} + 3t\,y(t) = t^2 x(t)$

For $x_1(t)$ i/p. $y_1(t)$ is an o/p

$\therefore \frac{dy_1(t)}{dt} + 3t\,y_1(t) = t^2 x_1(t)$ — ①

m^{ly} For $x_2(t)$ i/p, $y_2(t)$ is an o/p

$\therefore \frac{dy_2(t)}{dt} + 3t\,y_2(t) = t^2 x_2(t)$ — ②

Multiply eq ① LHS & RHS with 'a'.

& Multiply eq ② LHS & RHS with 'b'

$\therefore a\left[\frac{dy_1(t)}{dt}\right] + a3t\,y_1(t) = a t^2 x_1(t)$ — ③

$b\left[\frac{dy_2(t)}{dt}\right] + b\,3t\,y_2(t) = b t^2 x_2(t)$ — ④

then ③ + ④

$$\underbrace{\frac{d}{dt}\left[a y_1(t) + b y_2(t)\right]}_{\text{weighted sum of outputs}} + 3t\underbrace{\left[a y_1(t) + b y_2(t)\right]}_{\text{weighted sum of outputs}} = t^2\underbrace{\left[a x_1(t) + b x_2(t)\right]}_{\text{weighted sum of inputs.}}$$

2) m^{ly} of above ① problem

Ex:- Check whether the following Systems are linear & not

i) $y(t) = e^{x(t)}$ ii) $y(t) = x^2(t)$ iii) $y(t) = t^2 x(t)$

i) $y(t) = e^{x(t)}$

Let $y_1(t) = e^{x_1(t)}$ — ①

Illy $y_2(t) = e^{x_2(t)}$ — ②

Multiply ① LHS & RHS with 'a'..

&

② LHS & RHS with 'b'.

then.

$a y_1(t) = a e^{x_1(t)}$ — ③

$b y_2(t) = b e^{x_2(t)}$ — ④

③ + ④ is

$a y_1(t) + b y_2(t) \neq a e^{x_1(t)} + b e^{x_2(t)}$ — Non-linear.

Ex:- Check whether the following Systems are linear & not.

i) $\frac{dy}{dt} + t^2 y(t) = 2x(t)$ (linear) ii) $y(n) = x(n)\cos \omega n$ (linear)

Ex:- $2\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5y(t) = 5x(t)$, the given System is an LTI System.

Ex:- $\frac{d^2y(t)}{dt^2} + 4t\frac{dy(t)}{dt} + 5y(t) = x(t)$

→ It is a linear time varying System, since all co-efficients are not constant.

→ To test the time-invariant property of a System apply a signal $x(t)$ and find $y(t)$. Now delay the i/p signal by 'T' and find the o/p signal and denote it as

$$\boxed{y(t,T) = T[x(t-T)]}$$

→ Delay the o/p signal by T and denote it as $y(t-T)$.

→ If $y(t,T) = y(t-T)$ for any value of T, then the System is time-invariant. On the other hand if the o/p $\boxed{y(t,T) \neq y(t-T)}$ then the System is time-Variant

→ To test the time-invariant property of the given discrete-time System, first apply an arbitrary input sequence $x(n)$ and find $y(n)$.

→ Delay the input Sequence by 'K' Samples and find the Output Sequence, denote it as

$$y(n,k) = T[x(n-k)]$$

→ Delay the o/p sequence by 'K' Samples denote it as $y(n-k)$

if $y(n,k) = y(n-k)$ for all possible values of k, the System is time-invariant, otherwise the System is time-Variant.

Ex:- $y(n) - 4y(n-1) + 3y(n-2) = x(n)$ — A LTI System

Ex:- $y(n) - 3ny(n-1) + n^2 y(n-2) = x(n) + x(n-1)$ — A LT-Variant System.

Examples:- For each of the following Systems, determine whether or not the System in time-invariant.

i) $y(t) = t\,x(t)$ ii) $y(t) = e^{x(t)}$ iii) $y(t) = x(t)\cos 50\pi t$

iv) $y(n) = x(2n)$ v) $y(t) = x(t^2)$ vi) $y(n) = x(n) + n\,x(n-1)$

vii) $y(t) = x(-t)$ viii) $y(n) = x^2(n-1)$

① $y(t) = t\,x(t)$

we know $y(t) = T[x(t)] = t\,x(t)$

Step1:- The o/p due to delayed input is

$y(t,T) = T[x(t-T)] = t\,x(t-T)$ — ①

step2:- If the o/p is delayed by T, we get

$y(t-T) = (t-T)\,x(t-T)$ — ②

by Comparing ① & ②.

$y(t,T) \neq y(t-T)$ i.e; ① ≠ ② ∴ the System is time Variant

② $y(t) = x(t)\cos 50\pi t$

Step1:- The o/p due to the delayed i/p is

$y(t,T) = x(t-T)\cos 50\pi t$ — ①

Step2:- If the Output is delayed by T, we get

$y(t-T) = x(t-T)\cos 50\pi(t-T)$ — ②

∴ ① ≠ ② ∴ the System is time-Variant

3. $y(t) = x(t^2)$

$y(t) = T[x(t)] = x(t^2)$

The o/p due to the delayed input

$y(t,T) = T[x(t-T)] = x(t^2 - T)$ — ①

The delayed output

$y(t-T) = x[(t-T)^2]$ — ②

$\therefore$ ① ≠ ② the System is time-variant.

4. $y(t) = x(-t)$

we know $y(t) = T[x(t)]$

The o/p due to the delayed input

$y(t,T) = T[x(t-T)] = x(-t-T)$ — ①

The delayed output.

$y(t-T) = x[-(t-T)] = x[-t+T]$ — ②

$\because$ ① ≠ ② the System is time-variant.

5. $y(t) = e^{x(t)}$

we know $y(t) = T[x(t)]$

The o/p due to the delayed i/p

$y(t,T) = T[x(t-T)]$

$y(t,T) = e^{x(t-T)}$ — ①

The delayed output.

$y(t-T) = e^{x(t-T)}$ — ②

① = ② $\therefore$ System is time invariant.

6. $y(n) = x(n) + n\,x(n-1)$

we know $y(n) = T[x(n)]$

The o/p due to the delayed input

$y(n,k) = x(n-k) + n\,x(n-k-1)$ — ①

The delayed output

$y(n-k) = x(n-k) + (n-k)\,x(n-k-1)$ — ②

$\because$ ① = ②, So the System is time-variant.

Convolution:-

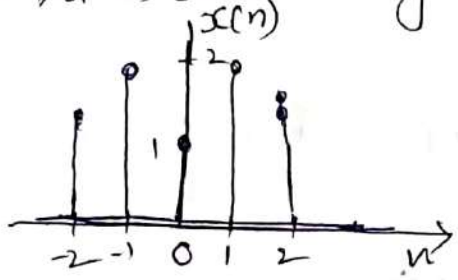
Representation of Discrete-Time Signals interms of impulses

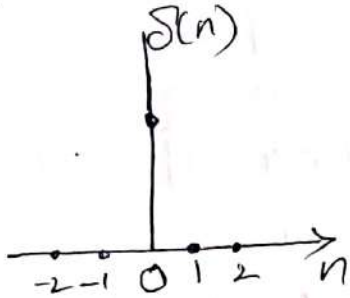
→ The unit impulse can be used as a basic signal to construct any discrete-time signal.

→ That is any discrete time signal can be represented as a sum of shifted, scaled unit impulse sequence.

→ The sample at $n=0$ can be obtained by multiplying $x(0)$, the magnitude with impulse $\delta(n)$

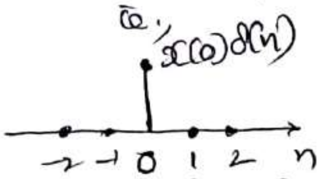
→ Let us consider any $x(n)$



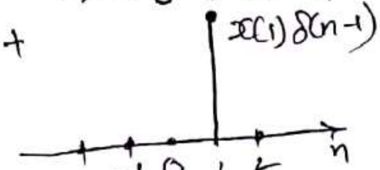


$=$ at $k=0$ $x(k)\delta(n-k) = x(0)\delta(n)$

i.e.,

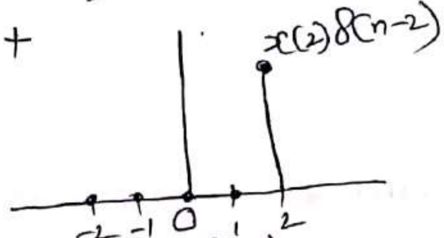


$+$



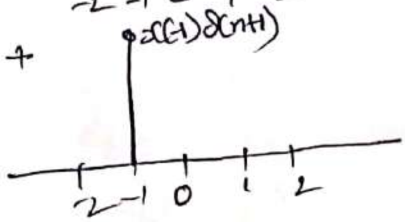
at $k=1$ $x(1)\delta(n-1)$

$+$



at $k=2$ $\delta(n-2)x(2)$

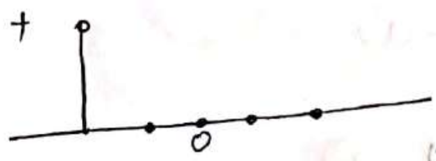
$+$



at $k=-1$ $x(-1)\delta(n+1)$

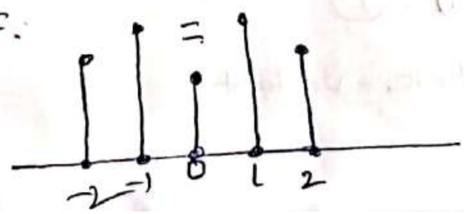
@ $k=-2$ $x(-2)\delta(n+2)$

$+$



Representation of a sequence as a sum of delayed impulses.

$=$



That is

$x(0)\delta(n) = x(0)$ for $n=0$

$= 0$ for $n \neq 0$.

Illy. the Sample $x(2)$ can be obtained by multiplying $x(2)$, the magnitude, with two Samples advanced unit impulse $\delta(n-2)$

So $x(2)\delta(n-2) = x(2)$ for $n=2$

$= 0$ for $n \neq 2$

Illy. $x(1)\delta(n-1) = x(1)$ for $n=1$

$= 0$ for $n \neq 1$

$x(-1)\delta(n+1) = x(-1)$ for $n=-1$

$= 0$ for $n \neq -1$

$x(-2)\delta(n+2) = x(-2)$ for $n=-2$

$= 0$ for $n \neq -2$

The Sum of the above Sequences is

$x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2)$

equals to the given Sequence $x(n)$

In general, we can write $x(n)$ for $-\infty \leq n \leq \infty$ as

$x(n) = ----x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + ----$

$$\boxed{x(n) = \sum_{k=-\infty}^{+\infty} x(k)\,\delta(n-k)}$$

where $\delta(n-k) = 1$ for $n=k$

$= 0$ for all other 'k'

Ex:- Representation of Sequence $x(n) = \{3, 2, -1, 2, 4, 1\}$ as Sum of Shifted unit impulses.

Step 1: Given. $x(n) = \{3, \underset{\uparrow}{2}, -1, 2, 4, 1\}$

Index k. $-1 \quad 0 \quad +1 \quad +2 \quad +3 \quad +4$.

Step 2:- we can find $x(-1) = 3$, $x(0) = 2$, $x(1) = -1$ $x(2) = 2$ $x(3) = 4$ $x(4) = 1$

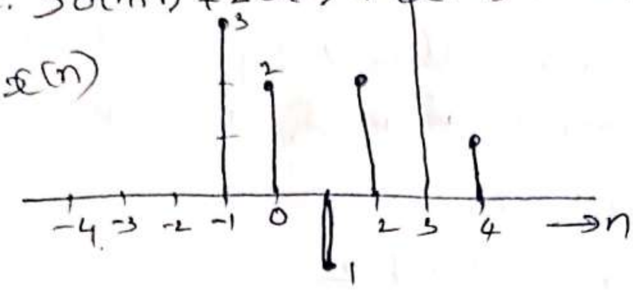
we know.

$$x(n) = \sum_{k=-\infty}^{+\infty} x(k).\delta(n-k)$$

$$= x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + x(3)\delta(n-3) + x(4)\delta(n-4)$$

∴ By Substituting the Sequence values yields.

$$x(n) = 3\delta(n+1) + 2\delta(n) - \delta(n-1) + 2\delta(n-2) + 4\delta(n-3) + 1\delta(n-4)$$



Impulse Response and convolution Sum :-

→ A DT systems performs an operation on an input signal based on predefined Criteria to produce a modified output signal.

→ The input signal to the System is $x(n)$ and the output of the System is $y(n)$.

This transform operation is

$$x(n) \longrightarrow \boxed{T} \longrightarrow y(n) = T[x(n)]$$

$$\delta(n) \longrightarrow \boxed{T} \longrightarrow T[\delta(n)] = h(n)$$

→ If the input to the System is a unit impulse $\delta(n)$ then the output of the System is known as an impulse response of the System denoted by $h(n)$. That is

$$\boxed{h(n) = T[\delta(n)]} \quad \text{——— (1)}$$

→ we know that any arbitrary Sequence $x(n)$ can be represented as a weighted Sum of discrete impulses given by

i/p. $$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)} \quad \text{——— (2)}$$

∴ $$\boxed{y(n) = T\left[\sum_{k=-\infty}^{+\infty} x(k)\,\delta(n-k)\right]} \quad \text{——— (3)}$$

→ Using linearity property (3) can be written as

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, T[\delta(n-k)] \quad \text{——— (4)}$$

The Response to the Shifted impulse Sequence can be denoted by $h(n,k)$

$$\boxed{\therefore\ h(n,k) = T[\delta(n-k)]} \quad \text{——— (5)}$$

Hence (4) can be written as

$$\boxed{y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n,k)} \quad \text{——— (6)}$$

For a time-invariant System

$$\boxed{h(n,k) = h(n-k)} \quad \text{—— (7)}$$

∴ (4) can be written as

$$\boxed{y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k)} \quad \text{—— (8)}$$

→ Thus the output of a LTI System is given by weighted sum of time-shifted responses.

∴ The Equation (8) is termed the Convolution Sum & can be represented as

$$\boxed{y(n) = x(n) * h(n)}$$ where * denotes the convolution operation.

Properties of Convolution:—

1) Distributive property:—

$$\boxed{x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)}$$

2) Associative property:—

$$\boxed{[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]}$$

3) Commutative property:—

$$\boxed{h_1(n) * h_2(n) = h_2(n) * h_1(n)}$$

4) Shifting property:—

$$x(n) * h(n) = y(n)$$

then $\boxed{x(n-k) * h(n-m) = y(n-k-m)}$

5) Convolution with an impulse

$$\boxed{x(n) * \delta(n) = x(n)}$$

6) Convolution with a shifted impulse.

$$\boxed{x(n) * \delta(n-n_0) = x(n-n_0)}$$

Convolution of two Sequences:-

→ Steps to be found. Convolution Sum of two Sequences.

Step 1:- Choose an initial value of 'n'; the starting time for evaluating the $y(n)$. If $x(n)$ start at $n=n_1$ and $h(n)$ starts at $n=n_2$ then $n=n_1+n_2$ is a good choice.

Step 2:- Express both Sequences in terms of the index 'k'

Step 3:- Fold $h(k)$ about $k=0$ to obtain $h(-k)$ and shift right by 'n' if 'n' is +ve and shift left if 'n' is -ve to obtain $h(n-k)$.

Step 4:- Multiply the two Sequences $x(k)$ & $h(n-k)$ element by element and Sum the products to get $y(n)$.

Step 5:- Increment the index 'n', shift the Sequence $h(n-k)$ to right by one Sample and do step 4.

Step 6:- Repeat step 5 until the Sum of product is '0' for all remaining Values of 'n'.

Ex:- Determine the convolution Sum of two Sequences.

$$x(n) = \{1, \underset{\uparrow}{4}, 3, 2\} \; ; \; h(n) = \{1, 3, 2, 1\}$$

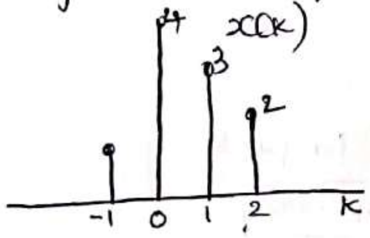
$x(n)$: indices $-1 \; 0 \; 1 \; 2$; $h(n)$: indices $0 \; 1 \; 2 \; 3$

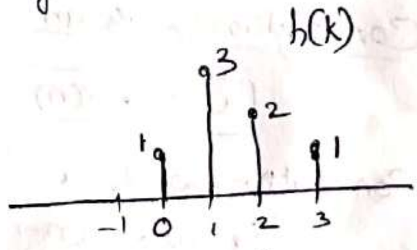
Step 1:- The Sequence $x(n)$ starts at $n_1=-1$ and $h(n)$ starts at $n_2=0$.

$\therefore$ The starting time for evaluating the o/p sequence $y(n)$ is

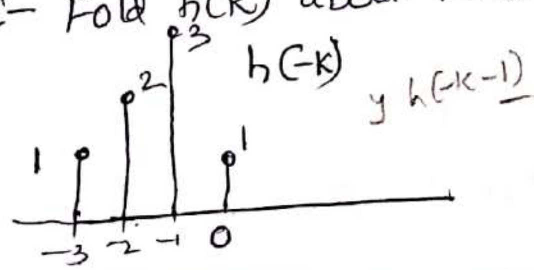
$n = n_1+n_2 = -1+0 = -1$

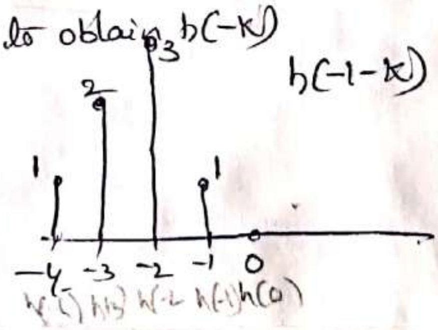
Step 2:- Express both Sequences in terms of the index 'k'.





Step 3:- Fold $h(k)$ about $k=0$ to obtain $h(-k)$





→ As starting time to evaluate $y(n)$ is -1, shift $h(k)$ by one unit to left to obtain $h(-1-k)$

(i) $y(-1) = \sum_{k=-\infty}^{+\infty} x(k)\,h(-1-k)$

$x(-4)h(-1+4) + x(-3)h(-1+3)$

$= x(-4)h(-1+4) + x(-3)h(-1+3) + x(-2)h(-1+2) + x(-1)h(-1+1) + x(0)h(-1) + x(1)h(-1-1) + x(2)h(-1-2)$

$= (0)(1) + (0)(2) + (0)(3) + (1)(1) + (4)(0) + (3)(0) + (2)(0) = 1$

$\boxed{y(-1) = 1}$

(ii) $y(0) = \sum_{k=-\infty}^{+\infty} x(k)\,h(0-k) = \sum_{k=-\infty}^{+\infty} x(k)\,h(-k)$

$= (0)(1) + (0)(2) + (1)(3) + (4)(1) = 3+4$
$= 7$

$\boxed{y(0) = 7}$

$y(1) = \sum_{k=-\infty}^{+\infty} h(1-k)\,x(k)$

$= x(-2)h(1-k) + x(-1)h(1-k) + x(0)h(1-k) + x(1)h(1-k) + x(2)h(1-k)$

$= (0)(1) + (1)(2) + (4)(3) + (3)(1)$

$= 0 + 2 + 12 + 3 = 17$

$\boxed{y(1) = 17}$

So $y(2) = \sum_{k=-\infty}^{+\infty} x(k)\,h(2-k)$

$= x(-1)h(2-k) + x(0)h(2-k) + x(1)h(2-k) + x(2)h(2-k)$

$= (1)(1) + (4)(2) + (3)(3) + (2)(1)$

$= 1 + 8 + 9 + 2 = 20$

$\boxed{y(2) = 20}$

$y(3) = \sum_{k=-\infty}^{+\infty} x(k)\,h(3-k)$

$= x(-1)h(3-k) + x(0)h(3-k) + x(1)h(3-k) + x(2)h(3-k) + x(3)h(3-k)$

$= (1)0 + (4)(1) + (3)(2) + (2)(3) + (0)(3)$

$= 0 + 4 + 6 + 6 + 0 = 16$

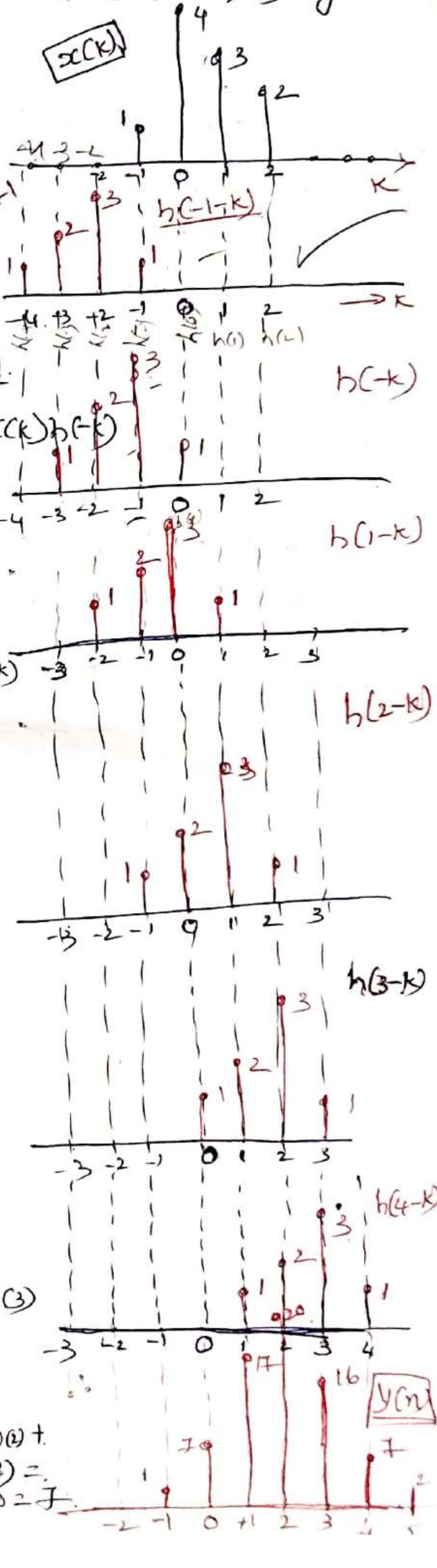
$\boxed{y(3) = 16}$

$y(4) = \sum_{k=-\infty}^{+\infty} x(k)\,h(n-k) = x(1)h(4-k) + x(2)h(4-k)$

$= (3)(1) + (2)(2) + (0)(3) + (0)(4) =$

$= 3 + 4 + 0 + 0 = 7$

$\boxed{y(4) = 7}$



Ex: - Determine the O/p response $y(n)$ if
$x(n) = \{1, 2, 3, 2\}$ $h(n) = \{1, 2, 2\}$

Ex:-. Find the convolution of $x_1(t)$ & $x_2(t)$ for the following Signals.

1) $x_1(t) = e^{-at}u(t)$; $x_2(t) = e^{-bt}u(t)$

2) $x_1(t) = u(t)$; $x_2(t) = u(t)$

3) $x_1(t) = t\,u(t)$; $x_2(t) = u(t)$

4) $x_1(t) = \sin t\, u(t)$; $x_2(t) = u(t)$

Sol:- 1) $x_1(t) = e^{-at}u(t)$; $x_2(t) = e^{-bt}u(t)$

We know $x_1(t) \circledast x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$.

→ A signal $x(t)$ is said to be causal, if $x(t) = 0$ for $t<0$

→ Above $x_1(t)$ & $x_2(t)$ are causal.

$\therefore$ $x_1(t) \circledast x_2(t) = \int_0^t x_1(\tau)\, x_2(t-\tau)\, d\tau$

$$= \int_0^t e^{-a\tau}\, e^{-b(t-\tau)}\, d\tau$$

$$= e^{-bt} \int_0^t e^{-a\tau} \cdot e^{+b\tau} \cdot d\tau$$

$$= e^{-bt} \cdot \int_0^t e^{-(a-b)\tau}\, d\tau$$

$$= e^{-bt} \cdot \frac{e^{-(a-b)\tau}}{-(a-b)}\Bigg|_0^t$$

$$= e^{-bt}\left[\frac{e^{-(a-b)t}}{-(a-b)} - \frac{e^{-(a-b)\cdot 0}}{-(a-b)}\right]$$

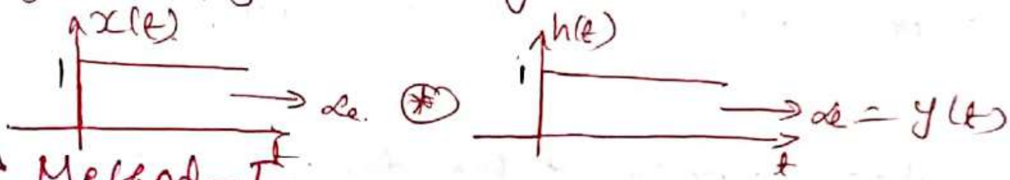
$$= e^{-bt}\left[\frac{e^{-(a-b)t}}{-(a-b)} - \frac{1}{-(a-b)}\right]$$

$$= e^{-bt}\left[\frac{e^{-(a-b)t} - 1}{(b-a)}\right] \text{ for } t \geq 0$$

$$= \frac{e^{-bt} \cdot e^{-at} \cdot e^{+bt} - e^{-bt}}{(b-a)}, \quad t \geq 0$$

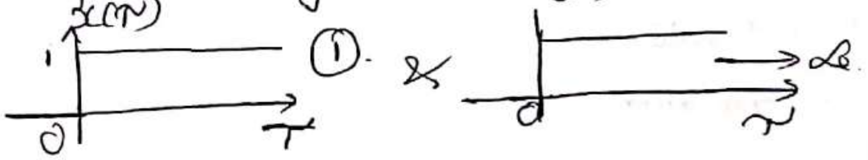
$$\boxed{x_1(t) * x_2(t) = \frac{e^{-at} - e^{-bt}}{(b-a)}\, u(t)}$$

Ex:- find out the $y(t)$ of an LTI System for the input & impulse response given.



M-II Method-I

Step1:- Replace 't' by 'τ'.



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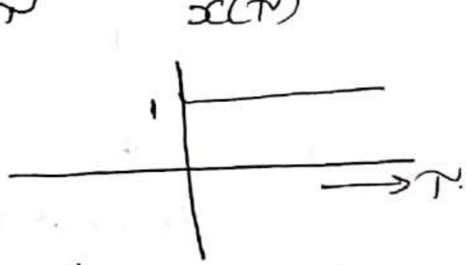
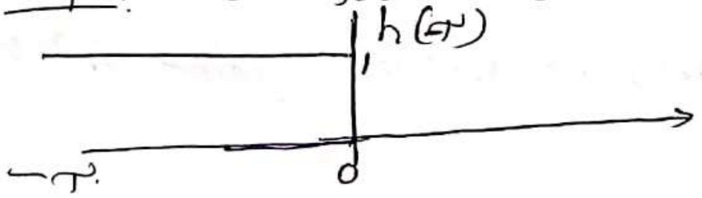
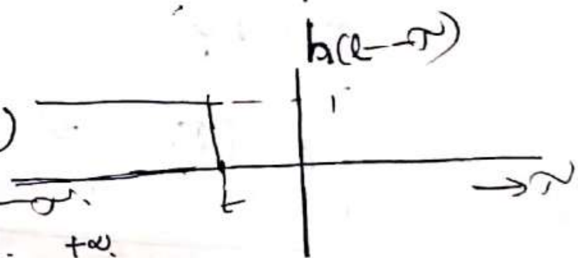
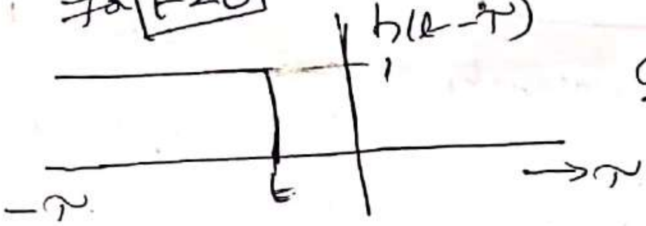


Step2:- Now fold $h(\tau)$ then.



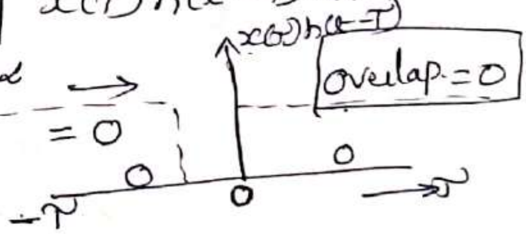


Step3:- perform time shifting $h(t-\tau)$
for $\boxed{t<0}$



Case I (a). $\int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$

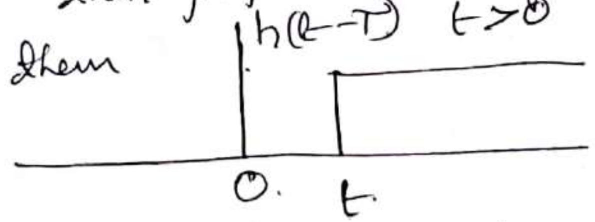
$\therefore = 0$



Step4:- for $t>0$

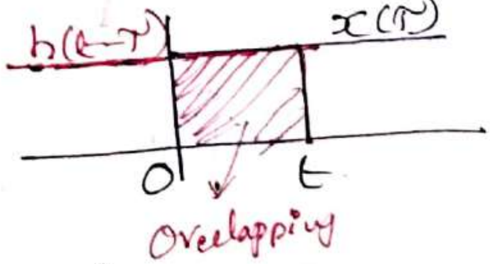
then perform time shifting $h(t-\tau)$

them



i.e; Case II

(b). $\int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)$ is



$$\therefore y(t) = \int_{-\infty}^{+\infty} 1 \cdot dt = \int_0^t 1 \cdot dt = t.$$

$\therefore \int_0^t x(\tau)\,h(t-\tau)\,d\tau = t.$

$$\therefore y(t) = \begin{cases} 0 & t<0 \\ t & t\geq 0 \end{cases}$$

$$\boxed{\therefore y(t) = r(t) \quad \forall t \geq 0}$$

Correlation of two Sequences:-

→ Correlation is basically used to Compare two Signals.
→ It occupies a significant place in Signal processing

Definition:-

Correlation is a measure of the degree to which two Signals are Similar

They are divided into
1) Cross Correlation
2) Auto Correlation

1) Cross Correlation:-

The Cross correlation between a pair of Signals $x(n)$ & $y(n)$ is given by

$$\boxed{\gamma_{xy}(l) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-l)} \quad l = 0, \pm 1, \pm 2, \pm 3 \ldots \quad \text{— (1)}$$

→ Index 'l' is Shift (lag) parameter
→ The order of Subscripts xy indicates that $x(n)$ is the Reference sequence that remains unshifted in time.
→ The Sequence $y(n)$ is Shifted 'l' units in time with respect to $x(n)$.
→ If $x(n)$ & $y(n)$ are interchanging then

$n-l = n$
$n = n+l$
$y(n+l)$

$$\gamma_{yx}(l) = \sum_{n=-\infty}^{+\infty} y(n)\, x(n-l)$$

$$\boxed{\gamma_{yx} = \sum_{n=-\infty}^{+\infty} y(n+l)\, x(n)} \quad \text{— (2)}$$

→ If the time Shifts $l=0$ then (1) becomes

$$\boxed{\gamma_{xy}(l) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n)} \quad \text{— (3)}$$

→ By Comparing (1) & (2) we find that

$$\boxed{\gamma_{xy}(l) = \gamma_{yx}(-l)} \quad \text{— (4)}$$

where $\gamma_{yx}(-l)$ is the folded version of $\gamma_{xy}(l)$ about $l=0$

then $$\gamma_{xy}(l) = \sum_{n=-\infty}^{+\infty} x(n)\, y[-(l-n)]$$

$$\boxed{\gamma_{xy}(l) = x(l) * y(-l)} \quad — (5)$$

∴ (5) shows that the Correlation process is essentially, the convolution of two data Sequences in which one of the Sequence has been reversed.

∴ the Same algorithm (or) procedure can be used to Compute Convolution and Correlation of two Sequences.

Auto - Correlation:-

→ The auto - correlation of a sequence is correlation of a sequence with itself.

→ The auto-correlation of a Sequence $x(n)$ is defined by

$$\boxed{\gamma_{xx}(l) = \sum_{n=-\infty}^{+\infty} x(n)\, x(n-l)} \quad — (1)$$

(or)

$$\boxed{\gamma_{xx}(l) = \sum_{n=-\infty}^{+\infty} x(n+l)\, x(n)} \quad — (2)$$

If the time Shift $l=0$, then we have.

$$\boxed{\gamma_{xx}(0) = \sum_{n=-\infty}^{+\infty} x^2(n)} \quad — (3)$$

Computation of Correlation :-

→ To Compute the Cross correlation of Sequence $x(n)$ and $y(n)$ the following Steps are followed.

1. Obtain the Sequence $y(n-l)$ by Shifting the Sequence right by the time lag 'l'.

2. Multiply the Shifted Sequence $y(n-l)$ with $x(n)$ and Sum all the Values to Obtain $\gamma_{xy}(l)$

3. Repeat Steps 1 and 2 for all values of the lag 'l'.

→ The Correlation of two Sequences $x(n)$ & $y(n)$ can be obtained by using the convolution procedure.

we know

$$\boxed{\gamma_{xy}(l) = x(l) * y(-l)}$$

To get $\gamma_{xy}(l)$ first fold Sequence $y(l)$ to obtain $y(-l)$. Then the convolution of $x(l) * y(-l)$ gives the value $\gamma_{xy}(l)$.

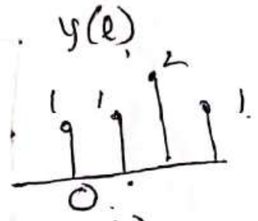
Ex:- Find the Cross-correlation of two finite length Sequences $x(n) = \{1, 2, 1, 1\}$ & $y(n) = \{1, 1, 2, 1\}$

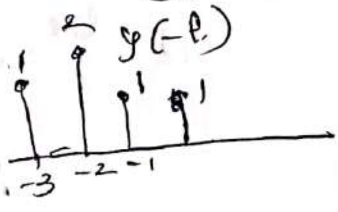
Sol:- Step 1:- Given Sequences with replaced with 'l'.

$\therefore x(l) = \{1, 2, 1, 1\}$

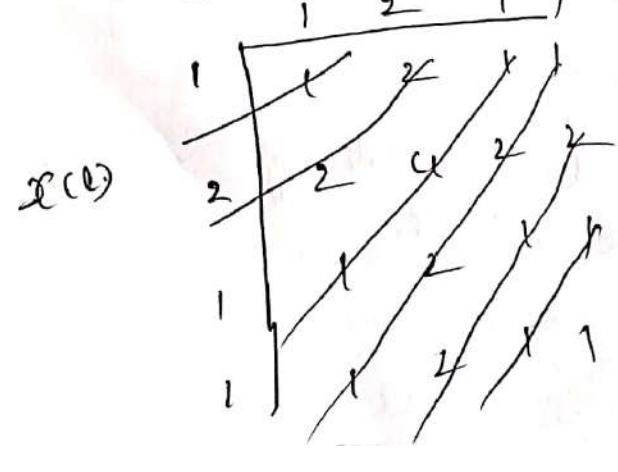
$y(l) = \{1, 1, 2, 1\}$

$y(-l) = \{1, 2, 1, 1\}$





Step 2:- $\gamma_{xy} = x(l) * y(-l)$



$\gamma_{xy} = \{1, 2+2, 1+4+1, 1+2+2+1, 2+1+2, 1+1, 1\}$

$$\boxed{\gamma_{xy} = \{1, 4, 6, 6, 5, 2, 1\}}$$

Fourier Series Representation:-

→ Periodic Signals

→ Fourier Series is used to represent time domain signal $x(t)$ into frequency domain signal $x(\omega)$.

→ Standard Signal for representation of periodic signal

$x(t) = \sin\omega_0 t$

$x(t) = e^{j\omega_0 t}$.

→ The Fourier Series of $x(t)$

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} a_n e^{j\omega_0 n t}} = \cdots a_{-3}e^{-j\omega_0 3t} + a_{-2}e^{-j\omega_0 2t} + a_{-1}e^{-j\omega_0 t} + a_0 e^{j\omega_0 0} + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_3 e^{j3\omega_0 t} + \cdots$$

where a_n → Magnitude

→ Now. a_n → which is fourier Series Co-efficient

now. $$x(t) = \sum_{K=-\infty}^{+\infty} a_K e^{j\omega_0 K t} \quad \text{——— (1)}$$

both RHS & LHS of above eq is multiply with $e^{-j\omega_0 K t}$.

∴ (1) becomes

$$x(t).e^{-j\omega_0 K t} = \sum_{K=-\infty}^{+\infty} a_K e^{j\omega_0 K t}.e^{-j\omega_0 K t}$$

$$\Rightarrow x(t)\, e^{-j\omega_0 n t} = \sum_{K=-\infty}^{+\infty} a_K\, e^{j(K-n)\omega_0 t} \quad \text{——— (2)}$$

→ Let integrate eq (2) from 0 to T on both sides

$$\Rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \int_0^T \sum_{K=-\infty}^{+\infty} a_K\, e^{j(K-n)\omega_0 t}$$

$$= \sum_{K=-\infty}^{+\infty} a_K \int_0^T e^{j(K-n)\omega_0 t}\, dt$$

$$\Rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{K=-\infty}^{+\infty} a_K \left[\int_0^T \cos(K-n)\omega_0 t + j\sin(K-n)\omega_0 t\right] dt$$

$$= \sum_{K=-\infty}^{+\infty} a_K \left[\int_0^T 1.dt + 0\right]$$

$$\int_0^T x(t).e^{-jn\omega_0 t}\, dt = \sum_{K=-\infty}^{+\infty} a_K T \quad \begin{cases} \text{if } K=n \\ 0 \quad K\neq n. \end{cases} \quad \int_0^T \cos(K-n) = T$$

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By Euler's formula, RHS can be written and get T.

So. $\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = a_n T$

ie; $\boxed{a_n = \frac{1}{T}\int_0^T x(t) \cdot e^{-jn\omega_0 t}\, dt}$

Trigonometric Fourier Series:-

a_n

→ We know that any function $f(t)$ (or) $x(t)$ can be

$\boxed{x(t) = f(t) = \sum_{n=1}^{\infty} a_n x_n(t)}$ —— (1) where $x_n(t)$ → orthogonal signal set.

(1) is called generalized fourier series.

→ We have seen that the set.

$\{1, \cos\omega_0 t, \cos 2\omega_0 t - \cos n\omega_0 t, \sin\omega_0 t, \sin 2\omega_0 t, ---- \sin n\omega_0 t ---\}$

is orthogonal over the period T_0.

→ Here ω_0 → angular frequency (or) fundamental frequency & $n\omega_0$ → n^{th} harmonic

→ At $n=0$, DC component ie, $\cos 0\omega_0 t = 1$

→ This signal set consists of sine & cosine terms. Hence it is called Trigonometric set.

→ for that set we can write

$f(t) = a_0 + a_1\cos\omega_0 t + a_2\cos 2\omega_0 t + ------ a_n\cos n\omega_0 t +$
$b_1\sin\omega_0 t + b_2\sin 2\omega_0 t + ------ b_n\sin n\omega_0 t$

$\boxed{\therefore f(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos n\omega_0 t + \sum_{n=1}^{\infty} b_n\sin n\omega_0 t}$

where
a_n → component of $\cos n\omega_0 t$
b_n → component of $\sin n\omega_0 t$

where $a_0 = \frac{1}{T_0}\int_t^{t+T_0} x(t)\, dt$

$a_n = \frac{2}{T_0}\int_t^{t+T_0} x(t)\cos(n\omega_0 t)\, dt$.

$b_n = \frac{2}{T_0}\int_t^{t+T_0} x(t)\sin(n\omega_0 t)\, dt$

Types of Fourier Series:- '3' types.

1. Trigonometric F.S.
2. Compact Trigonometric F.S
3. Exponential F.S.

Properties of Fourier Series:-

1. Linearity
2. Time Shifting
3. Frequency Shifting
4. Time Scaling
5. Multiplication & Convolution
6. Differentiation properties

we know $x(t) = \sum_{n=-\infty}^{+\infty} a_n e^{jnw_0 t}$

1. **Linearity property:-**

Let 2 signals $x(t)$ & $y(t)$ then

if $x(t) \xleftrightarrow{FS} F_x \cdot X(k)/X(\omega)$

$y(t) \xleftrightarrow{FS} F_y \; Y(k)/X(\omega)$

Then linearity property states that it obeys Scaling & Superposition i.e.,

$$\boxed{ax(t) + by(t) \xleftrightarrow{FS} aF_x + bF_y} \text{ (or) } aX(k) + bY(k)$$

2. **Time Shifting Property**

If $x(t) \xleftrightarrow{FS} F_x$.

Then if time shifting

$$\boxed{x(t-t_0) \xleftrightarrow{FS} e^{-jnw_0 t} F_x} \text{ (or) } e^{-jnw_0 t} X(K)$$

3. **Frequency Shifting Property:-**

If $x(t) \xleftrightarrow{FS} F_x$.

Then for Shifting frequency stats that.

$$\boxed{\begin{array}{l} e^{jw_0 t} x(t) \xleftrightarrow{FS} F(n-n_0) \\ e^{-jnw_0 t} x(t) \xleftrightarrow{FS} F(n+n_0) \end{array}}$$

(or) $X(n-n_0)$

(or) $X(n+n_0)$

Illy

4. **Time Scaling:-**

If $x(t) \xleftrightarrow{FS} F_x$.

then ~~FS{x(at)}~~ $FS\{x(at)\} = \frac{a}{T_0} F_x$

It states that the time scaling, Changes frequency component $\frac{1}{T_0}$ to $\frac{a}{T_0}$.

5. Multiplication & Convolution property.

If $x(t) \xleftrightarrow{FS} F_x$

$y(t) \xleftrightarrow{FS} F_y$

then.

$x(t)\, y(t) \xleftrightarrow{FS} F_x \circledast F_y$

Multiplication in time domain = Convolution in frequency domain

& if $x(t) * y(t) \xleftrightarrow{FS} F_x F_y$.

6. Differentiation property :-

If $x(t) \xleftrightarrow{FS} F_x$.

then

$$\frac{d}{dt}x(t) \xleftrightarrow{FS} j\frac{2\pi n}{T_0} F_x = j\omega_0 n F_x.$$

$x(t) = \sum_{n=-\infty}^{+\infty} a_n e^{jn\omega_0 t}$

$\frac{d}{dt}x(t) = \sum_{n=-\infty}^{+\infty} a_n \cdot \frac{e^{jn\omega_0 t}}{jn\omega_0}$

7. Symmetry property :-

Real.

$f(t) \xleftrightarrow{FS} F_n$.

$$\boxed{F_n^* = F_{-n}}$$

we know

$F_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t} dt$

$F_{-n} = \frac{1}{T}\int_0^T f(t)\, e^{jn\omega_0 t} dt \; (\because F_n^* = F_{-n})$

For real signal

i.e. if $x(t) = t$.

$x^*(t) = t$.

i.e. $\boxed{f(t) = f^*(t)}$

$= \frac{1}{T}\int_0^T [f^*(t)] [e^{-jn\omega_0 t}]^* dt.$

(or)

$F_{-n} = \frac{1}{T}\int_0^T [f^*(t) e^{-jn\omega_0 t}]^* dt.$

i.e. $\boxed{F_{-n} = F_n^*}$

Imaginary.

$f(t) \xleftrightarrow{FS} F_n$.

$$\boxed{F_n^* = -F_{-n}}$$

$F_n = \frac{1}{T}\int^T f(t)\, e^{-jn\omega_0 t} dt$

$F_n^* = \left[\frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t} dt\right]^*$

for imaginary

$x(t) = jt$

$x^*(t) = -jt$.

But $\boxed{f(t) = -f^*(t)}$

$\boxed{x^*(t) = -x(t)}$.

$\therefore F_n^* = \frac{1}{T}\int_0^T -f(t)\, e^{jn\omega_0 t} dt$

$= -\left[\frac{1}{T}\int_0^T f(t)\, e^{-j(-n)\omega_0 t} dt\right]$

$\boxed{F_n^* = -F_{-n}}$

8. Parseval's theorem:-

→ Let us consider two periodic signals $x_1(t)$ & $x_2(t)$ with the equal period 'T'.

→ If the Fourier Series of $\{x(t)\} \leftrightarrow$

$$= \frac{1}{T}\int_0^T x_1(t)\, x_2^*(t) = \frac{1}{T}\int_0^T \left[\sum_{n=-\infty}^{+\infty} a_n e^{jn\omega_0 t}\right]\left\{\sum_{m=-\infty}^{+\infty} b_m e^{jm\omega_0 t}\right\}^* dt$$

$$= \frac{1}{T}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} a_n b_m \int_0^T e^{jn\omega_0 t} \cdot e^{-jm\omega_0 t}\, dt$$

$$= \frac{1}{T}\cdot 0 \quad n \neq m.$$

$$= \sum_{n=-\infty}^{+\infty} a_n b_n^* \Rightarrow n = m.$$

If $x_1(t) = x_2(t) = x(t)$ then.

$$\boxed{\frac{1}{T}\int_0^T |x(t)|^2 = \sum_{n=-\infty}^{+\infty} |C_n|^2}$$

Sub:- Signals & Systems

UNIT-I

SIGNALS and SYSTEMS

Signals:- A function of one & more independent Variables which Contain Some information is called Signal.

Ex:- 1. Electric Voltage (or) Current, Such as radio signal, TV signal, telephone Signal, Computer Signals etc.,
2. pressure Signal, Sound Signal etc., are non-electric Signals.

Systems:- A system is a Set of elements (or) functional block that are Connected together and produces an o/p in response to an i/p Signal.

Ex:- Audio amplifier, attenuator, TV Set, Transmitter, Receiver etc., all are Systems.
Any machine (or) engine are also Systems.

Classification of Signals & Systems:-

→ Signals can be Classified into 2 parts depending upon independent Variables (time)

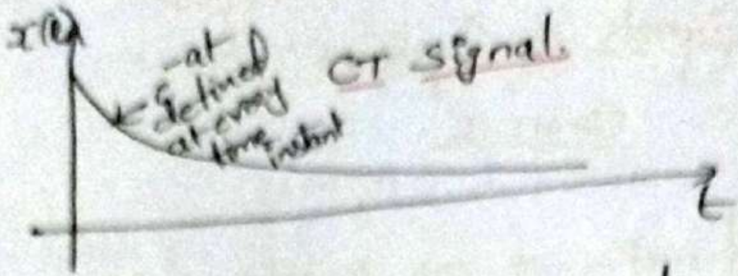
1. Continuous Time (CT) Signals
2. Discrete Time (DT) Signals.

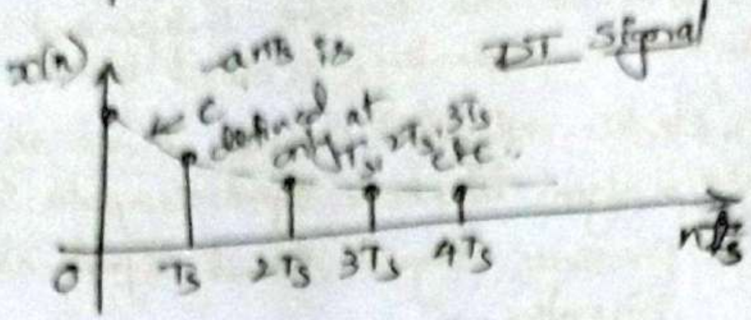
→ CT & DT both Signals are Classified into

1. periodic and Non-periodic/Aperiodic Signals
2. Even and odd Signals
3. Energy & power Signals.
4. Deterministic and Random Signals.

Continuous Time Signals & Discrete Time Signals:-

→ A signal is Said to be Continuous, if a Signal is Varied continuously w.r.t time whereas if a Signal is not Varied continuously w.r.t time (or) a Signal defined only at Specific (or) regular time instants defined as Discrete Time Signal





1. Periodic and Non-periodic Signals:-

Periodic:- A signal is said to be periodic if it repeats regular intervals. ie, $x(t) = x(t+T_0)$

Non-periodic:- Non-periodic signals do not repeat at regular intervals.

ie, $x(t) \neq x(t+T_0)$

Condition for periodicity of CT Signal.

The CT signal repeat after certain period T_0 ie,

$$\boxed{x(t) = x(t+T_0)}$$

Condition for periodicity of DT Signal.

Consider DT Cosine wave $x(n) = \cos(2\pi f_0 n)$

$$x(n+N) = \cos[2\pi f_0(n+N)]$$
$$= \cos(2\pi f_0 n + 2\pi f_0 N)$$

For periodicity $x(n) = x(n+N)$

$$\therefore \cos(2\pi f_0 n) = \cos[2\pi f_0 n + 2\pi f_0 N]$$

It is satisfied only if $2\pi f_0 N$ is integer multiple of 2π ie,

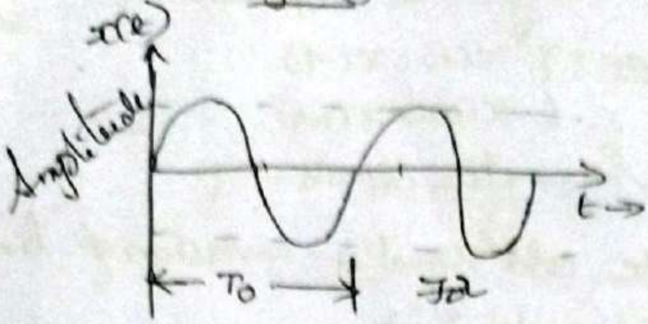
$2\pi f_0 N = 2\pi k$, where k is an integer.

$$\boxed{\therefore f_0 = \frac{k}{N}}$$

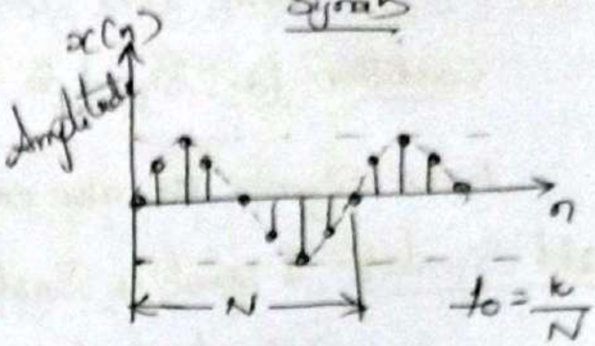
This shows DT signal is periodic only if its frequency is rational (function of two integers).

Examples of CT and DT periodic/Non-periodic Signals:-

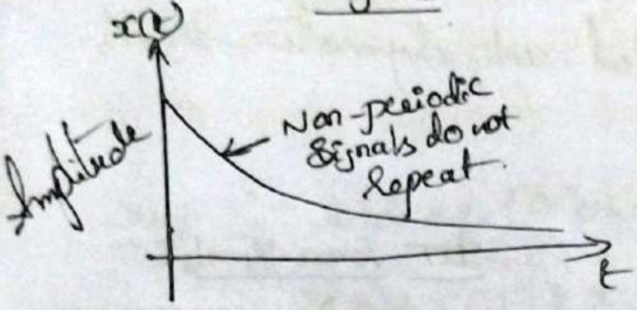
CT periodic Signals

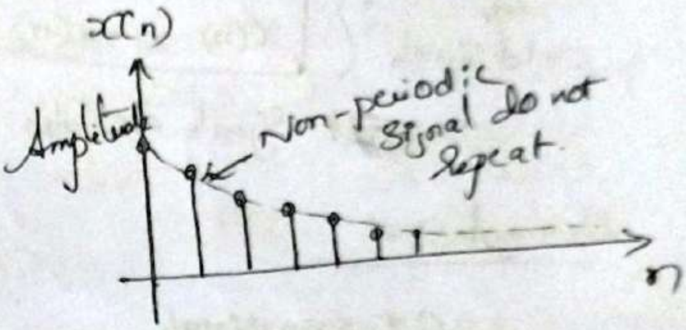


DT periodic Signals



CT Non-periodic Signals





→ Periodicity of Signal $x_1(t) + x_2(t)$

Let us consider that the signal $x(t) = x_1(t) + x_2(t)$

Then $x_1(t)$ will be periodic if,

$$x_1(t) = x_1(t + T_1) = x_1(t + 2T_1) = - - - -$$

(or) $x_1(t) = x_1(t + mT_1)$ Here 'm' is an integer.

ıly. $x_2(t) = x_2(t + T_2) = x_2(t + 2T_2) = - - - -$

(or) $x_2(t) = x_2(t + nT_2)$ Here 'n' is an integer.

Then $x(t)$ will be periodic if,

$$mT_1 = nT_2 = T_0 \quad \text{, Here } T_0 \text{ is period of } x(t)$$

This means 'T_0' is integer multiple of periods of $x_1(t)$ & $x_2(t)$.

$\boxed{\frac{T_1}{T_2} = \frac{n}{m}}$ ie; Ratio of two integers. The period of $x(t)$ will be least common multiple of T_1 & T_2.

This is the condition for periodicity.

→ Periodicity of $x_1(n) + x_2(n) = x(n)$ is periodic if

$\boxed{\frac{N_1}{N_2} = \frac{n}{m}}$ The period of $x(n)$ will be least common multiple of N_1 and N_2

2. Even and odd Signals:-

Even signals:- A Signal is Said to be even Signal if inversion of time axis does not change the amplitude i.e.,

Condition for Signal to be even: $\begin{cases} x(t) = x(-t) \\ x(n) = x(-n) \end{cases}$

Even Signals are also called Symmetric Signals.

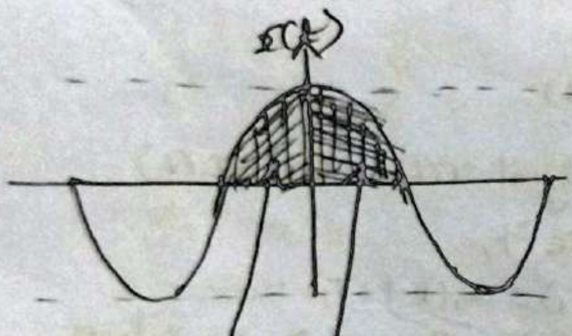
Odd Signals:- A signal is Said to be odd signal if inversion of time axis does change the amplitude i.e.,

Condition for odd signals: $\begin{cases} \boxed{x(t) = -x(-t)} \\ \boxed{x(n) = -x(-n)} \end{cases}$

Odd Signals are also called anti-Symmetric Signals.

Examples:-

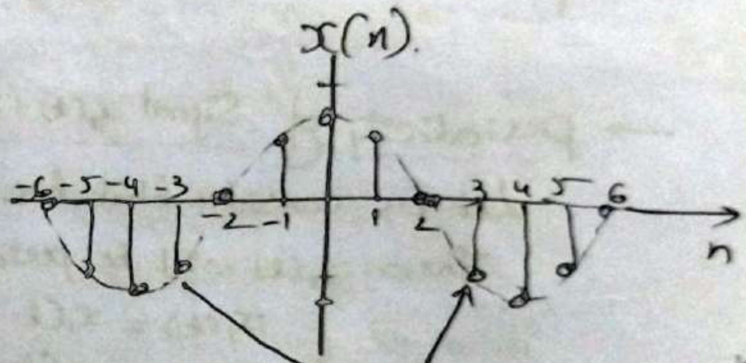
CT Even signals



x(t) is same as x(-t)

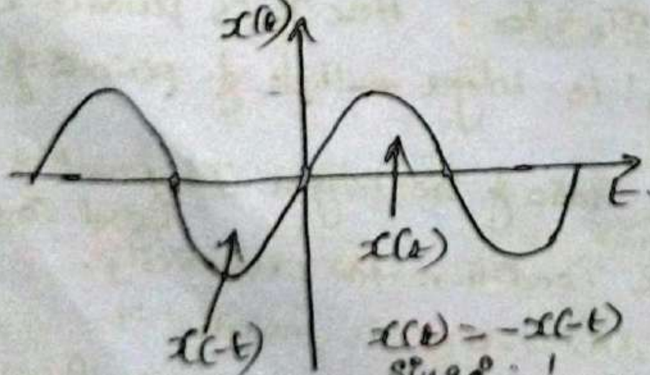
Cosine wave is even

DT Even Signals



$x(3) = x(-3)$

CT odd signals.



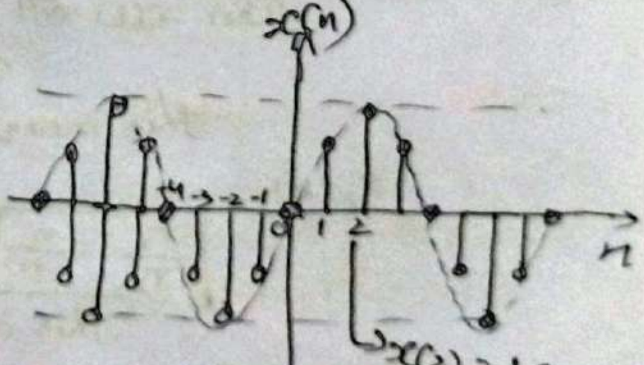
$x(t) = -x(-t)$

$\sin 90^\circ = 1$

$\sin(-90^\circ) = -1$

Sinewave is odd.

DT odd signals



$x(2) = 1$

$x(-2) = -1$

(3)

Representation of Signals in even and odd

Step 1:- Let the Signal be represented into its even and odd parts as

$$x(t) = x_e(t) + x_o(t) \quad \text{--- (1)}$$

$x_e(t) \rightarrow$ Even part of $x(t)$
$x_o(t) \rightarrow$ Odd " " "

Step 2:- Substitute $-t$ for t in above equation.

$$x(-t) = x_e(-t) + x_o(-t) \quad \text{--- (2)}$$

We know even Signal $x(t) = x(-t) \quad \therefore x_e(t) = x_e(-t)$

$||^{ly}$ " Odd signal $x(t) = -x(-t) \quad \therefore x_o(t) = -x_o(-t)$

$\therefore x_o(-t) = -x_o(t)$

Hence above equation will be

$$x(-t) = x_e(t) - x_o(t) \quad \text{--- (3)}$$

After Substitution of
$x(t) = x(-t)$
$x_e(-t) = x_e(t)$
$x_o(-t) = -x_o(t)$

Step 3:- Add eq (1) and (3)

$$x(t) + x(-t) = x_e(t) + x_o(t) + x_e(t) - x_o(t) = 2x_e(t)$$

$$\therefore \boxed{x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]} \quad \text{--- (4)}$$

$||^{ly}$ Subtract (1) from (3)

$$x(-t) - x(t) = x_e(t) - x_o(t) - x_e(t) - x_o(t)$$
$$= -2x_o(t)$$

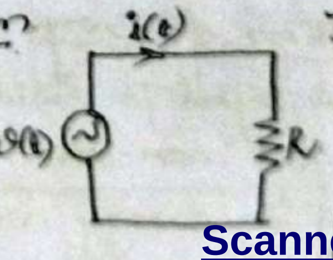
$$\boxed{\therefore x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]}$$

$||^{ly}$ for DT Signals we can write

$$\boxed{\begin{aligned} x_e(n) &= \frac{1}{2}\left[x(n) + x(-n)\right] \\ x_o(n) &= \frac{1}{2}\left[x(n) - x(-n)\right] \end{aligned}}$$

3. Energy and power Signals:-

- Instantaneous power dissipation



The instantaneous power dissipated in load Resistance 'R' will be given as

$$p(t) = \frac{v^2(t)}{R} = i^2(t)R$$

• Normalized power:- In this the power dissipated in $R = 1\Omega$

then $P(t) = \frac{v^2(t)}{1} = i^2(t) \cdot 1$

$$\boxed{P(t) = v^2(t) = i^2(t)}$$

• Significance of normalized power:-

Let $v(t)$ (or) $i(t)$ be denoted by $x(t)$. Then $\boxed{P(t) = x^2(t)}$ shows for voltage as well as current the normalized power is same.

Definition of Energy and power:-

1. Energy for CT and DT Signals.

$$\boxed{\begin{aligned} E &= \text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \text{for CT Signal} \\ E &= \text{Energy} = \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad \text{for DT Signal.} \end{aligned}}$$

2. Power for CT and DT Signals:-

$$\boxed{\begin{aligned} &\text{Power } P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \quad \text{for CT Signal.} \\ &\text{or} \quad P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \quad \text{for DT Signal.} \end{aligned}}$$

If the signal is periodic, then the period (T & N) have finite value. Hence, there is no need to take limits.

• Definition of power Signal:-

A Signal is said to be power Signal if its normalized power is non-zero and finite ie,

$$\boxed{\text{For power Signal, } 0 < P < \infty}$$

• Definition of Energy Signal:-

A signal is said to be energy signal if its total energy is finite and non-zero ie,

$$\boxed{\text{For Energy Signal } 0 < E < \infty}$$

Deterministic and Random Signals:-

- **Deterministic Signal:-**

A Signal which can be represented by any mathematical equation is called Deterministic Signal at any time.

Ex:- Sine wave, $x(t) = \cos \omega t$
$x(n) = \cos 2\pi f n$, exponential pulse, triangular wave, Square pulse etc.,

- **Random Signal:-**

A Signal which cannot be represented by any mathematical equation is called Random Signal.

Ex:- Noise generated in electronic Components, Transmission Channels, Cables etc.,

Examples:-

1. Determine whether the following DT Signals are periodic or not? If periodic, determine fundamental period.

i) $\cos(0.01n\pi)$ ii) $\cos(3\pi n)$ iii) $\sin 3n$

iv) $\cos\frac{2\pi n}{5} + \cos\frac{2\pi n}{7}$ v) $\cos\left(\frac{n}{8}\right)\cos\frac{n\pi}{8}$ vi) $\sin(\pi + 0.2n)$ vii) $e^{\left(\frac{i\pi}{4}\right)n}$

i) $\cos(0.01n\pi)$

Compare with $x(n) = 2\pi f n$

Here $2\pi f n = 0.01 n\pi$

$$f = \frac{0.01}{2} = \frac{1}{200} = \frac{K}{N} \quad \text{So } K = 1 \text{ \& } N = 200.$$

Hence the Signal is periodic with $\boxed{N = 200}$

ii) $\cos(3\pi n)$

Here $2\pi f n = 3\pi n$

$$f = \frac{3}{2} = \frac{k}{N} \quad \text{hence } K = 3, \text{ \& } N = 2$$

$\therefore$ periodic with period $\boxed{N = 2}$

iii) $\sin 3n = x(n)$

So $2\pi f n = 3n$

$f = \frac{3}{2\pi}$ which is not ratio of two integers

Hence this signal is non-periodic.

iv) $\cos\frac{2\pi n}{5} + \cos\frac{2\pi n}{7}$

$x(n) = \cos\frac{2\pi n}{5} + \cos\frac{2\pi n}{7}$

$= x_1(n) + x_2(n)$

$\therefore x_1(n) = \cos\frac{2\pi n}{5} \Rightarrow f_1 = \frac{1}{5} = \frac{k_1}{N_1} \quad \therefore \boxed{N_1 = 5}$

$x_2(n) = \cos\frac{2\pi n}{7} \Rightarrow f_2 = \frac{1}{7} = \frac{k_2}{N_2} \quad \therefore \boxed{N_2 = 7}$

Here, Since $\frac{N_1}{N_2} = \frac{5}{7}$ is the ratio of two integers, the Seque is periodic

The period of $x(n)$ is least common multiple of N_1 and N_2. Here least common multiple of $N_1 = 5$ and $N_2 = 7$

$N_1 N_2 = 7 \times 5 = 35$.

$\therefore$ This Sequence is periodic with $\boxed{N = 35}$

v) $x(n) = \cos\left(\frac{n}{8}\right)\cos\left(\frac{n\pi}{8}\right)$

Here $2\pi f_1 n = \frac{n}{8}$ $\quad f_1 = \frac{1}{16\pi}$ which is not rational

& $2\pi f_2 n = \frac{n\pi}{8}$ $\quad f_2 = \frac{1}{16}$ which is rational.

Hence $x(n)$ is non-periodic since it is the product of periodic and non-periodic signal.

vi) $x(n) = \sin(\pi + 0.2n)$

Compare with $x(n) = \sin(2\pi f n + \theta)$ $\quad \theta \to$ phase Shift

Hence $2\pi f n = 0.2n$ $\quad \therefore \theta = \pi$.

$f = \frac{0.2}{2\pi} = \frac{1}{10\pi}$ = not rational

$\therefore$ $\boxed{\text{non-periodic}}$.

vii)

vii) $x(n) = e^{(j\pi/4)n} = \cos\frac{\pi}{4}n + j\sin\frac{\pi}{4}n$. ⑤

Compare with $x(n) = \cos 2\pi f n + j\sin 2\pi f n$

Here $2\pi f n = \frac{\pi}{4}n \Rightarrow f = \frac{1}{8}$ which is Rational

$= \frac{k}{N} = \frac{1}{8}$ i.e. $K = 1$ is $\boxed{N = 8}$ periodic

$\therefore$ $\boxed{\text{periodic with } N = 8}$

Example 2:- Determine whether the following CT signals are periodic or not? If periodic determine fundamental period.

i) $\cos t + \sin\sqrt{2}t$ ii) $2\cos 100\pi t + 5\sin 50t$ iii) $\cos 100\pi t + \sin 50\pi t$

iv) $2\cos t + 3\cos\frac{t}{3}$ v) $\cos^2(2\pi t)$ vi) e^{-j2t}

i) $x(t) = \cos t + \sin\sqrt{2}t = \cos 2\pi f_1 t + \sin 2\pi f_2 t$.

by comparing $2\pi f_1 = 1$

$f_1 = \frac{1}{2\pi} = \frac{1}{T_1}$ means $\boxed{T_1 = 2\pi}$

Ily $2\pi f_2 = \sqrt{2}$

$f_2 = \frac{\sqrt{2}}{2\pi} = \frac{1}{\sqrt{2}\pi}$ means $\boxed{T_2 = \sqrt{2}\pi}$

The ratio of $\frac{T_1}{T_2} = \frac{2\pi}{\sqrt{2}\pi} = \sqrt{2}$ $\therefore$ ratio of $\frac{T_1}{T_2}$ is not ratio of two integers (i.e. not rational number), $\therefore$ The signal is $\boxed{\text{Non-periodic}}$

ii) $2\cos 100\pi t + 5\sin 50t = x(t)$ —①

let $2\cos 2\pi f_1 t + 5\sin 2\pi f_2 t = x(t)$ —②

By comparing ① & ②.

$2\pi f_1 = 100\pi$

$f_1 = \frac{100}{2\pi} = \frac{50}{\pi}$. $\therefore T_1 = \frac{1}{50}$

Ily $2\pi f_2 t = 50t \Rightarrow f_2 = \frac{50}{2\pi} = \frac{25}{\pi} = \frac{1}{T_2}$ i.e. $T_2 = \frac{\pi}{25}$

Then ratio of two periods $\frac{T_1}{T_2} = \frac{1/50}{\pi/25} = \frac{1}{50} \times \frac{25}{\pi} = \frac{1}{2\pi}$ which is not rational ... $\therefore$ This signal is $\boxed{\text{non-periodic}}$

iii) periodic.

iv) periodic.

$\cos^2(2\pi t) = x(t) = \frac{1+\cos 4\pi t}{2}$

$= \frac{1}{2} + \frac{1}{2}\cos 4\pi t.$ — periodic.

vi) $e^{-j2t} = x(t)$ is periodic with $T = \pi$.

~~for~~ iii) To determine fundamental period:-

I) Multiply T_1 and T_2 by a number such that they are ~~found~~ full integers. Here multiply by 100 i.e.,

$T_1 = \frac{1}{50} \times 100 = 2$ &

$T_2 = \frac{1}{25} \times 100 = 4.$

II) Now least common multiple of $T_1 = 2$ & $T_2 = 4$ is 4

$(2 \times 2 = 4)$

III) Now divide this least common multiple by 100.

It gives fundamental period i.e.,

$$\boxed{T = \frac{4}{100} = \frac{1}{25}}$$

Example 3:- Find and sketch the even and odd components of the followings.

i) $x(n) = e^{-(n/4)}u(n)$ ii) $x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \end{cases}$

iii) $x(t) = \cos^2\left(\frac{\pi t}{2}\right)$ iv) $I_m\left[e^{j(n\pi/4)}\right]$

1. $x(n) = e^{-(n/4)}u(n).$

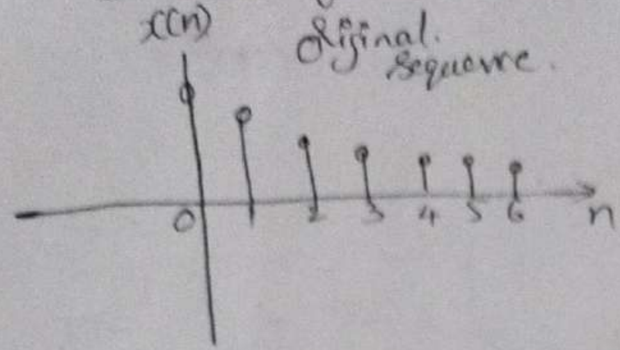
We know $u(n) = \begin{cases} 1 & n \ge 0 \\ 0 & \text{elsewhere} \end{cases}$

so

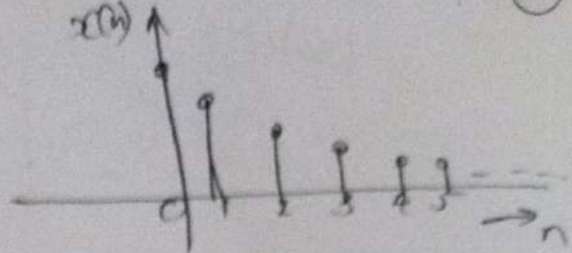
Even part $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$ &

Odd part $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$ at $n = 1, n = 2, \ldots$

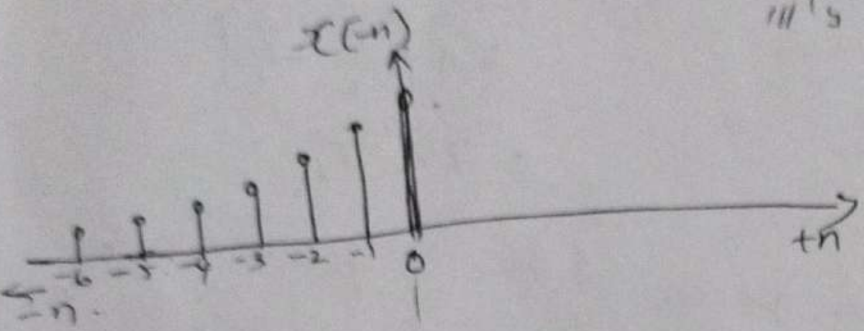
Even part of $x(n)$

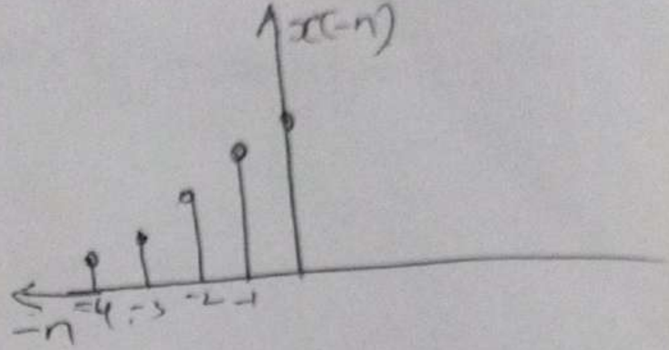


Odd part of $x(n)$

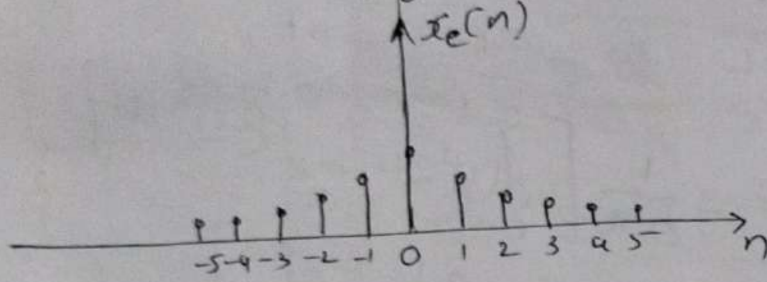


IIIly

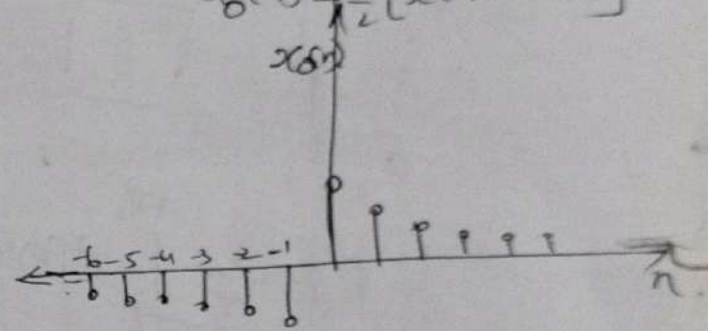




Now we know
$x_e(n)=\frac{1}{2}\left[x(n)+x(-n)\right]$



Here
$x_o(n)=\frac{1}{2}\left[x(n)-x(-n)\right]$



(2) $x(t)=\begin{cases} t & 0\le t\le 1 \\ 2-t & 1\le t\le 2. \end{cases}$

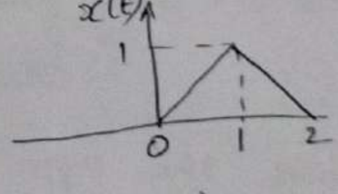
This is a Δle pulse &

$x_e(t)=\frac{1}{2}\left[x(t)+x(-t)\right]$

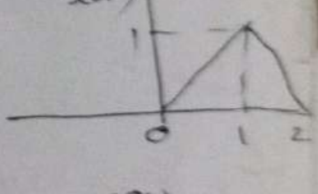
$x_o(t)=\frac{1}{2}\left[x(t)-x(-t)\right]$

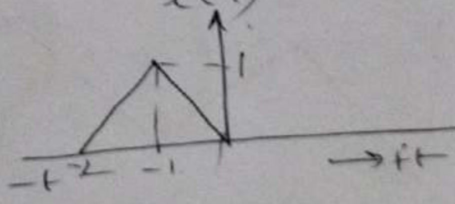
IIIly (3) & (4). Homework

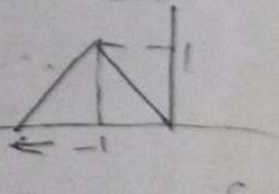
Even part



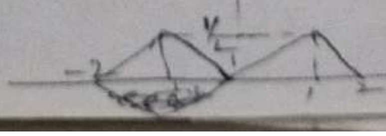
Odd part.





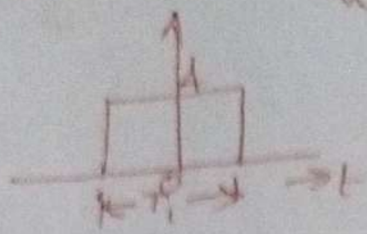


$x_e(t)=\frac{x(t)+x(-t)}{2}$



$x_o(t)=\frac{1}{2}\left[x(t)-x(-t)\right]$

Ex:- What is the total energy of the Rectangular pulse?



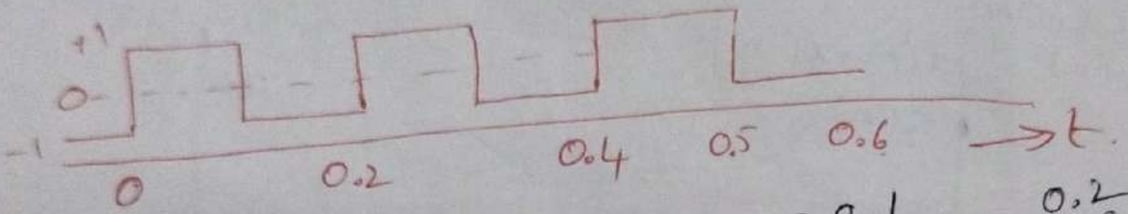
Here $x(t) =$.

$$\text{Energy } E = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-T_1/2}^{+T_1/2} A^2\,dt = A^2 \cdot t \Big|_{-T_1/2}^{+T_1/2}$$

$$= A^2\left[\frac{T_1}{2} + \frac{T_1}{2}\right]$$

$$= \cancel{2} A^2 T_1.$$

Ex:- What is the avg. Power of the Square wave



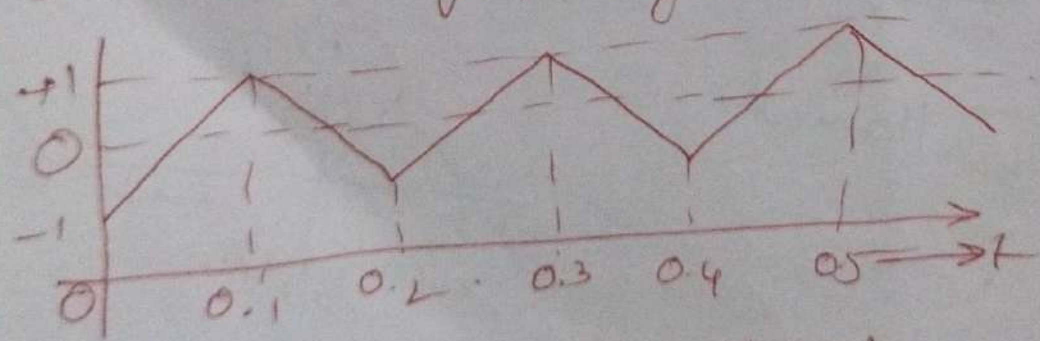
$$P = \frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\,dt = \frac{1}{0.2}\left[\int_0^{0.1} (1)^2\,dt + \int_{0.1}^{0.2} (-1)^2\,dt\right]$$

$$= \frac{1}{0.2}\left[t\Big|_0^{0.1} + t\Big|_{0.1}^{0.2}\right]$$

$$= \frac{1}{0.2}\left[0.1 + (0.2 - 0.1)\right]$$

$$= \frac{1}{\cancel{0.2}} \times \cancel{0.2} = 1.$$

Ex:- Determine the avg. power of the Δ^{le} wave



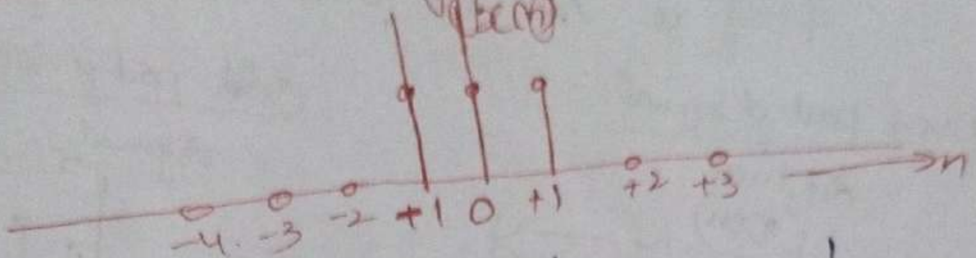
$x(t) = 20t - 1 \qquad 0 \le t \le 0.1$

$x(t) = 3 - 20t \qquad 0.1 \le t \le 0.2$

$$P = \frac{1}{0.2}\left[\int_0^{0.1}(20t-1)^2\,dt + \int_{0.1}^{0.2}(3-20t)^2\,dt\right]$$ (7)

$$\boxed{P = 1/3}$$

Ex:- Determine the total energy of the discrete time signal



$$E = \sum_{n=-\infty}^{\infty} x^2(n) = \sum_{-1}^{1} x^2(n) = \sum_{n=-1}^{1} 1 = 1+1+1 = \underline{3}$$

Previous ~~power~~ Even & ~~Energy~~ Odd signals example problem solution

ii) $x(t) = \cos^2\left(\frac{\pi t}{2}\right) = \frac{1+\cos 2\left(\frac{\pi t}{2}\right)}{2} = \frac{1}{2} + \frac{\cos \pi t}{2}$

$\frac{1}{2}$ is dc.

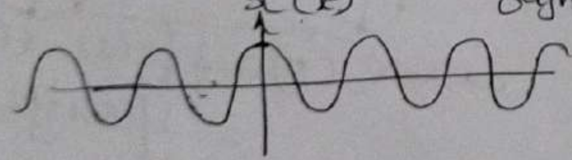
$x(t) = \frac{1}{2} + \frac{1}{2}\cos \pi t$.

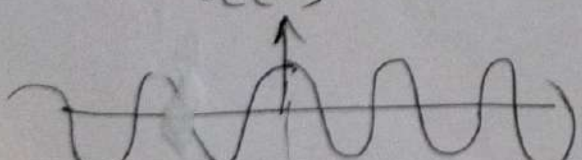
Hence, here $2\pi f t = \pi t$.

$f = \frac{1}{2}$ & $T = 2$.

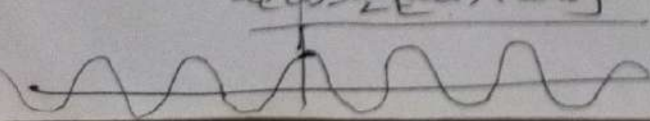
Even and odd parts of signals

Even Signal

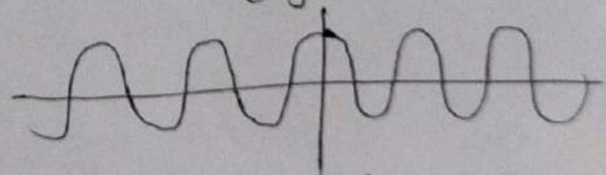


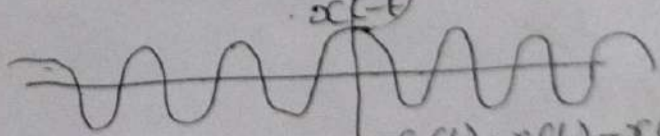


$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

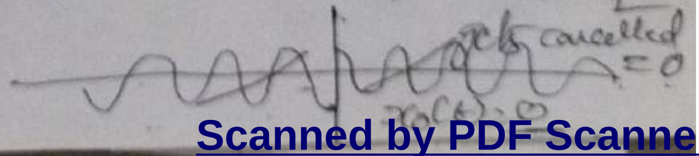


Odd signal





$x_o(t) = \frac{x(t) - x(-t)}{2}$



iv) $x(n) = Im\left[e^{jn\pi/4}\right]$

$= Im\left[\cos\frac{n\pi}{4} + j\sin\frac{n\pi}{4}\right]$

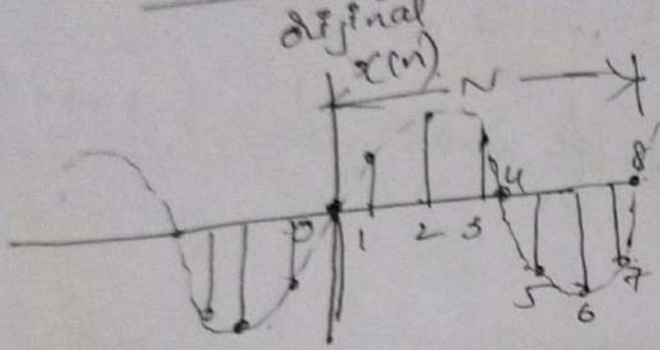
$\therefore \frac{\sin n\pi}{4}$ i.e., Imaginary part of $[\cos\theta + j\sin\theta]$ i.e. $\sin\theta$

So $x(n) = \sin\frac{n\pi}{4}$

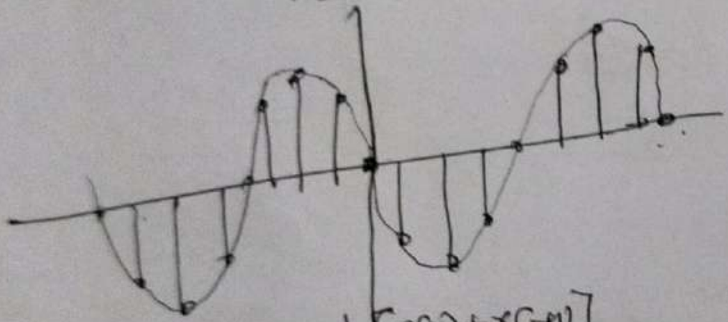
$\therefore 2\pi f n = \frac{n\pi}{4}$

$f = \frac{1}{8}$ i.e. $N = 8$, $k = 1$ means 8 samples are present

Even part of signal

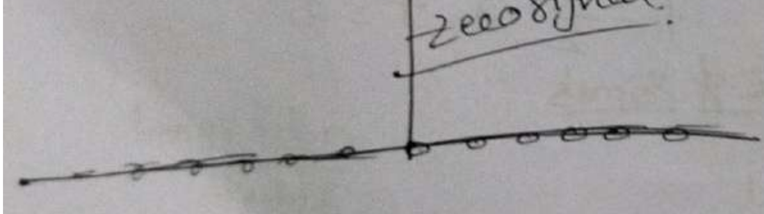


$x(-n)$

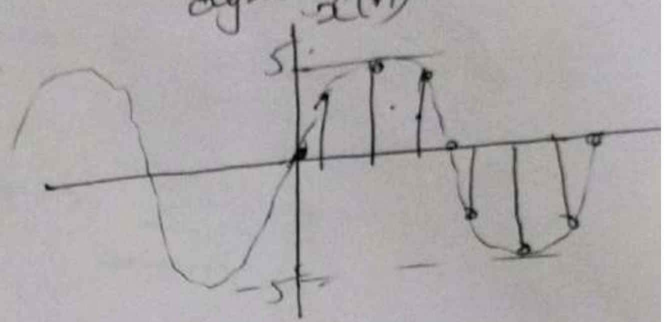


$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$

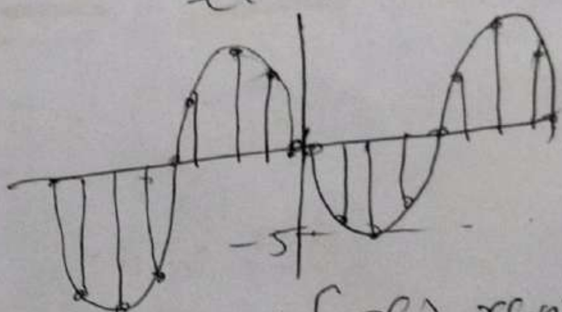
Zero signal



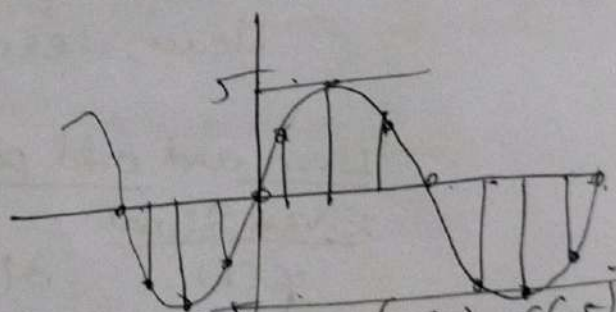
Odd part of signal



$x(-n)$



$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$



Ex:- $x_o(5) = \frac{1}{2}\left[x(5) - x(-5)\right]$

$= \frac{1}{2}\left[5 + 5\right] = 10$

Example:-

(8)

(a) Prove 'The Power of the Energy Signal is '0' over infinite time.

(b) 'The energy of the power signal is infinite over infinite time.'

(a) Power of the energy signal

→ Let $x(t)$ be an energy signal.

→ The power of $x(t)$ is

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \because T \to \infty$$

$$= \lim_{T\to\infty} \frac{1}{T} \cdot E \quad \left(\because \text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2 dt\right)$$

$$= \frac{1}{\infty} \times E = \underline{0}$$

$\boxed{\therefore \text{Power of the energy Signal is '0' over infinite time}}$

(b) The energy of the power Signal is '∞' over infinite time.

→ We know the energy of the Signal is

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

→ Change the integration as $-T/2, T/2$ & take $\lim_{T\to\infty}$. So

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$= \lim_{T\to\infty} \left[T \cdot \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \cdot dt \right]$$

$$= T \cdot \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \cdot dt$$

$$= T \cdot P$$

i.e. $E = \lim_{T\to\infty} T \cdot P$

$\boxed{E = \infty}$

Thus energy of power Signal is ∞ over infinite time.

Ex: Determine whether the following signals are energy signals & power signal.

(i) $x(n) = \left(\frac{1}{2}\right)^n u(n)$ (ii) $x(t) = rect(t/T_0)$ (iii) $x(t) = \cos^2 \omega_0 t$

(iv) $= rect(t/T_0) \cos \omega_0 t$ (v) $x(n) = u(n)$ (vi) $x(t) = A e^{-\alpha t} u(t)$

(i) $x(n) = \left(\frac{1}{2}\right)^n u(n)$

This is not periodic due to $u(n)$.

So directly we can calculate Energy of the signal.

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^n\right]^2 = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}.$$

$$\boxed{E = \frac{4}{3}}$$

Note:- Use $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ for $|a| < 1$ ∴

above eq $= \frac{1}{1-1/4} = \frac{4}{3}$

∴ the energy is finite & nonzero, it is energy signal with $\boxed{E = 4/3}$

If Power signal must satisfy :- periodic & infinite duration then it is power signal, can calculate power of that signal directly.

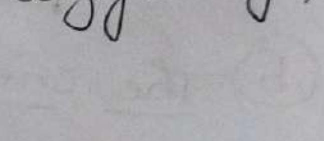
If energy signal must satisfy :- 1) If the signal is periodic but finite duration, then it can be energy signal. So directly we can calculate energy of the signal.

2) If the signal is not periodic, then it must be an energy signal means that directly we can calculate energy directly.

(ii) $x(t) = rect(t/T_0)$

The function $rect(t/T_0) = \begin{cases} 1 & \text{for } -\frac{T_0}{2} \le t \le T_0/2 \\ 0 & \text{elsewhere} \end{cases}$

$rect(t/T_0) =$



It is also not periodic. Hence, it can be energy signal so

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-T_0/2}^{T_0/2} |(1)|^2 dt = t \Big|_{-T_0/2}^{T_0/2}$$

$\therefore E = \left[\frac{T_0}{2} - \left(-T_0/2\right)\right] = T_0.$

The energy is finite and non-zero. So it is an energy Signal. with $\boxed{E = T_0}$

(3) $x(t) = \cos^2 \omega_0 t$

We know that $\cos\omega_0 t$ is periodic Hence, it is. power. Signal..

$$\text{So. } P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 . dt = \lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} \left[\cos^2\omega_0 t\right]^2 . dt$$

$$= \lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \left[\frac{1+\cos 2\omega_0 t}{2}\right]^2 . dt$$

$$= \lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} \frac{1}{4}\left[1+\cos 2\omega_0 t\right]^2 . dt = \lim_{T_0\to\infty} \frac{1}{4T_0} \int_{-T_0/2}^{+T_0/2} \left[1+\cos^2 2\omega_0 t + 2\cos 2\omega_0 t\right]. dt$$

$$= \lim_{T_0\to\infty} \frac{1}{4T_0} \int_{-T_0/2}^{+T_0/2} \left\{1 + \left(\frac{1+\cos 4\omega_0 t}{2}\right) + 2\cos 2\omega_0 t\right\} . dt$$

$$= \lim_{T_0\to\infty} \frac{1}{4T_0} \int_{-T_0/2}^{T_0/2} \left[\frac{3}{2} + \frac{\cos 4\omega_0 t}{2} + 2\cos 2\omega_0 t\right] . dt$$

$$= \lim_{T_0\to\infty} \frac{1}{4T_0} \left\{ \int_{-T_0/2}^{T_0/2} \frac{3}{2} . dt + \underbrace{\int_{-T_0/2}^{T_0/2} \frac{\cos 4\omega_0 t}{2} . dt}_{\text{'0' } \because \text{ it is integration of Cosine wave over full cycle}} + \underbrace{\int_{-T_0/2}^{T_0/2} 2\cos 2\omega_0 t \, dt}_{\text{'0'. } \because \text{ it is also integration of cosine wave over full cycle}} \right\}$$

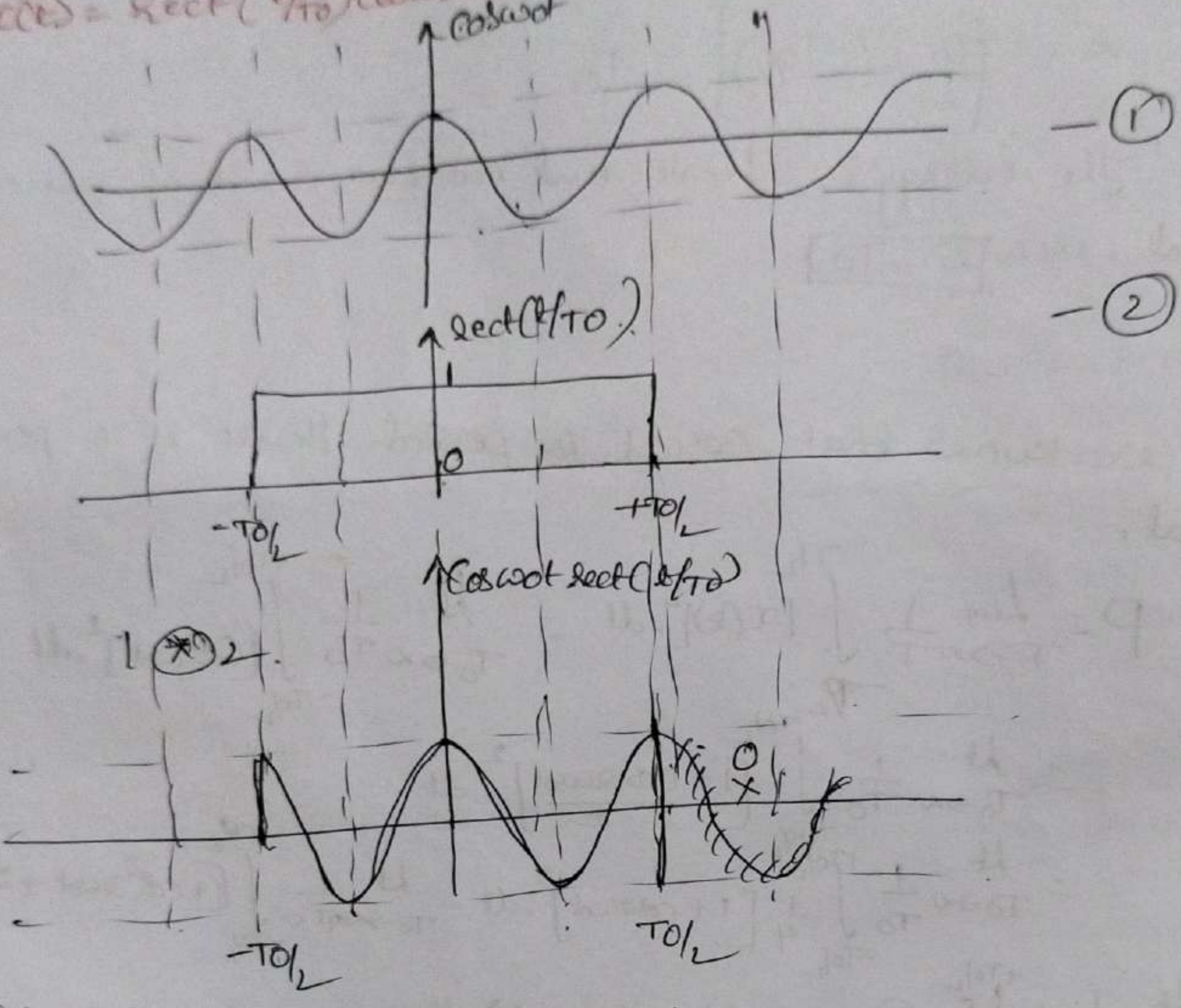
$$\therefore P = \lim_{T_0\to\infty} \frac{3}{8T_0} \, t \Big|_{-T_0/2}^{+T_0/2}$$

$$P = \lim_{T_0 \to \infty} \frac{1}{T_0} \cdot \frac{3}{8}\left[\frac{T_0}{2} + \frac{T_0}{2}\right]$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \times \frac{3}{8} \times \cancel{T_0} = 3/8.$$

$\therefore \boxed{P = 3/8}$ The power of the signal is finite & non zero hence power signal with power $\boxed{P = 3/8}$

(v) $x(t) = \text{Rect}(t/T_0)\cos\omega_0 t$



Note:- Originally $\cos\omega_0 t \rightarrow$ is periodic.
but rect(t/T_0) is not period.
$\therefore$ multiplied signal is not-periodic $\therefore$ it is an Energy signal
hence, we can calculate energy of the signal only

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt = \int_{-T_0/2}^{+T_0/2} [\cos\omega_0 t]^2 \cdot dt = \int_{-T_0/2}^{+T_0/2} \left[\frac{1+\cos 2\omega_0 t}{2}\right] \cdot dt$$

$$= \frac{1}{2}\int_{-T_0/2}^{T_0/2} dt + \frac{1}{2}\int_{-T_0/2}^{T_0/2} \cos 2\omega_0 t \cdot dt = \frac{1}{2} T_0 = T_0/2 + 0.$$

Elementary Signals:-

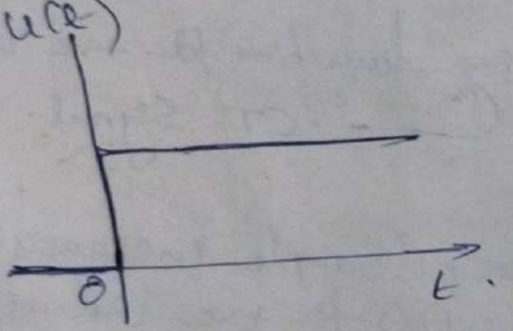
Standard Signals are used for the analysis of Systems. They are

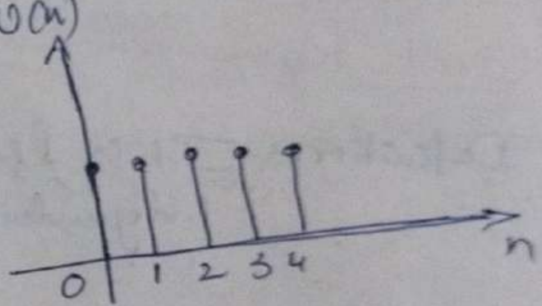
Unit-Step Function:-

Definition:- This Signal has amplitude of '1' for +ve values of independent Variable, and it has '0' amplitude for -ve Values of independent Variables.

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \quad \text{for CT Signal}$$

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \quad \text{for DT Signal}$$





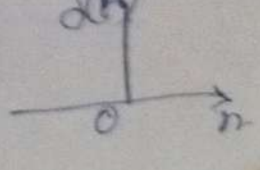
2) Unit impulse(δ). Delta function:-

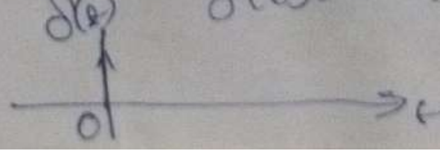
Definition:- Area under unit impulse approaches '1' as its width approaches zero. Thus it has zero value everywhere except $t = 0$.

Representation:- CT: $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$ and $t \to 0$

$\delta(t) = 0$ for $t \neq 0$.

DT: $\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$





Significance :- $\delta(n)$ is the sampled version of $\delta(t)$.
Difference is Area under $\delta(t) = 1$, Amplitude of $\delta(n) = 1$.

→ Unit impulse & Unit Sample functions are used determine unit impulse response of the System

1) Shifting property :- $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$

2) Replication property :- $\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$

(or) $x(t) * \delta(t) = x(t)$
Means that convolution of any function with delta function leaves that function unchanged.

⇒ Unit Ramp Function :-

Definition :- It is linearly growing function for +ve values of independent Variable. —— CT Signal.

The amplitude of every Sample increases linearly with its number (n) for +ve Values of 'n'.

$$r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } < 0 \end{cases}$$

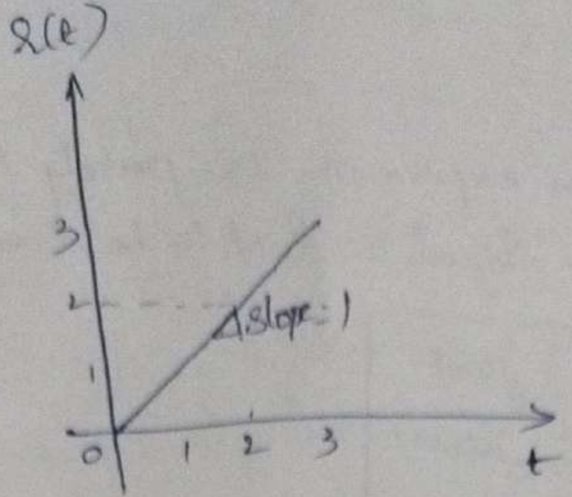
$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

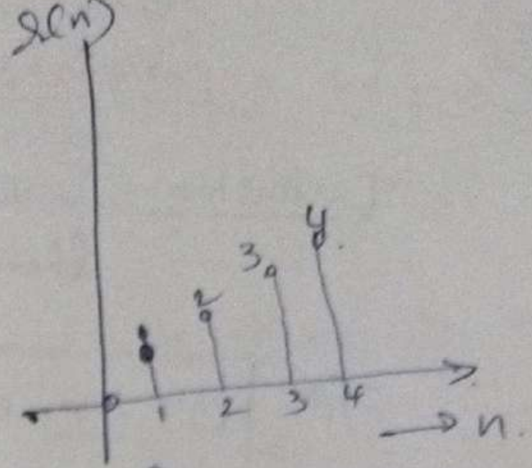
$$\boxed{r(t) = t\,u(t)}$$

$$\boxed{r(n) = n\,u(n)}$$

∵ $u(t) = 1$ for $t \geq 0$ &
$u(t) = 0$ for $t < 0$.

Since $u(n) = 1$ for $n \geq 0$ &
$u(n) = 0$ for $n < 0$
$r(n) = \{\underset{\uparrow}{0}, 1, 2, 3, 4, 5 \ldots\}$





→ The Ramp function indicates linear relationship

→ It indicates constant current charging of the capacitor.

4) Complex Exponential & Sinusoidal Signals :-

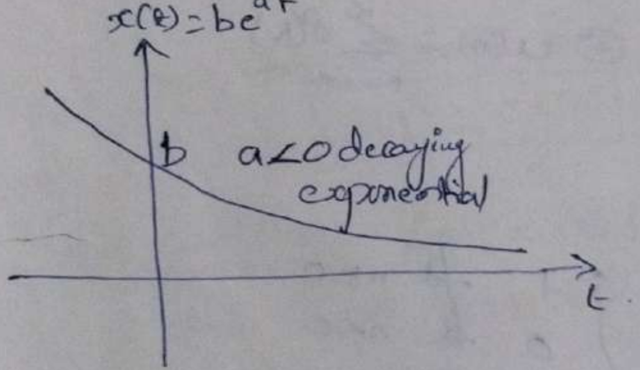
Exponential :- It is exponentially growing or decaying signal.

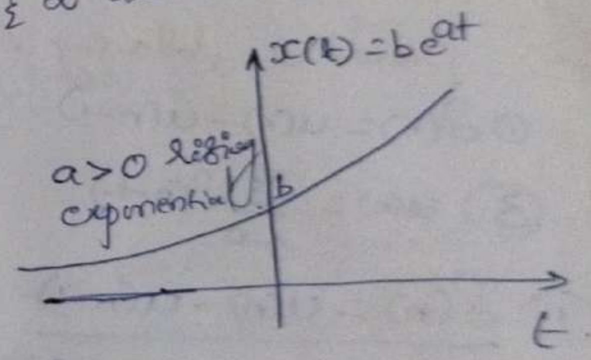
$x(t) = b e^{at}$ b & a are real. —— (CT Signal)

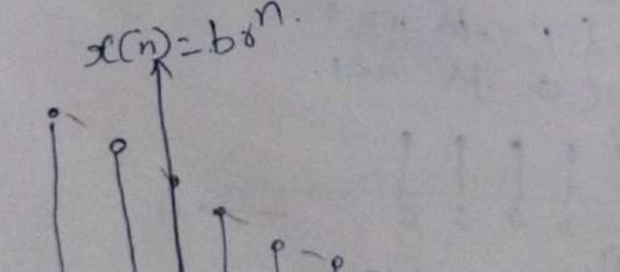
$x(n) = b r^{n}$ if $r = e^{\alpha}$ then —— (DT Signal)

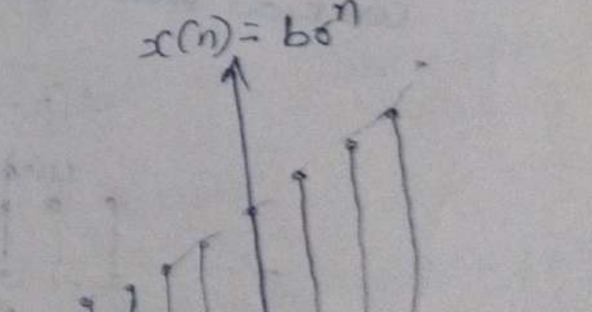
$x(n) = b \cdot e^{\alpha n}$ Here b & α are real.

CT Signal









Complex exponential:-

Definition:- When an exponent is purely imag. then the Signal is Said to be Complex e

for $\boxed{\begin{array}{l} CT \rightarrow x(t) = e^{j\omega_0 t} \\ DT \rightarrow x(n) = e^{j\omega_0 n} \end{array}}$

The Sinusoidal Signal is given by

$CT \rightarrow x(t) = \cos(\omega_0 t + \phi)$

$DT \rightarrow x(n) = \cos(\omega_0 n + \phi)$

Complex exponential Can be written in terms of Sinusoidal Signals as

$CT = e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$

$DT = e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$

Ex:- prove the following relationships b/w functions

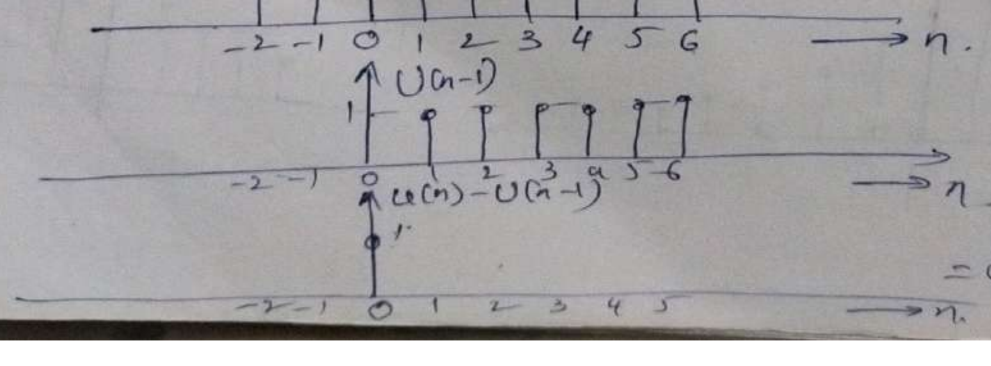
① $\delta(n) = u(n) - u(n-1)$ ② $u(n) = \sum_{k=-\infty}^{\infty} \delta(k)$

③ $u(n) = \sum_{k=0}^{\infty} \delta(n-k)$

① $\delta(n) = u(n) - u(n-1)$

We know that $u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$

$u(n-1) = \begin{cases} 1 & \text{for } n \geq 1 \\ 0 & \text{for } n < 1 \end{cases}$



$\therefore u(n) - u(n-1) = \delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$

Basic operations on Signals:-

The operations performed on signals can be classified into two types.

1. Dependent Variables
2. Independent Variables.

I. Dependent Variables:-

a) **Amplitude Scaling:-**

→ Let $x(t)$ denote a CT signal. Then $y(t)$ is

$\boxed{y(t) = C\,x(t)}$ where $C \rightarrow$ the scaling factor.

So. o/p $y(t)$ = multiplication of scaling factor C with $x(t)$.

→ For discrete time signals, we write

$\boxed{y(n) = C\,x(n)}$

(b) **Addition:-** Let $x_1(t)$ & $x_2(t)$ denote a pair of CT signals.

The signal $y(t)$ obtained by the addition of $x_1(t)$ and $x_2(t)$ is defined as.

$\boxed{y(t) = x_1(t) + x_2(t)}$ — Ex:- Audio mixer / voice signals

For DT signals $\boxed{y[n] = x_1[n] + x_2[n]}$

(c) **Multiplication:-** Let $x_1(t)$ and $x_2(t)$ denote a pair of CT signal.

→ The signal $y(t)$ resulting from multiplication of $x_1(t)$ & $x_2(t)$ & is defined by

$\boxed{y(t) = x_1(t)\,x_2(t)}$

For discrete time Signals, we write

$$\boxed{y[n] = x_1[n]\,x_2[n]}$$

(d) Differentiation:-

Let us consider a CT Signal $x(t)$ with respect to time is defined by

$$\boxed{y(t) = \frac{d}{dt}x(t)}$$

(e) Integration:- The CT Signal $x(t)$ integrable w.r.t. time 't' is defined by

$$y(t) = \int_{-\infty}^{\infty} x(\tau).d\tau$$

τ is an integral Variable

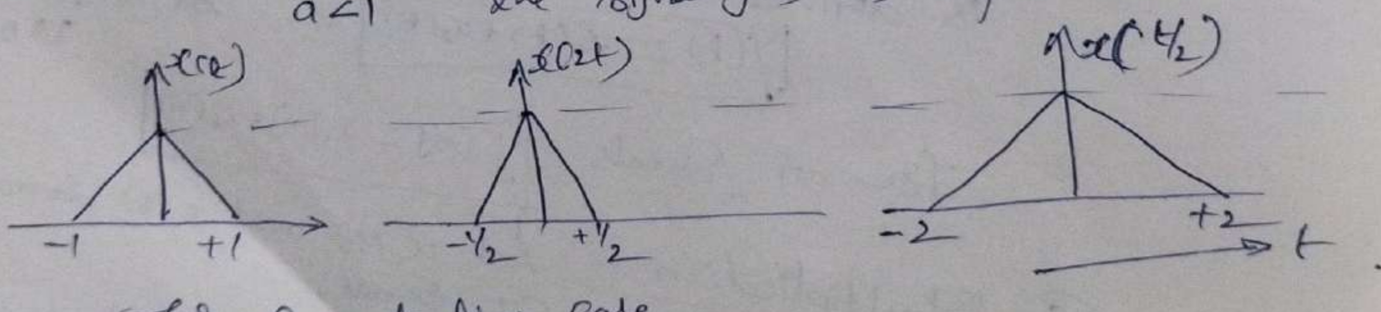
~~Note~~

II. Independent Variables:-

(1) Time Scaling:- Let $x(t)$ be a CT Signal, then Signal $y(t)$ is obtained by Scaling the independent Variable 't' by a factor "a" & is defined by

$$\boxed{y(t) = x(at)}$$

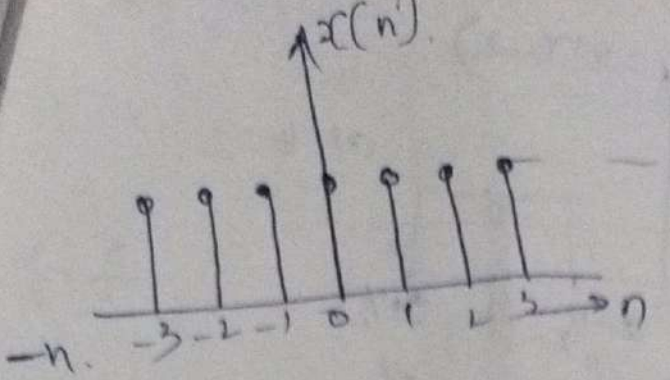
if $a > 1$ the Signal $y(t)$ is compressed & if $a < 1$ the Signal $y(t)$ is expanded.

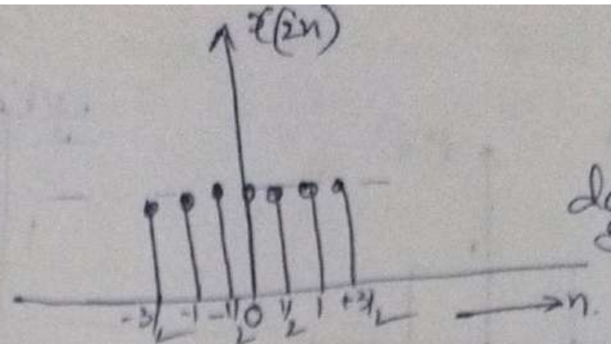


For discrete time case

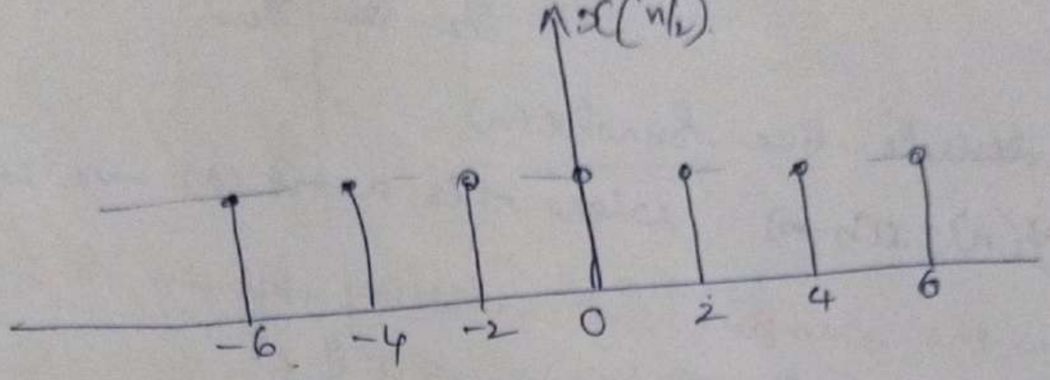
$y(n) = x(an)$ $a > 0$ down Sampling

$= x(an)$ $a < 0$ up sampling





down Sampling



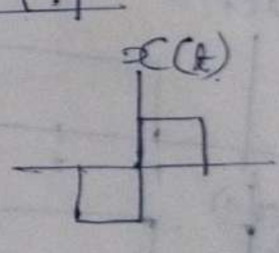
(2) Reflection:-

Let $x(t)$ be a CT Signal, then the $y(t)$ is obtained by replacing t by $-t$ by reflecting of a CT Sinal $x(t)$

$\therefore \boxed{y(t) = x(-t)}$ Reflected/folded/Mirrored

for even Signal $x(t) = x(-t) =$

for odd Signal $y(t) = x(t) = -x(-t) =$



11ly for discrete - time Signal

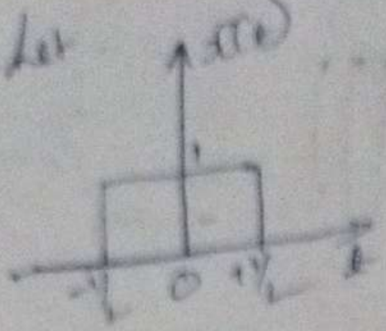
$$\boxed{y(n) = x(-n)}$$

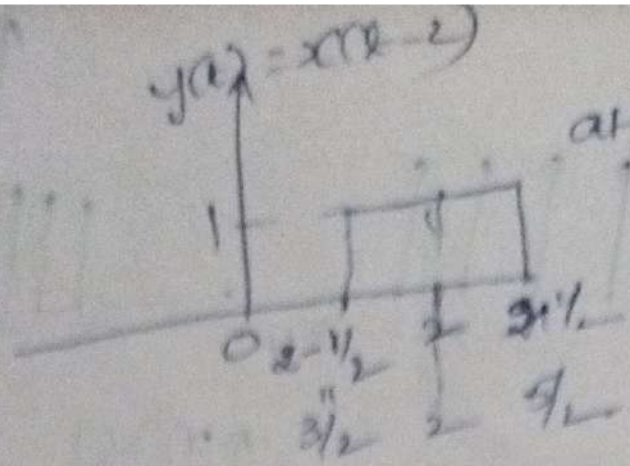
(3) Time shifting:- The time-shifted version of $x(t)$ is defined by

$\boxed{y(t) = x(t - t_0)}$ $t_0 \rightarrow$ time shift towards right

If $t_0 > 0$, $y(t)$ is shifted version of $x(t)$ towards the right
$t_0 < 0$ $y(t)$ is shifted version of $x(t)$ towards the left.

Let





For a discrete-time signal $x(n)$

$y(n) = x(n-m)$ where m is a +ve (or) -ve integer.

Ex:- Draw the waveforms represented by following step function

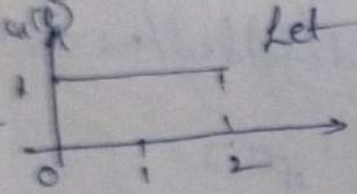
① $f_1(t) = 2u(t-1)$ ② $f_2(t) = -2u(t-2)$

③ $f(t) = f_1(t) + f_2(t)$ ④ $f(t) = f_1(t) - f_2(t)$

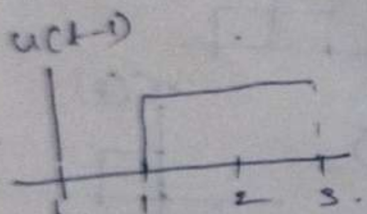
① $f_1(t) = 2u(t-1)$:-

Let

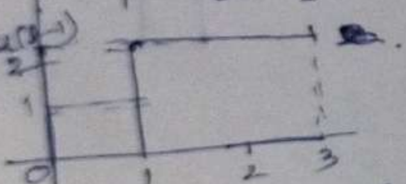
step1



Step2

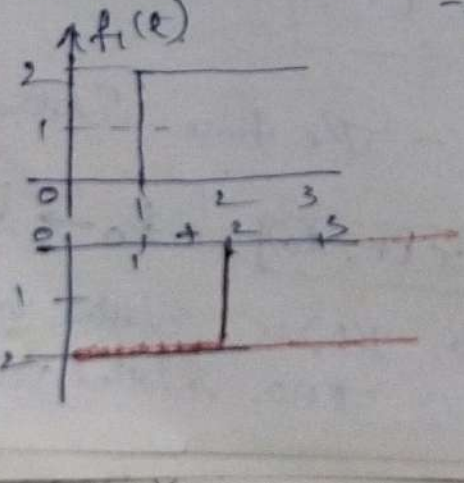


Step3

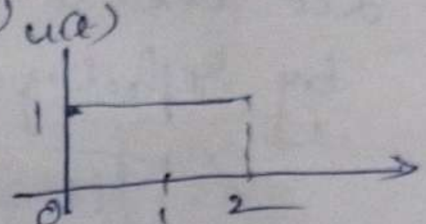


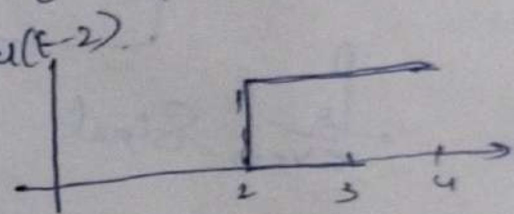
$f_1(t) = 2u(t-1)$

③ $f(t) = f_1(t) + f_2(t)$

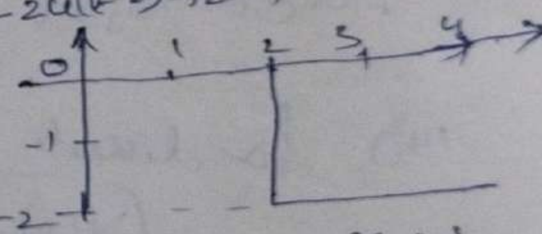


11ly ②



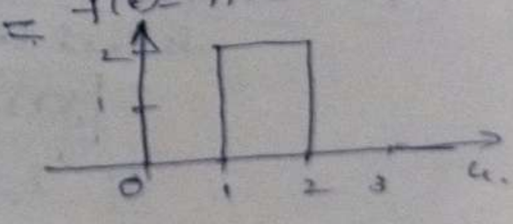


$-2u(t-2) = f_2(t)$



$f_2(t) = -2u(t-2)$

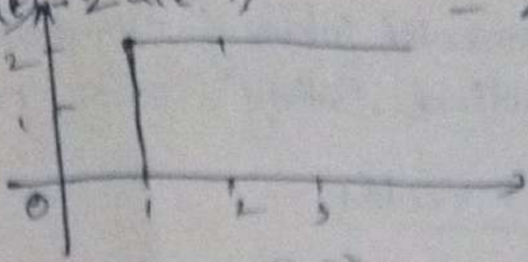
$f(t) = f_1(t) + f_2(t)$



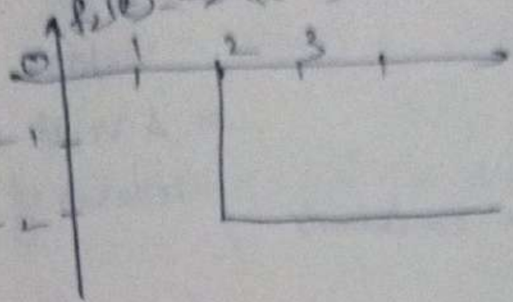
(14)

(4) $f(t) = f_1(t) - f_2(t) = 2u(t-1) - (-2u(t-2)) = 2u(t-1) + 2u(t-2)$

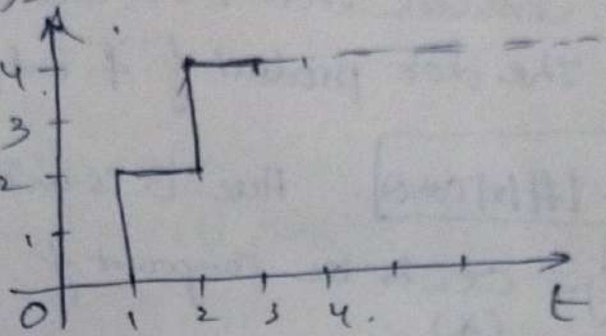
$f_1(t) = 2u(t-1)$



$f_2(t) = -2u(t-2)$



$f(t) = 2u(t-1) + 2u(t-2)$



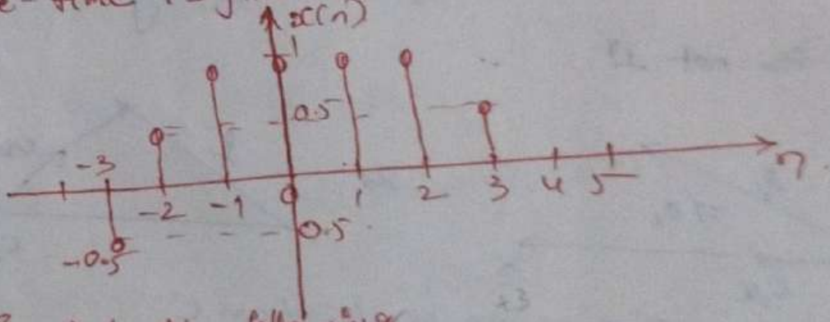
(2) Draw the waveforms of the following

(1) $f_1(t) = Ku(t-1)$ — delayed

(2) $f_2(t) = u(2-t)$ — folded

(3) $f(t) = f_1(t) - f_2(t)$

(3) A discrete-time Signal.



Sketch the following

(1) $x(n-3)$ (2) $x(3-n)$ (3) $x(2n)$ (4) $x(n)u(3-n)$ (5) $x[(n-1)^2]$

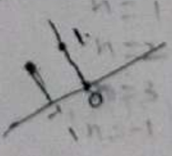
$x(-3) = -0.5$
$x(-2) = 0.5$
$x(-1) = 1$
$x(0) = 1$
$x(1) = 1$
$x(2) = 1$
$x(3) = 0.5$

at $n=0$. $x(0) = 1$

at $n=1$ $x(2) = 1$ $x(-1) = 1$

at $n=2$. $x(4) = 1$ $x(-2) = 0.5$

at $n=3$ $x(3) = 0.5$ $x(-3) = -0.5$



at $n=0$ $x(3)$ $n=4 \to x(-1)$
$n=1$ $x(2)$ $n=5$ $x(-2)$
$n=2$ $x(1)$ $n=6$ $x(-3)$
$n=3$ $x(0)$
$n=-1$ $x(4)$
$n=-2$ $x(5)$
$n=-3$ $x(6)$
$n=-4$ $x(7)$

$x(2) \neq x(-2)$
$x(2) = -x(-2)$
$x(3) \neq x(-3)$
$x(3) = -x(-3)$

at $n=0$ $y(0) = x$
$n=1$ $y(1) = x$
$n=2$ $y(2) = x$
$n=3$ $y(3) = x$
$n=-1$ $y(-1) = x$
$n=-2$ $y(-2) = x$

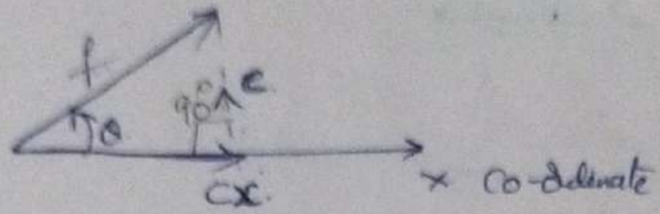
Analogy b/w Vectors & Signals:-

→ Signals can be represented in terms of orthogonal functions.
→ These specific functions satisfy specific properties.

Orthogonality Concept in Vectors:-

→ All the signals are basically vectors.

→ A vector can be represented in terms of its co-ordinates.



Where c is component.

For ex:- Consider vector 'f' & another vector x.
The dot product of 'f' and x is

$$\boxed{\bar{f}\cdot\bar{x} = |f||x|\cos\theta}$$

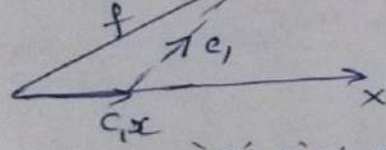
Here 'θ' is an angle b/w 'f' and x

→ In the fig., cx is the component of vector 'f' along 'x'.
(or)
cx is the projection of f on 'x'.
Here 'f' can be expressed as vector addition as

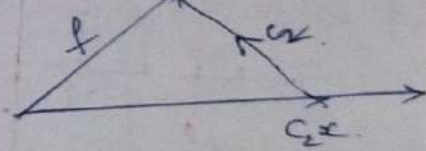
$$\boxed{f = cx + e}$$

e → error vector
'e' is min only when it is $\perp^r$ to x.

→ If 'e' is not $\perp^r$.



&



'e_1' & 'e_2' are greater than 'e'

→ The component of 'f' along x is cx.

So $\boxed{c|x| = |f|\cos\theta}$

Multiplying bothsides by $|x|$,

$\therefore c|x|^2 = |f||x|\cos\theta$

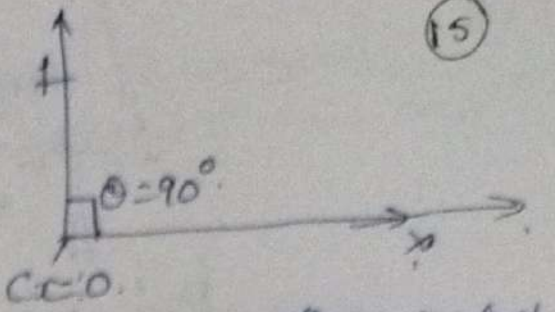
$c = \frac{|f||x|\cos\theta}{|x||x|} = \frac{\bar{f}\cdot\bar{x}}{|x||x|}$ from Ist above eq

$$\boxed{c = \frac{f\cdot x}{x\cdot x}}$$

when $\theta = 90°$.

$\bar{f} \cdot \bar{x} = |f||x| \cos 90° = 0$.

i.e. $\boxed{\bar{f} \cdot \bar{x} = 0}$



→ Vectors f & x are said to be orthogonal if their · (dot) product is '0'.

→ Vectors are orthogonal if they are mutually ⊥r.

Orthogonality in Signals:-

→ Let us consider a signal f(t) to be represented in terms of x(t) over an interval t_1 and t_2.

$f(t) = C\,x(t) + e(t)$

$e(t) = f(t) - C\,x(t)$ shows if e(t) is min then f(t) can be approximated in terms of x(t).

→ for min e(t). min energy of e(t) & mean square value of e(t) gives appropriate measure.

∴ for min energy of e(t), representation of f(t) in x(t) will be better. Energy of e(t) will be

$$E_e = \int_{t_1}^{t_2} |e^2(t)| \cdot dt$$

Then mean square value of e(t) is

$$\overline{e^2(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} e^2(t) \cdot dt = \frac{E_e}{t_2 - t_1}$$

$$\overline{e^2(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f(t) - C\,x(t)]^2 dt$$

we know for min e(t) i.e. min E_e, $\frac{dE_e}{dC} = 0$

$$\therefore \frac{d}{dC}\left[\int_{t_1}^{t_2} [f(t) - C\,x(t)]^2 \right] dt = 0$$

$$\underbrace{\frac{d}{dC}\int_{t_1}^{t_2} f^2(t) \cdot dt}_{0} + \frac{d}{dC}\int_{t_1}^{t_2} [C^2 x^2(t)]\, dt - \frac{d}{dC}\int [2f(t)\, C\,x(t)]\, dt = 0$$

$$+ \; 2C\int_{t_1}^{t_2} x^2(t)\,dt - 2\int_{t_1}^{t_2} f(t)\,x(t) \cdot dt = 0.$$

$$\therefore\ C = \frac{\int_{t_1}^{t_2} f(t)\,x(t).dt}{\int_{t_1}^{t_2} x^2(t).dt}.$$ This is $\parallel$lar to $C = \frac{f \cdot x}{|x|^2}$

above $\int_{t_1}^{t_2} x^2(t)\,dt$ gives Energy of $x(t)$. so Cannot be $=0$.

Hence numerator must be 0 to make $C = 0$

if $C=0$ there is no component of $f(t)$ in terms of $x(t)$.

Then $f(t)$ & $x(t)$ are said to be orthogonal over an interval $[t_1, t_2]$ ie.,

For orthogonality $\int_{t_1}^{t_2} f(t)\,x(t).dt = 0$.

$|||^{ly}$ if $f(t)$ & $x(t)$ are complex signals, then they are orthogonal over an interval $[t_1, t_2]$ if,

$$\int_{t_1}^{t_2} f(t)\,x^*(t).dt = 0 \quad (or) \quad \int_{t_1}^{t_2} f^*(t)\,x(t)\,dt = 0.$$

$x^*(t)$ = Complex conjugate of $x(t)$

$f^*(t)$ = " " $f(t)$

Ex:- Show that the following signals are orthogonal over an interval $[0, 1]$.

$f(t) = 1$

$x(t) = \sqrt{3}(1-2t)$

We know, if $f(t)$ & $x(t)$ are orthogonal then $\int_{t_1}^{t_2} f(t)x(t) = 0$

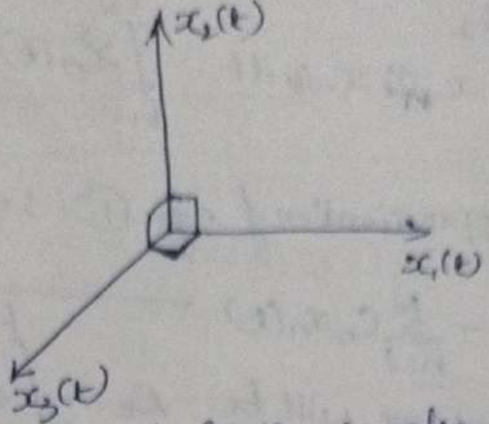
$$\Rightarrow \int_{t_1}^{t_2} \sqrt{3}(1-2t).dt = \sqrt{3}\int_0^1 dt - 2\sqrt{3}\int_0^1 t.dt$$

$$= \sqrt{3}\times 1 - 2\sqrt{3}.\frac{t^2}{2}\Big|_0^1$$

$$= \sqrt{3} - \sqrt{3}.1 = \underline{\underline{0}}$$

Orthogonal Signal Space:-

→ Let $x_1(t)$, $x_2(t)$ & $x_3(t)$ be orthogonal to each other. means that '3'-signals are mutually ⊥r to each other.



Three Dimensional Signal Space.

→ Such signal space is also called orthogonal signal space.

→ This signal space is used to represent any signal lying in that space.

→ If there are 'N' such mutually orthogonal signals $x_1(t), x_2(t), x_3(t) - - - - - x_N(t)$ then they form N-dimensional orthogonal signal space.

→ Any signal $f(t)$ can be represented in this N-dimensional signal space.

Signal Approximation using orthogonal functions:-

→ Let us consider the set of signals which are mutually orthogonal over an interval $[t_1, t_2]$. These signals can represents any signal $f(t)$ as,

$$f(t) \cong c_1 x_1(t) + c_2 x_2(t) + - - - - - - c_N x_N(t)$$

$$\boxed{f(t) \cong \sum_{n=1}^{N} c_n x_n(t)} \quad \text{—— } 1$$

→ In the above equation any two signals, $x_m(t)$ and $x_n(t)$ are orthogonal over an interval $[t_1, t_2]$

$$\int_{t_1}^{t_2} x_m(t) x_n(t) \cdot dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases} \quad \text{—— } 2$$

$$\int_{t_1}^{t_2} [f(n)]^2 \cdot dt = \text{Energy}$$

$$\int_{t_1}^{t_2} [f(n)]^2 dt = \text{Energy}$$

→ In the above equation, observe that any two different signals are orthogonal.

when $m=n$, $\int_{t_1}^{t_2} x_m(t)\, x_n(t)\, dt = \int_{t_1}^{t_2} x_n^2(t).dt = E_n$ Gives energy of the signal.

→ Error $e(t)$ in the approximation of eq ① is given as from previous
we know $e(t) = f(t) - \sum_{n=1}^{N} C_n x_n(t)$ ——— ③

Then error energy will be $E_e = \int_{t_1}^{t_2} e^2(t).dt$.

$$= \int_{t_1}^{t_2} \left[f(t) - \sum_{n=1}^{N} C_n x_n(t)\right]^2 .dt \quad (4)$$

Here E_e is the function of $C_1, C_2, C_3 --- C_N$.

Hence E_e will be min w.r.to C_i, if

$$\frac{\partial E_e}{\partial C_i} = 0.$$

i.e. $\frac{\partial}{\partial C_i}\left\{\int_{t_1}^{t_2} \left[f(t) - \sum_{n=1}^{N} C_n x_n(t)\right]^2 .dt\right\} = 0.$

$\frac{\partial}{\partial C_i} \int_{t_1}^{t_2} f^2(t).dt + \frac{\partial}{\partial C_i} \int_{t_1}^{t_2} C_n^2 x_n^2(t).dt - \frac{\partial}{\partial C_i} \int_{t_1}^{t_2} \sum_{n=1}^{N} 2 f(t) C_n x_n(t).dt = 0$

becomes (0)

$\therefore \frac{\partial}{\partial C_i} \int_{t_1}^{t_2} \sum_{n=1}^{N} C_n^2 x_n^2(t).dt - f(t) \int_{t_1}^{t_2} \sum_{n=1}^{N} C_n x_n(t).dt = 0$

i.e. $\frac{\partial}{\partial C_i} \int_{t_1}^{t_2} \sum_{i=n=1}^{N} C_i^2 x_i^2(t)dt - f(t) 2 \int_{t_1}^{t_2} \sum_{i=n=1}^{N} x_i(t).dt = 0$

i.e. $2C_i \int_{t_1}^{t_2} x_i^2(t).dt - f(t) 2 \int_{t_1}^{t_2} x_i(t).dt = 0$ if $i = n = 1,2,3 --- N$.

$$\boxed{\therefore C_i = \frac{\int_{t_1}^{t_2} f(t)\, x_i(t).dt}{\int_{t_1}^{t_2} x_i(t).x_i(t).dt} \quad i = 1,2,3 --- N} \quad — (5)$$

We know that

$$\int_{t_1}^{t_2} x_i^2(t).dt = E_i \text{ ie., energy.. Hence,}$$

Hence eq (5) becomes.

$$\boxed{c_i = \frac{1}{E_i}\int_{t_1}^{t_2} f(t)\, x_i(t).dt} \quad i = 1, 2, 3 \ldots N \quad (6)$$

Mean Square Error:-

From fundamentals

if

$$e(t) = f(t) - \sum_{n=1}^{N} c_n x_n(t) \text{ then.}$$

$$\overline{e^2(t)} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[f(t) - \sum_{n=1}^{N} c_n x_n(t)\right]^2 . dt.$$

$$\boxed{\overline{e^2(t)} = \frac{E_e}{t_2 - t_1}} \text{ Ans} \qquad \text{Where } E_e = \int_{t_1}^{t_2} |e^2(t)| . dt$$

Proof:-

$$\text{We know. } E_e = \int_{t_1}^{t_2}\left[f(t) - \sum_{n=1}^{N} c_n x_n(t)\right]^2 . dt$$

$$= \int_{t_1}^{t_2} f^2(t).dt + \int_{t_1}^{t_2}\sum_{n=1}^{N} c_n^2 x_n^2(t).dt - \int_{t_1}^{t_2}\sum_{n=1}^{N} 2f(t)\, c_n x_n(t)\, dt.$$

Interchanging the order of summation & integration

$$E_e = \int_{t_1}^{t_2} f^2(t).dt - 2\sum_{n=1}^{N} c_n \int_{t_1}^{t_2} f(t)x_n(t)dt + \sum_{n=1}^{N} c_n^2 \underbrace{\int_{t_1}^{t_2} x_n^2(t).dt}$$

$$= \int_{t_1}^{t_2} f^2(t).dt - 2\sum_{n=1}^{N} c_n . c_n E_n + \sum_{n=1}^{N} c_n^2 E_n \quad \left(\because c_n = \frac{\int_{t_1}^{t_2} f(t)x_n(t)dt}{E_n}, \; E_n = \int_{t_1}^{t_2} x_n^2(t)dt\right)$$

$$= \int_{t_1}^{t_2} f^2(t)\, dt - 2\sum_{n=1}^{N} c_n^2 E_n + \sum_{n=1}^{N} c_n^2 E_n.$$

7

$$\therefore E_e = \int_{t_1}^{t_2} f^2(t).dt - \sum_{n=1}^{N} C_n^2 E_n.$$

Then Mean Square error & error energy are related as

$$\overline{e^2(t)} = \frac{E_e}{t_2-t_1} = \frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2} f^2(t).dt - \sum_{n=1}^{N} C_n^2 E_n\right] \quad (8)$$

→ In (8) $C_n^2 E_n$ is always +ve. Hence error energy can be reduced if number of terms 'N' used for representation are increased.

Ideally $E_e \to 0$ as $N \to \infty$. Under this condition, the orthogonal signal set is said to be complete.

Closed (or) Complete Set of Orthogonal Functions:-

→ From above $\overline{e^2(t)}$ (mean square error) → 0 as $C_n^2 E_n$ no. of terms are made ∞.

→ Under this condition, we can write

$$0 = \frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2} f^2(t).dt - \sum_{n=1}^{\infty} C_n^2 E_n\right] \text{ with } \overline{e^2(t)} = 0 \text{ as } N = \infty.$$

$$\int_{t_1}^{t_2} f^2(t).dt = \sum_{n=1}^{\infty} C_n^2 E_n \quad (9)$$

with $N \to \infty$,

$$f(t) = \sum_{n=1}^{N} C_n x_n(t) \text{ becomes } = \sum_{n=1}^{\infty} C_n x_n(t) \quad (10)$$

Here $x_1(t), x_2(t) - - - x_N(t)$ are mutually orthogonal functions.

→ It is said to be complete & closed set if there exists no function $p(t)$ for which

$$\int_{t_1}^{t_2} p(t)\, x_n(t)\, dt = 0 \quad \text{for } n = 1, 2 - - -$$

→ If $p(t)$ exists & above integral is zero, then obviously p(t) must be a member of set $[x_n(t)]$

→ For the set of mutually orthogonal signal $x_n(t)$ over an interval (t_1, t_2), (18)

$$\int_{t_1}^{t_2} x_m(t)\, x_n(t)\,dt = \begin{cases} 0 & m \neq n \\ E_n & m = n. \end{cases}$$

→ For this complete set, the function f(t) is expressed as

$$f(t) = C_1 x_1(t) + C_2 x_2(t) + \cdots$$

Here $C_i = \dfrac{\int_{t_1}^{t_2} f(t)\, x_i(t)\, dt}{\int_{t_1}^{t_2} x_i^2(t)\, dt}$

$$\boxed{C_i = \frac{1}{E_i} \int_{t_1}^{t_2} f(t)\, x_i(t)\, dt}$$

→ Eq (18) is called generalised Fourier series. ~~And~~ is

→ The set of $x_n(t)$ is called Orthogonal basis functions.

Orthogonality in Complex Functions:–

we know $\int_{t_1}^{t_2} x_m(t)\, x_n^*(t) = \int_{t_1}^{t_2} x_n^*(t)\, x_m(t)\, dt = \begin{cases} 0 & m \neq n \\ E_n & \text{for } m = n \end{cases}$ (1)

(2)

Then f(t) can be expressed as

$$\boxed{f(t) = \sum_{n=1}^{\infty} C_n x_n(t)}$$

in the similar fashion of eq (6) of above

$n = 1, 2 \ldots$ (3)

Where C_n is the

$$\boxed{C_n = \frac{1}{E_n} \int_{t_1}^{t_2} f(t)\, x_n^*(t)\, dt}$$ (4)

$$\boxed{\text{Where } E_n = \int_{t_1}^{t_2} x_n(t) \cdot x_n^*(t)\, dt}$$

Ex:- Show that the Signal set,

$\{1, \cos\omega_0 t, \cos 2\omega_0 t, \ldots, \cos n\omega_0 t, \sin\omega_0 t, \sin 2\omega_0 t, \ldots, \sin n\omega_0 t, \ldots\}$

are orthogonal over an interval $T_0 = \frac{2\pi}{\omega_0}$

I) To Check orthogonality of cosine waves;-

Consider the orthogonality of $\cos n\omega_0 t$ and $\cos m\omega_0 t$ i.e.,

$$\int_t^{t+T_0} \cos n\omega_0 t \cos m\omega_0 t \cdot dt.$$

$\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$

$\cos(A+B) = \cos A \cdot \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

We know $\cos x \cdot \cos y = \left[\frac{\cos(x+y) + \cos(x-y)}{2}\right]$

$$\therefore \int_t^{t+T_0} \cos n\omega_0 t \cos m\omega_0 t \cdot dt = \frac{1}{2}\int_t^{t+T_0}\left[\cos(n-m)\omega_0 t + \cos(n+m)\omega_0 t\right]$$

$$= \frac{1}{2}\int_t^{t+T_0} \cos(n-m)\omega_0 t \, dt + \frac{1}{2}\int_t^{t+T_0} \cos(n+m)\omega_0 t \cdot dt.$$

For $n \neq m$, the integration of $(n-m)$ full cycles of cosine wave is taken over one period. Hence integration will be '0'.

IIIly The integration of $(n+m)$ full cycles of cosine wave over one period. Hence its integration is also '0'.

For $n = m$, $\cos(n-m)\omega_0 t = 1$

$\therefore$ above equation

$$\int_t^{t+T_0} \cos n\omega_0 t \cdot \cos m\omega_0 t \, dt = \frac{T_0}{2}$$

Thus we obtain.

$$\boxed{\int_t^{t+T_0} \cos n\omega_0 t \cos m\omega_0 t \, dt = \begin{cases} 0 & \text{for } n \neq m. \\ \frac{T_0}{2} & \text{for } n = m. \end{cases}}$$

This equation shows that two cosine waves of the given set are orthogonal over one period.

To check orthogonality of Sinewaves:-

$$\int_t^{t+T_0} \sin n\omega_0 t . \sin m\omega_0 t . dt$$

$\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$

$$= \int_t^{t+T_0} \cos(n-m)\omega_0 t . dt - \int_t^{t+T_0} \cos(n+m)\omega_0 t . dt$$

(lar to the previous

if $n \neq m$ the integration of full $(n-m)$ & $(n+m)$ cycles over one period $= 0$

if $n = m$ the $\int_t^{t+T_0} \cos(n-m)\omega_0 t . dt = \frac{T_0}{2}$

Thus we obtained

$$\boxed{\int_t^{t+T_0} \sin n\omega_0 t . \sin m\omega_0 t . dt = \begin{cases} 0 & n \neq m \\ T_0/2 & n = m \end{cases}}$$

III. To check the orthogonality of Sine & Cosine wave

$\sin A \cos B = \left[\frac{\sin(A+B)}{2} + \frac{\sin(A-B)}{2}\right]$

$$\int_t^{t+T_0} \sin n\omega_0 t \cos m\omega_0 t . dt = \int_t^{t+T_0} \frac{\sin(n-m)\omega_0 t}{2} dt + \int_t^{t+T_0} \frac{\sin(n+m)\omega_0 t}{2} . dt$$

$= 0$. $\because$ integration of $(n-m)$ & $(n+m)$ full cycles of Sinewave over one period will be '0'.

Hence both the above integrals will be '0' i.e.,

$$\boxed{\int_t^{t+T_0} \sin n\omega_0 t \cos m\omega_0 t . dt = 0 \text{ for all values of } n \,\&\, m}$$

Trigonometric Fourier Series:—

We know that any function $f(t)$ can be

$$\boxed{f(t) = \sum_{n=1}^{\infty} C_n x_n(t)} \quad \text{—— (1)}$$

where $x_n(t) \rightarrow$ orthogonal signal set

→ eq (1) is called generalized Fourier Series.

→ We have seen that the set

$\{1, \cos\omega_0 t, \cos 2\omega_0 t, \ldots \cos n\omega_0 t, \ldots \sin\omega_0 t, \sin 2\omega_0 t, \ldots \sin n\omega_0 t, \ldots\}$ ($n=0$, $n=1$, $n=2$)

is orthogonal over the period T_0.

→ Here $\omega_0 \rightarrow$ angular frequency (or) fundamental frequency

& $n\omega_0 \Rightarrow n^{th}$ harmonic

→ At $n=0$, DC component i.e; $\cos 0\omega_0 t = 1$.

→ This signal set consists of sine & cosine terms.

→ This signal set is called Trigonometric set.

Hence it is called

→ For that set we can write

$$f(t) = a_0 + a_1\cos\omega_0 t + a_2\cos 2\omega_0 t + \ldots a_n\cos n\omega_0 t + b_1\sin\omega_0 t + b_2\sin 2\omega_0 t + \ldots b_n\sin n\omega_0 t.$$

$$\boxed{f(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos n\omega_0 t + \sum_{n=1}^{\infty} b_n\sin n\omega_0 t}$$

where $a_n \rightarrow$ component of $\cos n\omega_0 t$

$b_n \rightarrow$ component of $\sin n\omega_0 t$

$$\therefore a_n = \frac{\int_t^{t+T_0} f(t)\, x_n(t)\,dt}{\int_t^{t+T_0} x_n^2(t)\,dt}$$

$x_n(t) = \cos n\omega_0 t$

$$= \frac{\int_t^{t+T_0} f(t)\cos n\omega_0 t\,dt}{\int_t^{t+T_0} \cos^2 n\omega_0 t\,dt}$$

(20)

$$a_n = \frac{\int_t^{t+T_0} f(t)\cos n\omega_0 t \, dt}{\int_t^{t+T_0} \left[\frac{1+\cos 2n\omega_0 t}{2}\right] dt} = \frac{\int_t^{t+T_0} f(t)\cos n\omega_0 t \, dt}{\int_t^{t+T_0} \frac{1}{2}\, dt + \underbrace{\int_t^{t+T_0} \frac{\cos 2n\omega_0 t \, dt}{2}}_{0}}$$

'0' ∵ integration over period $T_0 = 0$.

$$a_n = \frac{\int_t^{t+T_0} f(t)\cdot\cos n\omega_0 t\cdot dt}{\frac{t}{2}\Big|_t^{t+T_0}} = \frac{\int_t^{t+T_0} f(t)\cdot\cos n\omega_0 t\cdot dt}{\frac{t+T_0}{2} - \frac{t}{2}}.$$

$$= \frac{2}{T_0}\int_t^{t+T_0} f(t)\cdot\cos n\omega_0 t \, dt$$

$$\boxed{\therefore a_n = \frac{2}{T_0}\int_t^{t+T_0} f(t)\cdot\cos n\omega_0 t\cdot dt.}$$

Ily.

$$b_n = \frac{\int_t^{t+T_0} f(t)\cdot\sin n\omega_0 t}{\int_t^{t+T_0} \sin^2 n\omega_0 t\cdot dt} = \frac{\int_t^{t+T_0} f(t)\sin n\omega_0 t}{\int_t^{t+T_0} \frac{1}{2}\cdot dt - \underbrace{\int_t^{t+T_0} \cos 2n\omega_0 t \, dt}_{'0'}}$$

$$\boxed{\therefore b_n = \frac{2}{T_0}\int_t^{t+T_0} f(t)\sin n\omega_0 t\cdot dt}$$

Value of a_0 is obtained if $n=0$ in the a_n: then

$$\boxed{a_0 = \frac{2}{T_0}\int_t^{t+T_0} f(t)\cdot\cos 0\,\omega t\cdot dt = \frac{2}{T_0}\int_t^{t+T_0} f(t)\cdot dt}.$$

Thus any periodic signal $x(t)$ can be expressed in Fourier Series as

Trigonometric Fourier Series

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t).$$

where

$$a_0 = \frac{2}{T_0}\int_{t}^{t+T_0} x(t).dt$$

$$a_n = \frac{2}{T_0}\int_{t}^{t+T_0} x(t)\cos(n\omega_0 t)\,dt.$$

$$b_n = \frac{2}{T_0}\int_{t}^{t+T_0} x(t)\sin(n\omega_0 t)\,dt.$$

Necessity of Fourier Series:-

→ FS is used to analyse periodic Signals.

→ The harmonic constant of the Signals is analyzed with the help of FS.

→ FS can be developed for C.T as well as D.T.

Types of Fourier Series:-

Depending upon the representation '3' types

1. Trigonometric F.S.
2. Compact Trigonometric F.S (or) polar Fourier Series.
3. Exponential Fourier series.

$x(t) = e^{jn\omega_0 t}$

Exponential form of Fourier Series:-

Definition:-

The exponential form of Fourier Series of a periodic Signal $x(t)$ with period 'T' is defined as

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}}$$

$n\omega_0$ → Harmonic frequency

C_n → Fourier co-efficient of exponential form of FS.

Then the Fourier Co-efficients C_n can be evaluated using the following equation.

(21)

$$C_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)e^{-jn\omega_0 t}.dt$$

(or)

$$C_n = \frac{1}{T}\int_{0}^{T} x(t)e^{-jn\omega_0 t}.dt.$$

Negative freq i.e. ($-\infty$ to 0):—

→ In this the $x(t)$ has complex exponential harmonic components for both +ve & -ve frequencies.

→ When the +ve & -ve Complex exponential Components of Same harmonic are added, it gives rise to real Sine (or) cosine Signals.

→ Alternatively, when the real Sine (or) Cosine Signal has to be represented in terms of Complex exponentials then a Signal with -ve frequency is reqd.

→ This -ve freq is only for Mathematical representation of real signals in terms of complex exponentials signals.

Derivation:-

The Fourier Co-efficient C_n is given by

$$\boxed{C_n = \frac{1}{T}\int_0^T x(t).e^{-jn\omega_0 t}.dt}$$

Proof:- Let us Consider

$$x(t) = C_0 + C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t} + \cdots +$$
$$C_{-k} e^{-jk\omega_0 t} + \cdots C_{-2} e^{-j2\omega_0 t} + C_{-1} e^{-j\omega_0 t}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t}$$ ∴ By multiply $e^{-jk\omega_0 t}$ on both sides

$$x(t)e^{-jk\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}.e^{-jk\omega_0 t}$$

Let us integrate the above equation b/w limits 0 to T with

$$\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt = \dots + \int_0^T c_{-2}\, e^{-j2\omega_0 t} \cdot e^{-jk\omega_0 t} dt + \int_0^T c_{-1}\, e^{-j\omega_0 t} \cdot e^{-jk\omega_0 t} dt + \int_0^T c_0\, e^{-jk\omega_0 t} dt + \int_0^T c_1\, e^{j\omega_0 t} \cdot e^{-jk\omega_0 t} dt + \dots + \int_0^T c_k \cdot dt + \dots$$

$$\boxed{\int_0^T x(t)\, e^{-jk\omega_0 t} \cdot dt = \int_0^T c_k \cdot dt}$$

(∵ all the integrals = 0

ex: $\int_0^T c_1\, e^{j\omega_0 t} \cdot e^{-j\omega_0 t}$ at $k=1$.

$= \int_0^T c_1 \cdot dt$. III^{ly} $\int_0^T c_2 \cdot dt$.

$$\int_0^T x(t) \cdot e^{-jk\omega_0 t} = c_k \cdot t \Big|_0^T = c_k T$$

$$\boxed{\therefore c_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t} \cdot dt}$$ for k^{th} co-efficient

for n^{th} coefficient $$\boxed{C_n = \frac{1}{T} \int_0^T x(t) \cdot e^{-jn\omega_0 t} \cdot dt}$$

Relation b/w Fourier Co-efficients of Trigonometric & Exponential form:-

They are.

$C_0 = \frac{a_0}{2}$ $\quad C_n = \frac{a_n - jb_n}{2}$ for $n = 1, 2, 3, 4 \dots$ $\quad C_{-n} = \frac{a_n + jb_n}{2}$ for $n = -1, -2, -3 \dots$

$$\therefore |C_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

Proof:-

Consider the trigonometric form of F.S.

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left[\frac{e^{+j\theta} + e^{-j\theta}}{2}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{e^{+j\theta} - e^{-j\theta}}{2j}\right]$$

$$\cos\theta = \frac{e^{+j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{+j\theta} - e^{-j\theta}}{2j}$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n e^{jn\omega_0 t}}{2} + \sum_{n=1}^{\infty} \frac{a_n e^{-jn\omega_0 t}}{2}$$

$$+ \sum_{n=1}^{\infty} \frac{b_n e^{jn\omega_0 t}}{2j} - \sum_{n=1}^{\infty} \frac{b_n e^{-jn\omega_0 t}}{2j}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{(a_n - jb_n)}{2} e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{(a_n + jb_n)}{2} e^{-jn\omega_0 t}$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{-n=1}^{\infty} C_{-n} e^{jn\omega_0 t}$$

So $C_0 = \frac{a_0}{2}$, $C_n = \left[\frac{a_n - jb_n}{2}\right]$, $C_{-n} = \left[\frac{a_n + jb_n}{2}\right]$

$n = 1, 2, 3 \ldots \infty$ $\quad -n = -1, -2, -3 \ldots \ldots -\infty$

$$\therefore |C_n| = \sqrt{\frac{a_n^2 + b_n^2}{2}} = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

Frequency Spectrum (or Line Spectrum) of Periodic Continuous Time Signals:—

→ Let $x(t)$ be a periodic continuous time signal. Now, exponential form of Fourier Series of $x(t)$ is

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t}$$

→ The Fourier Co-efficient C_n is a complex quantity and so it can be expressed in the polar form as

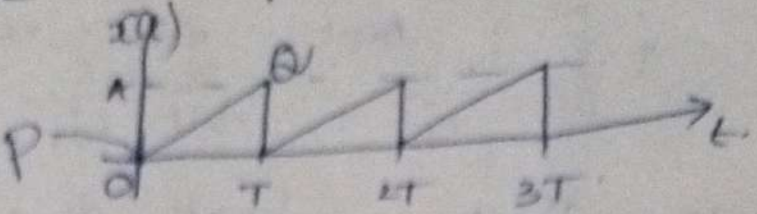
$$C_n = |C_n| \angle C_n \qquad |C_n| \rightarrow \text{Magnitude}$$
$$\angle C_n \rightarrow \text{phase of } C_n$$

→ The plot of harmonic magnitude / phase of a signal versus 'n' (or harmonic frequency $n\omega_0$) is called Frequency Spectrum (or Line Spectrum).

The plot of harmonic magnitude Vs 'n' ($n\omega_0$) is called magnitude (Line) Spectrum

The plot of phase of a signal Vs "n" ($n\omega_0$) is called phase (line) Spectrum.

Let us conside a Ramp waveform.



Slope / Equation of straight line :- $\dfrac{y-y_1}{y_1-y_2} = \dfrac{x-x_1}{x_1-x_2}$

Here $y = x(t)$, $x = t$

coordinates of point P $[\overset{x_1}{0}, \overset{y_1}{0}] = [t_1, x(t_1)]$

" Q $[\overset{x_2}{T}, \overset{y_2}{A}] = [t_2, x(t_2)]$

$\therefore$ Equation of Straightline $= \dfrac{\overset{y}{x(t)} - \overset{y_1}{0}}{\underset{y_1}{0} - \underset{y_2}{A}} = \dfrac{\overset{x - x_1}{t-0}}{\underset{x_1 \quad x_2}{0-T}}$

$= \dfrac{x(t)}{-A} = \dfrac{t}{-T}$

$\therefore x(t) = \dfrac{At}{T}$ for $t = 0$ to T

So. we know

$$C_n = \frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t} dt.$$

when $n=0$ $\quad C_0 = \dfrac{1}{T}\int_0^T x(t) \cdot e^0 \cdot dt = \dfrac{1}{T}\int_0^T x(t) dt.$

$$= \frac{1}{T}\int_0^T \frac{At}{T} \cdot dt = \frac{A}{T^2} \cdot \frac{t^2}{2}\Big|_0^T$$

$$= \frac{A}{\cancel{T^2}} \cdot \frac{\cancel{T^2}}{2} = \frac{A}{2}$$

$$\boxed{C_0 = \frac{A}{2}}$$

For $n \neq 0$.

$$C_n = \frac{1}{T}\int_0^T x(t) \cdot e^{-jn\omega_0 t} \cdot dt = \frac{1}{T}\int_0^T \frac{At}{T} \cdot e^{-jn\omega_0 t} \cdot dt.$$

$\int uv = \cancel{u \int v} - u\int v - \int (v \cdot du)$.

$u = t$, $v = e^{-jn\omega_0 t}$

$du = dt$, $dv =$

$$= \frac{A}{T^2}\int_0^T t \cdot e^{-jn\omega_0 t} \cdot dt.$$

$$= \frac{A}{T^2}\left[t \cdot \frac{e^{-jn\omega_0 t}}{-jn\omega_0} - \int \frac{e^{-jn\omega_0 t}}{-jn\omega_0} \cdot dt\right]_0^T$$

$$= \frac{A}{T^2}\left[\frac{-t e^{-jn\omega_0 t}}{jn\omega_0}\Big|_0^T - \frac{e^{-jn\omega_0 t}}{(jn\omega_0)^2}\Big|_0^T\right]$$

The Signal $x(t) = \cos 200\pi t + 0.25 \cos 700\pi t$ is Sampled at the rate of 400 Samples/second. Sampled waveform is then passed through an ideal LPF with 200Hz Bandwidth. Write an expression for filter output. Sketch the frequency Spectrum of Sampled waveform.

Ans:- Given $x(t) = \cos 200\pi t + 0.25 \cos 700\pi t$.

$$f_s = 400\,Hz$$

So $x(t) = \cos(2\times 100\pi t) + 0.25\cos(2\times 350\pi t)$

$\therefore A_1 = 1,\quad A_2 = 0.25 \quad f_1 = 100,\quad f_2 = 350$

We know

$$FT\{\cos 2\pi f_c t\} = \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right] \quad \text{— (1)}$$

$\therefore$ Spectrum of given signal is.

$$X(f) = \frac{A_1}{2}\left[\delta(f-100) + \delta(f+100)\right] + \frac{A_2}{2}\left[\delta(f-350) + \delta(f+350)\right] \quad \text{— (2)}$$

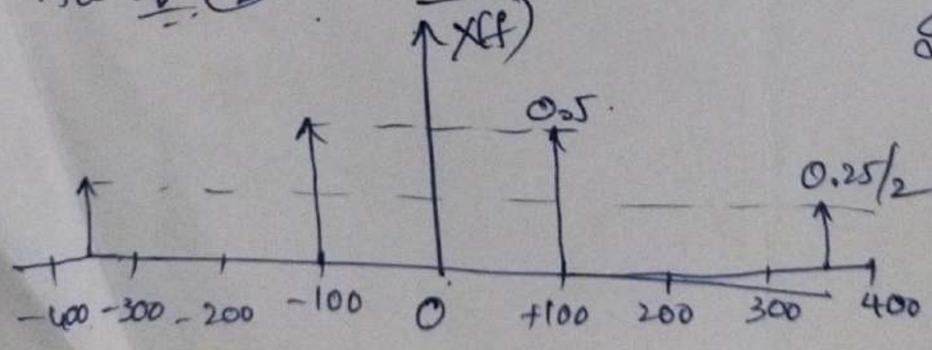
then:

$$S(f) = f_s \sum_{n=-\infty}^{+\infty} X(f - nf_s) \quad \text{— (3)}$$

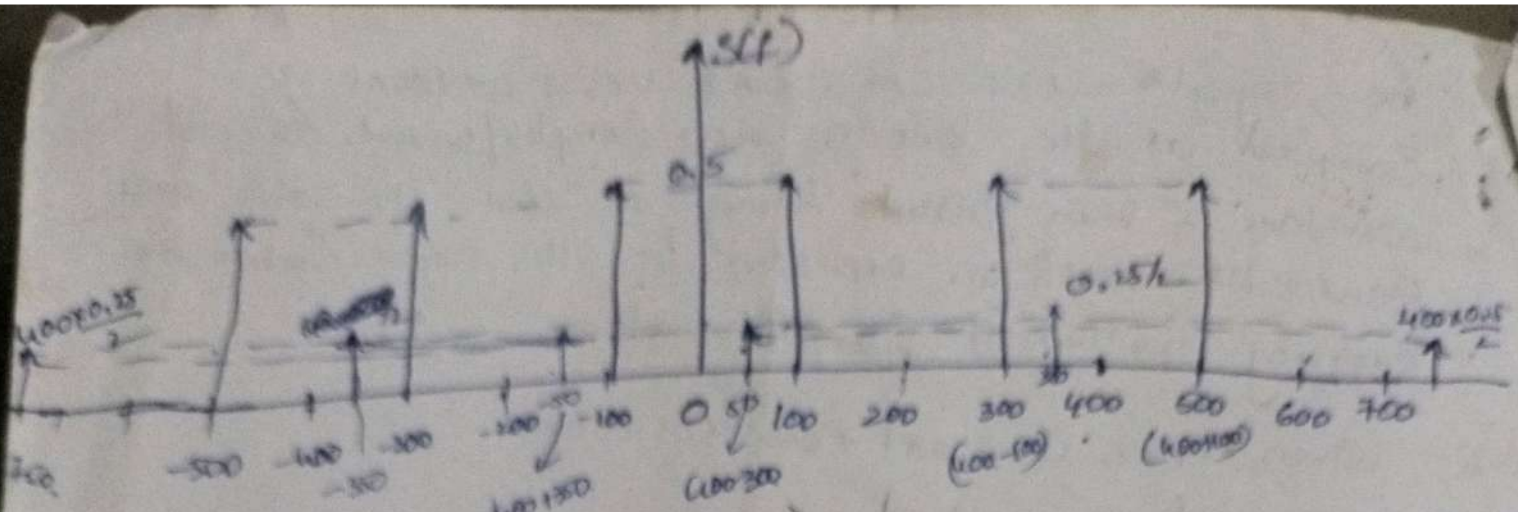
$$= 400 \sum_{n=-\infty}^{+\infty} X(f - 400n)$$

by expanding we get $S(f)$

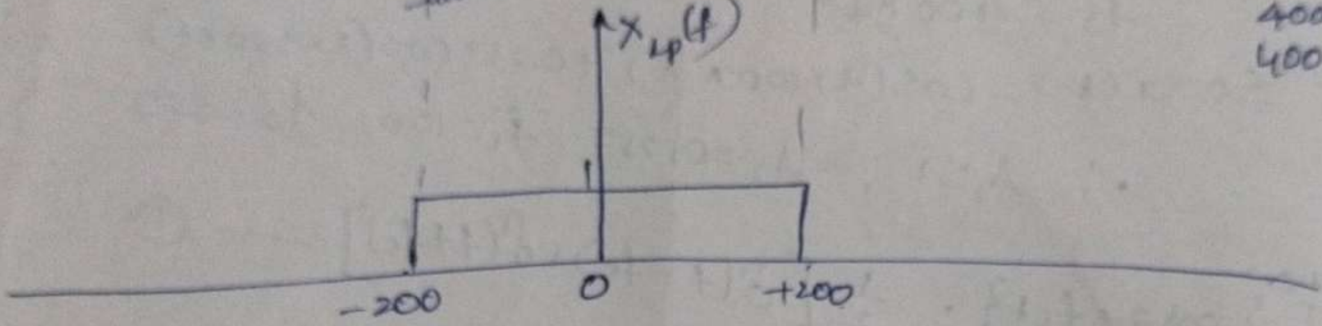
for eq (2) ie: $X(f)$



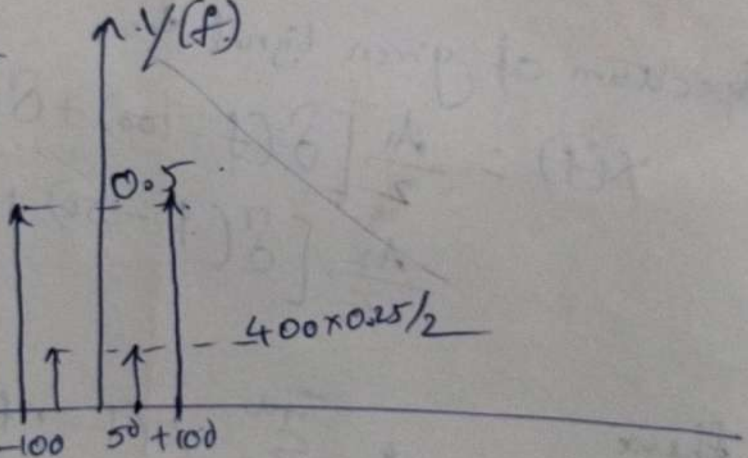
Spectrum of $x(t)$



400-100=300
400+100=500

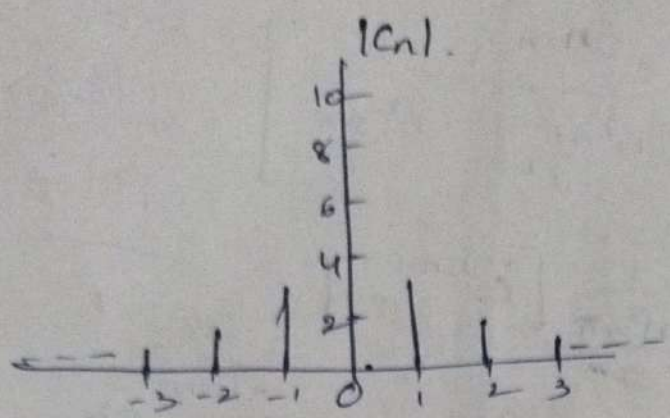


then we get

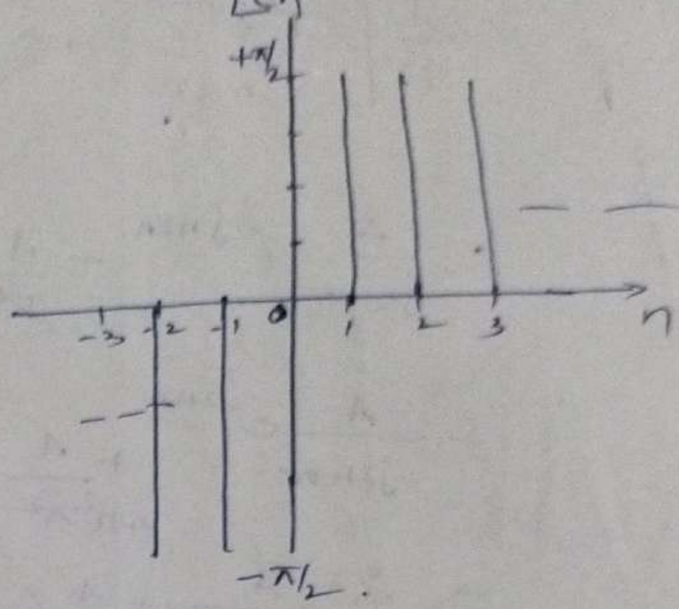


Using above calculations

Magnetic Spectrum.



Phase Spectrum



I SlipTest:- on Ist Unit.

Dt: 6/8/16

Dec; 2014 / Reg exams -R13
compulse

1) (a) Discrete function in terms of Unit step-function. Also show a Graphical illustration } -10

(b) Explain about Time Independence Variables.

(2) Expand $f(t) = At$ by exponential FS over (0, 1) interval. (10)

(3) Define Elementary Signals with each one example — (10)

$$\frac{A}{T^2}\left[\frac{t\,e^{-jn\frac{2\pi}{T}t}}{-jn\frac{2\pi}{T}} + \frac{e^{-jn\frac{2\pi}{T}t}}{n^2\frac{4\pi^2}{T^2}}\right]_0^T$$

(83)

Representation in calci

pol(0, 10/π)

pol(0, 5/π)

$$= \frac{A}{T^2}\left[\frac{T\,e^{-jn2\pi}}{-jn\frac{2\pi}{T}} + 0 + \frac{e^{-jn2\pi}}{n^2\frac{4\pi^2}{T^2}} - \frac{1}{n^2\frac{4\pi^2}{T^2}}\right]$$

$$= \frac{A}{-j2n\pi}e^{-jn2\pi} + \frac{A}{T^2}\times\frac{1}{n^2\frac{4\pi^2}{T^2}}\left[e^{-j2n\pi}-1\right]$$

$$= \frac{A}{-j2n\pi}e^{-j2n\pi} + \frac{A}{4n^2\pi^2}e^{-j2n\pi} - \frac{A}{4n^2\pi^2}$$

$e^{-j2n\pi} = \cos 2n\pi - j\sin 2n\pi$

$= 1 - j0 = 1$

$$\Rightarrow \therefore \frac{A}{-j2n\pi} + \cancel{\frac{A}{4n^2\pi^2}} - \cancel{\frac{A}{4n^2\pi^2}} = \frac{-A}{j2n\pi} = \frac{jA}{2n\pi}$$

$$\boxed{C_n = \frac{jA}{2n\pi}}$$

at $n=-3$ $\quad c_{-3} = \frac{-jA}{6\pi}$ if $A=20$ $\quad C_{-3} = \frac{-j10}{3\pi}$

$= -j1.061 = 1.061\angle -90$

$n=-2$ $\quad C_{-2} = \frac{-A}{j2\times(-2)\pi}$ $\quad C_{-2} = \frac{-j10}{2\pi} = -j1.592 = 1.592\angle -90°$

$n=-1$ $\quad C_{-1} = \frac{-j20}{2\pi} = \frac{-j10}{\pi} = -j3.183 = 3.183\angle -90°$

$n=0$ $\quad c_0 = \frac{20}{2} = 10 = 10\angle 0°$

$n=1$ $\quad c_1 = j\frac{10}{\pi} = j3.183 = 3.183\angle +90°$

$n=2$ $\quad c_2 = \frac{j10}{2\pi} = j1.592 = 1.592\angle +90°$

$n=3$ $\quad c_3 = \frac{j10}{3\pi} = j1.061 = 1.061\angle +90°$

UNIT-II

:- Continuous Time Fourier Transform :-

Introduction:-

→ Non-periodic / Aperiodic Signals can be represented with the help of Fourier Transform.

→ Fourier Transform provides effective reversible link between frequency domain and time domain representation of the Signal.

→ For non-periodic Signals $T_0 \to \infty$. Hence $\omega_0 = 0$. ∴ Spacing b/w the Spectral Components become infinitesimal and hence Spectrum appears to be Continuous.

Discrete Spectrum become Continuous for aperiodic signals.

Definition of Fourier Transform:-

→ The Fourier Transform of $x(t)$ is defined as

$$\boxed{FT\{x(t)\} = X(\omega) = X(f) = X(j\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}.dt \quad (\text{or}) \quad X(f) = \int_{-\infty}^{+\infty} x(t).e^{-j2\pi ft}.dt.} \quad (2.1)$$

Here $x(t)$ is time domain representation of the Signal. and '$X(\omega)$' & $X(f)$ is frequency domain representation of the Signal. 'ω' is the frequency.

→ Similarly, $x(t)$ can be obtained from $X(\omega)$ by Inverse Fourier Transform ie,

$$\boxed{IFT\{X(\omega)\} = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega).e^{j\omega t}.d\omega = \int_{-\infty}^{\infty} X(f).e^{j2\pi ft}.df} \quad (2.2)$$

$T \to \infty$
$\omega_0 \to 0$
$\omega_0 = \frac{2\pi}{T}$
$\frac{1}{T} = \frac{\omega_0}{2\pi}$

→ A Fourier Transform pair is represented as,

$$x(t) \xleftrightarrow{FT} X(\omega) \quad (or) \quad x(t) \xleftrightarrow{FT} X(f)$$

Existence of Fourier Transform. - Dirichlet Conditions :-

→ The following conditions should be satisfied by the function $x(t)$ for Fourier Transform to exist.

1. Single Valued property :-

$x(t)$ must have only one value at any time instant over a finite time interval T.

2. Finite discontinuous :-

$x(t)$ must have at the most finite no. of discontinuous over a finite time interval T.

3. Finite peaks :-

$x(t)$ should have finite no. of maxima & minima over a finite time interval T.

4. Absolute Integrability :-

$x(t)$ should ~~have~~ be absolutely integrable i.e.,

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty.$$

Note :- The above conditions are sufficient, but not necessary for the signal to be Fourier Transformable.

Properties of Fourier Transform.

1. Linearity

If $x(t) \xleftrightarrow{FT} X(\omega)$ &

$y(t) \xleftrightarrow{FT} Y(\omega)$

then.

$$z(t) = [a x(t) + b y(t)] \xleftrightarrow{FT} [a X(\omega) + b Y(\omega)]$$

The Fourier Transform of linear combination of the signals is equal to linear combination of their Fourier Transforms.

It is also called Superposition.

Proof:-

Let $z(t) = a\,x(t) + b\,y(t)$. then.

$$FT\{z(t)\} = \int_{-\infty}^{\infty} [a\,x(t) + b\,y(t)].\,e^{-j\omega t}.\,dt = Z(\omega) = Z(f)$$

$$= a\int_{-\infty}^{\infty} x(t).\,e^{-j\omega t}.\,dt + b\int_{-\infty}^{\infty} y(t).\,e^{-j\omega t}.\,dt.$$

$$\boxed{Z(\omega) = a.X(\omega) + b.Y(\omega)} \quad \because \int_{-\infty}^{\infty} x(t).\,e^{-j\omega t}.\,dt = X(\omega)$$

$$\int_{-\infty}^{\infty} y(t).\,e^{-j\omega t}.\,dt = Y(\omega)$$

2. Time Shifting:-

If. $x(t) \xleftrightarrow{FT} X(\omega)$ then.

$$\boxed{y(t) = x(t-t_0) \xleftrightarrow{FT} Y(\omega) = e^{-j\omega t_0}X(\omega)}$$

In this a shift of 't_0' in time domain is equivalent to introducing a phase shift of $-\omega t_0$ but amplitude remains same.

Proof:-

$$Y(\omega) = FT\{y(t)\} = \int_{-\infty}^{\infty} [x(t-t_0)].\,e^{-j\omega t}.\,dt. \quad \text{at } t - t_0 = \tau.$$

$dt = d\tau.$

$t = \tau + t_0.$

Limits at $t = -\infty$

$\tau = -\infty$

at $t = +\infty$

$\tau = +\infty$

$$\therefore Y(\omega) = \int_{-\infty}^{\infty} x(\tau).\,e^{-j\omega(t_0+\tau)}.\,d\tau.$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau).\,e^{-j\omega\tau}.\,d\tau.$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(t-t_0).\,e^{-j\omega(t-t_0)}.\,dt.$$

$$\boxed{Y(\omega) = e^{-j\omega t_0}.\,X(\omega)}$$

3. Frequency Shift:-

If $x(t) \xleftrightarrow{FT} X(\omega)$ then.

$$y(t) = e^{j\beta t}.x(t) \xleftrightarrow{FT} Y(\omega) = X(\omega-\beta)$$

(or)

$$y(t) = e^{j2\pi f_0 t}.x(t) \longleftrightarrow Y(\omega) = X(\omega-\omega_0) = X(f-f_0)$$

Definition:-

It states that by shifting the frequency by ω_0 in frequency domain. which is equivalent to multiplying the time domain signal by $e^{j\omega_0 t}$.

Proof:-

$$FT\{y(t)\} = Y(\omega) = \int_{-\infty}^{\infty} y(t).e^{-j2\pi f t}.dt.$$

$$= \int_{-\infty}^{\infty} [e^{j\omega_0 t}.x(t)]\, e^{-j\omega t}.dt.$$

$$= \int_{-\infty}^{\infty} x(t).e^{-j(\omega-\omega_0)t}.dt.$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f-f_0)}..dt.$$

$$= X(\omega \to (\omega-\omega_0))$$

$$\boxed{\therefore Y(\omega) = X(\omega-\omega_0) = X(f-f_0)}$$

4. Time Scaling:-

If $x(t) \xleftrightarrow{FT} X(\omega)$ then.

$$x(at) \xleftrightarrow{FT} Y(\omega) = \frac{1}{|a|}X\left(\frac{f}{a}\right) = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

... a signal. in time domain is

Proof:-

$$Y(\omega) = \int_{-\infty}^{\infty} x(t).e^{-j2\pi ft}.dt$$

$$= \int_{-\infty}^{\infty} x(at).e^{-j2\pi ft}.dt$$

put $at = T$
$t = T/a$
$dt = \frac{1}{a}.dT$

$$= \int_{-\infty}^{\infty} x(T).e^{-j2\pi f(T/a)}.\left(\frac{1}{a}\right)dT$$

at $t=-\infty, T=-\infty$
$t=+\infty\ T=+\infty$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(T).e^{\frac{-j2\pi fT}{a}}.dT$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(T).e^{-j2\pi f_1 T}.dT$$

$$= \frac{1}{a}.X(f_1) = \frac{1}{a}X(f/a) \quad \because f \rightarrow f/a = f_1.$$

we can write

$$\boxed{Y(\omega) = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)}$$

V : Frequency - Differentiation

If $x(t) \xleftrightarrow{FT} X(\omega)$. then

$$\boxed{-jt\ x(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(\omega)}$$

Definition:- Differentiation in frequency Spectrum is equivalent to multiply the time domain Signal by Complex number $-jt$.

Proof:-

we know

$$FT\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}.dt.$$

Differentiating above equation.

$$\frac{d}{d\omega} X(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\cdot dt$$

$$= \int_{-\infty}^{\infty} x(t)\cdot \frac{d}{d\omega} e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(t)(-jt)\, e^{-j\omega t}\cdot dt$$

$$= -jt\cdot \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\cdot dt$$

$$\boxed{\frac{d}{d\omega}(X(\omega)) = -jt\cdot X(j\omega)}$$

6) Time-Differentiation:-

Definition:- Differentiation in time domain corresponds to multiplying by $j\omega$ in frequency domain. It accentuates high frequency components of the signal.

If $x(t) \xleftrightarrow{FT} X(\omega)$, then

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega\, X(\omega)$$

$$FT\{x(t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\cdot dt$$

$$FT\left\{\frac{d}{dt}x(t)\right\} = \int_{-\infty}^{\infty} \frac{d}{dt}x(t)\cdot e^{-j\omega t}\cdot dt$$

$$= \int_{-\infty}^{\infty} x(t)\cdot(-j\omega)\cdot e^{-j\omega t}\cdot dt$$

$$= -j\omega \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\cdot dt$$

$$= -j\omega\cdot X(\omega)$$

(OR)

We know that

$$IFT\{X(j\omega)\} = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\frac{d}{dt}\cdot e^{j\omega t}\cdot d\omega$$

$$= \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\cdot e^{j\omega t}\cdot d\omega$$

$$\frac{d}{dt}x(t) = j\omega\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}.d\omega\right]$$ 4

$$\boxed{\frac{d}{dt}x(t) = j\omega X(j\omega)}$$

(7) Convolution :-

If $x(t) \xleftrightarrow{FT} X(\omega)$ &

$y(t) \xleftrightarrow{FT} Y(\omega)$,

then

$$z(t) = x(t) * y(t) \xleftrightarrow{FT} Z(\omega) = X(\omega).Y(\omega)$$

Definition:-

→ A convolution operation is transformed to modulation in frequency domain.

Proof:-

$$Z(\omega) = \int_{-\infty}^{\infty} z(t).e^{-j\omega t}.dt = \int_{-\infty}^{\infty}[x(t) * y(t)]e^{-j\omega t}dt.$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)d\tau\right]e^{-j\omega t}.dt.$$

$$= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} y(t-\tau)\,e^{-j\omega t}.dt.\right]d\tau.$$

at $t-\tau = \alpha$, then $t = \tau+\alpha$

$dt = d\alpha$

at $t = -\infty$. $\alpha = -\infty$.

$t = +\infty$ $\alpha = +\infty$

$$\therefore Z(\omega) = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} y(\alpha)\,e^{-j\omega(\tau+\alpha)}.d\alpha\right]d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau).e^{-j\omega\tau}\left[\int_{-\infty}^{\infty} y(\alpha)\,e^{-j\omega\alpha}d\alpha\right].d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau).e^{-j\omega\tau}.d\tau \int_{-\infty}^{\infty} y(\alpha)\,e^{-j\omega\alpha}d\alpha$$

$$\boxed{Z(\omega) = X(\omega).Y(\omega)}$$

8) Integration:-

If $x(t) \xleftrightarrow{FT} X(\omega)$, then.

$$\boxed{\int_{-\infty}^{t} x(\tau).d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(\omega)}$$

Definition:-

→ Integration in time represents smoothing in time domain.

→ This smoothing in time corresponds to de-emphasizing the high frequency components of the signal.

Proof:-

Let $x(t)$ be expressed as

$$x(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau).d\tau\right]$$

$$F\{x(t)\} = F\left\{\frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau).d\tau\right]\right\}.$$

By using differentiating property RHS. of above equation becomes.

$$F\{x(t)\} = j\omega\left\{F\left[\int_{-\infty}^{t} x(\tau)d\tau\right]\right\}.$$

$$X(j\omega) = j\omega.\left[F\int_{-\infty}^{t} x(\tau).d\tau\right]$$

$$\boxed{\frac{X(j\omega)}{j\omega} = F\left\{\int_{-\infty}^{t} x(\tau).d\tau\right\}.}$$

$\frac{d}{dt}x(t) = j\omega\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega\right]$ 4

Ex:-9.1 :- Obtain the FT of the Signal $e^{-at}u(t)$ & plot its magnitude & phase Spectrum 5

Sol:- $x(t) = e^{-at}u(t)$

If $x(t) \xleftrightarrow{FT} X(\omega)$

then

$$FT\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty}\left[e^{-at}\cdot u(t)\right]e^{-j\omega t}\cdot dt$$

$$= \int_{-\infty}^{0} e^{-at}\cdot 0\cdot e^{-j\omega t}\cdot dt + \int_{0}^{\infty} e^{-at}\cdot e^{-j\omega t}\cdot dt$$

$$= 0 + \int_{0}^{\infty} e^{-(a+j\omega)t}\cdot dt$$

$$= \frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\Bigg|_{0}^{\infty}$$

$$= \frac{e^{-\infty}}{-(a+j\omega)} - \frac{1}{-(a+j\omega)}$$

$$X(j\omega) = 0 + \frac{1}{a+j\omega}$$

Then

$$X(j\omega) = \frac{1}{a+j\omega}\times\frac{a-j\omega}{a-j\omega}$$

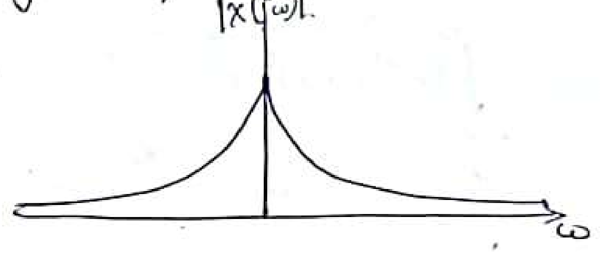
$$= \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}$$

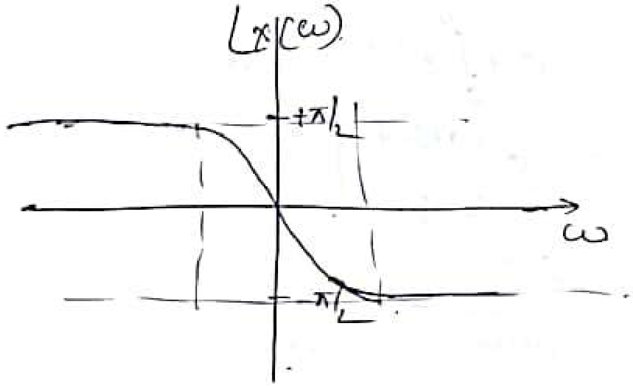
$$|X(j\omega)| = \sqrt{\left[\frac{a}{a^2+\omega^2}\right]^2 + \left[\frac{\omega}{a^2+\omega^2}\right]^2} = \sqrt{\frac{\cancel{a^2+\omega^2}}{(a^2+\omega^2)^{\cancel{2}}}}$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\angle X(j\omega) = \tan^{-1}\frac{\left(\frac{\omega}{a^2+\omega^2}\right)}{\frac{a}{a^2+\omega^2}} = \tan^{-1}\frac{\omega}{\cancel{a^2+\omega^2}}\times\frac{\cancel{a^2+\omega^2}}{a} = \tan^{-1}(\omega/a)$$

Magnitude plot:-





Ex:- $x(t) = \delta(t)$

$$FT\{x(t)\} = X(\omega) = \int_{-\alpha}^{\alpha} \delta(t) \cdot e^{-j\omega t} \cdot dt.$$

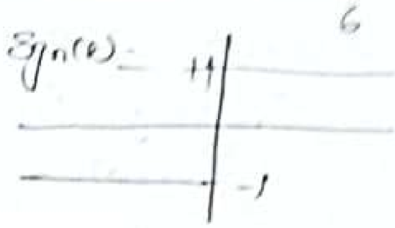
By Shifting property:-

$$\int_{-\alpha}^{\alpha} x(t) \cdot \delta(t - t_0) = x(t_0)$$ in the above $x(t) = e^{-j\omega t}$ & $t_0 = 0$.

Hence $X(\omega) = e^{-j\omega t}\Big|_{t=0} = 1.$

Thus $\boxed{\delta(t) \xleftrightarrow{FT} 1}$

Ex:- $x(t) = \mathrm{Sgn}(t)$



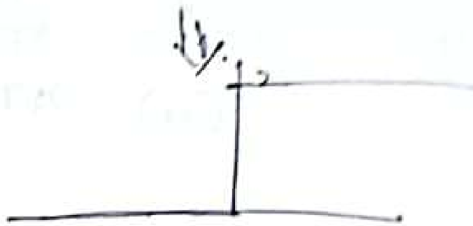
So. Signum function can be expressed as

$x(t) = 2u(t) - 1$

Diff both the sides

$\therefore \frac{d}{dt} x(t) = 2\frac{d}{dt} u(t)$

$= 2\delta(t)$



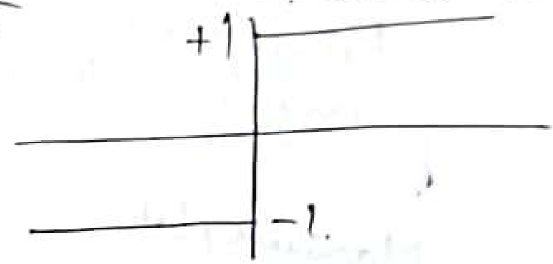
(−) 1.

Taking FT on both sides

$FT\{\frac{d}{dt} x(t)\} = 2F\{\delta(t)\}$.

$= 2u(t) - 1$

By differentiating property



$\frac{d}{dt} x(t) \xleftrightarrow{FT} j\omega X(\omega)$ &

$\delta(t) \xleftrightarrow{FT} 1$.

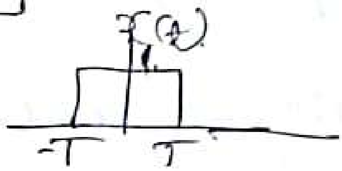
$\therefore$ above eq can be written as

$j\omega X(\omega) = 2$.

$X(\omega) = \frac{2}{j\omega}$.

$$\boxed{\therefore \mathrm{Sgn}(t) \xleftrightarrow{FT} \frac{2}{j\omega}}$$

Ex:- FT{Rectangular pulse}.



$x(t) = 1$

$X(\omega) = \int_{-T}^{T} e^{-j\omega t}\, dt = \frac{e^{-j\omega t}}{-j\omega}\Big|_{-T}^{+T}$.

$= \frac{1}{-j\omega}\left[e^{-j\omega T} + e^{j\omega T}\right]$

$= \frac{+2}{\omega}\left[\frac{e^{j\omega T} - e^{-j\omega T}}{2j}\right] = \frac{2}{\omega}\cdot \sin(\omega T)$

8) We know $\mathrm{Sinc}\,\theta = \frac{\sin \pi\theta}{\pi\theta}$

for getting Sinc function, modify the above equation

$\therefore X(\omega) = \cancel{\frac{2}{\frac{\pi}{T}}} \cancel{\sin\left(\pi \cdot \frac{\omega T}{\pi}\right)}$

$$X(\omega) = 2T \cdot \frac{\sin\left(\pi \cdot \frac{\omega T}{\pi}\right)}{\pi \cdot \frac{\omega T}{\pi}}$$

$$= 2T\,\mathrm{Sinc}\left(\frac{\omega T}{\pi}\right)$$

Thus.

$$\boxed{\text{Rectangular pulse amplitude 1, period } 2T \xleftrightarrow{FT} 2T\,\mathrm{Sinc}\left(\frac{\omega T}{\pi}\right)}$$

Magnitude plot.

$$|X(\omega)| = 2T\,\mathrm{Sinc}\left(\frac{\omega T}{\pi}\right)$$

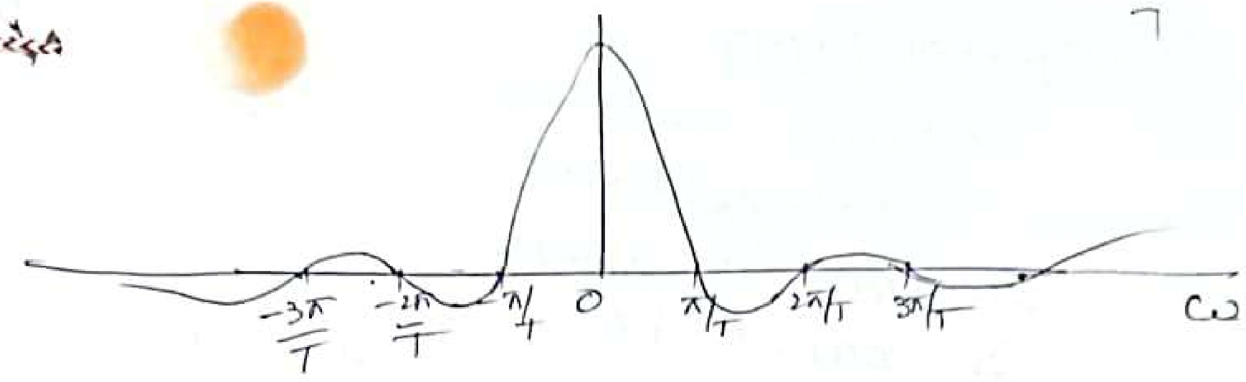
$$\& \angle X(\omega) = 0.$$

at $\omega = \pm\frac{\pi}{T}, \pm\frac{2\pi}{T}, \pm\frac{3\pi}{T}, \pm - - - - -$

By L-Hospital Rule,

$$\lim_{\omega \to 0} \frac{2}{\omega} \sin(\omega T) = 2T$$

Hence $X(\omega) = 2T$ at $\omega = 0$.



(9) Modulation property:-

$$FT\{x(t)y(t)\} = \frac{1}{2\pi}[X(\omega) * Y(\omega)]$$

(10) Duality:-

If $x(t) \xleftrightarrow{FT} X(\omega)$.

$X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$.

Proof:- we know that:

$$IFT\{X(\omega)\} = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \cdot d\omega.$$

interchange 't' by 'ω' we get.

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t) e^{j\omega t} \cdot dt.$$

interchage 'ω' by $-\omega$. then

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t) \cdot e^{-j\omega t} \cdot dt$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} \cdot dt$$

$$2\pi x(-\omega) = X(t)$$

$$\therefore \boxed{FT\{X(t)\} = 2\pi x(-\omega)}$$

(11) Symmetry property :-

Let $x(t)$ be Real Signal &

$$X(\omega) = X_R(\omega) + jX_I(\omega)$$

then $x_e(t) \xleftrightarrow{FT} X_R(\omega)$.

& $x_o(t) \xleftrightarrow{FT} jX_I(\omega)$

where $x_e(t)$ & $x_o(t)$ are even & odd parts of $x(t)$

Proof:- $x(t) \xleftrightarrow{FT} X_R(\omega) + jX_I(\omega)$

$\because$ $x(t)$ is real,

$$x(-t) \xleftrightarrow{FT} X^*(\omega) = X_R(\omega) - jX_I(\omega)$$

Even part is given by.

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

$$FT\{x_e(t)\} = \frac{1}{2}\left[X(\omega) + X^*(\omega)\right]$$

$$x_e(t) \xleftrightarrow{FT} \frac{1}{2}\left[X_R(\omega) + \cancel{jX_I(\omega)} + X_R(\omega) - \cancel{jX_I(\omega)}\right]$$

$$x_e(t) \longleftrightarrow \frac{1}{\cancel{2}}\cancel{2}X_R(\omega)$$

i.e. $\boxed{x_e(t) \longleftrightarrow X_R(\omega)}$

Odd part is given by

$$x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

$$F\{x_o(t)\} = \frac{1}{2}\left[\cancel{X_R(\omega)} + jX_I(\omega) - \cancel{X_R(\omega)} + jX_I(\omega)\right]$$

$$FT\{x_o(t)\} = jX_I(\omega)$$

$$x_o(t) \xleftrightarrow{FT} jX_I(\omega)$$

$$\boxed{\therefore FT\{x(t)\} = X_R(\omega) + jX_I(\omega)}$$

Parseval's Theorem (or) Rayleigh's Theorem:-

If $x(t) \xleftrightarrow{FT} X(\omega)$ then.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(f)|^2 df.$$

Definition:-

Energy of the signal can be obtained by interchanging its energy spectrum.

Proof:-

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) . x^*(t) . dt \quad — (1)$$

We know

$$IFT\{X(\omega)\} = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) . e^{j\omega t} . d\omega \quad — (2)$$

Taking conjugate of both the sides

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} . d\omega . \quad — (3)$$

Substitute in (3) then (1) becomes

$$E = \int_{-\infty}^{\infty} x(t) . \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} . d\omega\right] dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t) . e^{-j\omega t} . dt .\right] d\omega .$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) . X(\omega) . d\omega$$

$$\boxed{E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega}$$

Fourier Transform of Some important Signals:-

(a) Fourier Transform of Unit Impulse Signal:-

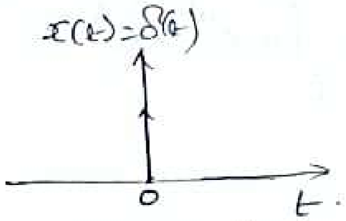
The impulse Signal is defined as

$$x(t) = \delta(t) = \infty \;;\; t = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t).dt = 1$$
$$= 0 \;;\; t \neq 0$$

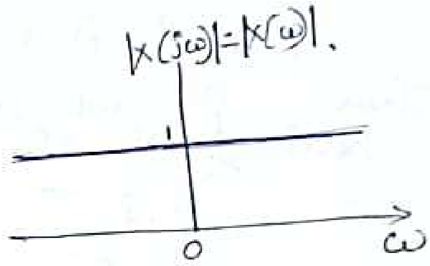
Now.

$$F\{\delta(t)\} = X(\omega) = \int_{-\infty}^{\infty} \delta(t).e^{-j\omega t}.dt = 1 \times e^{-j\omega t}\Big|_{t=0} = 1 \times e^{0} = 1$$

$$\boxed{\therefore F\{\delta(t)\} = 1}$$



Impulse Signal.



Magnitude Spectrum of impulse Signal.

(b) Fourier Transform of Single sided Exponential Signal:-

A Single sided exponential Signal is defined as

$$x(t) = Ae^{-at} \;;\; \text{for } t \geq 0$$

(or)

$$x(t) = Ae^{-at}u(t)$$

By definition of Fourier Transform.

$$X(\omega) = \int_{-\infty}^{\infty} Ae^{-at}.u(t).e^{-j\omega t}.dt = A\int_{0}^{\infty} e^{-at}.e^{-j\omega t}.dt. = A.\int_{0}^{\infty} e^{-(a+j\omega)t}.dt.$$

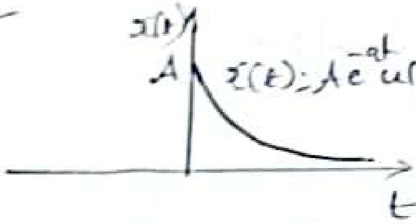
$$= A.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\Big|_{0}^{\infty} = \left[A.(0) - \frac{Ae^{0}}{-(a+j\omega)}\right]$$

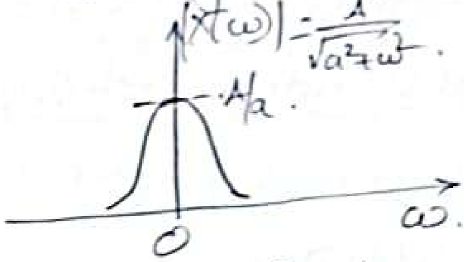
$$= \frac{A}{a+j\omega}.$$

ie; $F\{x(t)\} = \dfrac{A}{a+j\omega} = X(\omega)$

$$|X(\omega)| = \left|\frac{A}{a+j\omega}\right| = \frac{A}{\sqrt{a^2+\omega^2}}$$

The plot of exponential signal & its magnitude spectrum are





at $\omega = 0$

$|X(\omega)| = \frac{A}{a}$.

(c) Fourier Transform of Double Sided Exponential Signal:-

The double sided exponential signal is defined as

$x(t) = A e^{-a|t|}$: for all t.

$\therefore x(t) = A e^{+at}$ for $t = -\infty$ to 0

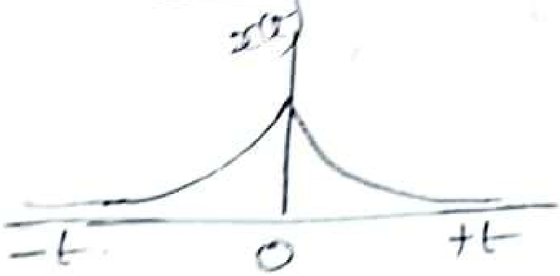
$= A e^{-at}$ for $t = 0$ to ∞.

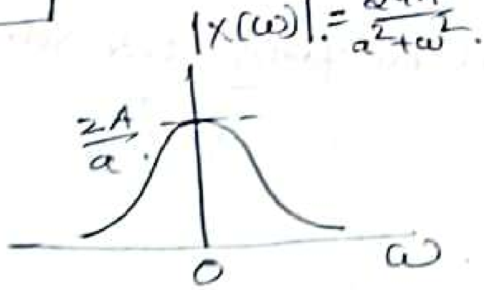
Then

$$X(\omega) = \int_{-\infty}^{0} A \cdot e^{+at} \cdot e^{-j\omega t} \cdot dt + \int_{0}^{\infty} A \cdot e^{-at} \cdot e^{-j\omega t} \cdot dt.$$

$$= \frac{2aA}{a^2+\omega^2}.$$

$$\boxed{\therefore F\{A e^{-a|t|}\} = \frac{2aA}{a^2+\omega^2}}$$





(d) Fourier Transform of a Constant

Let $x(t) = A$, where A is a constant.

If definition of FT is directly applied, the constant will not satisfy the condition,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

Hence, the constant can be viewed as a doubled sided exponential with limit 'a' tends to '0' as shown below

$x_1(t) = A\, e^{-a|t|}$

$\therefore\ x_1(t) = A\, e^{at}$; for $t = -\infty$ to 0

$= A\, e^{-at}$; for $t = 0$ to $+\infty$

$\therefore\ x(t) = A \lim\limits_{a \to 0} x_1(t)$

On taking FT of the above equation we get.

$$F\{x(t)\} = F\{\lim_{a\to 0} x_1(t)\}$$

$$= \lim_{a\to 0} F\{x_1(t)\}$$

$$X(\omega) = \lim_{a\to 0} X_1(\omega)$$

$$X(\omega) = \lim_{a\to 0} \frac{2aA}{a^2+\omega^2}$$

The above equation is '0' for all values of ω except at $\omega=0$.

At $\omega=0$, the above equation represents an impulse of magnitude 'k'.

$$\therefore\ X(\omega) = k\delta(\omega)\ ;\ \omega=0$$
$$= 0\ ;\ \omega \neq 0.$$

The magnitude 'k' can be evaluated as shown below

$$k = \int_{-\infty}^{\infty} \frac{2aA}{\omega^2+a^2}\cdot d\omega = 2aA\left[\frac{1}{a}\cdot \tan^{-1}\frac{\omega}{a}\right]_{-\infty}^{\infty}$$

$\because \left[\int \frac{dx}{x^2+a^2} = \frac{1}{a}\cdot \tan^{-1} x/a\right]$

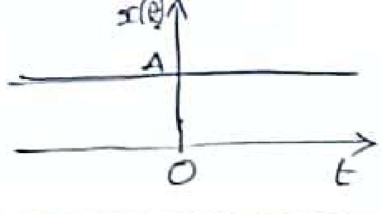
$$= 2aA\left[\frac{1}{a}\tan^{-1}(\infty) - \frac{1}{a}\tan^{-1}(-\infty)\right]$$

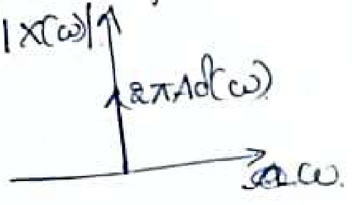
$$= 2aA\left[\frac{1}{a}\cdot\frac{\pi}{2} - \frac{1}{a}\cdot\left(-\frac{\pi}{2}\right)\right]$$

$$= 2aA\left[\frac{\pi}{2a} + \frac{\pi}{2a}\right] = 2aA\cdot\frac{\pi}{a}$$
$$= 2\pi A$$

$$\boxed{\therefore\ F\{A\} = 2\pi A\,\delta(\omega)}$$

The plot of constant & its magnitude spectrum are.





(c) Fourier Transform of Signum function:-

→ The Signum function is defined as

$$x(t) = Sgn(t) = 1 \quad ; \quad t>0$$
$$= -1 \quad ; \quad t<0$$

→ The Signum function can be expressed as a Sum of two one sided exponential Signal & taking limit 'a' tends to '0' as Shown below.

$$\therefore Sgn(t) = \lim_{a\to 0}\left[e^{-at}u(t) - e^{at}u(-t)\right]$$

$$FT\{Sgn(t)\} = X(\omega) = \int_{-\infty}^{\infty} \lim_{a\to 0}\left[e^{-at}u(t) - e^{+at}u(-t)\right] e^{-j\omega t}.dt.$$

$$= \lim_{a\to 0}\int_{0}^{\infty} e^{-at}\cancel{u(t)}.e^{-j\omega t}.dt - \lim_{a\to 0}\int_{-\infty}^{0} e^{at}\cancel{u(-t)}.e^{-j\omega t}.dt$$

Since already limits are applied

$$= \lim_{a\to 0}\int_{0}^{\infty} e^{-(a+j\omega)t}.dt - \lim_{a\to 0}\int_{-\infty}^{0} e^{+(a-j\omega)t}.dt.$$

$$= \lim_{a\to 0}. \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty} - \lim_{a\to 0} \left.\frac{e^{+(a-j\omega)t}}{(a-j\omega)}\right|_{-\infty}^{0}.$$

$$= \lim_{a\to 0}\left[+\left(\frac{1}{a+j\omega}\right) - \frac{1}{(a-j\omega)}\right]$$

After applying $a \to 0$.

$$FT\{Sgn(t)\} = \left[\frac{1}{j\omega} - \left(\frac{1}{-j\omega}\right)\right] = \frac{1}{j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega}.$$

$$\boxed{\therefore F\{Sgn(t)\} = \frac{2}{j\omega}}$$

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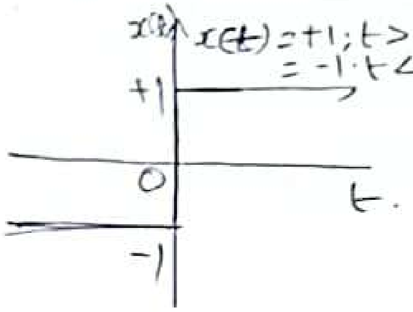
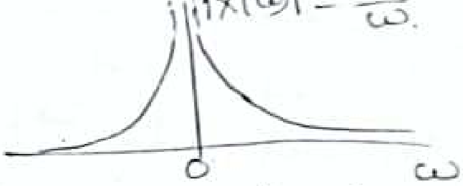
Signum function.

$x(t) = +1; t>0$
$= -1; t<0$



Magnitude Spectrum

$|X(\omega)| = \frac{2}{\omega}$



at $\omega = 0$ $|X(\omega)| = \infty$ means amplitude is infinite.

at $\omega = 1$

at $\omega = 2$ - - - - .

(f) Fourier Transform of Unit Step Signal:-

The unit step signal is defined as

$$u(t) = 1 \; ; \; t \geq 0$$
$$= 0 \; ; \; t < 0$$

It can be proved that $Sgn(t) = 2u(t) - 1$

$\therefore u(t) = \frac{[Sgn(t)+1]}{2}$

$$\therefore x(t) = u(t) = \frac{[Sgn(t)+1]}{2}$$

Then take FT of the above equation we get,

$$F\{x(t)\} = F\left\{\frac{1}{2}[1 + Sgn(t)]\right\}$$

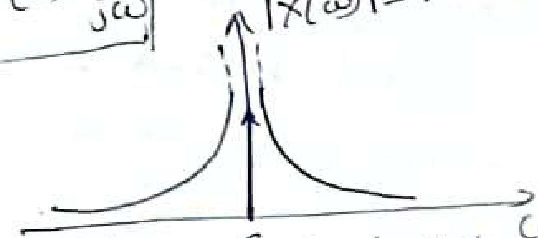
$$X(\omega) = F\left\{\frac{1}{2}\right\} + F\left\{\frac{Sgn(t)}{2}\right\} = \frac{1}{2}F\{1\} + \frac{1}{2}F\{Sgn(t)\}$$

$$= \frac{1}{2}[2\pi\delta(\omega)] + \frac{1}{2}\left[\frac{2}{j\omega}\right]$$

$$= \pi\delta(\omega) + \frac{1}{j\omega}$$

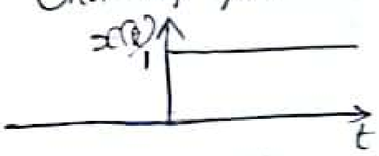
$$\boxed{\therefore F\{u(t)\} = \pi\delta(\omega) + \frac{1}{j\omega}}$$

$|X(\omega)| = \pi\delta(\omega) + \frac{1}{\omega}$



Magnitude Spectrum of Unit step Signal.

Unit step - plot.



(9) Fourier Transform of Complex Exponential Signal:-

The complex exponential signal is defined as

$$x(t) = A\,e^{j\omega_0 t}.$$

$$F\{x(t)\} = F\{A\,e^{j\omega_0 t}\}$$

$$= \int_{-\infty}^{\infty} A\,e^{j\omega_0 t}\cdot e^{-j\omega t}\,dt.$$

$$= \int_{-\infty}^{\infty} A\cdot e^{-j(\omega-\omega_0)t}\,dt.$$

$$= F\{A\}_{\text{at } \omega=\omega-\omega_0}$$

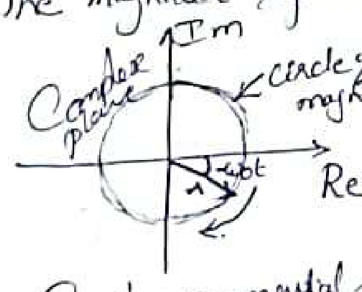
$$\boxed{\therefore\; X(\omega) = 2\pi\delta(\omega-\omega_0)}$$

$$\boxed{\text{i.e. } F\{A\,e^{j\omega_0 t}\} = 2\pi A\delta(\omega-\omega_0)}$$

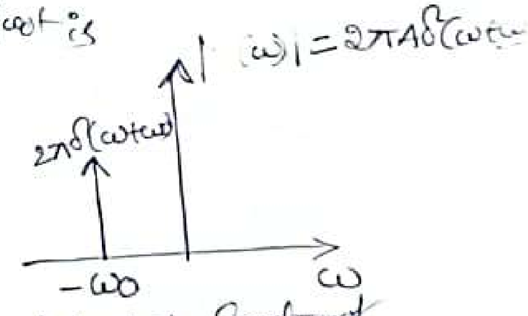
$$\text{11}^{\text{ly}}\; \boxed{F\{e^{-j\omega_0 t}\cdot A\} = 2\pi A\delta(\omega+\omega_0)}$$

→ The signal $Ae^{-j\omega_0 t}$ can be represented by a rotating vector of magnitude, 'A' in clockwise direction in a complex plane with an angular speed of $\omega_0 t$.

→ The magnitude spectrum of $Ae^{-j\omega_0 t}$ is

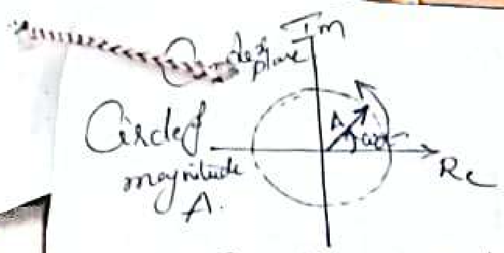


Complex exponential signal.



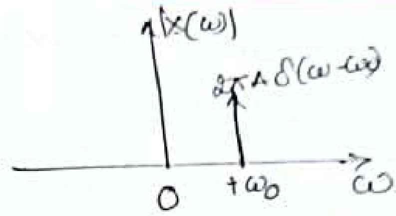
Magnitude Spectrum of $Ae^{-j\omega_0 t}$.

→ The signal $Ae^{+j\omega_0 t}$ can be represented by a rotating vector of magnitude A in anticlockwise direction in a complex plane with an angular speed of $\omega_0 t$ is



Circle of magnitude A.

Complex exponential Signal.



Magnitude Spectrum of $Ae^{+j\omega_0 t}$.

(5) Fourier Transform of Sinusoidal Signal:—

The Sinusoidal Signal is defined as

$$x(t) = A\sin\omega_0 t = A.\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right]$$

Take FT, we get

$$F\{x(t)\} = \int_{-\infty}^{\infty} A\sin\omega_0 t . e^{-j\omega t} . dt$$

$$= \int_{-\infty}^{\infty} \frac{A}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] . e^{-j\omega t} . dt$$

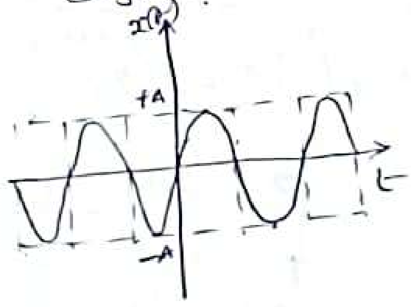
$$= \frac{A}{2j} \int_{-\infty}^{\infty} \left[e^{-j(\omega_0 - \omega_0)t} - e^{-j(\omega + \omega_0)t}\right] . dt$$

$$= \frac{A}{2j}\left[2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)\right]$$

$$= \frac{A\pi}{j}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$$

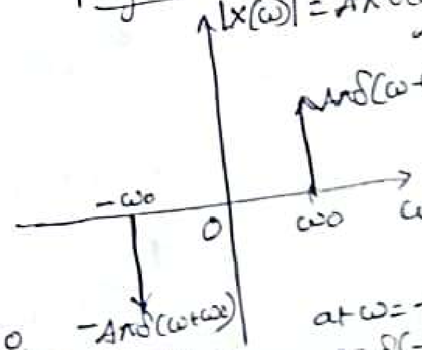
$$\therefore F\{A\sin\omega_0 t\} = \frac{A\pi}{j}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$$

Plot of Sinusoidal Signal.



Magnitude Spectrum

$|X(\omega)| = A\pi\delta(\omega - \omega_0) - A\pi\delta(\omega + \omega_0)$



at $\omega = 0$
$A\pi\delta(-\omega_0) - A\pi\delta(\omega_0)$

at $\omega = \omega_0$
$A\pi\delta(0) - A\pi\delta(2\omega_0)$

at $\omega = -\omega_0$
$2\pi\delta(-2\omega_0) - 2\pi\delta(0)$

(i) Fourier Transform of Cosinusoidal Signal:-

It is defined as

$$x(t) = A\cos\omega_0 t = \frac{A}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

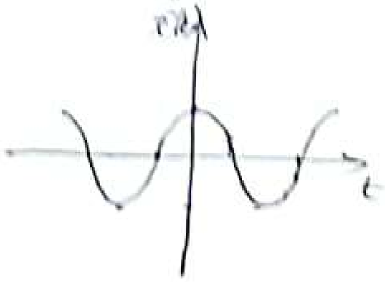
$$F\{x(t)\} = X(\omega) = \frac{A}{2}\int_{-\infty}^{\infty} e^{j\omega_0 t}\cdot e^{-j\omega t}\cdot dt + \frac{A}{2}\int_{-\infty}^{\infty} e^{-j\omega_0 t}\cdot e^{-j\omega t}\cdot dt$$

$$= \frac{A}{2}\cdot\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\cdot dt + \frac{A}{2}\int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)t}\cdot dt$$

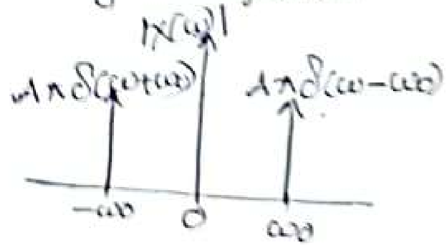
$$= \frac{A}{2}\cdot\left[2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right]$$

$$X(\omega) = A\pi\delta(\omega-\omega_0) + A\pi\delta(\omega+\omega_0)$$

Cosinusoidal Signal

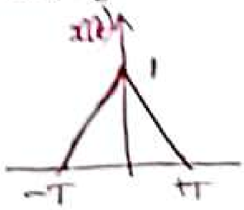


Magnitude Spectrum



(j) Fourier Transform of a triangular pulse Signal.

The mathematical equation of Δ^{le} pulse is



$$x(t) = 1 + t/T \;;\; \text{for } t = -T \text{ to } 0$$
$$= 1 - t/T \;;\; \text{for } t = 0 \text{ to } T$$

$$\therefore F\{x(t)\} = \int_{-T}^{0}(1+t/T)\,e^{-j\omega t}\cdot dt + \int_{0}^{T}(1-t/T)\cdot e^{-j\omega t}\cdot dt$$

$u = t$
$du = dt$
$dv = e^{-j\omega t}$
$\int v = \frac{e^{-j\omega t}}{-j\omega}$

→ Continued in the next page

$$= \int_{-T}^{0} e^{-j\omega t}\cdot dt + \int_{-T}^{0} t/T\cdot e^{-j\omega t}\cdot dt + \int_{0}^{T} e^{-j\omega t}\cdot dt - \int_{0}^{T}\frac{t}{T}e^{-j\omega t}\cdot dt$$

$$= \frac{e^{-j\omega t}}{-j\omega}\Big|_{-T}^{0} + \frac{1}{T}\left[t\cdot\frac{e^{-j\omega t}}{-j\omega} - \int\frac{e^{-j\omega t}}{-j\omega}\cdot dt\right]_{-T}^{0} + \frac{e^{-j\omega t}}{-j\omega}\Big|_{0}^{T} - \frac{1}{T}\left[t\,\frac{e^{-j\omega t}}{-j\omega} - \int\frac{e^{-j\omega t}}{-j\omega}\cdot dt\right]_{0}^{T}$$

$$= \frac{1}{-j\omega}\left[1 - e^{j\omega T}\right] + \frac{1}{T}\left[0 + \frac{1}{j\omega}\cdot\frac{e^{-j\omega t}}{-j\omega}\Big|_{-T}^{0} - T e^{j\omega T}\right]$$

Continuation.

$$\int uv = u\int v - \int v.du = u\int v - \int[du\int v]$$

$$F\{x(t)\} = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}.dt$$

$$= \int_{-T}^{0}(1+t/T)e^{-j\omega t}.dt + \int_{0}^{T}(1-t/T)e^{-j\omega t}.dt.$$

$$= \int_{-T}^{0} e^{-j\omega t}.dt + \frac{1}{T}\int_0^T t\,e^{-j\omega t}.dt + \int_0^T e^{-j\omega t}.dt - \frac{1}{T}\int_0^T t\,e^{-j\omega t}.dt.$$

$$= \frac{e^{-j\omega t}}{-j\omega}\Big|_{-T}^{0} + \frac{1}{T}\left[t\frac{e^{-j\omega t}}{-j\omega} - \int\frac{e^{-j\omega t}}{-j\omega}.dt\right]_{-T}^{0} + \frac{e^{-j\omega t}}{-j\omega}\Big|_0^T - \frac{1}{T}\left[t\frac{e^{-j\omega t}}{-j\omega} - \int 1.\frac{e^{-j\omega t}}{-j\omega}dt\right]_0^T$$

$$= \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{-T}^{0} - \frac{1}{j\omega T}\left[t\,e^{-j\omega t} - \int e^{-j\omega t}.dt\right]_{-T}^{0} - \frac{1}{j\omega}\left[e^{-j\omega t}\right]_0^T$$

$$+ \frac{1}{j\omega T}\left[t\,e^{-j\omega t} - \int e^{-j\omega t}.dt\right]_0^T$$

$\int uv$, $u = t$, $\int dv = e^{-j\omega t}$, $dv = e^{-j\omega t}dt$, $v = \int e^{-j\omega t} = \frac{e^{-j\omega t}}{-j\omega}$

$$= \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{-T}^{0} - \frac{1}{j\omega T}\left[t\,e^{-j\omega t} - \frac{e^{-j\omega t}}{-j\omega}\right]_{-T}^{0} - \frac{1}{j\omega}\left[e^{-j\omega t}\right]_0^T$$

$$+ \frac{1}{j\omega T}\left[t\,e^{-j\omega t} - \frac{e^{-j\omega t}}{-j\omega}\right]_0^T$$

$$= \frac{-1}{j\omega}\left[e^{0} - e^{-j\omega(-T)}\right] - \frac{1}{j\omega T}\left[0 - \frac{e^0}{-j\omega} + Te^{j\omega T} + \frac{e^{j\omega T}}{-j\omega}\right]$$

$$- \frac{1}{j\omega}\left[e^{-j\omega T} - e^0\right] + \frac{1}{j\omega T}\left[Te^{-j\omega T} - \frac{e^{-j\omega T}}{-j\omega} - 0 + \frac{e^0}{-j\omega}\right]$$

$$= \cancel{\frac{-1}{j\omega}} + \cancel{\frac{e^{j\omega T}}{j\omega}} + \frac{1}{\omega^2 T} - \cancel{\frac{e^{j\omega T}}{j\omega}} - \frac{1}{\omega^2 T}e^{j\omega T} - \cancel{\frac{e^{-j\omega T}}{j\omega}} + \cancel{\frac{1}{j\omega}} + \cancel{\frac{e^{-j\omega T}}{j\omega}} - \frac{e^{-j\omega T}}{\omega^2 T} + \frac{1}{\omega^2 T}$$

$$= \frac{1}{\omega^2 T}\left[2 - e^{j\omega T} - e^{-j\omega T}\right] = \frac{2}{\omega^2 T} - \frac{2}{\omega^2 T}\cos\omega T$$

$$= \frac{2}{\omega^2 T} - \frac{1}{\omega^2 T}\left[e^{j\omega T} + e^{-j\omega T}\right] = \frac{2}{\omega^2 T}\left[1 - \cos\omega T\right]$$

(d).

$$F\{x(t)\} = \frac{2}{\omega^2 T}(1-\cos\omega T) = \frac{2}{\omega^2 T}\left[1-\cos 2\left(\frac{\omega T}{2}\right)\right]$$

$$= \frac{2}{\omega^2 T}\left(2\sin^2\frac{\omega T}{2}\right)$$

$$= T \cdot \frac{4}{T^2\omega^2}\sin^2\frac{\omega T}{2} = T \cdot \frac{\sin^2\frac{\omega T}{2}}{(\omega T/2)^2}.$$

$$= T\left(\operatorname{sinc}\frac{\omega T}{2}\right)^2.$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\because \frac{\sin\theta}{\theta} = \operatorname{sinc}(\theta)$$

i.e. $\boxed{F\{x(t)\} = T\left(\operatorname{sinc}\frac{\omega T}{2}\right)^2}$

Ex:- ① Determine the IFT of the following function, using partial fraction expansion technique.

1) $X(\omega) = \dfrac{3(\omega)+14}{\omega^2+7\omega+12}$.   ② $X(\omega) = \dfrac{\omega+7}{(\omega+3)^2}$.

① $X(\omega) = \dfrac{3\omega+14}{\omega^2+7\omega+12} = \dfrac{3\omega+14}{\omega^2+3\omega+4\omega+12}$

$= \dfrac{3\omega+14}{\omega(\omega+3)+4(\omega+3)} = \dfrac{3\omega+14}{(\omega+3)(\omega+4)}$

ie: $X(\omega) = \dfrac{3\omega+14}{(\omega+3)(\omega+4)} = \dfrac{A}{(\omega+3)} + \dfrac{B}{(\omega+4)} = \dfrac{3\omega+14}{(\omega+3)(\omega+4)}$

$A(\omega+4)+B(\omega+3) = 3\omega+14$.

at $\omega = -3$

$A(1)+0 = -9+14$

$\boxed{A = 5}$

at $\omega = -4$

$B(-4+3) = -12+14$

$-B = +2$

$\therefore \boxed{B = -2}$

$\therefore X(\omega) = \dfrac{5}{\omega+3} + \dfrac{2}{\omega+4}$.

$IFT\{X(\omega)\} = x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \dfrac{5}{\omega+3} \cdot e^{j\omega t} \cdot d\omega +$

$\dfrac{1}{2\pi}\int_{-\infty}^{\infty} \dfrac{2}{\omega+4} \cdot e^{j\omega t} \cdot d\omega$

we know that $F\{e^{-at}u(t)\} = \dfrac{1}{a+i\omega}$.

$\therefore IFT\left\{\dfrac{1}{a+\omega}\right\} = e^{-at}u(t)$

$\therefore$ ① term becomes $5 \cdot e^{-3t}u(t) - 2e^{-4t} \cdot u(t) = x(t)$

&

② term becomes.

(2) $X(\omega) = \frac{\omega+7}{(\omega+3)^2} = \frac{A}{(\omega+3)} + \frac{B}{(\omega+3)^2} = \frac{\omega+7}{(\omega+3)^2}$

$$A(\omega+3) + B = \omega+7.$$

at $\omega = -3$
$B = 4$

at ~~B=4~~ $\omega = 0$.
$3A + B = 7.$
$3A + 4 = 7$
$3A = 3 \therefore A = 1$

$$X(\omega) = \frac{1}{\omega+3} + \frac{4}{(\omega+3)^2}$$

$$x(t) = F^{-1}\left\{\frac{1}{\omega+3}\right\} + 4F^{-1}\left\{\frac{1}{(\omega+3)^2}\right\}$$

$$\boxed{x(t) = e^{-3t}u(t) + 4t\,e^{-3t}.u(t)}$$

Ex:- Determine the convolution of $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = e^{-6t}u(t)$ using FT

Let
$X_1(\omega) = FT\{x_1(t)\}$
$X_2(\omega) = FT\{x_2(t)\}$

Using Convolution property.

$$F\{x_1(t) * x_2(t)\} = X_1(\omega).X_2(\omega)$$

Here $X_1(\omega) = \int_{-\infty}^{\infty} e^{-2t}.u(t)\,e^{-j\omega t}.dt = \int_0^{\infty} e^{-(2+j\omega)t}.dt.$

$= \frac{1}{2+j\omega}$ we know from previous examples

$X_2(\omega) = \int_0^{\infty} e^{-6t}.u(t)\,e^{-j\omega t}.dt = \frac{1}{6+j\omega}$ "

$\therefore F\{x_1(t) * x_2(t)\} = X(\omega) = X_1(\omega).X_2(\omega)$

$= \frac{1}{2+j\omega}.\frac{1}{6+j\omega} = \frac{1}{(2+j\omega)(6+j\omega)}$

By taking partial fractions

$$\frac{A}{2+j\omega} + \frac{B}{6+j\omega} = \frac{1}{(2+j\omega)(6+j\omega)}$$

$A(6+j\omega) + B(2+j\omega) = 1$

at $j\omega = -2$

$A(6-2) + 0 = 1$

$A = 1/4$

at $j\omega = -6$

$0 + B(2-6) = 1$

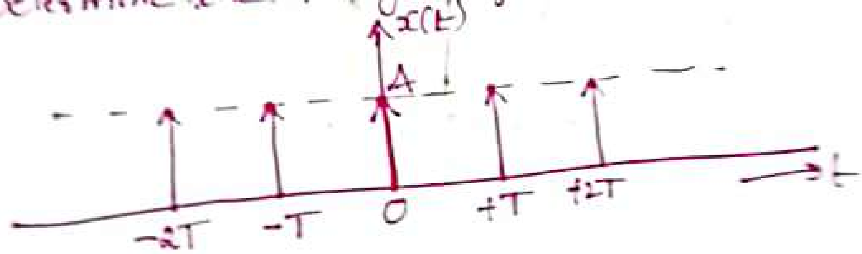
$B = -1/4$

$$\text{IFT}\{X(\omega)\} = \frac{\frac{1}{4}}{2+j\omega} + \frac{-1/4}{6+j\omega}$$

$$\text{IFT}\{X(\omega)\} = x(t) = \frac{1}{4} \cdot e^{-2t} u(t) - \frac{1}{4} \cdot e^{-6t} \cdot u(t)$$

$$\boxed{x(t) = 0.25\left[e^{-2t} - e^{-6t}\right] u(t)}$$

Ex:- Determine the FT of the periodic impulse function.



→ The mathematical equation for one period of the periodic impulse function is

$$x(t) = A\,\delta(t) \; ; \; \text{for } t = -T/2 \text{ to } +T/2$$

The fourier co-efficient C_n is given by

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} A\,\delta(t)\, e^{-jn\omega t}\, dt$$

$$= \frac{A}{T} \cdot e^{-jn\omega t}\Big|_{t=0} = \frac{A}{T}$$

The exponential fourier series representation of the periodic impulse train is —

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

∴ On taking FT of the above equation we get

$$F\{x(t)\} = F\left\{\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right\}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} C_n F\{e^{jn\omega_0 t}\}$$

$$= \sum_{n=-\infty}^{\infty} C_n . \delta(\omega - n\omega_0) \times 2\pi$$

$$= \sum_{n=-\infty}^{\infty} \frac{A}{T} \delta(\omega - n\omega_0) \times 2\pi$$

$$\therefore \sum_{n=-\infty}^{\infty} A.\omega_0 \delta(\omega - n\omega_0) \quad \left(\because 2\pi \times \frac{1}{T} = 2\pi f_0 = \omega_0\right)$$

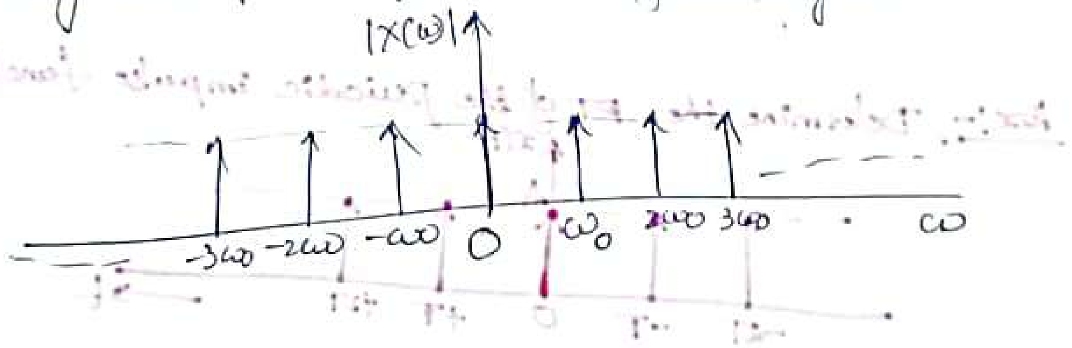
$$F\{e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} . dt$$

$$= \int_{-\infty}^{\infty} e^{-j(\omega - \omega_0)t} . dt$$

$$= \delta(\omega - \omega_0)$$

$$\frac{1}{T_0} = f$$

Magnitude Spectrum of $X(\omega)$ which also a periodic impulse function of ω.



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UNIT-III

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Distortionless Transmission Through System:-

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→ A distortionless transmission means o/p of the System is an exact Replica of the i/p Signal.

→ The difference b/w i/p & o/p of Such system is that,

1. Amplitude of the o/p Signal may increase (or) decrease by Some factor with respect to i/p and
2. The o/p of the Signal may be delayed in time w.r.t i/p Signal. i.e of System delay.

∴ O/p $y(t)$ can be written in terms of i/p $x(t)$ as.

$y(t) = K\,x(t-t_0)$ —① K → Constant represents change in amplitude.

t_0 → time delay in transmission of Signal through a system.

Take FT on both sides.

$FT\{y(t)\} = Y(\omega) = FT\{K\,x(t-t_0)\}$

$= K.\,e^{-j2\pi f t_0}.\,X(\omega)$ as per time Shifting property.

(or)

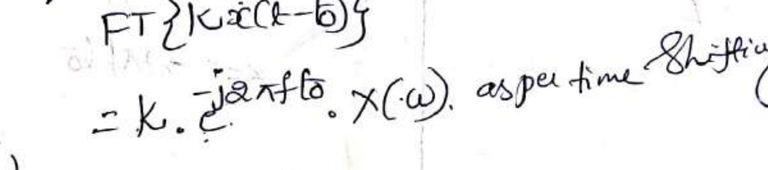
$$Y(f) = K.\,e^{-j2\pi f t_0}\,X(f) \quad —②$$

Now, the transfer function $H(f)$ is given as

$$\boxed{H(f) = \frac{Y(f)}{X(f)} = K\,e^{-j2\pi f t_0}} \quad —③$$

This is the Transfer function for a distortionless System.

In the above, 'K' is magnitude which is independent of frequency. i.e, the transfer function $H(f)$ has constant amplitude at all frequencies.



→ The phase Shift of above equation is.

$\therefore \theta(f) = -2\pi f t_0$

$= (-2\pi t_0) f$ → ④

$K.\cos(2\pi f t_0)$
$-j\sin(2\pi f t_0)$

④ Shows the phase Shift is linearly proportional to frequency. Hence the phase Shift is linear at all frequencies.

$K.\tan\theta = \frac{\sin 2\pi f t_0}{\cos 2\pi f t_0}$

$\theta = \tan^{-1}\tan 2\pi f t_0$

For the phase shift, the example is $x(t) = \cos 2\pi f t$

$y(t) = \cos[2\pi f(t - t_0)]$ as time shifted 'to' sec

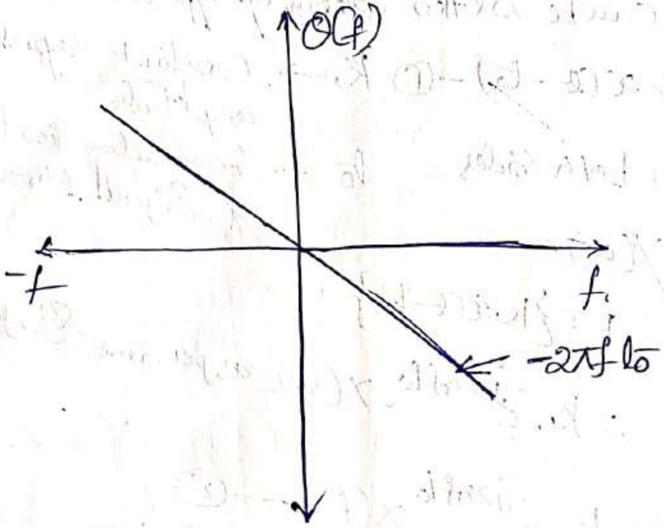
$\therefore y(t) = \cos[2\pi f t - 2\pi f t_0]$

$= \cancel{\cos 2\pi f t \cdot \cos 2\pi f t_0}$

$= \cos[2\pi f t + \theta(f)]$ $\because \theta(f) = -2\pi f t_0$

Thus phase shift of $y(t)$ is

$\boxed{\theta(f) = -2\pi f t_0}$ which is proportional to 'f'.



Phase Spectrum of distortionless transmission

Filter Characteristics of Linear System:-

(a) Ideal Low pass Filters:-

→ An ideal LPF transmits (passes to o/p) all of the Signals below certain frequency 'ω' H_z w/o any distortion.

→ The range of frequencies from 0 H_z to 'ω' H_z is called passband of the low pass filter.

→ It rejects all the Signals which lie Outside of the passband.

→ The frequency 'ω' H_z is called cut-off frequency of the ideal LPF.

→ Since the filter is ideal and distortionless the phase should follow.

$\boxed{\theta(f) = -2\pi f t_0}$

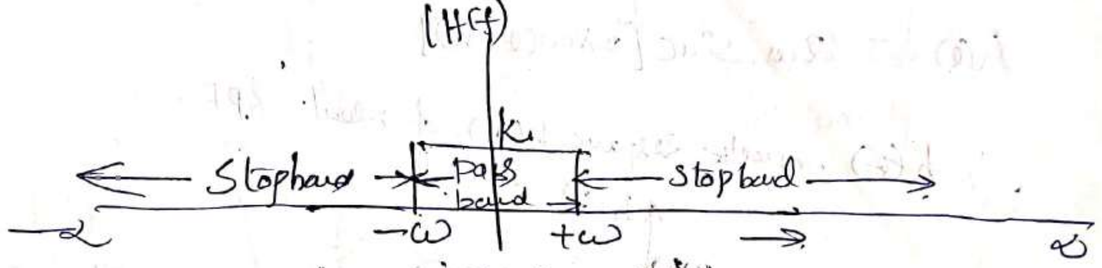
The below shows the magnitude & phase response of the Transfer function. 24

$\therefore$ Transfer function of ideal LPF can be written as

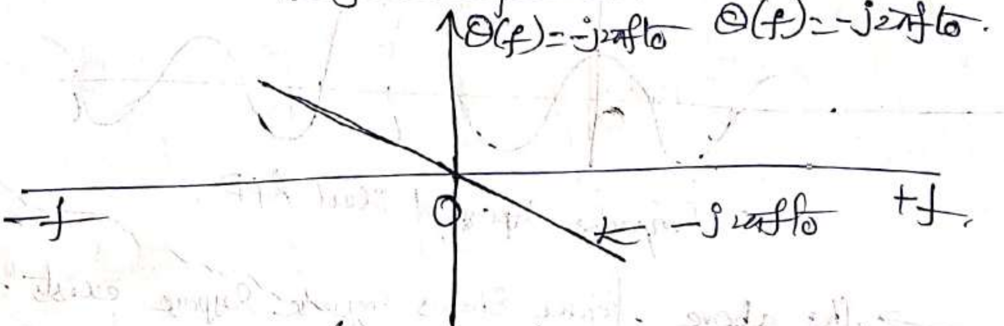
$$H(f) = K \cdot e^{-j2\pi f t_0} \quad ; \quad -\omega \le f < \omega$$
$$= 0 \quad ; \quad |f| > B$$

$$\therefore |H(f)| = K \cdot (1) \quad ; \quad -\omega \le f < \omega$$
$$= 0 \quad ; \quad |f| > \omega$$

$\because e^{-j2\pi f t_0} = \cos 2\pi f t_0 - j\sin 2\pi f t_0$
$= \sqrt{\cos^2 + \sin^2} = 1$



Magnitude Spectrum



$\theta(f) = -j2\pi f t_0$.

Phase Spectrum

→ If $K=1$ in above eq, then the transfer function

$$H(f) = e^{-j2\pi f t_0} \quad ; \quad -\omega \le f \le \omega$$
$$= 0 \quad ; \quad |f| > \omega$$

→ The $H(f)$ is FT of $h(t)$ i.e., impulse response can be obtained for ideal LPF.

$\therefore h(t) = IFT\{H(f)\}$

$$h(t) = \frac{1}{2\pi}\int_{-\omega}^{\omega} e^{-j2\pi f t_0} \cdot e^{j2\pi f t} \, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega}^{\omega} e^{+j2\pi f(t-t_0)} \, d\omega$$

$$= \frac{1}{2\pi} \frac{e^{+j2\pi f(t-t_0)}}{j2\pi(t-t_0)} \Big|_{-\omega}^{+\omega}$$

$$= \frac{1}{j2\pi(t-t_0)}\left[e^{j2\pi\omega(t-t_0)} - e^{-j2\pi\omega(t-t_0)}\right]$$

$$= \frac{1}{\pi(t-t_0)}\left[\frac{e^{j2\pi\omega(t-t_0)} - e^{-j2\pi\omega(t-t_0)}}{2j}\right]$$

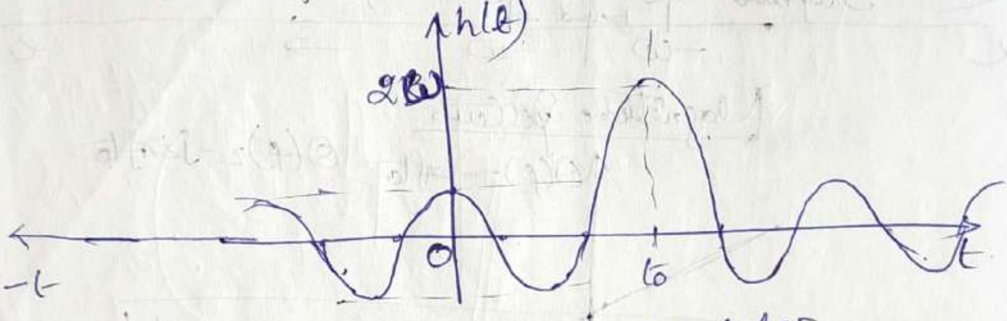
$$= \frac{1}{\pi(t-t_0)} \cdot \sin\left[2\pi\omega(t-t_0)\right]$$

$$= \frac{2\omega}{2\omega} \cdot \frac{1}{\pi(t-t_0)} \cdot \sin\left[2\pi\omega(t-t_0)\right]$$

$$= 2\omega \cdot \left[\frac{\sin 2\pi\omega(t-t_0)}{2\pi\omega(t-t_0)}\right]$$

$$h(t) = 2\omega \cdot \operatorname{Sinc}\left[2\pi\omega(t-t_0)\right]$$

$\therefore$ $h(t)$ impulse response $h(t)$ of ideal LPF



Impulse response of ideal LPF.

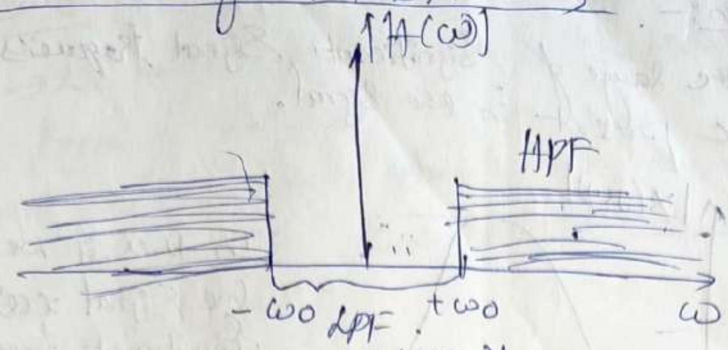
→ The above figure shows impulse response exists for -ve values of 't'. But actually unit impulse is applied at $t=0$ always.

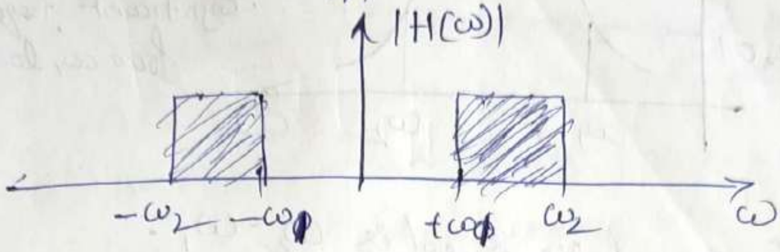
→ Practically it is impossible to implement such a system. And $h(t)=0$ for $t<0$ for causal system.

→ $\therefore$ it is clear that although ideal LPF is very desirable it cannot be physically realizable.

→ Practically unit impulse to the filter is applied at $t=0$, then impulse response of the filter should start at the most at $t \geq 0$. & not at $t<0$, i.e., -ve value of 't'.

Ideal filters of HPF, BPF etc.,





→ From previous LPF impulse response begins before i/p is applied, means that ideal LPF is anticipatory.

→ Hence LPF is not physically realizable.

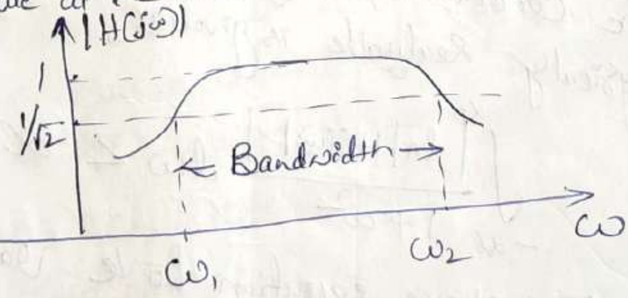
→ Similarly, other ideal filters like HPF & BPF have frequency response.

→ These filters have sharp transition at cut-off frequencies.

→ These sharp transition in frequency response results in non-causal impulse response (i.e., it begins before i/p is applied). This means all ideal filters are physically not realizable since their impulse response is non-causal.

Bandwidth of a System :— (a) System Bandwidth

→ It is nothing but the interval of frequencies over which the magnitude $|H(j\omega)|$ (or) $|H(f)|$ & $|H(\omega)|$ remains within $\frac{1}{\sqrt{2}}$ times of its value at the midband.

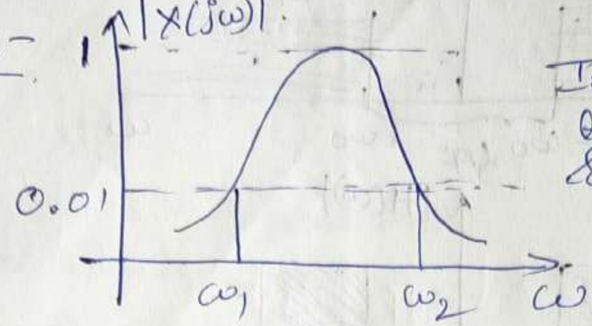


$$\boxed{BW = \omega_2 - \omega_1}$$

* Signal Bandwidth:-

→ It is the range of significant signal frequencies which are present in the signal.

Ex:-



In this if we observe the signal $x(t)$ has significant frequency from ω_1 to ω_2.

The BW of the signal is $\omega_2 - \omega_1$.

→ Normally, all the physically obtained signals have limited BW.

1. The transmission through the system will be distortionless if it has infinite BW.
2. When the signal has finite BW, then the system BW must be sufficiently high to accommodate the signal for distortionless transmission.

Causality and Paley-Wiener Criterion:-

→ Paley-Wiener Criterion gives the condition for causality in frequency domain.

→ Causality relates to physical realizability.

→ The condition for magnitude function $|H(j\omega)|$ to be physically realizable is given as

$$\int_{-\infty}^{\infty} \frac{|\ln|H(j\omega)||}{1+\omega^2}\, d\omega < \infty$$

→ This equation is known as Paley Wiener Criterion.

For above equation to be valid, $|H(j\omega)|$ must be square integrable i.e.,

$$\int_{-\infty}^{\infty} |H(j\omega)|^2\, d\omega < \infty$$

(a) Causality Condition & physical Realizability :-

→ We know paley-wiener condition for physical realizability in frequency domain.

→ Its time domain equivalent can be stated as

1. The System which violates paley-wiener Criterion is anticipatory.

Such system produces Response before i/p is applied.

∴ Such System is physically not realizable.

2. The System which satisfies paley-wiener Criterion has causal unit Sample Response i.e.,

$h(t) = 0$ for $t < 0$ Such systems are physically realizable.

(b) Interpretations of paley-Wiener Criterion:-

1. For a physically realizable System, $|H(j\omega)|$ may be '0' at certain discrete frequencies. But it cannot be '0' over a finite band of frequencies, Otherwise

$$\int_{-\infty}^{\infty} \frac{|\ln|H(j\omega)||}{1+\omega^2}\, d\omega < \infty$$ would become infinite

2. For a physically realizable system, $|H(j\omega)|$ cannot decay faster than a function of exponential order.

For ex:- $H(j\omega) = e^{-\alpha|\omega|}$ represents realizable system

but $H(j\omega) = e^{-\alpha\omega^2}$ is not realizable.

Rise Time & System BW:-

→ System BW can be obtained from rise time.

→ Let a unit step function $u(t)$ be applied to an ideal LPF.

The response $r(t)$ is Observed.

→ The time taken by the response to rise to its final value depends upon cut-off frequency (ω_m) of the filter, i.e., System BW.

→ The t_r – rise time is the time required by the response to reach its final value from initial value.

Step 1:- The Transfer function of ideal LPF is given as.

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$$

$= G(\omega) e^{-j\omega t_0}$ from rectangular pulse as magnitude of $|H(j\omega)|$

$G(\omega) \rightarrow$ Rectangular pulse (gate function) its magnitude is K. for $-\omega \le f \le +\omega$

i.e; $-\omega_m \le \omega < +\omega_m$

where $\omega_m = 2\pi f$

$\theta(\omega) = -2\pi f t_0 = -\omega t_0$

Step 2:- $FT\{u(t)\} = \pi\delta(\omega) + \frac{1}{j\omega}$

Step 3:- FT of response $R(\omega)$, i/p & system function $H(\omega)$ are related as

$$R(\omega) = \left[\pi\delta(\omega) + \frac{1}{j\omega}\right] H(\omega)$$

$$= \pi\delta(\omega) H(\omega) + \frac{1}{j\omega} H(\omega)$$

$= \pi\delta(\omega) + \frac{1}{j\omega} H(\omega)$ [$\because$ $\delta(\omega)$ exists only for $\omega = 0$, & $H(\omega)\big|_{\omega=0} = 1$, hence $\delta(\omega)H(\omega) = \delta(\omega)$

Step 4:- IFT of $R(\omega) = r(t)$

$$\therefore r(t) = IFT\left\{\pi\delta(\omega) + \frac{1}{j\omega} H(\omega)\right\}$$

$$= IFT\left\{\pi\delta(\omega) + \frac{1}{j\omega} G(\omega) \cdot e^{-j2\pi f t_0}\right\}.$$

we know $FT\{1\} = 2\pi\delta(\omega)$, $\therefore IFT\{\pi\delta(\omega)\} = \frac{1}{2}$

$$r(t) = \frac{1}{2} + IFT\left\{\frac{1}{j\omega} G(\omega) \cdot e^{-j2\pi f t_0}\right\}$$

$$= \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{j\omega} G(\omega) \cdot e^{-j2\pi f t_0}\right] e^{j\omega t} \cdot d\omega$$

we know. $G(\omega) = 1$ for $-\omega_m \le \omega < \omega_m$. Hence.

$$\therefore r(t) = \frac{1}{2} + \frac{1}{2\pi} \int_{-\omega_m}^{\omega_m} \frac{1}{j\omega} \cdot e^{j\omega(t-t_0)} \cdot d\omega.$$

$$= \frac{1}{2} + \frac{1}{2\pi} \cdot \int_{-\omega_m}^{+\omega_m} \frac{\cos\omega(t-t_0) + j\sin\omega(t-t_0)}{j\omega}$$

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$$= \frac{1}{2} + \frac{1}{2\pi}\frac{1}{j\omega}\times 0 + \frac{1}{2\pi}\int_{-\omega_m}^{\omega_m} \frac{\sin\omega(t-t_0)}{\omega}\,d\omega$$ $\because \int$ integration of $\cos\omega(t-t_0)$ $= 0 \because$ it is an odd.

$$= \frac{1}{2} + \frac{1}{2\pi}\times 2\int_{0}^{\omega_m} \frac{\sin\omega(t-t_0)}{\omega}\,d\omega$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[Si\ \omega(t-t_0)\right]_0^{\omega_m}$$ from standard integration rules.

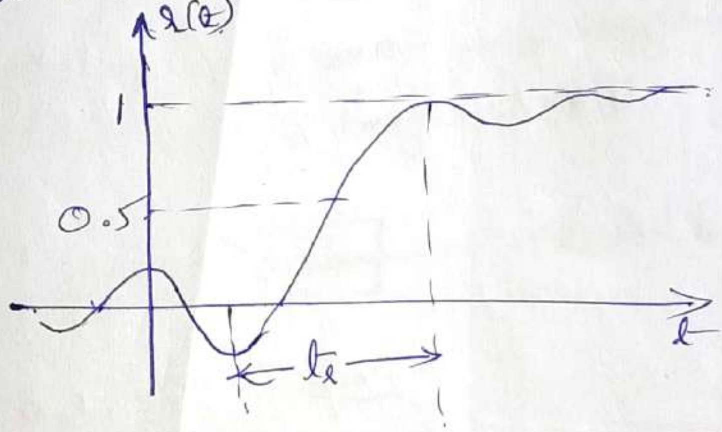
$$\boxed{r(t) = \frac{1}{2} + \frac{1}{\pi} Si\ \omega_m(t-t_0)}$$ is the response of LPF system having BW ω_m.

The rise time is

$$t_r = \frac{2\pi}{\omega_m} = \frac{2\pi}{2\pi f_m} = \frac{1}{f_m} = \frac{1}{B} = \frac{1}{\omega_m}.$$

As $\omega_m\uparrow$, $t_r\downarrow$ & $r(t)$ increases slowly.

The ω_m bandwidth & rise time are inversely proportional to each other.



$$Si(x) = \int_0^x \frac{\sin t}{t}\,dt.$$

$\frac{\sin t}{t} \rightarrow$ sinc function & also zeroth spherical bessel function.

sinc is an even entire function

$Si \rightarrow$ is entire, odd, & the integral in its definition can be taken along any path connecting the endpoints

$Si(x)$ is the antiderivative.

UNIT-IV

RANDOM PROCESSES - TEMPORAL CHARACTERISTICS

Introduction:-

→ We know, the real world of Engineering and Sciences be able to deal with time waveforms.

→ In our practical Systems, we frequently encountered random time waveforms.

→ Means that the desired Signal from the system in Some-times is random.

For ex:- The bit stream in a Binary Communication System is a random message because each bit in the Stream occurs randomly.

→ On the other hand, the desired Signal is often accompanied by an undesired random waveform, which is noise.

→ when this noise interferes with message Signal (or) Information Signal. which will limits the performance of the System.

→ So, in dealing any System with random waveforms is nothing but. the ability to describe and deal with Such waveforms.

RANDOM PROCESS CONCEPT:-

→ This process is based on enlarging the random variable concept to include time.

Random Variable:- (X)

It is a function of the possible outcomes 's' of an experiment.

It is a function of both 's' and time.

∴ it is like $x(t,s)$ to every outcome 's'.

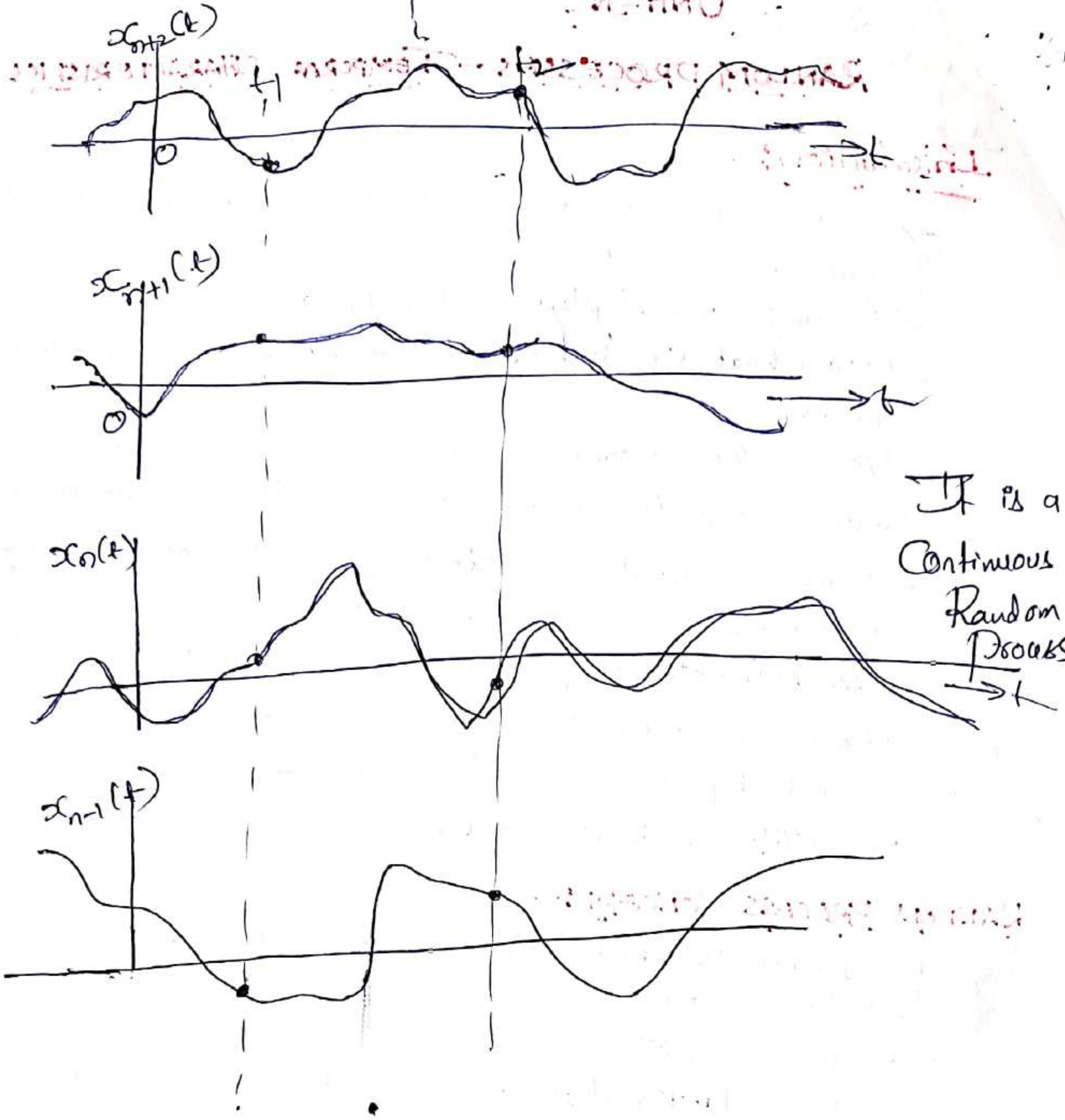
→ The family of all Such functions denoted $X(t,s)$ is called random process

'X' is a Random Variable.

'x' is a Specific Value of 'X' (RV).

→ In this $x(t)$ represents a Specific waveform of a Random process $X(t)$.

→ So a Random process $X(t,S)$ represents a family or ensemble of time functions when 't' and 'S' are Variables.



It is a Continuous Random Process.

→ A Random process also represents a Random Variable, when t is fixed and 'S' is a variable. For ex:- A RV $X(t_1, S) = X(t_1)$ is obtained when time is "frozen" at the value 't_1'.

→ We often use the notation X_1 to denote the RV associated with the $X(t)$ at time 't_1'.

X_1 → corresponds to a vertical "slice" through the ensemble at time 't_1'.

→ The Statistical properties of $X_1 = X(t_1)$ describes the statistical properties of the Random process at time 't_1'.

→ The expected value of X_1 is called the ensemble average as well as the expected (or) mean value of the random process (at time t_1).

We easily shows any number of random variables X_i derived from a Random process $X(t)$ at times t_i, $i=1,2 \ldots$

$$\boxed{X_i = X(t_i; S) = X(t_i)}$$

A <u>Random Process</u> can also called as <u>Stochastic process</u>

Classification of processes:-

→ The classification of random process is done according to the Characteristics of 't' and the random Variable $X = X(t)$ at time 't'.

→ we considered only 4 cases based on t and x having values in the ranges $-\infty < t < \infty$ and $-\infty < x < +\infty$.

→ They are
1. Continuous Random processes.
2. Discrete Random processes.
3. Continuous Random Sequence
4. Discrete Random Sequence.

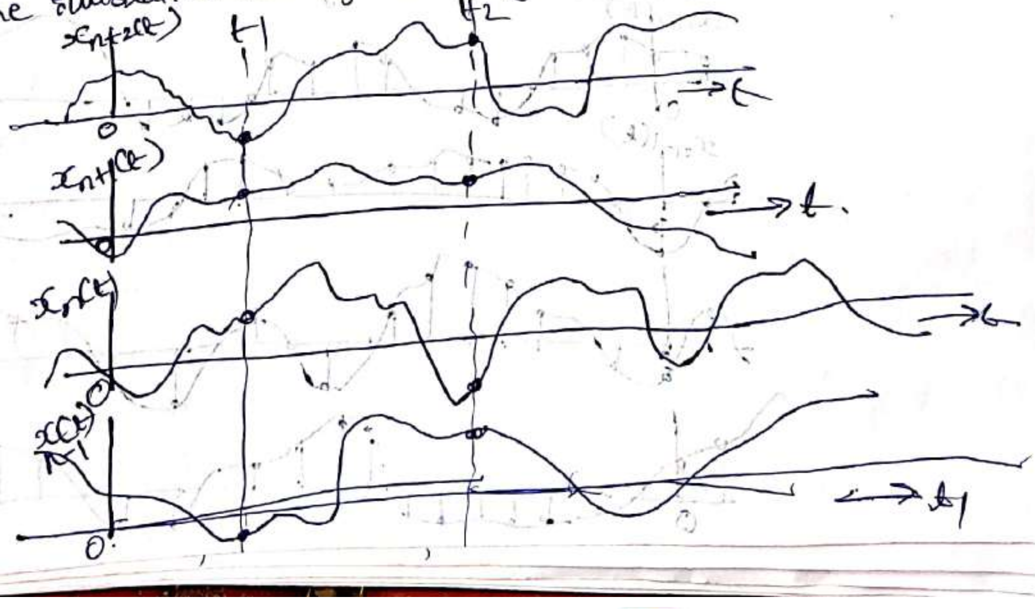
1. <u>Continuous Random Processes</u>:-

→ In this Random process $X(t)$, if the RV 'x' is continuous for all values of time 't', then it is termed as Continuous Random process.

Ex:- The Thermal noise which is generated by any realizable network is a practical example of a waveform

→ Each network establishes a sample function, and all sample functions form the process.

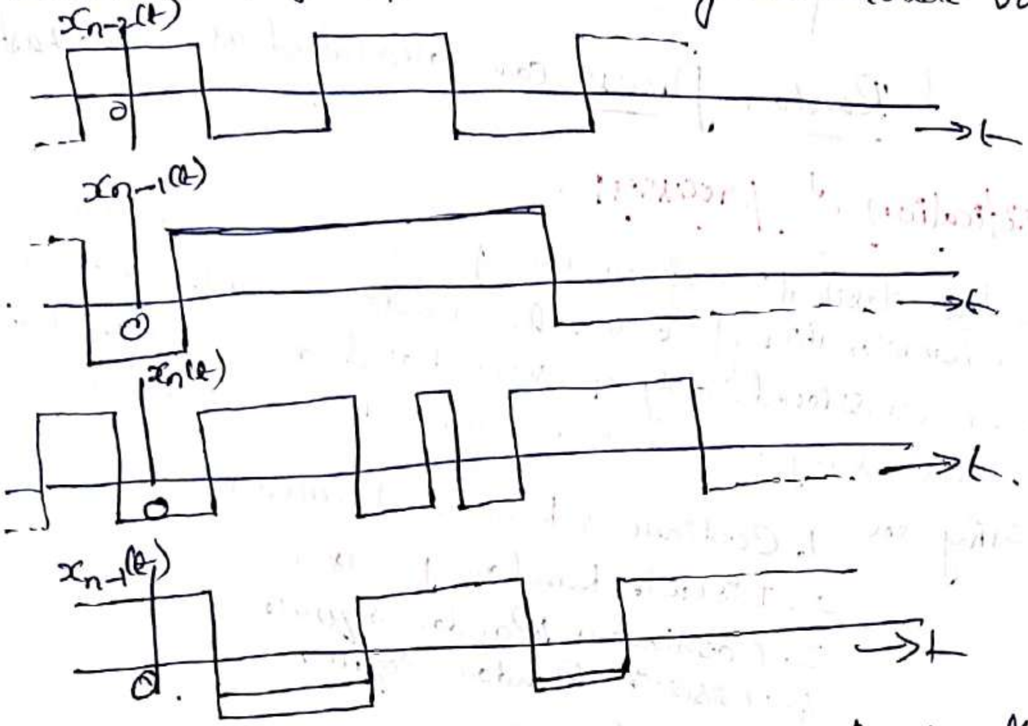
→ The illustration is explained by



2. Discrete Random process:-

→ This Random process corresponds to the Random variable X having only discrete values while 't' is continuous

→ In this the sample functions have only two discrete values.

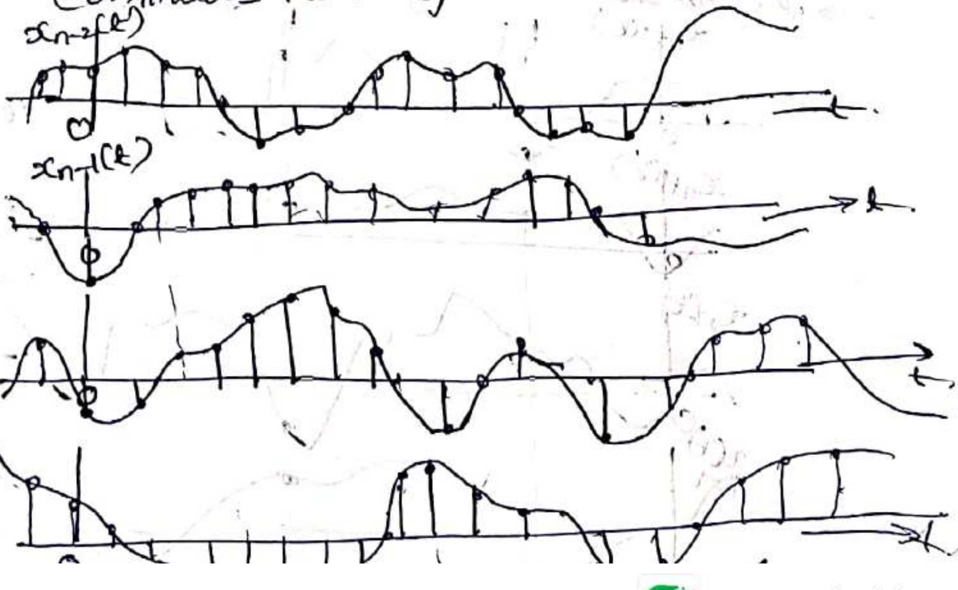


which is heavily limiting the waveforms of Continuous Random process figures.

3. Continuous Random Sequence:-

A Random process for which X is continuous but time has only discrete values is called a continuous Random sequence

Ex:- This example of such a sequence can be formed by periodically sampling the ensemble members of Continuous Random process.



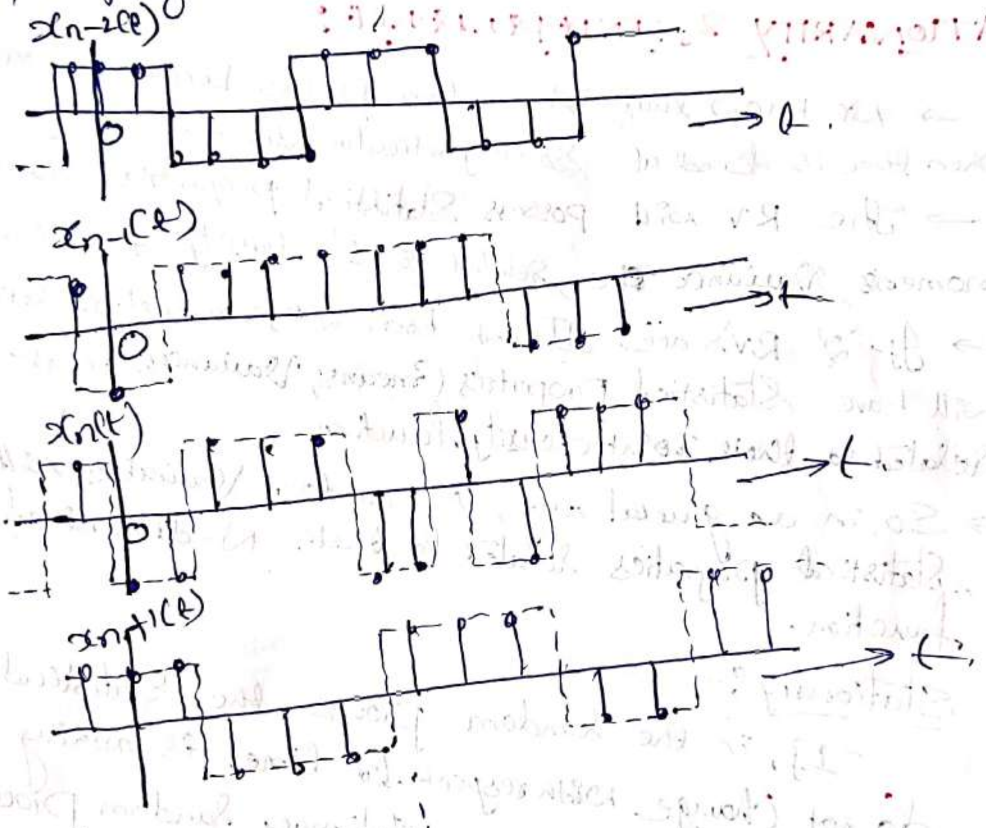
Scanned with OKEN Scanner

∴ a continuous Random Sequence is defined at only discrete (Sample) times, it is also frequently called a discrete-time (DT) random process, it's sample functions are often referred to as a DT random signals.

4. Discrete Random sequence :-

→ In this both time and random variable being discrete.

→ The figure which is used as an example is Discrete random process where discrete random sequence developed by Sampling the Sample function.



Deterministic & Non-deterministic processes:-

→ In addition to the above Classification, the random process can be described by the form of its Sample functions.

→ "IF FUTURE VALUES OF ANY SAMPLE FUNCTION CANNOT BE PREDICTED EXACTLY FROM OBSERVED PAST VALUES" then this process is called non-deterministic otherwise deterministic. Example is Same as Continuous Random process figures, and it is defined by

$$\boxed{X(t) = A\cos(\omega_0 t + \Theta)}$$

→ Here A, Θ, or ω_0 (or all) may be random variables.

→ Any one sample function corresponds to the above equation, with particular values of these random variables

→ ∴ Prior to any time instant, the knowledge of the sample function, automatically allows prediction of the sample function's future values because its form is known.

STATIONARITY & INDEPENDENCE:-

→ We know that 'A random process becomes a random variable when time is fixed at some particular value'.

→ The RV will possess statistical properties, such as a mean value, moments, variance etc., related to its density function.

→ If '2' RV's are obtained from '2' time instants like t_1 & t_2, they will have statistical properties (means, variances, joint moments etc.,) related to their joint density function.

→ So, in a general way, N-random variables will possess statistical properties related to their N-dimensional joint density function.

Stationary:-

If, in the random process, the statistical properties do not change with respect to time is nothing but stationary random process. Otherwise called as non-stationary random process

Distribution and Density functions:-

→ To define stationarity, firstly define distribution and density function, as they apply to a random process $X(t)$.

→ For a particular time t_1, the distribution function associated with the random variable $X_1 = X(t_1)$ will be denoted with $F_X(x_1; t_1)$. It is defined as where x_1 = real number.

$$\boxed{F_X(x_1; t_1) = P\{X(t_1) \le x_1\}}$$

→ For two random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$, the second order joint distribution function is the 2-D extension of above equation

$$\boxed{F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}}$$

→ In a similar manner for N random variables $X_i = X(t_i)$ $i = 1, 2 \dots N$ the N^{th}-order joint distribution function is

$$\boxed{\begin{aligned} F_X(x_1, x_2 \dots x_N; t_1, t_2 \dots t_N) &= P\{X(t_1) \le x(t_2) \\ &= P\{X(t_1) \le x_1, X(t_2) \le x_2 \dots \dots X(t_N) \le x_N\} \end{aligned}}$$

→ The joint density functions of interest are found from appropriate derivatives of the above '3' equations.

$$f_X(x_1; t_1) = dF_X(x_1; t_1)/dx_1$$

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2}{(\partial x_1 \partial x_2)}\left[F_X(x_1, x_2; t_1, t_2)\right]$$

$$f_X(x_1, \dots x_N; t_1, t_2 \dots t_N) = \frac{\partial^N}{(\partial x_1 \dots \partial x_N)}\left[F_X(x_1, \dots x_N; t_1, \dots t_N)\right]$$

Statistical Independence:-

→ Two processes $X(t)$ and $Y(t)$ are statistically independent if the random variable group $X(t_1), X(t_2) \dots X(t_N)$ is independent of the group $Y(t_1'), Y(t_2') \dots Y(t_M')$ for any choice of terms of times $t_1, t_2 \dots t_N$, $t_1', t_2' \dots t_M'$.

Independence requires that the joint density be factorable by groups.

$$f_{X,Y}(x_1 \dots x_N, y_1 \dots y_M; t_1 \dots t_N, t_1' \dots t_M')$$
$$= f_X(x_1 \dots x_N; t_1 \dots t_N)\, f_Y(y_1 \dots y_M; t_1' \dots t_M')$$

First-order Stationary Processes:-

→ A random process is called first-order Stationary, if its first-order density function doesn't change with a Shift in time origin. Other way

$$\boxed{f_X(x_1; t_1) = f_X(x_1; t_1+\Delta)} \quad — (1)$$

must be true for any t_1 and real number Δ. if $X(t)$ is to be first-order Stationary process.

→ The process mean value $E[X(t)]$ is a constant

$$\boxed{E[X(t)] = \bar{X} = \text{Constant}} \quad — (2)$$

Every Challenge is an opportunity

For X_1:

The mean values of the RV's $X_1 = X(t_1)$ and $X_2 = X_2(t_2)$

$$\boxed{E[X_1] = E[X(t_1)] = \int_{-\infty}^{+\infty} x_1 f_X(x_1; t_1)\, dx_1} \quad — (3)$$

For X_2:

$$\boxed{E[X_2] = E[X(t_2)] = \int_{-\infty}^{+\infty} x_2 f_X(x_2; t_2)\, dx_2} \quad — (4)$$

Now by letting $t_2 = t_1+\Delta$ in the above (4) substituting above first equation (1) using (3) then we get

$$\boxed{E[X(t_1+\Delta)] = E[X(t_1)]} \quad — (5)$$

which must be a constant because t_1 and Δ are arbitrary.

Second-order and Wide-Sense Stationarity:-

→ A Second order Stationary process is also first order Stationary because the 2nd order density function determines the lower, first order density.

→ A process is called Stationary to order two if its 2nd order density function Satisfies

$$\boxed{f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1+\Delta, t_2+\Delta)} \quad \forall\ t_1, t_2 \,\&\, \Delta \quad — (1)$$

→ The correlation $E[X_1 X_2] = E[X(t_1) X(t_2)]$ of a Random process will generally be a function of t_1 and t_2.

→ Let us denote this function by $R_{XX}(t_1, t_2)$ and call it the Autocorrelation function of the random process $X(t)$;

$$\boxed{R_{XX}(t_1, t_2) = E[X(t_1) X(t_2)]} \quad (2)$$

→ The autocorrelation function of a 2^{nd} order Stationary process is a function only of time differences and not absolute time. i.e., if

$$\boxed{\tau = t_2 - t_1} \quad (3)$$

then (2) becomes

$$\boxed{R_{XX}(t_1, t_1+\tau) = E[X(t_1) X(t_1+\tau)] = R_{XX}(\tau)} \quad (4)$$

→ Above processes is more restrictive than necessary, so that a more relaxed form of Stationarity is desirable.

→ The most useful form is the Wide-sense Stationary processes defined as that for which 2 conditions are true.

$$E[X(t)] = \bar{X} = \text{constant}$$

$$E[X(t)\, X(t+\tau)] = R_{XX}(\tau)$$

A process Stationary to order 2 is clearly wide-sense Stationary. However the converse is not necessarily true.

N-order and Strict-sense Stationarity:-

→ By extending the above reasoning to N random variables $X_i = X(t_i)$, $i = 1, 2, \ldots N$ we say a random process is Stationary to order N if its N^{th} order density function is invariant to a time-origin shift i.e., if

$$f_X(x_1, \ldots x_N; t_1, \ldots t_N) = f_X(x_1, \ldots x_N; t_1+\Delta, \ldots t_N+\Delta)$$
$$\forall\ t_1, \ldots t_N \ \&\ \Delta$$

→ Stationarity of order N implies Stationarity to all orders $k \leq N$.

→ A process Stationary to all orders $N = 1, 2, \ldots$ is called strict-sense Stationary.

Time Averages and Ergodicity:-

The time average of a quantity is defined as

$$A[\cdot] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} [\cdot]\, dt \quad \text{--- (1)}$$

where 'A' is to denote time average

→ Let the quantity is a sample function $x(t)$, then its time Average is $A[x(t)]$ and mean is $\bar{x}$ such that

$$\bar{x} = A[x(t)]$$

$$\boxed{\bar{x} = \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{+\infty} x(t)\, dt} \quad \text{--- (2)}$$

→ And its time auto-correlation function is given by

$$R_{xx}(\tau) = A[x(t)\, x(t+\tau)]$$

$$\boxed{R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} [x(t)\, x(t+\tau)]\, dt} \quad \text{--- (3)}$$

→ $\bar{x}$ and $R_{xx}(\tau)$ become random variables only when all the sample functions are considered in the random process.

→ Thus the mean values are given as

$$\therefore \boxed{E(\bar{x}) = \bar{X}} \text{ and } \quad \text{--- (4)}$$

$$\boxed{E[R_{xx}(\tau)] = R_{XX}(\tau)}$$

→ Let the variances of $\bar{x}$ and $R_{xx}(\tau)$ are zero,

$\therefore \bar{x}$ and $R_{xx}(\tau)$ become constants

$$\Rightarrow \boxed{\bar{x} = \bar{X} \text{ and } R_{xx}(\tau) = R_{XX}(\tau)} \quad \text{--- (5)}$$

→ In other words the time averages $\bar{x}$ and $R_{xx}(\tau)$ equal the statistical averages $\bar{X}$ and $R_{XX}(\tau)$ respectively.

Ergodicity:-

Ergodicity is defined as the feature & characteristic of an ergodic random process

If its time averaged properties are identical to its ensemble-averaged properties, then a system is said to be ergodic

Ergodic Random process:-

1. A random process is said to be ergodic if it satisfies the following two conditions

- The time average and Statistical of the process must be equal.

i.e., $\bar{x} = \overline{X}$

where $A[x(t)] = \bar{x}$ and

$E[x(t)] = \overline{X}$

2. The time auto correlation function & the Statistical autocorrelation function must be same i.e.,

$R_{xx}(\tau) = R_{XX}(\tau)$

where

$R_{xx}(\tau) = A[x(t)+x(t+\tau)]$ and

$R_{XX}(\tau) = E[x(t)x(t+\tau)]$

where $\bar{x}$ and $R_{xx}(\tau)$ are time averages.

whereas $\overline{X}$ and $R_{XX}(\tau)$ are Statistical averages.

In this case process $X(t)$ has variance '0'.

Mean Ergodic process:-

→ If the Statistical average $\overline{X}$ of a Random process $X(t)$ is equal to the time average $\bar{x}$ of any Sample function $x(t)$ with probability $=1$ then the process $X(t)$ called Mean Ergodic process.

i.e., $\boxed{E[x(t)] = \overline{X} = A[x(t)] = \bar{x}}$

In order to prove the above relation, the following 4 conditions are assumed.

1. $X(t)$ must possess a finite constant mean $\overline{X}$ for all values of 't'.
2. $X(t)$ must be bounded i.e., $|x(t)| < \infty$ for all 't' and all $x(t)$ consisting $X(t)$
3. $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[|X(t)|]\,dt < \infty$.
4. $X(t)$ must be a regular process i.e., $E[|X(t)|^2] = R_{XX}(t,t) < \infty$.

Correlation Ergodic Process:-

→ If $X(t)$ is a Stationary Continuous Process with auto-correlation function $R_{XX}(\tau)$ then it is called Correlation ergodic process if and only if it satisfies the below equation for all values of 'τ'.

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [X(t)\, X(t+\tau)]\, dt = R_{XX}(\tau)$$
$$= E[X(t)\, X(t+\tau)]$$

i.e., auto-correlation in Statistical Sense is equal to auto-correlation in time-sense.

→ The condition for a Wide-sense Stationary Sequence $X(k)$ to be auto-correlation ergodic for all 'n' is

$$\lim_{N \to \infty} \frac{1}{2N+1} \int_{k=-N}^{+N} [X(k)\, X(k+n)]\, dk = R_{XX}(n)$$

Auto Correlation function:-

→ The auto correlation function is defined as the function that gives the relation between time delayed and undelayed Version of Same Random process. 'τ' is a function delay

→ The autocorrelation of the random process $X(t)$ is given as

$$R_{XX}(\tau) = R_X(\tau)$$
$$= R_X(t_1, t_2)$$
$$= E[X(t)\, X(t+\tau)]$$

$$R_{XX}(\tau) = \int_{-\infty}^{+\infty} [X(t)\, X(t+\tau)]\, f_X(x)\, dx.$$

$R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]$

$t_1 = t$ & $t_2 = t_1 + \tau$

$R_{XX}(t, t+\tau) = E[X(t)\, X(t+\tau)]$

$\tau = t_2 - t_1$, Thus for widesense Stationaries process

$R_{XX}(\tau) = E[X(t)X(t+\tau)]$

→ The Average time of auto-correlation

$$\langle R_{xy}(\tau) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)\, X(t+\tau)\, dt$$

(or)

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)\, x(t+\tau)\, dt$$

Properties of Auto-correlation function

1) $|R_{XX}(\tau)| \le R_{XX}(0)$ Shows $R_{XX}(\tau)$ bounded by its value at the origin

2) $R_{XX}(-\tau) = R_{XX}(\tau)$ shows even symmetry

3) $R_{XX}(0) = E[X^2(t)]$ i.e; $t_1 = t_2$ shows that this bound is equal to the mean-squared value called the power in the process.

4) If $E[X(t)] = \overline{X} \ne 0$ and $X(t)$ is ergodic with no periodic components then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \overline{X}^2$$

5) If $X(t)$ has a periodic component, then $R_{XX}(\tau)$ will have a periodic component with the same period

6) If $X(t)$ is ergodic, zero-mean and has no periodic component, then $\lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$

7) $R_{XX}(\tau)$ cannot have an arbitrary shape → shows that arbitrary function cannot be an autocorrelation function

Cross-correlation function & its properties:-

→ The cross-correlation function of two random processes $X(t)$ and $Y(t)$ was defined as

$$\boxed{|R_{XY}(\tau)| \le \frac{1}{2}\left[R_{XX}(0) + R_{YY}(0)\right]} \quad (1)$$

Setting $t_1 = t$ and $\tau = t_2 - t_1$ we can write above equation as

$$\boxed{R_{XY}(t, t+\tau) = E[X(t)\, Y(t+\tau)]} \quad (2)$$

→ If $X(t)$ and $Y(t)$ are at least jointly wide-sense stationary $R_{XY}(t, t+\tau)$ is independent of absolute time & we can write

$$\boxed{R_{XY}(\tau) = E[X(t)\,Y(t+\tau)]} \quad \text{--- (3)}$$

If

$R_{xy}(t, t+\tau) = 0$ then $X(t)$ & $Y(t)$ are called orthogonal processes.

→ If the two processes are statistically independent, the cross-correlation function becomes

$$\boxed{R_{xy}(t, t+\tau) = E[X(t)]\,E[Y(t+\tau)]} \quad \text{--- (4)}$$

→ If $X(t)$ & $Y(t)$ are independent and are at least wide-sense stationary then (4) becomes

$$\boxed{R_{xy}(\tau) = \overline{X}\,\overline{Y}} \quad \text{--- (5)}$$ which is a constant.

→ Some properties of the cross-correlation function applicable to processes that are at least wide-sense stationary

1) $R_{XY}(-\tau) = R_{YX}(\tau)$ → from eq (2) says symmetry

2) $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0) R_{YY}(0)}$

3) $|R_{XY}(\tau)| \leq \frac{1}{2}\left[R_{XX}(0) + R_{YY}(0)\right]$

Covariance functions:-

→ The covariance of two random variables is.

$$\boxed{C_{XY} = E[(X-\overline{X})(Y-\overline{Y})] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\overline{X})(y-\overline{Y}) f_{XY}(x,y)\,dx\,dy} \quad \text{--- (1)}$$

extended to random processes.

→ The autocovariance function is defined by

$$C_{XX}(t, t+\tau) = E[\{X(t) - E[X(t)]\}\{X(t+\tau) - E[X(t+\tau)]\}] \quad \text{--- (2)}$$

$$C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - E[X(t)].E[X(t+\tau)] \quad \text{--- (3)}$$

The Cross-covariance function for two processes $X(t)$ and $Y(t)$ is defined by

$$C_{XY}(t, t+\tau) = E\left[\{X(t) - E[X(t)]\}\{Y(t+\tau) - E[Y(t+\tau)]\}\right]$$

(or)

$$C_{XY}(t, t+\tau) = R_{XY}(t, t+\tau) - E[X(t)]\,E[Y(t+\tau)] \quad \text{—— (4)}$$

→ For processes that are atleast jointly wide-sense stationary and reduce above eq to

$$\left.\begin{aligned} C_{XX}(\tau) &= R_{XX}(\tau) - \bar{X}^2 \quad \& \\ C_{XY}(\tau) &= R_{XY}(\tau) - \bar{X}\,\bar{Y} \end{aligned}\right\} \text{—— (5)}$$

→ The variance of a Random process in general is eq (2) with $\tau = 0$.

→ For a wide-sense stationary process, variance does not depend on time is given in the eq (5) with $\tau = 0$

$$\sigma_X^2 = E\left[\{X(t) - E[X(t)]\}^2\right] = R_{XX}(0) - \bar{X}^2$$

For two random processes, if

$C_{XY}(t, t+\tau) = 0$ they are called uncorrelated.

then eq (4) means that

$$R_{XY}(t, t+\tau) = E[X(t)]\,E[Y(t+\tau)] \quad \text{—— (6)}$$

Eq (6) is same as cross correlation which applies to independent processes, we conclude that independent processes are uncorrelated.

UNIT-V

RANDOM PROCESS - SPECTRAL CHARACTERISTICS

Introduction:-

→ All the Random process in IV unit is involved the time domain means characterized random process by means of Auto-correlation, Cross correlation, and covariance without any consideration of Spectral properties.

→ We know both time domain and frequency domain analysis methods are for analyzing linear systems and deterministic waveforms.

→ In this Chapter we are going to study the most important concepts that apply to characterize the random process in the frequency domain

→ For deterministic waveform, the spectral description is obtained by Fourier Transforming the waveform..

→ So, Fourier Transforms play an important role in Spectral characterization of random waveforms.

→ This Spectral analysis of random processes requires a bit more adopt than do deterministic signals.

POWER DENSITY SPECTRUM AND IT'S PROPERTIES:-

→ The Spectral properties of a deterministic Signal $x(t)$ are contained in its Fourier Transform $X(\omega)$ given by

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt} \quad \text{--- (1)}$$

where $X(\omega)$ is the Spectrum of $x(t)$ has the units of volts/Hertz when $x(t)$ is a voltage and describes the way in which relative signal voltage is distributed with frequency.

→ ∴ The Fourier Transform can be considered to be a Voltage density spectrum applicable to $x(t)$.

→ We know already that if $X(\omega)$ is known then $x(t)$ can be recovered by means of the Inverse Fourier Transform.

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega} \quad \text{--- (2)}$$

→ If equation (1) is applied to a Random processes, it immediately encounters with problems.

→ The Principal problem is the fact that $X(\omega)$ may not exist for most Sample functions of the process.

→ Thus above may conclude that a Spectral description of a Random process utilizing a voltage density Spectrum (Fourier transform) is not feasible because such a Spectrum may not exist.

→ While, if we liven our Random process to the description of power in the random process as a function of frequency, instead of voltage, then that result shows that this function does exist.

→ The development of this function is called power density Spectrum of the Random Process.

Power density Spectrum:-

→ For a Random Process $X(t)$, let $x_T(t)$ be defined as that portion of a Sample function $x(t)$ exist between $-T$ and T, i.e.,

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{elsewhere} \end{cases} \quad \text{—— (3)}$$

Now, as long as T is finite, we presume that $x_T(t)$ has bounded Variation, will Satisfy

$$\boxed{\int_{-\infty}^{+\infty} |x_T(t)|\,dt < \infty} \quad \text{—— (4)}$$

→ So, the above $x_T(t)$ Fourier Transform denoted as $X_T(\omega)$ given by

$$\boxed{X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j\omega t} dt = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt} \quad \text{—— (5)}$$

→ The energy contained in $x(t)$ in the interval $(-T, T)$ is

$$\boxed{E(T) = \int_{-T}^{T} |x(t)|^2 dt = \int_{-T}^{T} x^2(t).dt} \quad \text{—— (6)}$$

→ So, the energy of $x_T(t)$ is absolutely integrable and also related to parseval's theorem. Thus

$$\boxed{E(T) = \int_{-T}^{T} x^2(t).dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega} \quad \text{—— (7)}$$

→ Now, the average power $P(T)$ in $x(t)$ over the interval $(-T, T)$;

$$\boxed{P(T) = \frac{1}{2T}\int_{-T}^{T} x_T^2(t).dt = \frac{1}{2\pi}\cdot\frac{1}{2T}\int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|X_T(\omega)|^2}{2T}.d\omega} \quad — 8$$

→ In the eq (8) $\frac{|X_T(\omega)|^2}{2T}$ is a power density spectrum because power results through its integration, but eq (8) doesn't represent the power in an entire sample function and is one of the reason that eq (8) is not the function and another one is that eq (8) is only the power in one sample and doesn't represent the process.

→ In an another words $P(T)$ is a random variable with respect to the random process.

By taking the expected value in eq (8) we can obtain an average power P_{xx} for the random process.

→ To obtain a suitable power density spectrum, it is clear that we must still form the limit as $T \to \infty$ and take the expected value. This may take at the last of the operation.

→ After these operations are performed, (8) can be written

$$\boxed{P_{xx} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[x^2(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\lim_{T\to\infty}\frac{|X(\omega)|^2}{2T}d\omega} \quad — (9)$$

(9) shows P_{xx} in a random process $X(t)$ is given by the time average of its second

$$\boxed{P_{xx} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[x^2(t)]\,dt = A\{E[x^2(t)]\}} \quad — (10)$$

→ For a process that is at least Wide-Sense Stationarity (WSS),

$$E[x^2(t)] = \overline{x}^2, \text{ a constant, and}$$

$$P_{xx} = \overline{x^2} = \overline{x}^2.$$

→ P_{xx} can be obtained by a frequency domain integration.

→ If we define the power density spectrum for the random process by

$$\boxed{\mathcal{S}_{xx}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|]^2}{2T}} \quad \text{— (11)}$$

the applicable integral, we call the power formula is

$$\boxed{P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{S}_{xx}(\omega)\, d\omega.} \quad \text{— (12)}$$

Properties of the power Density Spectrum:-

→ The power density spectrum possesses a number of important properties.

(1) $\mathcal{S}_{xx}(\omega) \geq 0$ - Mean value of a Non -ve function is non -ve.

(2) $\mathcal{S}_{xx}(-\omega) = \mathcal{S}_{xx}(\omega)$ $\quad x(t)$ real — symmetry also.

(3) $\mathcal{S}_{xx}(\omega)$ is real. — True from $|X_T(\omega)|^2$ is real

(4) $\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{S}_{xx}(\omega)\, d\omega = A\{E[x^2(t)]\}$

(5) $\mathcal{S}_{\dot{x}\dot{x}}(\omega) = \omega^2 \mathcal{S}_{xx}(\omega)$ says. power density spectrum of the derivatives $\dot{x}(t) = \frac{d}{dt}x(t)$ is ω^2 times the power spectrum of $x(t)$.

(6) $\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{S}_{xx}(\omega) e^{j\omega\tau} d\omega = A[R_{xx}(t, t+\tau)]$

$$\mathcal{S}_{xx}(\omega) = \int_{-\infty}^{+\infty} A[R_{xx}(t, t+\tau)]\, e^{-j\omega\tau}\, d\tau$$

Shows the power density spectrum and the time averages of the auto correlation function form a Fourier Transform pair.

If $X(t)$ is at least wide sense stationarity

$A[R_{XX}(t, t+\tau)] = R_{XX}(\tau)$ so

$$\left.\begin{aligned} S_{XX}(\omega) &= \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau \\ R_{XX}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega \end{aligned}\right\} \text{for a wide sense stationary process.}$$

Relation between power spectrum and Autocorrelation function:—

→ In the above topic we saw the.

IFT{power density spectrum} $= A[R_{XX}(t, t+\tau)]$
= Time average of the Auto-Correlation function.

i.e, $$\boxed{\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega = A[R_{XX}(t, t+\tau)]} \quad \text{— (1)}$$

→ If we use the definition of $X_T(\omega)$ in the defining equation of the power spectrum eq (11) in the above topic then we have

'S' (S' from IIIrd style Alphabets)

$$S_{XX}(\omega) = \lim_{T\to\infty} E\left[\frac{1}{2T}\int_{-T}^{T} X(t_1) e^{j\omega t_1} dt_1 \cdot \int_{-\infty}^{+\infty} X(t_2) e^{-j\omega t_2} dt_2\right]$$

$$\boxed{S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E[X(t_1)X(t_2)] e^{-j\omega(t_2-t_1)} dt_2\, dt_1} \quad \text{— (2)}$$

The expectation (or) Mean in the above integral is defined as the autocorrelation function of $X(t)$

$$\boxed{E[X(t_1)X(t_2)] = R_{XX}(t_1, t_2) \qquad -T < (t_1 \& t_2) < T} \quad \text{— (3)}$$

∴ eq (2) becomes.

$$\boxed{S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{T} R_{XX}(t_1, t_2) e^{-j\omega(t_2-t_1)} dt_1\, dt_2} \quad \text{— (4)}$$

(4) shows the transform will exist since the power spectrum is real, non-negative & is absolutely integrable because we presume $P_{XX} < \infty$.

Step 2:-

The inverse transform is

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$\boxed{IFT\{S_{xx}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{xx}(t, t_1)\, e^{-j\omega(t_1 - t)}\, dt\, dt_1\, e^{j\omega\tau} d\omega} \quad (5)$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t, t_1)\, \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\omega(\tau - t_1 + t)}\, d\omega\, dt_1\, dt \quad (6)$$

in the above $\int_{-\infty}^{+\infty} e^{j\omega(\tau - t_1 + t)} \cdot dt/dt_1/d\tau$ is as the impulse $2\pi\delta(\tau - t_1 + t) = 2\pi\delta(t_1 - t - \tau)$. due to the even symmetry; so, eq (6) is

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{xx}(\omega)\cdot e^{j\omega\tau} d\omega = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{xx}(t, t_1)\,\delta(t_1 - t - \tau)\, dt_1\, dt$$

$t_1 - t - \tau = 0$
i.e. $t_1 = t + \tau$

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{xx}(\omega)\, e^{j\omega\tau} d\omega = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t, t+\tau)\, dt \quad -T < t+\tau < T} \quad (7)$$

So. RHS of (7) is time average of the autocorrelation.

i.e; $$\boxed{A[R_{xx}(t, t+\tau)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} R_{xx}(t, t+\tau)\, dt} \quad (8)$$

by substituting (8) in (7) the RHS of (7) becomes

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{xx}(\omega)\, e^{j\omega\tau} d\omega = A\, R_{xx}(t, t+\tau)} \quad (9)$$

$$\therefore \boxed{S_{xx}(\omega) = FT\{A[R_{xx}(t, t+\tau)]\}} \quad (10)$$

i.e; $$\boxed{A[R_{xx}(t, t+\tau)] \xleftrightarrow{FT} S_{xx}(\omega)}$$

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→ For the wide Sense Stationary, $A[R_{xx}(t, t+\tau)] = R_{xx}(\tau)$ & we get.

$$S_{xx}(\omega) = \int_{-\infty}^{+\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau \quad \text{——— (11)}$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega \quad \text{——— (12)}$$

Are called Wiener-Khintchine Relation.

(or) $\boxed{R_{xx}(\tau) \longleftrightarrow S_{xx}(\omega)}$ ——— (13)

CROSS-POWER DENSITY SPECTRUM AND ITS PROPERTIES:-

→ Consider a real random process $W(t)$ given by the sum of two other real processes $X(t)$ and $Y(t)$:

$$\boxed{W(t) = X(t) + Y(t)} \text{ ——— (1)}$$

→ The auto-correlation function of $W(t)$ is

$$R_{WW}(t, t+\tau) = E[W(t)\, W(t+\tau)]$$
$$= E\{[X(t)+Y(t)][X(t+\tau)+Y(t+\tau)]\}$$
$$= E[X(t)\,X(t+\tau)] + E[X(t)\,Y(t+\tau)] + E[Y(t)\,X(t+\tau)] + E[Y(t)\,Y(t+\tau)]$$

$$\boxed{R_{WW}(t,t+\tau) = R_{xx}(t,t+\tau) + R_{xy}(t,t+\tau) + R_{yx}(t,t+\tau) + R_{yy}(t,t+\tau)} \text{ ——— (2)}$$

→ If we take the time average of both sides of eq (2) and Fourier Transform the resulting expression is.

$$S_{WW}(\omega) = S_{xx}(\omega) + S_{xy}(\omega) + S_{yx}(\omega) + S_{yy}(\omega) \text{ ——— (3)}$$

but the $S_{yx}(\omega)$ and $S_{xy}(\omega)$ are not yet defined as power spectrum so (3) becomes

$$\boxed{S_{WW}(\omega) = S_{xx}(\omega) + S_{yy}(\omega) + FT\{A[R_{xy}(t,t+\tau)]\} + FT\{A[R_{yx}(t,t+\tau)]\}} \text{ ——— (4)}$$

So, FT of time average process are defined as the Cross-power density spectrums.

The Cross-power Density Spectrum:—

For two real random processes $X(t)$ and $Y(t)$, we define $x_T(t)$ and $y_T(t)$ as truncated ensemble members, that is

$$x_T(t) = \begin{cases} x(t) & -T<t<T \\ 0 & \text{otherwise} \end{cases} \quad \text{—— (5)}$$

&

$$y_T(t) = \begin{cases} y(t) & -T<t<T \\ 0 & \text{elsewhere} \end{cases} \quad \text{—— (6)}$$

we know by taking FT to eq (5) and (6)

$$x_T(t) \longleftrightarrow X_T(\omega) \quad \text{—— (7)}$$
$$y_T(t) \longleftrightarrow Y_T(\omega) \quad \text{—— (8)}$$

Now define cross power $P_{xy}(T)$ in the two processes within the interval $(-T, T)$ by

$$\boxed{P_{xy}(T) = \frac{1}{2T}\int_{-T}^{T} x_T(t)\, y_T(t)\, dt = \frac{1}{2T}\int_{-T}^{T} x(t)y(t)\,dt.} \quad \text{—— (9)}$$

Thus we may write from Parseval's theorem

$$\boxed{P_{xy}(T) = \frac{1}{2T}\int_{-T}^{T} x(t)y(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\frac{X_T^*(\omega)\, Y_T(\omega)}{2T}\right] d\omega} \quad \text{—— (10)}$$

→ This cross power is a random quantity since its value will vary depending on which ensemble member is considered.

→ we form the average cross power, denoted $\overline{P}_{xy}(T)$ by taking the expected value in the above eq (10). The result is

$$\boxed{\overline{P}_{xy}(T) = \frac{1}{2T}\int_{-T}^{T} R_{xy}(t,t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{E\left[X_T^*(\omega)\, Y_T(\omega)\right]}{2T}\, d\omega} \quad \text{(11)}$$

→ The total average cross power P_{xy} by letting $T \to \infty$

$$\boxed{P_{xy} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{xy}(t,t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{E\left[X_T^*(\omega)\, Y_T(\omega)\right]}{2T}\, d\omega} \quad \text{(12)}$$

→ It is clear that the integrand involving ω can be defined as a cross-power density spectrum; it is a function of ω which we denote preservation

$$\boxed{\mathcal{S}_{XX}(\omega) = \lim_{T\to\infty} \frac{E[X_T^*(\omega)\, Y_T(\omega)]}{2T}} \quad (13)$$

Cross power is given by

$$\boxed{P_{YX} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{S}_{YX}(\omega)\, d\omega = P_{XY}^*} \quad (14)$$

where $\mathcal{S}_{YX}(\omega) = \lim_{T\to\infty} \frac{E[Y_T^*(\omega)\, X_T(\omega)]}{2T}$ is the

Cross power formula. $\boxed{P_{XY} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{S}_{XY}(\omega)\, d\omega}$ (15)

→ Now, the total cross power $P_{XY} + P_{YX}$ can be interpreted as the additional power two processes are capable of generating, over and above their individual powers, due to the fact that they are not orthogonal.

Properties of the Cross-power Density Spectrum:-

Some properties of the cross-power spectrum of real random process $X(t)$ and $Y(t)$ are listed below without formal proof-so

1. $\mathcal{S}_{XY}(\omega) = \mathcal{S}_{YX}(-\omega) = \mathcal{S}^*_{YX}(\omega)$ follows from eqn (13) & (14)
2. $Re[\mathcal{S}_{XY}(\omega)]$ and $Re[\mathcal{S}_{YX}(\omega)]$ are even functions of ω } Symmetry possess
3. $Im[\mathcal{S}_{XY}(\omega)]$ and $Im[\mathcal{S}_{YX}(\omega)]$ are odd functions of ω } for real process.
4. $\mathcal{S}_{XY}(\omega) = 0$ and $\mathcal{S}_{YX}(\omega) = 0$ if $X(t)$ and $Y(t)$ are orthogonal.
5. If $X(t)$ and $Y(t)$ are uncorrelated and have constant means $\bar{X}$ and $\bar{Y}$

$$\mathcal{S}_{XY}(\omega) = \mathcal{S}_{YX}(\omega) = 2\pi \bar{X}\bar{Y}\,\delta(\omega)$$

6. $A[R_{XY}(t, t+\tau)] \longleftrightarrow \mathcal{S}_{XY}(\omega)$

$A[R_{YX}(t, t+\tau)] \longleftrightarrow \mathcal{S}_{YX}(\omega)$

(suraksha) ...

→ In the following discussion we show that

$$\boxed{S_{XY}(\omega) = \int_{-\infty}^{+\infty}\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R_{XY}(t, t+\tau)\,dt\right\}e^{-j\omega\tau}d\tau} \quad (1)$$

→ The development consists of using the transforms of the truncated processes given by

$$\boxed{X_T(\omega) = \int_{-T}^{T} X(t)\,e^{-j\omega t}dt} \quad (2)$$

$$\boxed{Y_T(\omega) = \int_{-T}^{+T} Y(t_1)\,e^{-j\omega t_1}dt_1} \quad (3)$$

and

$$\boxed{X_T^*(\omega)\,Y_T(\omega) = \int_{-T}^{+T} X(t)\,e^{j\omega t}dt \int_{-T}^{+T} Y(t_1)\,e^{-j\omega t_1}dt_1} \quad (4)$$

This result is used in above eq (13) to form the Cross-power density Spectrum

$$S_{XY}(\omega) = \lim_{T\to\infty}\frac{E[X_T^*(\omega)Y_T(\omega)]}{2T}$$

$$= \lim_{T\to\infty}\frac{1}{2T}E\left[\int_{-T}^{T}X(t)e^{j\omega t}dt\int_{-T}^{+T}Y(t_1)e^{-j\omega t_1}dt_1\right]$$

$$\boxed{S_{XY}(\omega) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_{XY}(t,t_1)\,e^{-j\omega(t_1-t)}dt\,dt_1} \quad (5)$$

→ We IFT both sides of above eq (5) and identify one integral as an impulse function.

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}S_{XY}(\omega)\,e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_{XY}(t,t_1)\,e^{-j\omega(t_1-t)}dt\,dt_1\,e^{j\omega\tau}d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_{XY}(t,t_1)\,\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{j\omega(t-t_1+\tau)}d\omega\,dt_1\,dt$$

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XY}(\omega)e^{j\omega\tau}d\omega = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_{XY}(t,t_1)\,\delta(t_1-\tau-t)\,dt_1\,dt} \quad (6)$$

→ The definition of the impulse allows the immediate solution for the integral over t_1.

... refers to the Exchange

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}(t, t+\tau)\,dt} \quad (7)$$

(7) gives the Cross-power Spectrum, the cross-correlation function cannot in general be recovered; only its time-average can.

→ Fd jointly wide-sense stationary processes, however, the cross-correlation function $R_{XY}(\tau)$ can be found from $S_{XY}(\omega)$ since its time average is just $R_{XY}(\tau)$

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