

UNIT-I

Solution of Algebraic & Transcendental Equations

Introduction-Bisection method-Iterative method-Regula falsi method-Newton Raphson method System of Algebraic equations: Gauss Jordan method-Gauss Siedal method.

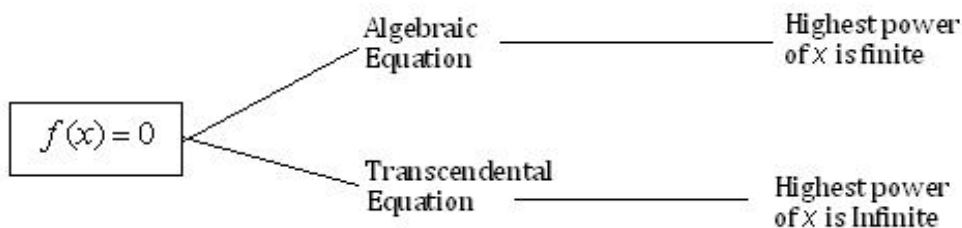
Polynomial Expression: An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} \dots\dots\dots a_n$ is called a polynomial in x of degree n , provided $a_0 \neq 0$ where $a_0, a_1, a_2, \dots\dots\dots a_n$ are constants (real or Complex)

Algebraic equation: A polynomial in x of degree n equated to zero i.e., $f(x) = 0$ is called an algebraic equation of degree n .

ex: $x^4 - x + 4 = 0$

Transcendental equation: An equation $f(x) = 0$ which involves trigonometric, logarithmic, exponential functions etc is called as a Transcendental equation.

ex: $xe^x + 4 = 0, x \sin x = 2 \& x \log x - 2 = 0$



Intermediate value theorem:-

State: If $f(x)$ is continuous on $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs. Then atleast one root lies between a and b .

- (1) Bisection method
- (2) Regula falsi method or false position method
- (3) Newton Raphson Method

To find the root of the equation $f(x) = 0$, we start with a known approximate solution and apply any one of the following methods :

Method 1: Bisection method:

The bisection method of Bolzano is popular, simple and useful for the computation of root of non linear equations. This method, also known as binary chapping or half-interval method. This method consists in locating the root of the equation $f(x) = 0$ between a and b . If $f(x)$ is real and continuous in the interval $a < x < b$, and $f(a)$ and $f(b)$ are of opposite signs, that is,

$f(a)f(b) < 0$. Then there is at least one real root in the interval between a and b . The desired root is approximately defined by the midpoint.

$$x_1 = \frac{a+b}{2}$$

If $f(x_1) = 0$, then is the desired root of $f(x) = 0$. However, if $f(x_1) \neq 0$, then the root may be between a and x_1 or x_1 and b . Now, we define the approximation by

$$x_2 = \frac{a+x_1}{2},$$

provided $f(a)f(x_1) < 0$, then the root may be between a and x_1 or by

$$x_2 = \frac{x_1+b}{2},$$

provided $f(x_1)f(b) < 0$, then the root may be between x_1 and b etc.

Thus, at each step, we either find the desired root to the required accuracy

Problem 1: find out the square root of 25 given 2.0 and 7.0 using bisection method

Solution : Consider $f(x) = x^2 - 25 = 0$.

Here $a = 2$ and $b = 7$, we have $f(a) = -21$ and $f(b) = 24$

x_i	$f(x) = x^2 - 25$	New interval (with signs)
$x_1 = \frac{a+b}{2} = 4.5$	$f(x_1) = f(4.5) = (4.5)^2 - 25 = -4.75 < 0$	(4.5, 7) (-, +)
$x_2 = \frac{4.5+7}{2} = 5.75$	$f(x_2) = f(5.75) = 8.0625 > 0$	(4.5, 5.75) (-, +)
$x_3 = \frac{4.5+5.75}{2} = 5.125$	$f(x_3) = f(5.125) = 1.2656 > 0$	(4.5, 5.125) (-, +)
$x_4 = \frac{x_2+x_3}{2} = 4.8125$	$f(x_4) = f(4.8125) = -1.83984 < 0$	(4.8125, 5.125) (-, +)
$x_5 = \frac{x_3+x_4}{2} = 4.96875$	$f(x_5) = f(4.96875) = -0.31152 < 0$	(4.96875, 5.125) (-, +)
$x_6 = \frac{x_4+x_5}{2} = 5.0468$	$f(x_6) = f(5.0468) = 0.47019 > 0$	(4.96875, 5.0468) (-, +)
$x_7 = \frac{x_5+x_6}{2} = 5.0077$	$f(x_7) = f(5.0077) = 0.0778 > 0$	(5.0468, 5.125) (-, +)
$x_8 = \frac{x_6+x_7}{2} = 4.988225$	$f(x_8) = f(4.988225) = -0.1176 < 0$	(5.0468, 5.125) (-, +)
$x_9 = \frac{x_7+x_8}{2} = 4.9979625$	$f(x_9) = f(4.9979625) = -0.0203 < 0$	(5.0468, 5.125) (-, +)

Since $x_8 \approx x_9 = 4.9979625$

Thus the approximation root is $x_8 \approx x_9 = 4.9979625$.

Problem 2: Find a real root of $x^3 - 5x + 3 = 0$ using bisection method

Solution : Consider $f(x) = x^3 - 5x + 3$.

$$\begin{array}{r} X: \quad 0 \quad 1 \quad 2 \\ Y: \quad 3 \quad -1 \quad 1 \end{array}$$

So a root lies between 1 and 2.

Choose $a = 1$ and $b = 2$.

x_i	$f(x) = x^2 - 25$	New interval (with signs)
$x_1 = \frac{a+b}{2} = 1.5$	$f(x_1) = f(1.5) = -1.125 < 0$	(1.5, 2) (-, +)
$x_2 = \frac{1.5+2}{2} = 1.75$	$f(x_2) = f(1.75) = -0.39 < 0$	(1.75, 2) (-, +)
$x_3 = \frac{4.5+5.75}{2} = 5.125$	$f(x_3) = f(1.875) = 0.216 > 0$	(1.75, 1.875) (-, +)
$x_4 = \frac{x_2+x_3}{2} = 4.8125$	$f(x_4) = f(1.8125) = -0.108 < 0$	(1.8125, 1.875) (-, +)
$x_5 = \frac{x_3+x_4}{2} = 4.96875$	$f(x_5) = f(1.84375) = 0.049 > 0$	(1.8125, 1.84375) (-, +)
$x_6 = \frac{x_4+x_5}{2} = 1.828125$	$f(x_6) = -0.03095 < 0$	(1.828125, 1.84375) (-, +)

Since $x_8 \approx x_9 = 4.9979625$

Hence an approximation root of $f(x)$ is $x_7 = \frac{1.828125+1.84375}{2} = 1.8359375$.

Method 2: Iteration Method:

Consider the equation $f(x) = 0$ (1)

Whose roots are to be determined. The equation (1) can be rewritten as

$$x = \phi(x) \text{ ----- (2)}$$

Let be x_0 an approximation to the root of $f(x) = 0$

Putting $x = x_0$ in RHS of (2), we get the first approximation

$$x_1 = \phi(x_0)$$

The successive approximations are given by

$$x_2 = \phi(x_1)$$

$$x_3 = \phi(x_2)$$

.....

$$x_{n+1} = \phi(x_n)$$

When the successive approximations $x_1, x_2, x_3, \dots, x_n$ are always converges to the root of $x = \phi(x)$. And it can be shown that, if $|\phi'(x)| < 1$, when is sufficient close to the exact value α of the root and $x_n \rightarrow \alpha$ as $n \rightarrow \infty$.

Problem 1: Find a real root of the equation $x^3 + x - 1 = 0$ by the Iteration method, given that the root lies near $x = 1$ correct to three decimal places.

Solution : Let $f(x) = x^3 + x - 1 = 0$ ----- (1)

then $f(0) = -1 < 0$ and $f(1) = 1 > 0$.

i.e., A root lies between 0 and 1.

The equation (1) can be written as

$$x = \frac{1}{1+x^2} = \phi(x), (\text{say})$$

Now, $\phi'(x) = \frac{-2x}{(1+x^2)^2}$ and $x_0 = 1$

$$|\phi'(x)| = \left| \frac{-2x}{(1+x^2)^2} \right| = \left| \frac{-2}{(1+1)^2} \right| = \frac{1}{2} < 1$$

So, the iteration method can be applied.

Let $x_0 = 1$ be the initial approximation of the root of (1).

By Iteration method, we have

$$x_{n+1} = \phi(x_n), n = 0, 1, 2, 3, \dots$$

$$\text{Put } n = 0, x_1 = \phi(x_0) = \frac{1}{1+x_0^2} = \frac{1}{2} = 0.5$$

$$\text{Put } n = 1, x_2 = \phi(x_1) = \frac{1}{1+x_1^2} = \frac{1}{1+(0.5)^2} = 0.6666$$

$$\text{Put } n = 2, x_3 = \phi(x_2) = \frac{1}{1+x_2^2} = \frac{1}{1+(0.6666)^2} = 0.6923$$

$$\text{Put } n = 3, x_4 = \phi(x_3) = \frac{1}{1+x_3^2} = \frac{1}{1+(0.6923)^2} = 0.6760$$

$$\text{Put } n = 4, x_6 = \phi(x_5) = \frac{1}{1+x_5^2} = \frac{1}{1+(0.6760)^2} = 0.6863$$

$$\text{Put } n = 5, x_4 = \phi(x_3) = \frac{1}{1+x_3^2} = \frac{1}{1+(0.6863)^2} = 0.6798$$

$$\text{Put } n = 6, x_7 = \phi(x_6) = \frac{1}{1+x_6^2} = \frac{1}{1+(0.6798)^2} = 0.6798$$

Sine $x_6 \approx x_7$

Hence the root of the equation is $x = 0.6798$.

Problem 2: Find a real root of the equation $x^3 + x^2 - 1 = 0$ by the Iteration method, given that the root lies near $x = 1$ correct to three decimal places.

Solution : Let $f(x) = x^3 + x^2 - 1 = 0$ ----- (1)

then $f(0) = -1 < 0$ and $f(1) = 1 > 0$.

i.e., A root lies between 0 and 1.

The equation (1) can be written as

$$x = \frac{1}{\sqrt{1+x}} = \phi(x), (\text{say})$$

Now, $\phi'(x) = -\frac{1}{2} \frac{1}{(1+x)^{3/2}}$ and $x_0 = 0$

$$|\phi'(x)| = \left| \frac{-1}{2} \frac{1}{(1+0)^{3/2}} \right| = \left| \frac{-1}{2} \right| = 0.5 < 1$$

So, the iteration method can be applied.

Let $x_0 = 1$ be the initial approximation of the root of (1).

By Iteration method, we have

$$x_{n+1} = \phi(x_n), n = 0, 1, 2, 3, \dots$$

$$\text{Put } n = 0, x_1 = \phi(x_0) = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+0}} = 1$$

$$\text{Put } n = 1, x_2 = \phi(x_1) = \frac{1}{\sqrt{1+x_1}} = \frac{1}{\sqrt{1+1}} = 0.7071$$

$$\text{Put } n = 2, x_3 = \phi(x_2) = \frac{1}{\sqrt{1+x_2}} = \frac{1}{\sqrt{1+(0.7071)}} = 0.7514$$

$$\text{Put } n = 3, x_4 = \phi(x_3) = \frac{1}{\sqrt{1+x_3}} = \frac{1}{\sqrt{1+(0.7514)}} = 0.7556$$

$$\text{Put } n = 4, x_5 = \phi(x_4) = \frac{1}{\sqrt{1+x_4}} = \frac{1}{\sqrt{1+(0.7556)}} = 0.7547$$

$$\text{Put } n = 5, x_6 = \phi(x_5) = \frac{1}{\sqrt{1+x_5}} = \frac{1}{\sqrt{1+(0.7547)}} = 0.7549$$

$$\text{Put } n = 6, x_7 = \phi(x_6) = \frac{1}{\sqrt{1+x_6}} = \frac{1}{\sqrt{1+(0.7549)}} = 0.7549$$

Sine $x_6 \approx x_7$

Hence the root of the equation is $x = 0.7549$.

Problem 3: Find a real root of the equation $x - 0.5 \cos x = 0$ by the Iteration method.

Solution : Let $f(x) = x - 0.5 \cos x = 0$ ----- (1)

then $f(0) = -0.5 < 0$ and $f(1) = 0.729 > 0$.

i.e., A root lies between 0 and 1.

The equation (1) can be written as

$$x = \frac{1}{2} \cos x = \phi(x), (\text{say})$$

Now, $\phi'(x) = -\frac{\sin x}{2}$ and $x_0 = 0$

$$\text{At } x=0, |\phi'(x)| = \left| -\frac{\sin 0}{2} \right| = |0| = 0 < 1$$

So, the iteration method can be applied.

Let $x_0 = 0$ be the initial approximation of the root of (1).

By Iteration method, we have

$$x_{n+1} = \phi(x_n), n = 0, 1, 2, 3, \dots$$

$$\text{Put } n = 0, x_1 = \phi(x_0) = 0.5 \cos(0) = 0.5$$

$$\text{Put } n = 1, x_2 = \phi(x_1) = 0.5 \cos(0.5) = 0.4387$$

$$\text{Put } n = 2, x_3 = \phi(x_2) = 0.5 \cos(0.4387) = 0.4526$$

$$\text{Put } n = 3, x_4 = \phi(x_3) = 0.5 \cos(0.4526) = 0.4496$$

$$\text{Put } n = 4, x_5 = \phi(x_4) = 0.5 \cos(0.4496) = 0.4503$$

$$\text{Put } n = 5, x_6 = \phi(x_5) = 0.5 \cos(0.4503) = 0.4501$$

$$\text{Put } n = 6, x_7 = \phi(x_6) = 0.5 \cos(0.4501) = 0.4502$$

Sine $x_6 \approx x_7$

Hence the root of the equation is $x = 0.4502$.

Method (3): Regula Falsi method or False Position Method:

Consider the equation $f(x) = 0$

Let be two values such that $f(x_0)$ & $f(x_1)$ are of opposite signs. Also let $x_0 < x_1$, the graph $y = f(x)$ will meet at some point between x_0 & x_1 . The equation of the chord joining the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is

$$\frac{(y - f(x_0))}{(x - x_0)} = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \dots\dots\dots(1)$$

If the chord AB meet x-axis

So, putting $y=0$ in (1), we get

$$\Rightarrow \frac{-f(x_0)}{(x - x_0)} = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$$

$$\Rightarrow x - x_0 = \frac{-f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x = x_0 + \frac{-f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = x_2 \text{ (say)}$$

If $f(x_0)$ & $f(x_2)$ have opposite signs, then the root lies between x_0 and x_2 . Otherwise the root lies between x_1 and x_2 .

If the root lies between x_0 and x_2 , then the next approximation is

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)}$$

Or

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

The above method is applied repeatedly till the desired accuracy is obtained.

Problem 1: Find a real root of the equation $x^4 - x = 10$ by the method of false position correct to three decimal places.

Solution : Let $f(x) = x^4 - x - 10$

So that $f(1) = -10 = -ve$, $f(2) = 4 = +ve$ i.e., A root lies between 1 and 2.

\therefore taking $x_0 = 1, x_1 = 2$, and using method of false position,

the first approximation is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{1 \times (4) - 2(-10)}{4 - (-10)} = \frac{24}{14} = 1.714286$$

Now $f(x_2) = f(1.714286) = -3.0778 < 0$. Since $f(x_2)$ and $f(x_1)$ have opposite signs, i.e., the root lies between 1.714286 and 2.

The second approximation to the root is given as

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2 \times (-3.0778) - 1.714286 \times (4)}{-3.0778 - (4)} = 1.838529$$

and $f(x_3) = -0.649414$. Now, since $f(x_3)$ and $f(x_1)$ have opposite signs.

i.e., the root lies between 1.838529 and 2.

The third approximation is obtained from

$$x_4 = \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)} = \frac{2 \times (-0.649414) - 1.838529 \times (4)}{-0.649414 - 4} = 1.861083$$

and $f(x_4) = 0.1356494$. Since $f(x_3)$ and $f(x_4)$ have opposite signs.

The fourth approximation is given by

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} = \frac{1.838529 \times (0.135649) - 1.861083 \times (-0.649414)}{0.135649 - (-0.649414)} = 1.857186$$

And $f(x_5) = 0.039379$. Since $f(x_3)$ and $f(x_5)$ have opposite signs.

The fifth approximation is given by

$$x_6 = \frac{x_3 f(x_5) - x_5 f(x_3)}{f(x_5) - f(x_3)} = \frac{1.838529 \times (0.039379) - 1.857186 \times (-0.649414)}{0.039379 - (-0.649414)} = 1.856119$$

and $f(x_6) = 0.013130$. Since $f(x_3)$ and $f(x_6)$ have opposite signs.

The sixth approximation is given by

$$x_7 = \frac{x_3 f(x_6) - x_6 f(x_3)}{f(x_6) - f(x_3)} = \frac{1.838529 \times (0.013130) - 1.856119 \times (-0.649414)}{0.013130 - (-0.649414)} = 1.856770$$

Since $x_6 = x_7$.

Hence, the desired root is 1.856 correct to three decimal places.

Problem 2: Find the real root of the equation $x^4 - x = 10$ by Regula-Falsi method correct to 4 decimal places.

Method 4: Newton Raphson Method:

Consider the equation $f(x) = 0$

Let be x_0 an approximation of a root of $f(x) = 0$

If $x_1 = x_0 + h$, where $h \ll 0$, is the exact root,

then $f(x_1) = f(x_0 + h) = 0$

we know that, $f(x_0 + h)$ can be extended by Tailors theorem as

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \dots = 0$$

Since h is very small, the highest power of h of order ≥ 2 can be neglected.

$$f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow h = \frac{-f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \text{ since } h = x - x_0$$

x_1 is a better approximation than x_0 .

The successive approximations are given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

This is called Newton Raphson method.

Problem 1: Find a real root of the equation $x \tan x + 1 = 0$ using Newton's raphson Method.

Solution: Let $f(x) = x \tan x + 1$

Then we have $f'(x) = x \sec^2 x + \tan x$

$x:$	0	1	2	3
$f(x):$	1	2.5574	-3.3700	0.57237

Here $a = 2$ and $b = 3$

So that $f(a) = -3.3700$ and $f(b) = 0.57237$

Therefore, the root lies between $a = 2$ and $b = 3$

By bisection method, the first approximation x_1 is

$$x_0 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$$

Using Newton Raphson method, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

The first approximation root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f(x_0) = f(2.5) = (2.5) \tan(2.5) + 1 = -0.86755$$

$$f'(x_0) = (2.5) \sec^2(2.5) + \tan(2.5) = 3.14808$$

$$\therefore x_1 = 2.5 + \frac{0.86755}{3.14808} = 2.7755$$

The second approximation root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$f(x_1) = (2.7755) \tan(2.7755) + 1 = -0.064056$$

$$f'(x_1) = (2.7755) \sec^2(2.7755) + \tan(2.7755) = 2.80005$$

$$\therefore x_2 = 2.7755 + \frac{0.064056}{2.80005} = 2.79837$$

The third approximation root is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$f(x_2) = (2.79837) \tan(2.79837) + 1 = -0.000044$$

$$f'(x_2) = (2.79837) \sec^2(2.79837) + \tan(2.79837) = 2.79837$$

$$\therefore x_3 = 2.79837 + \frac{0.00004}{2.79837} = 2.79838$$

Since $x_2 = x_3$

Hence the required root of the equation is $x = 2.79838$

Problem 2: By using Newton's Raphson Method, find a positive root of the equation $xe^x - \cos x = 0$ that lies between 1.8 and 2.

Solution: Let $f(x) = xe^x - \cos x$

Then we have $f'(x) = (x+1)e^x + \sin x$

$x:$	0	0.5	1
$f(x):$	-1	-0.05322	2.17797

Here $a = 0.5$ and $b = 1$

So that $f(a) = -0.05322 < 0$ and $f(b) = 2.17797$

Therefore, the root lies between $a = 0.5$ and $b = 1$

By bisection method, the first approximation x_1 is

$$x_0 = \frac{0.5+1}{2} = 0.75$$

Using Newton Raphson method, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

The first approximation root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f(x_0) = f(0.75) = 0.75e^{0.75} - \cos(0.75) = 0.85606$$

$$f'(x_0) = f'(0.75) = (0.75+1)e^{0.75} + \sin(0.75) = 4.38638$$

$$\therefore x_1 = 0.75 - \frac{0.85606}{4.38638} = 0.55483$$

The second approximation root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$f(x_0) = f(0.55483) = 0.55483e^{0.55483} - \cos(0.55483) = 0.85606$$

$$f'(x_0) = f'(0.55483) = (0.55483 + 1)e^{0.55483} + \sin(0.55483) = 4.38638$$

$$\therefore x_2 = 2.7755 + \frac{0.064056}{2.80005} = 2.79837$$

The third approximation root is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$f(x_0) = f(0.55483) = 0.55483e^{0.55483} - \cos(0.55483) = 0.85606$$

$$f'(x_0) = f'(0.55483) = (0.55483 + 1)e^{0.55483} + \sin(0.55483) = 4.38638$$

$$\therefore x_3 = 2.79837 + \frac{0.00004}{2.79837} = 2.79838$$

Since $x_2 = x_3$

Hence the required root of the equation is $x = 2.79838$

Home work:

Problem 3: Find a real root of the equation $x + \log_{10}^x - 2 = 0$ using Newton's Raphson Method.

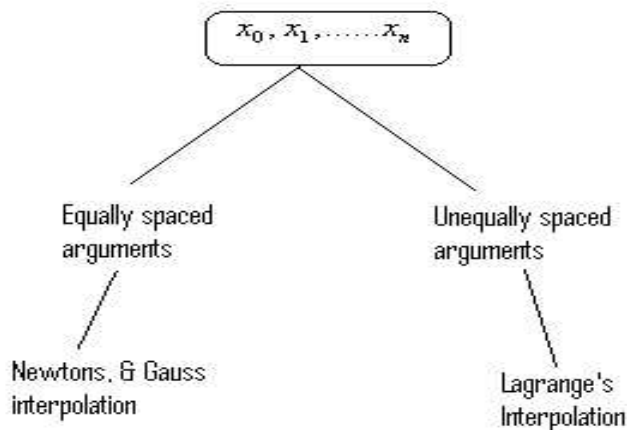
Ans: $x_6 = 1.796$

Problem 4: using Newton's Raphson Method, find a positive root of $x^4 - x - 9 = 0$.

Unit-II INTERPOLATION

Def: The process of finding the curve passing through the points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ is called as Interpolation and the curve obtained is called as Interpolating curve. Interpolating curve passing through the given set of points is unique.

Let x_0, x_1, \dots, x_n be given set of observations and $y = f(x)$ be the given function, then the method to find $f(x_m), \forall x_0 \leq x_m \leq x_n$ is called as an Interpolation. If x_m is not in the range of x_0 and x_n , then the method to find x_m is called as Extrapolation.



The Interpolation depends upon finite difference concept.

If x_0, x_1, \dots, x_n be given set of observations and let $y_0 = f(x_0), y_1 = f(x_1), \dots, y_n = f(x_n)$ be their corresponding values for the curve $y = f(x)$, then $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ is called as finite difference.

When the arguments are equally spaced i.e. $x_i - x_{i-1} = h, \forall i$ then we can use one of the following differences.

Note: If h is the common difference of the observations, then $h = x_i - x_{i-1}, \forall i$

- Forward differences
- Backward differences
- Central differences

Forward Difference:

Let us consider x_0, x_1, \dots, x_n be given set of observations and let y_0, y_1, \dots, y_n are corresponding values of the curve $y = f(x)$, then the Forward difference operator Δ is denoted by and is defined as $\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \dots, \Delta y_n = y_n - y_{n-1}$. In this case are $\Delta y_0, \Delta y_1, \dots, \Delta y_n$ called as First Forward differences of y .

The difference of first forward differences will give us Second forward differences and it is denoted Δ^2 by and is defined as $\Delta^2 y_0 = \Delta(\Delta y_0)$

$$\begin{aligned}
&= \Delta(y_1 - y_0) \\
&= \Delta y_1 - \Delta y_0 \\
&= (y_2 - y_1) - (y_1 - y_0) \\
&= (y_2 - 2y_1 + y_0)
\end{aligned}$$

Similarly, the difference of second backward differences will give us third backward difference and it is denoted by Δ^3 .

Forward Difference Table

x	y = f(x)	First Forward Differences	Second Forward Differences	Third Forward Differences	Forth Forward Differences
x_0	y_0				
x_1	y_1	$\Delta y_0 = y_1 - y_0$			
x_2	y_2		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
.	.	$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
.	.		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
.	.	$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_{n-1}	y_{n-1}	$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
x_n	y_n			
		$\Delta y_{n-1} = y_n - y_{n-1}$			

Note: $\Delta^n y_k = \sum_{i=0}^n (-1)^i {}^n C_i y_{k-i}$

Backward Difference:

Let us consider x_0, x_1, \dots, x_n be given set of observations and let y_0, y_1, \dots, y_n are corresponding values of the curve $y = f(x)$, then the backward difference operator ∇ is denoted by and is defined as $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \dots, \nabla y_n = y_n - y_{n-1}$. In this case, $\nabla y_1, \nabla y_2, \dots, \nabla y_n$ are called as First backward differences of y .

The difference of first forward differences will give us Second forward differences and it is denoted ∇^2 by and is defined as $\nabla^2 y_2 = \nabla(\nabla y_2)$

$$\begin{aligned}
&= \nabla(y_2 - y_1) \\
&= \nabla y_2 - \nabla y_1 \\
&= (y_2 - y_1) - (y_1 - y_0) \\
&= (y_2 - 2y_1 + y_0)
\end{aligned}$$

Similarly, the difference of second backward differences will give us third backward difference and it is denoted by ∇^3 .

Backward Difference Table

x	y = f(x)	First Forward Differences	Second Forward Differences	Third Forward Differences	Forth Forward Differences
x ₀	y ₀	∇y ₁ = y ₁ - y ₀			
x ₁	y ₁		∇ ² y ₂ = ∇y ₂ - ∇y ₁		
		∇y ₂ = y ₂ - y ₁		∇ ³ y ₃ = ∇ ² y ₃ - ∇ ² y ₂	
x ₂	y ₂		∇ ² y ₃ = ∇y ₃ - ∇y ₂		∇ ⁴ y ₄ = ∇ ³ y ₄ - ∇ ³ y ₃
		∇y ₃ = y ₃ - y ₂		∇ ³ y ₄ = ∇ ² y ₄ - ∇ ² y ₃	
x ₃	y ₃		∇ ² y ₄ = ∇y ₄ - ∇y ₃		
		∇y ₄ = y ₄ - y ₃			
x ₄	y ₄				

Note: $\nabla^n y_k = \sum_{i=0}^n (-1)^i {}^n C_i y_{k-i}$

Newton's forward interpolation formula:

Definition: Let x_0, x_1, \dots, x_n are the given set of observations with common difference h and let y_0, y_1, \dots, y_n be their corresponding values for the curve $y = f(x)$, then the Newton's Forward interpolation formula is given by

$$f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots + \frac{p(p-1)(p-2)\dots(p-(n-1))}{n!} \Delta^n y_0, \text{ where } p = \frac{x-x_0}{h}.$$

Newton's backward interpolation formula:

Definition: Let x_0, x_1, \dots, x_n are the given set of observations with common difference h and let y_0, y_1, \dots, y_n be their corresponding values for the curve $y = f(x)$, then the Newton's Backward interpolation formula is given by

$$f(x) = f(x_0 + ph) = y_0 + p\nabla y_0 + \frac{p(p+1)}{2!} \nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_0 + \dots + \frac{p(p+1)(p+2)\dots(p-(n-1))}{n!} \nabla^n y_0, \text{ where } p = \frac{x-x_n}{h}.$$

Problem 1:

For $x = 0, 1, 2, 3, 4$, $f(x) = 1, 14, 15, 5, 6$. Find $f(3)$ using forward difference table

Solution: Given

$x :$	0	1	2	3	4
$y = f(x) :$	1	14	15	5	6

The difference table is shown below

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$y_0 = 1$				
		$\Delta y_0 = 13$			
1	14		$\Delta^2 y_0 = -12$		
		1		$\Delta^3 y_0 = 1$	
2	15		-11		$\Delta^4 y_0 = 21$
		-10		22	
3	5		11		
		1			
4	6				

By Newton's Forward interpolation formula is given by

If we take $x_0 = 0, h = 1$ then $p = \frac{x - x_0}{h} = \frac{3 - 0}{1} = 3$

$$\therefore f(3) = 1 + 3(13) + \frac{3(3-1)}{2!}(-12) + \frac{3(3-1)(3-2)}{3!}1 + \frac{3(3-1)(3-2)(3-3)}{4!}21$$

$$\Rightarrow f(3) = 1 + 39 - 36 + 1$$

$$\therefore f(3) = 5$$

Problem 2: Find a polynomial by using Newton's forward difference interpolation formula for the following data. Also find $f(2.5)$

x	0	1	2	3
$y = f(x)$	1	3	7	13

Solution: The difference table is shown below

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	$y_0 = 1$			
		$\Delta y_0 = 2$		
1	3		$\Delta^2 y_0 = 2$	
		4		$\Delta^3 y_0 = 0$
2	7		2	
		6		
3	13			

By Newton's Forward interpolation formula is given by

If we take $x_0 = 0, h = 1$ then $p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$

$$f(x) = 1 + x(2) + \frac{x(x-1)}{2!}(2) + \frac{x(x-1)(x-2)}{3!}(0) + \dots$$

$$= 1 + 2x + x^2 - x$$

$$\therefore f(x) = x^2 + x + 1$$

Also, $f(2.5) = (2.5)^2 + 2.5 + 1 = 10.125$

- (1) Given $\sin 45^\circ = 0.7071, \sin 50^\circ = 0.7660, \sin 55^\circ = 0.8192,$ and $\sin 60^\circ = 0.8660,$
 $\sin 52^\circ$, Using Newton's interpolation formula.

Ans: $\sin 52^\circ = 0.7880$ correct to four decimal places

- (2) The population of a country in the decennial census is as shown below. Estimate for the year 1925

Years x	1891	1901	1911	1921	1931
Population in thousands y	46	66	81	93	101

Ans: $h=10, y(1925): 96.8368$

- (3) Find the missing term in the table

x	16	18	20	22	24	26
y	39	85	-	151	264	388

Ans: 96.4

Find the missing terms in the table

x	45	50	55	60	65
y	3.0	-	2.0	-	-2.4

Ans $y_1=2.925$ $y_3=0.225$

- (6) The following data give the melting point of an alloy of lead and zinc, where “ t ” is the temperature in $^{\circ}\text{C}$ and x is the % of lead in the alloy

x	60	70	80	90
t	226	250	276	304

Find melting point of the alloy containing 84% of lead using Newton’s interpolation formula.

Lagrange’s interpolation Formula:

If x_0, x_1, \dots, x_n is the given set of observations, which are need not be equally spaced and let y_0, y_1, \dots, y_n be their corresponding values of the curve $y = f(x)$, then

$$y = f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1 + \dots + \frac{(x-x_1)(x-x_2)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)\dots(x_n-x_{n-1})} y_n$$

This is known as Lagrange’s interpolation formula for unequal intervals.

Problem 1: Evaluate $f(10)$ given $f(x) = 168, 192, 336$ at $x = 1, 7, 15$ respectively, Use Lagrange’s interpolation.

Solution: Here $x = 10, x_0 = 1, x_1 = 7, x_2 = 15$ and $y_0 = 168, y_1 = 192, y_2 = 336$

Lagrange’s interpolation formula is given by

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$\therefore y = f(10) = \frac{(10-7)(10-15)}{(1-7)(1-15)} 168 + \frac{(10-1)(10-15)}{(7-1)(7-15)} 192 + \frac{(10-1)(10-7)}{(15-1)(15-7)} 336$$

$$= \frac{-15}{84} 168 + \frac{-45}{-48} 192 + \frac{27}{112} 336$$

$$= -30.005 + 180 + 81.01 = 231.005$$

Problem 2: Given $u_0 = 580, u_1 = 556, u_2 = 520, u_3 = 385$ Use Lagrange’s interpolation.

Solution: By Lagrange’s interpolation formula

$$u(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} u_0 + \frac{(x-x_0)(x-x_2)(x_0-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} u_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} u_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} u_3$$

Here $x = 3, x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$ and

$$f(x_0) = u_0 = 580$$

$$f(x_1) = u_1 = 556$$

$$f(x_2) = u_2 = 520$$

$$f(x_3) = u_3 = 385$$

$$\begin{aligned} \therefore f(4) &= \frac{(4-1)(4-2)(4-3)}{(0-1)(0-2)(0-3)} 580 + \frac{(4-0)(4-2)(4-3)}{(1-0)(1-2)(1-3)} 556 \\ &\quad + \frac{(4-0)(4-1)(4-3)}{(2-0)(2-1)(2-3)} 520 + \frac{(4-0)(4-1)(4-2)}{(3-0)(3-1)(3-2)} 385 \\ &= \frac{3 \times 2 \times 1}{-1 \times -2 \times -3} 580 + \frac{4 \times 2 \times 1}{1 \times -1 \times -2} 556 + \frac{4 \times 3 \times 1}{2 \times 1 \times -1} 520 + \frac{4 \times 3 \times 2}{3 \times 2 \times 1} 385 \\ &= -1 \times 580 + 4 \times 556 - 6 \times 520 + 4 \times 385 = -580 + 2224 - 3120 + 1540 = 64 \end{aligned}$$

Problem 3 : Find the unique polynomial of degree 2 or less such that $P(1)=1$, $P(3)=27$, $P(4)=64$
Using Lagrange's interpolation formula.

Solution : Given $x_0 = 1, x_1 = 3, x_2 = 4$

$$f(x_0) = P(1) = 1$$

$$\text{and } f(x_1) = P(3) = 27$$

$$f(x_2) = P(4) = 64$$

By Lagrange's interpolation formula for three points,

$$\begin{aligned} f(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \\ &= \frac{(x-3)(x-4)}{(1-3)(1-4)} \times 1 + \frac{(x-1)(x-4)}{(3-1)(3-4)} \times 27 + \frac{(x-1)(x-3)}{(4-1)(4-3)} \times 64 \\ &= \frac{x^2 - 7x + 12}{6} + \frac{x^2 - 5x + 4}{-2} + \frac{x^2 - 4x + 3}{3} \\ &= \frac{1}{6}[48x^2 - 114x + 72] = 8x^2 - 19x + 12 \end{aligned}$$

\therefore The required polynomial is $P(x) = 8x^2 - 19x + 12$

Problem 4 : Find the interpolating polynomial $f(x)$ from the table

x	0	1	4	5
$f(x)$	4	3	24	39

Solution : Given

$$x_0 = 0, x_1 = 1, x_2 = 4, x_3 = 5 \text{ and}$$

$$f(x_0) = 4, f(x_1) = 3, f(x_2) = 24, f(x_3) = 39$$

Using Lagrange's interpolation formula,

$$\begin{aligned} f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\ &\quad + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \end{aligned}$$

$$\begin{aligned} \therefore f(x) &= \frac{(x-1)(x-4)(x-5)}{(0-1)(0-4)(0-5)} \times 4 + \frac{(x-0)(x-4)(x-5)}{(1-0)(1-4)(1-5)} \times 3 \\ &+ \frac{(x-0)(x-1)(x-5)}{(4-0)(4-1)(4-5)} \times 24 + \frac{(x-0)(x-1)(x-4)}{(5-0)(5-1)(5-4)} \times 39 \end{aligned}$$

On simplification, $f(x) = 2x^2 - 3x + 4$

Problem 4 : Find the polynomial $f(x)$ by using Lagrange's interpolation formula and hence find $f(3)$ for

x	0	1	2	5
$f(x)$	2	3	12	147

Solution : Here $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$ and

$$f(x_0) = 2, f(x_1) = 3, f(x_2) = 12, f(x_3) = 147.$$

Lagrange's interpolation formula is

$$\begin{aligned} f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\ &+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \\ \therefore f(x) &= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \times 2 + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \times 3 \\ &+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \times 12 + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \times 147 \end{aligned}$$

$$\text{Hence } f(x) = x^3 + x^2 - x + 2$$

$$\therefore f(3) = 27 + 9 - 3 + 2 = 35$$

Problem 4 : A curve passes through the points (0, 18), (1, 10), (3, -18) and (6, 90). Find the slope of the curve at $x = 2$.

Solution : Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 6$ and $y_0 = 18, y_1 = 10, y_2 = -18, y_3 = 90$.

Since the values of x are unequally spaced, we use Lagrange's interpolation formula:

$$\begin{aligned} y = f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 \\ &+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 \\ \therefore y &= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)} \times 18 + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)} \times 10 \\ &+ \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-5)} \times -18 + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)} \times 90 \\ &= (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x) + (x^3 - 7x^2 + 6x) + (x^3 - 4x^2 + 3x) \end{aligned}$$

$$\text{i.e., } y = 2x^3 - 10x^2 + 18$$

$$\text{Thus, the slope of the curve at } x = 2 \left(\frac{dy}{dx} \right)_{x=2} = (6x^2 - 20x)_{x=2} = -16.$$

Problem 5: Using Lagrange's interpolation formula, express the function

$$\frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}$$
 as a sum of partial fractions.

Solution: Let us evaluate $y = 3x^2 + x + 1$ for $x = 1, x = 2, x = 3$

These values are $x: x_0 = 1, x_1 = 2, x_2 = 3$ and

$$y: y_0 = 5, y_1 = 15, y_2 = 31$$

Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

Substituting the above values, we get

$$\begin{aligned} \therefore y &= \frac{(x-2)(x-3)}{(1-2)(1-3)} \times 5 + \frac{(x-1)(x-3)}{(2-1)(2-3)} \times 15 + \frac{(x-1)(x-2)}{(3-1)(3-2)} \times 31 \\ &= \frac{(x-2)(x-3)}{2} \times 5 + \frac{(x-1)(x-3)}{-1} \times 15 + \frac{(x-1)(x-2)}{2} \times 31 \end{aligned}$$

Thus

$$\begin{aligned} \frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)} &= \frac{5}{2} \frac{(x-2)(x-3)}{(x-1)(x-2)(x-3)} - 15 \frac{(x-1)(x-3)}{(x-1)(x-2)(x-3)} + \frac{31}{2} \frac{(x-1)(x-2)}{(x-1)(x-2)(x-3)} \\ &= \frac{5}{2} \frac{1}{(x-1)} - 15 \frac{1}{(x-2)} + \frac{31}{2} \frac{1}{(x-3)} \end{aligned}$$

EXERCISE 5.3

(1). Using Lagrange's interpolation formula, find $y(10)$ from the following table

x	5	6	9	11
y	12	13	14	16

Ans: $y(10) = 14.67$

(2). Find by using Lagrange's interpolation formula, the interpolation polynomial

Which corresponds to the following data

$$x: -1 \quad 0 \quad 2 \quad 5$$

$$y: 9 \quad 5 \quad 3 \quad 10$$

Ans: $x^2 - 3x + 5$

(3). Using Lagrange's interpolation formula, fit a polynomial to the data

$$x: \quad 0 \quad 1 \quad 3 \quad 4$$

$$y: -12 \quad 0 \quad 6 \quad 12$$

Ans: $y(x) = x^3 - 7x^2 + 18x - 12$

(4). Given $u_0 = 580, u_1 = 556, u_2 = 520, u_4 = 385$ Find u_3

Ans: $u_3 = 465.25$

(5). Find $y(5)$ given that $y(0) = 1, y(1) = 3, y(3) = 13$ and $y(8) = 123$, using Lagrange's interpolation formula

Ans: $y(5) = 51.7142$

Central Difference Interpolation formula:

We have to discussed that Newton's interpolation formulae and suitable for interpolation near the beginning and end values of the data. For interpolation near the middle of a difference table, Central difference formulae are preferable.

Let x be the middle of the data. If x takes the values $x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h$ and their the corresponding values are $y_{-2}, y_{-1}, y_0, y_1, y_2, \dots$ of $y=f(x)$.

x	$y = f(x)$	First Forward Differences	Second Forward Differences	Third Forward Differences	Forth Forward Differences
$x_0 - 2h$	y_{-2}	$\Delta y_{-2} = y_{-1} - y_{-2}$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} = \Delta y_{-2} - \Delta y_{-1}$		
x_0	y_0	$\Delta y_{-1} = y_0 - y_{-1}$		$\Delta^3 y_{-2} = \Delta^2 y_{-1} - \Delta^2 y_{-2}$	
$x_0 + h$	y_1		$\Delta^2 y_{-1} = \Delta y_0 - \Delta y_{-1}$		$\Delta^4 y_{-2} = \Delta^3 y_{-1} - \Delta^3 y_{-2}$
$x_0 + 2h$	y_2	$\Delta y_0 = y_1 - y_0$		$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$	
			$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$			

Gauss's forward interpolation formula:

The Gauss's forward interpolation formula is given by

$$f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \dots$$

The central difference table is given below

x	$y = f(x)$	First Forward Differences	Second Forward Differences	Third Forward Differences	Forth Forward Differences
$x_0 - 2h$	y_{-2}	$\Delta y_{-2} = y_{-1} - y_{-2}$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} = \Delta y_{-2} - \Delta y_{-1}$		
x_0	y_0	$\Delta y_{-1} = y_0 - y_{-1}$		$\Delta^3 y_{-2} = \Delta^2 y_{-1} - \Delta^2 y_{-2}$	
		$\Delta y_0 = y_1 - y_0$	$\Delta^2 y_{-1} = \Delta y_0 - \Delta y_{-1}$		$\Delta^4 y_{-2} = \Delta^3 y_{-1} - \Delta^3 y_{-2}$
$x_0 + h$	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$	
$x_0 + 2h$	y_2	$\Delta y_1 = y_2 - y_1$			

Gauss's forward interpolation formula:

The Gauss's Backward interpolation formula is given by

$$f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \dots$$

The central difference table is given below

x	$y = f(x)$	First Forward Differences	Second Forward Differences	Third Forward Differences	Forth Forward Differences
$x_0 - 2h$	y_{-2}	$\Delta y_{-2} = y_{-1} - y_{-2}$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} = \Delta y_{-2} - \Delta y_{-1}$		
		$\Delta y_{-1} = y_0 - y_{-1}$		$\Delta^3 y_{-2} = \Delta^2 y_{-1} - \Delta^2 y_{-2}$	
x_0	y_0		$\Delta^2 y_{-1} = \Delta y_0 - \Delta y_{-1}$		$\Delta^4 y_{-2} = \Delta^3 y_{-1} - \Delta^3 y_{-2}$
		$\Delta y_0 = y_1 - y_0$		$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$	
$x_0 + h$	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
$x_0 + 2h$	y_2	$\Delta y_1 = y_2 - y_1$			

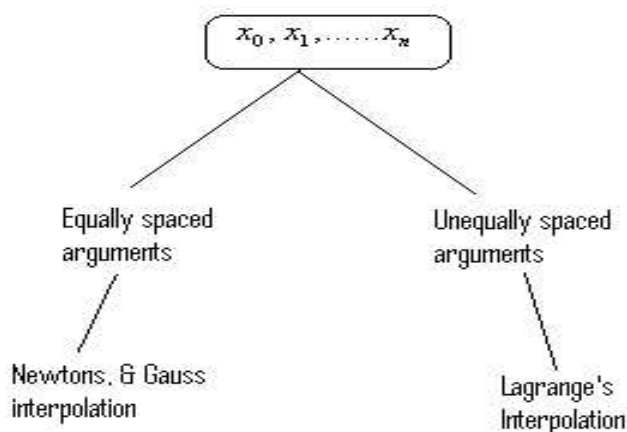
UNIT-II

Interpolation:

Finite differences-Newton's forward and backward interpolation formulae – Lagrange's formulae. Gauss forward and backward formula, Stirling's formula, Bessel's formula.

INTERPOLATION: The process of finding the curve passing through the points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ is called as Interpolation and the curve obtained is called as Interpolating curve. Interpolating curve passing through the given set of points is unique.

Let x_0, x_1, \dots, x_n be given set of observations and $y = f(x)$ be the given function, then the method to find $f(x_m), \forall x_0 \leq x_m \leq x_n$ is called as an Interpolation. If x_m is not in the range of x_0 and x_n , then the method to find x_m is called as Extrapolation.



The Interpolation depends upon finite difference concept.

If x_0, x_1, \dots, x_n be given set of observations and let $y_0 = f(x_0), y_1 = f(x_1), \dots, y_n = f(x_n)$ be their corresponding values for the curve $y = f(x)$, then $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ is called as finite difference.

When the arguments are equally spaced i.e. $x_i - x_{i-1} = h, \forall i$ then we can use one of the following differences.

Note: If h is the common difference of the observations, then $h = x_i - x_{i-1}, \forall i$

- Forward differences
- Backward differences
- Central differences

Forward Difference:

Let us consider x_0, x_1, \dots, x_n be given set of observations and let y_0, y_1, \dots, y_n are corresponding values of the curve $y = f(x)$, then the Forward difference operator Δ is denoted by and is defined as $\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \dots, \Delta y_n = y_n - y_{n-1}$. In this case are $\Delta y_0, \Delta y_1, \dots, \Delta y_n$ called as First Forward differences of y .

The difference of first forward differences will give us Second forward differences and it is denoted Δ^2 by and is defined as $\Delta^2 y_0 = \Delta(\Delta y_0)$

$$\begin{aligned}
 &= \Delta(y_1 - y_0) \\
 &= \Delta y_1 - \Delta y_0 \\
 &= (y_2 - y_1) - (y_1 - y_0) \\
 &= (y_2 - 2y_1 + y_0)
 \end{aligned}$$

Similarly, the difference of second backward differences will give us third backward difference and it is denoted by Δ^3 .

Forward Difference Table

x	$y = f(x)$	First Forward Differences	Second Forward Differences	Third Forward Differences	Fourth Forward Differences
x_0	y_0				
x_1	y_1	$\Delta y_0 = y_1 - y_0$			
x_2	y_2		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
\cdot	\cdot				
\cdot	\cdot	$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
\cdot	\cdot				
x_{n-1}	y_{n-1}		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
x_n	y_n	$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
				

		$\Delta y_{n-1} = y_n - y_{n-1}$
--	--	----------------------------------

Note: $\Delta^n y_k = \sum_{i=0}^n (-1)^i {}^n C_i y_{k-i}$

Backward Difference:

Let us consider x_0, x_1, \dots, x_n be given set of observations and let y_0, y_1, \dots, y_n are corresponding values of the curve $y = f(x)$, then the backward difference operator ∇ is denoted by and is defined as $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \dots, \nabla y_n = y_n - y_{n-1}$. In this case, $\nabla y_1, \nabla y_2, \dots, \nabla y_n$ are called as First backward differences of y .

The difference of first forward differences will give us Second forward differences and it is denoted ∇^2 by and is defined as $\nabla^2 y_2 = \nabla(\nabla y_2)$

$$\begin{aligned}
 &= \nabla(y_2 - y_1) \\
 &= \nabla y_2 - \nabla y_1 \\
 &= (y_2 - y_1) - (y_1 - y_0) \\
 &= (y_2 - 2y_1 + y_0)
 \end{aligned}$$

Similarly, the difference of second backward differences will give us third backward difference and it is denoted by ∇^3 .

Backward Difference Table

		First	Second	Third	Forth
x	$y = f(x)$	Forward	Forward	Forward	Forward
		Differences	Differences	Differences	Differences

x_0	y_0			
x_1	y_1	$\nabla y_1 = y_1 - y_0$		
			$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
x_2	y_2	$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_3	y_3		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	$\nabla^4 y_4 = \nabla^3 y_4 - \nabla^3 y_3$
x_4	y_4	$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$
			$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$	
		$\nabla y_4 = y_4 - y_3$		

Note: $\nabla^n y_k = \sum_{i=0}^n (-1)^i {}^n C_i y_{k-i}$

Newton's forward interpolation formula:

Definition: Let x_0, x_1, \dots, x_n are the given set of observations with common difference h and let y_0, y_1, \dots, y_n be their corresponding values for the curve $y = f(x)$, then the Newton's Forward interpolation formula is given by

$$f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots$$

$$+ \frac{p(p-1)(p-2)\dots(p-(n-1))}{n!} \Delta^n y_0, \text{ where } p = \frac{x - x_0}{h}.$$

Newton's backward interpolation formula:

Definition: Let x_0, x_1, \dots, x_n are the given set of observations with common difference h and let y_0, y_1, \dots, y_n be their corresponding values for the curve $y = f(x)$, then the Newton's Backward interpolation formula is given by

$$f(x) = f(x_0 + ph) = y_0 + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \dots$$

$$+ \frac{p(p+1)(p+2)\dots(p-(n-1))}{n!}\nabla^n y_0, \text{ where } p = \frac{x-x_0}{h}.$$

Problem 1: For $x = 0, 1, 2, 3, 4$, $f(x) = 1, 14, 15, 5, 6$. Find $f(3)$ using forward difference table

Solution: Given

$x :$	0	1	2	3	4
$y = f(x) :$	1	14	15	5	6

The difference table is shown below

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$y_0 = 1$				
		$\Delta y_0 = 13$			
1	14		$\Delta^2 y_0 = -12$		
		1		$\Delta^3 y_0 = 1$	
2	15		-11		$\Delta^4 y_0 = 21$
		-10		22	
3	5		11		
		1			
4	6				

By Newton's Forward interpolation formula is given by

If we take $x_0 = 0, h = 1$ then $p = \frac{x-x_0}{h} = \frac{3-0}{1} = 3$

$$\therefore f(3) = 1 + 3(13) + \frac{3(3-1)}{2!}(-12) + \frac{3(3-1)(3-2)}{3!}1 + \frac{3(3-1)(3-2)(3-3)}{4!}21$$

$$\Rightarrow f(3) = 1 + 39 - 36 + 1$$

$$\therefore f(3) = 5$$

Problem 2: Find a polynomial by using Newton's forward difference interpolation formula for the following data. Also find $f(2.5)$

x	0	1	2	3
$y = f(x)$	1	3	7	13

Solution: The difference table is shown below

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	$y_0 = 1$			
		$\Delta y_0 = 2$		
1	3		$\Delta^2 y_0 = 2$	
		4		$\Delta^3 y_0 = 0$
2	7		2	
		6		
3	13			

By Newton's Forward interpolation formula is given by

If we take $x_0 = 0, h = 1$ then $p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$

$$f(x) = 1 + x(2) + \frac{x(x-1)}{2!}(2) + \frac{x(x-1)(x-2)}{3!}(0) + \dots$$

$$= 1 + 2x + x^2 - x$$

$$\therefore f(x) = x^2 + x + 1$$

Also, $f(2.5) = (2.5)^2 + 2.5 + 1 = 10.125$

Problem 3. The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

$x = \text{height:}$	100	150	200	250	300	350	400
$y = \text{distance:}$	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when $x = 160 \text{ ft.}$

Solution:

The difference table is as under:

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		- 0.39		
		2.01		0.15	
200	15.04		- 0.24		- 0.07
		1.77		0.08	
250	16.81		- 0.16		- 0.05
		1.61		0.03	
300	18.42		- 0.13		- 0.01
		1.48		0.02	
350	19.90		- 0.11		
		1.37			
400	21.27				

Since $x = 160$ and $h = 50$, $\therefore p = \frac{x - x_0}{h} = \frac{10}{50} = 0.2$

\therefore Using Newton's forward interpolation formula, we get

$$y_{218} = y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots$$

$$y_{160} = 13.03 + 0.402 + 0.192 + 0.0384 + 0.00168 = 13.46 \text{ nautical miles}$$

Home work:

Problem 4. Given $\sin 45^\circ = 0.7071, \sin 50^\circ = 0.7660, \sin 55^\circ = 0.8192,$ and $\sin 60^\circ = 0.8660, \sin 52^\circ$, Using Newton's interpolation formula.

Ans: $\sin 52^\circ = 0.7880$ correct to four decimal places

Problem 5. The population of a country in the decennial census is as shown below. Estimate for the year 1925

Years x	1891	1901	1911	1921	1931
Population in thousands y	46	66	81	93	101

Ans: Here $h=10, y(1925): 96.8368.$

Lagrange's interpolation Formula:

If x_0, x_1, \dots, x_n is the given set of observations, which are need not be equally spaced and let y_0, y_1, \dots, y_n be their corresponding values of the curve $y = f(x)$, then

$$y = f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1 + \dots + \frac{(x-x_1)(x-x_2)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)\dots(x_n-x_{n-1})} y_n$$

This is known as Lagrange's interpolation formula for unequal intervals.

Example 1: Evaluate $f(10)$ given $f(x) = 168, 192, 336$ at $x = 1, 7, 15$ respectively, Use Lagrange's interpolation.

Solution: Here $x = 10, x_0 = 1, x_1 = 7, x_2 = 15$ and $y_0 = 168, y_1 = 192, y_2 = 336$

Lagrange's interpolation formula is given by

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$
$$\therefore y = f(10) = \frac{(10-7)(10-15)}{(1-7)(1-15)} 168 + \frac{(10-1)(10-15)}{(7-1)(7-15)} 192 + \frac{(10-1)(10-7)}{(15-1)(15-7)} 336$$
$$= \frac{-15}{84} 168 + \frac{-45}{-48} 192 + \frac{27}{112} 336$$
$$= -30.005 + 180 + 81.01 = 231.005$$

Example 2: Given $u_0 = 580, u_1 = 556, u_2 = 520, u_3 = 385$ Use Lagrange's interpolation.

Solution: By Lagrange's interpolation formula

$$u(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} u_0 + \frac{(x-x_0)(x-x_2)(x_0-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} u_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} u_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} u_3$$

Here $x = 3, x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$ and

$$f(x_0) = u_0 = 580$$

$$f(x_1) = u_1 = 556$$

$$f(x_2) = u_2 = 520$$

$$f(x_3) = u_3 = 385$$

$$\begin{aligned}
\therefore f(4) &= \frac{(4-1)(4-2)(4-3)}{(0-1)(0-2)(0-3)} 580 + \frac{(4-0)(4-2)(4-3)}{(1-0)(1-2)(1-3)} 556 \\
&\quad + \frac{(4-0)(4-1)(4-3)}{(2-0)(2-1)(2-3)} 520 + \frac{(4-0)(4-1)(4-2)}{(3-0)(3-1)(3-2)} 385 \\
&= \frac{3 \times 2 \times 1}{-1 \times -2 \times -3} 580 + \frac{4 \times 2 \times 1}{1 \times -1 \times -2} 556 + \frac{4 \times 3 \times 1}{2 \times 1 \times -1} 520 + \frac{4 \times 3 \times 2}{3 \times 2 \times 1} 385 \\
&= -1 \times 580 + 4 \times 556 - 6 \times 520 + 4 \times 385 = -580 + 2224 - 3120 + 1540 = 64
\end{aligned}$$

Example 3 : Find the unique polynomial of degree 2 or less such that $P(1)=1$, $P(3)=27$, $P(4)=64$ Using Lagrange's interpolation formula.

Solution : Given $x_0 = 1, x_1 = 3, x_2 = 4$

$$f(x_0) = P(1) = 1$$

$$\text{and } f(x_1) = P(3) = 27$$

$$f(x_2) = P(4) = 64$$

By Lagrange's interpolation formula for three points,

$$\begin{aligned}
f(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \\
&= \frac{(x-3)(x-4)}{(1-3)(1-4)} \times 1 + \frac{(x-1)(x-4)}{(3-1)(3-4)} \times 27 + \frac{(x-1)(x-3)}{(4-1)(4-3)} \times 64 \\
&= \frac{x^2 - 7x + 12}{6} + \frac{x^2 - 5x + 4}{-2} + \frac{x^2 - 4x + 3}{3} \\
&= \frac{1}{6} [48x^2 - 114x + 72] = 8x^2 - 19x + 12
\end{aligned}$$

\therefore The required polynomial is $P(x) = 8x^2 - 19x + 12$

Example 4 : Find the interpolating polynomial $f(x)$ from the table

x	0	1	4	5
$f(x)$	4	3	24	39

Solution : Given

$$x_0 = 0, x_1 = 1, x_2 = 4, x_3 = 5 \text{ and}$$

$$f(x_0) = 4, f(x_1) = 3, f(x_2) = 24, f(x_3) = 39$$

Using Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\ + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$\therefore f(x) = \frac{(x-1)(x-4)(x-5)}{(0-1)(0-4)(0-5)} \times 4 + \frac{(x-0)(x-4)(x-5)}{(1-0)(1-4)(1-5)} \times 3 \\ + \frac{(x-0)(x-1)(x-5)}{(4-0)(4-1)(4-5)} \times 24 + \frac{(x-0)(x-1)(x-4)}{(5-0)(5-1)(5-4)} \times 39$$

On simplification, $f(x) = 2x^2 - 3x + 4$

Home work:

Problem 5: A curve passes through the points (0, 18), (1, 10), (3, -18) and (6, 90). Find the slope of the curve at $x = 2$.

Problem 6: Find by using Lagrange's interpolation formula, the interpolation polynomial which corresponds to the following data

$$x: -1 \quad 0 \quad 2 \quad 5 \\ y: 9 \quad 5 \quad 3 \quad 10$$

Ans: $x^2 - 3x + 5$

Problem 7: Using Lagrange's interpolation formula, fit a polynomial to the data

$$x: \quad 0 \quad 1 \quad 3 \quad 4 \\ y: -12 \quad 0 \quad 6 \quad 12$$

Ans: $y(x) = x^3 - 7x^2 + 18x - 12$

Central Difference Interpolation formula:

We have to discussed that Newton's interpolation formulae and suitable for interpolation near the beginning and end values of the data. For interpolation near the middle of a difference table, Central difference formulae are preferable.

Let x be the middle of the data. If x takes the values the corresponding values of $y=f(x)$.

x	$y = f(x)$	First Forward Differences	Second Forward Differences	Third Forward Differences	Fourth Forward Differences
$x_0 - 2h$	y_{-2}				
$x_0 - h$	y_{-1}	$\Delta y_{-2} = y_{-1} - y_{-2}$			
x_0	y_0		$\Delta^2 y_{-2} = \Delta y_{-2} - \Delta y_{-1}$		
$x_0 + h$	y_1	$\Delta y_{-1} = y_0 - y_{-1}$		$\Delta^3 y_{-2} = \Delta^2 y_{-1} - \Delta^2 y_{-2}$	
$x_0 + 2h$	y_2		$\Delta^2 y_{-1} = \Delta y_0 - \Delta y_{-1}$		$\Delta^4 y_{-2} = \Delta^3 y_{-1} - \Delta^3 y_{-2}$
		$\Delta y_0 = y_1 - y_0$		$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$	
			$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$			

Gauss's forward interpolation formula:

The Gauss's forward interpolation formula is given by

$$f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \dots$$

The central difference table is given below

x	$y = f(x)$	First Forward Differences	Second Forward Differences	Third Forward Differences	Forth Forward Differences
$x_0 - 2h$	y_{-2}	$\Delta y_{-2} = y_{-1} - y_{-2}$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} = \Delta y_{-2} - \Delta y_{-1}$		
x_0	y_0	$\Delta y_{-1} = y_0 - y_{-1}$		$\Delta^3 y_{-2} = \Delta^2 y_{-1} - \Delta^2 y_{-2}$	
		$\Delta y_0 = y_1 - y_0$	$\Delta^2 y_{-1} = \Delta y_0 - \Delta y_{-1}$		$\Delta^4 y_{-2} = \Delta^3 y_{-1} - \Delta^3 y_{-2}$
$x_0 + h$	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$	
$x_0 + 2h$	y_2	$\Delta y_1 = y_2 - y_1$			

Gauss's forward interpolation formula:

The Gauss's Backward interpolation formula is given by

$$f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \dots$$

The central difference table is given below

x	$y = f(x)$	First Forward Differences	Second Forward Differences	Third Forward Differences	Forth Forward Differences
$x_0 - 2h$	y_{-2}	$\Delta y_{-2} = y_{-1} - y_{-2}$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} = \Delta y_{-2} - \Delta y_{-1}$		
		$\Delta y_{-1} = y_0 - y_{-1}$		$\Delta^3 y_{-2} = \Delta^2 y_{-1} - \Delta^2 y_{-2}$	
x_0	y_0		$\Delta^2 y_{-1} = \Delta y_0 - \Delta y_{-1}$		$\Delta^4 y_{-2} = \Delta^3 y_{-1} - \Delta^3 y_{-2}$
		$\Delta y_0 = y_1 - y_0$		$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$	
$x_0 + h$	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
$x_0 + 2h$	y_2	$\Delta y_1 = y_2 - y_1$			

Problem 1.

Find $f(22)$ from the Gauss forward formula:

$x:$	20	25	30	35	40	45
$f(x):$	354	332	291	260	231	204

Solution:

Taking $x_0 = 25$, $h = 5$, we have to find the value of $f(x)$ for $x = 22$.

$$\text{i.e., for } p = \frac{x - x_0}{h} = \frac{22 - 25}{5} = -0.6$$

The difference table is as follows:

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
20	-1	354 ($= y_{-1}$)	-22				
25	0	332 ($= y_0$)	-41	-19	29		
30	1	291 ($= y_1$)	-31	10	-8	-37	45
35	2	260 ($= y_2$)	-29	2	0	8	
40	3	231 ($= y_3$)	-27	2			
45	4	204 ($= y_4$)					

Gauss forward formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} \\ + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} \\ + \frac{(p+1)(p-1)p(p-2)(p+2)}{5!}\Delta^5 y_{-2} + \dots$$

$$\begin{aligned} \therefore f(22) &= 332 + (0.6)(-41) + \frac{(-0.6)(-0.6-1)}{2!}(-19) \\ &\quad + \frac{(-0.6+1)(-0.6)(-0.6-1)}{3!}(-8) \\ &\quad + \frac{(-0.6+1)(-0.6)(-0.6-1)(-0.6-2)}{4!}(-37) \\ &\quad + \frac{(-0.6+1)(-0.6)(-0.6-1)(-0.6-2)(-0.6+2)}{5!}(45) \\ &= 332 + 24.6 - 9.12 - 0.512 + 1.5392 - 0.5241 \end{aligned}$$

Hence $f(22) = 347.983$.

Problem 2:

Use Gauss's forward formula to evaluate y_{30} , given that $y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070$, $y_{33} = 16.3432$ and $y_{37} = 15.5154$.

Solution

Taking $x_0 = 29$, $h = 4$, we require the value of y for $x = 30$

i.e., for $p = \frac{x - x_0}{h} = \frac{30 - 29}{4} = 0.25$

The difference table is given below:

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$
21	-2	18.4708				
			-0.6564			
25	-1	17.8144		-0.0510		
			-0.7074		-0.7074	
29	0	17.1070		-0.0564		-0.0022
			-0.7638		-0.0076	
33	1	16.3432		-0.0640		
			-0.8278			
37	2	15.5154				

Gauss's forward formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p+1)}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\Delta^4 y_{-2} + \dots$$

$$y_{30} = 17.1070 + (0.25)(-0.7638) + \frac{(0.25)(-0.75)}{2}(-0.0564)$$

$$+ \frac{(1.25)(0.25)(-0.75)}{6}(-0.0076) + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24} \times (-0.0022)$$

$$= 17.1070 - 0.19095 + 0.00529 + 0.0003 - 0.00004 = 16.9216 \text{ approx.}$$

Problem 3:

Using Gauss backward difference formula, find $y(8)$ from the following table.

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Solution:

Taking $x_0 = 10$, $h = 5$, we have to find y for $x = 8$, i.e., for

$$p = \frac{x - x_0}{h} = \frac{8 - 10}{5} = -0.4.$$

The difference table is as follows:

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
0	2	7					
			4				
5	1	11		-1			
			3		2		
10	0	14		1		-1	
			4		1		0
15	1	18		2		-1	
			6		0		
20	2	24		2			
			8				
25	3	32					

Gauss backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)p(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \dots$$
$$y(8) = 14 + (-0.4)(3) + \frac{(-0.4+1)(-0.4)}{2!}(1) + \frac{(-0.4+1)(-0.4)(-0.4-1)}{3!}(2) + \frac{(-0.4+2)(-0.4+1)(-0.4)(-0.4-1)}{4!}(-1)$$
$$= 14 - 1.2 - 0.12 + 0.112 + 0.034$$

Hence $y_{(8)} = 12.826$

Problem 4:

Interpolate by means of Gauss's backward formula, the population of a town for the year 1974, given that:

Year:	1939	1949	1959	1969	1979	1989
Population: (in thousands)	12	15	20	27	39	52

Solution:

Taking $x_0 = 1969$, $h = 10$, the population of the town is to be found for

$$p = \frac{1974 - 1969}{10} = 0.5$$

The Central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1939	-3	12	3	2	0	3	-10
1949	-2	15					
			5				
1959	-1	20		2			
			7		3		
1969	0	27		5		-7	
			12		-4		
1979	1	39		1			
			13				
1989	2	52					

Gauss's backward formula is

$$\begin{aligned}
 y_p = & y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} \\
 & + \frac{(p+2)p(p+1)p(p-1)}{4!}\Delta^4 y_{-2} \\
 & + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-3} + \dots
 \end{aligned}$$

$$\begin{aligned}
 y_{0.5} = & 27 + (0.5)(7) + \frac{(1.5)(0.5)}{2}(5) + \frac{(1.5)(0.5)(-0.5)}{6} \\
 & + \frac{(2.5)(1.5)(-0.5)}{24}(-7) + \frac{(2.5)(1.5)(0.5)(-0.5)(1.5)}{120}(-10) \\
 = & 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172 \\
 = & 32.532 \text{ thousands approx.}
 \end{aligned}$$

Stirling's Formula

Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \dots \quad (1)$$

Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \dots \quad (2)$$

Taking the mean of (1) and (2), we obtained

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \dots \quad (3)$$

Which is called Stirling's formula.

NOTE: This formula involves means of the odd differences just above and below the central line and even differences on this line as shown below:

$$\dots y_0 \dots \left(\frac{\Delta y_{-1}}{\Delta y_0}\right) \dots \Delta^2 y_{-1} \dots \left(\frac{\Delta^3 y_{-2}}{\Delta^3 y_{-1}}\right) \dots \Delta^4 y_{-2} \dots \left(\frac{\Delta^5 y_{-1}}{\Delta^5 y_0}\right) \dots \Delta^6 y_{-3} \dots \text{Central line.}$$

Problem 1:

Given

θ° :	0	5	10	15	20	25	30
$\tan \theta$:	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Using Stirling's formula, estimate the value of $\tan 16^\circ$.**Solution:**

Taking the origin at $\theta^\circ = 15^\circ$, $h = 5^\circ$ and $p = \frac{\theta - 15}{5}$, we have the following central difference table:

p	$y = \tan \theta$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-3	0.0000					
		0.0875				
-2	0.0875		0.0013			
		0.0888		0.0015		
-1	0.1763		0.0028		0.0002	
		0.0916		0.0017		-0.0002
0	0.2679		0.0045		0.0000	
		0.0961		0.0017		0.0009
1	0.3640		0.0062		0.0009	
		0.1023		0.0026		
2	0.4663		0.0088			
		0.1111				
3	0.5774					

At $\theta = 16^\circ$, $p = \frac{16 - 15}{5} = 0.2$

Stirling's formula is

$$y_p = y_0 + \frac{p}{1} \cdot \frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p^2(p^2 - 1)}{3!} \cdot \frac{\Delta^2 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2 - 1)}{4!} \Delta^4 y_{-2} + \dots$$

$$\begin{aligned} \therefore y_{0.2} &= 0.2679 + 0.2 \left(\frac{0.0916 + 0.0916}{2} \right) + \frac{(0.2)^2}{2} (0.0045) + \dots \\ &= 0.2679 + 0.01877 + 0.00009 + \dots = 0.28676 \end{aligned}$$

Hence, $\tan 16^\circ = 0.28676$.

Problem 2:

Employ Stirling's formula to compute $y_{12.2}$ from the following table ($y_x = 1 + \log_{10} \sin x$):

x° :	10	11	12	13	14
$10^5 y_x$:	23,967	28,060	31,788	35,209	38,368

Solution:

Taking the origin at $x_0 = 12^\circ$, $h = 1$ and $p = x - 12$, we have the following central difference table:

p	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
$-2 = x_{-2}$	$0.23967 = y_{-2}$				
		$0.04093 = \Delta y_{-2}$			
$-1 = x_{-1}$	$0.28060 = y_{-1}$		$-0.00365 = \Delta^2 y_{-2}$		
		$0.03728 = \Delta y_{-1}$		$0.00058 = \Delta^3 y_{-2}$	
$0 = x_0$	$0.31788 = y_0$		$-0.00307 = \Delta^2 y_{-1}$		$-0.00013 = \Delta^4 y_{-2}$
		$0.03421 = \Delta y_0$		$-0.00045 = \Delta^3 y_{-1}$	
$1 = x_1$	$0.35209 = y_1$		$-0.00062 = \Delta^2 y_0$		
		$0.03159 = \Delta y_1$			
$2 = x_2$	$0.38368 = y_2$				

At $x = 12.2$, $p = 0.2$. (As p lies between $-\frac{1}{4}$ and $\frac{1}{4}$, the use of Stirling's formula will be Quite suitable.)

Stirling's formula is

$$y_p = y_0 + \frac{p}{1} \frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2 - 1)}{3!} \cdot \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2 - 1)}{4!} \Delta^4 y_{-2} + \dots$$

When $p = 0.2$, we have

$$\begin{aligned} y_{0.2} &= 0.3178 + 0.2 \left(\frac{0.03728 + 0.03421}{2} \right) + \frac{(0.2)^2}{2} (-0.00307) \\ \therefore &+ \frac{(0.2)^2 [(0.2)^2 - 1]}{6} \left(\frac{0.00058 + 0.00054}{2} \right) + \frac{(0.2)^2 [(0.2)^2 - 1]}{24} (-0.00013) \\ &= 0.31788 + 0.00715 - 0.00006 - 0.000002 + 0.0000002 \\ &= 0.32497. \end{aligned}$$

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Bessel's Formula

Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p+1)}{4!}\Delta^4 y_{-2} + \dots$$

$$\text{We have } \Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1} \quad (1)$$

$$\text{i.e., } \Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1} \quad (2)$$

$$\text{Similarly } \Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \text{ etc.}$$

Now (1) can be written as

$$\begin{aligned} y_p &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{1}{2}\Delta^2 y_{-1} + \frac{1}{2}\Delta^2 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} \\ &\quad + \frac{p(p^2-1)(p-2)}{4!}\left(\frac{1}{2}\Delta^4 y_{-2} + \frac{1}{2}\Delta^4 y_{-2}\right) + \dots \\ &= y_0 + p\Delta y_0 + \frac{1}{2}\frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}(\Delta^2 y_0 + \Delta^3 y_{-1}) \\ &\quad + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!} \\ &\quad \times (\Delta^4 y_{-1} - \Delta^5 y_{-1}) + \dots \end{aligned}$$

$$\text{Hence } y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p - \frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1} \quad (4)$$

Which is known as Bessel's formula.

NOTE: This is a very useful formula for practical purposes. It involves odd differences below the central line and means of even differences of and below this line as shown below

$$y_0 \quad \Delta y_0 \left\{ \begin{array}{l} \Delta^2 y_{-1} \\ \Delta^2 y_0 \end{array} \right\} \quad \Delta^3 y_{-1} \left\{ \begin{array}{l} \Delta^4 y_{-2} \\ \Delta^4 y_{-1} \end{array} \right\} \quad \Delta^5 y_{-2} \left\{ \begin{array}{l} \Delta^6 y_{-1} \\ \Delta^6 y_0 \end{array} \right\} \quad \Delta^7 y_{-3} \quad \text{Central line}$$

Problem 1:

Apply Bessel's formula to obtain y_{25} , given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$.

Solution:

Taking the origin at $x_0 = 24$, $h = 4$, we have $p = (x - 24)$.

∴ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	2854			
		308		
0	3162		74	
		382		-8
1	3544		66	
		448		
2	3992			

At $x = 25$, $p = \frac{(25 - 24)}{4} = \frac{1}{4}$. ∴ (As p lies between $\frac{1}{4}$ and $\frac{3}{4}$, the use of Bessel's formula will yield accurate results)

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p - \frac{1}{2}\right)p(p-1)}{2!} \Delta^3 y_{-1} + \dots \quad (1)$$

When $p = 0.25$, we have

$$\begin{aligned} y_p &= 3162 + 0.25 \times 382 + \frac{0.25(-0.75)}{2!} \left(\frac{74 + 66}{2} \right) + \frac{(0.25)0.25(-0.75)}{2!} - 8 \\ &= 3162 + 95.5 - 6.5625 - 0.0625 \\ &= 3250.875 \text{ approx.} \end{aligned}$$

Problem 2:

Apply Bessel's formula to find the value of $f(27.5)$ from the table:

x :	25	26	27	28	29	30
$f(x)$:	4.000	3.846	3.704	3.571	3.448	3.333

Solution:

Taking the origin at $x_0 = 27$, $h = 1$, we have $p = x - 27$

The central difference table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
25	-2	4.000				
			-0.154			
26	-1	3.846		0.012		
			-0.142		-0.003	
27	0	3.704		0.009		0.004
			-0.133		-0.001	
28	1	3.571		0.010		-0.001
			-0.123		-0.002	
29	2	3.448		0.008		
			-0.115			
30	3	3.333				

At $x = 27.5$, $p = 0.5$ (As p lies between $1/4$ and $3/4$, the use of Bessel's formula will yield an accurate result),

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p - \frac{1}{2}\right)p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!} \left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \dots$$

When $p = 0.5$, we have

$$\begin{aligned} y_p &= 3.704 - \frac{(0.5)(0.5-1)}{2} \left(\frac{0.009+0.010}{2}\right) + 0 \\ &\quad + \frac{(0.5+1)(0.5)(0.5-1)(0.5-2)}{2} \frac{(-0.001-0.004)}{2} \\ &= 3.704 - 0.11875 - 0.00006 = 3.585 \end{aligned}$$

Hence $f(27.5) = 3.585$.

Exercises

- Find $y(25)$, given that $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$, using Gauss forward difference formula.
- Using Gauss's forward formula, find a polynomial of degree four which takes the following values of the function $f(x)$:

x :	1	2	3	4	5
$f(x)$:	1	-1	1	-1	1

- Using Gauss's forward formula, evaluate $f(3.75)$ from the table:

x :	2.5	3.0	3.5	4.0	4.5	5.0
Y :	24.145	22.043	20.225	18.644	17.262	16.047

- From the following table:

x :	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x :	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Find $e^{1.17}$, using Gauss forward formula.

- Apply Gauss's backward formula to find $\sin 45^\circ$ from the following table:

θ° :	20	30	40	50	60	70	80
$\sin \theta$:	0.34202	0.502	0.64279	0.76604	0.86603	0.93969	0.98481

- Using Stirling's formula find y_{35} , given $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$, where y_x represents the number of persons at age x years in a life table.
- Use Stirling's formula to evaluate $f(1.22)$, given

x :	1.0	1.1	1.2	1.3	1.4
$f(x)$:	0.841	0.891	0.932	0.963	0.985

- Calculate the value of $f(1.5)$ using Bessel's interpolation formula, from the table

x :	0	1	2	3
$f(x)$:	3	6	12	15

- Use Bessel's formula to obtain y_{25} , given $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$.

Unit-V

Binomial, Poisson & Normal distribution

Bernoulli's theorem: If probability of the occurrence of the event in a single Bernoullian trial is p , then the probability of the occurrence r success out of n trials is

$$P(X = r) = {}^n C_r p^r q^{n-r}, p + q = 1, r = 0, 1, 2, 3, \dots, n.$$

Note: $(x + y)^n = {}^n C_0 p^n + {}^n C_1 p^{n-1} q^1 + {}^n C_2 p^{n-2} q^2 + \dots + {}^n C_n q^n$
(or)

$$(x + y)^n = {}^n C_0 q^n + {}^n C_1 q^{n-1} p^1 + {}^n C_2 q^{n-2} p^2 + \dots + {}^n C_n p^n$$

Definition: If a random variable X has the binomial distribution, then its [probability](#) mass function is given by:

$$P(X = r) = {}^n C_r p^r q^{n-r}, r = 0, 1, 2, 3, \dots, n.$$

What are the necessary conditions for a binomial distribution?

- 1) there are a fixed number of trials, n
- 2) there are 2 possible outcomes, success and failure
- 3) there is a fixed probability of success p for all trials
- 4) the trials are independent.

Constants of the Binomial distribution:

(i) **Mean of the Binomial distribution:**

$$\begin{aligned} \text{Mean}(\mu) = E(x) &= \sum_{x=0}^n xP(x) \\ &= \sum_{x=0}^{\infty} xP(x) = 0.P(0) + 1.P(1) + 2.P(2) + \dots + n.P(n) \\ &= 1.P(1) + 2.P(2) + \dots + n.P(n) \\ &= 1.{}^n C_1 p^1 q^{n-1} + 2.{}^n C_2 p^2 q^{n-2} + \dots + n.{}^n C_n p^n \\ &= np^1 q^{n-1} + 2. \frac{n(n-1)}{2} p^2 q^{n-2} + \dots + n.p^n \\ &= np(q^{n-1} + (n-1)pq^{n-2} + \dots + p^{n-1}) \\ &= np(p + q)^{n-1} = np.1 = np \end{aligned}$$

$$\text{Mean}(\mu) = E(x) = np$$

(ii) **Variance of the Binomial distribution:**

$$\text{Variance}(\sigma^2) = E(x^2) - E(x)$$

$$\begin{aligned}
&= \sum_{x=0}^{\infty} x^2 P(x) - \mu^2 \\
&= \sum_{x=0}^{\infty} [x(x-1) + x] P(x) - \mu^2 \\
&= \sum_{x=0}^{\infty} [x(x-1)] P(x) + \sum_{x=0}^{\infty} x P(x) - \mu^2 \\
&= \sum_{x=0}^{\infty} [x(x-1)] P(x) + \mu - \mu^2 \\
&= 2.1.P(2) + 3.2.P(3) + \dots + n(n-1)P(n) + \mu - \mu^2 \\
&= 2.1.nc_2 p^2 q^{n-2} + 3.2.nc_{32} p^3 q^{n-3} + \dots + n.(n-1)p^n + \mu - \mu^2 \\
&= 2. \frac{n(n-1)}{2} p^2 q^{n-2} + 6. \frac{n(n-1)(n-2)}{2} p^3 q^{n-3} + \dots + n.(n-1)p^n + \mu - \mu^2 \\
&= n(n-1)p \left(q^{n-2} + (n-2)p.q^{n-3} + \dots + p^{n-2} \right) + \mu - \mu^2 \\
&= n(n-1)p^2 (p+q)^{n-2} + \mu - \mu^2 \\
&= n^2 p^2 - np^2 + np - n^2 p^2 = np(1-p) \\
&= npq
\end{aligned}$$

Variance (σ^2) = npq

Recurrence Relation for the Poisson distribution:

$$\begin{aligned}
P(r) &= nc_r p^r q^{n-r} \\
P(r+1) &= nc_{r+1} p^{r+1} q^{n-(r+1)}
\end{aligned}$$

$$\begin{aligned}
\text{Now } \frac{P(r+1)}{P(r)} &= \frac{nc_{r+1} p^{r+1} q^{n-(r+1)}}{nc_r p^r q^{n-r}} \\
&= \frac{n!/[n-(r+1)]!.r!}{n!/(n-r)! .r!} \frac{p^{r+1} q^{n-(r+1)}}{p^r q^{n-r}} \\
&= \frac{(n-r)}{(r+1)} \cdot \frac{p}{q}
\end{aligned}$$

$$\therefore P(r+1) = \frac{(n-r)}{(r+1)} \cdot \frac{p}{q} P(r), \text{ which is the required recurrence relation.}$$

Problem 1: Find the binomial distribution for which mean is 4 and variance is 3.

Solution: (a) Given mean (μ) = $np = 4$

$$\text{Variance } (\sigma^2) = npq = 3$$

$$\frac{npq}{np} = \frac{3}{4} \Rightarrow q = \frac{3}{4}$$

$$\therefore p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

and $np = 4 \Rightarrow n \cdot \frac{1}{4} = 4 \Rightarrow n = 16$.

The probability of the binomial distribution is

$$\therefore P(X = x) = {}^{12}C_x p^x q^{12-x} = {}^{12}C_x \times \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^x$$

Problem 2: The probability that a person can achieve a target is 3/4. The count of tries is 5. What is the probability that he will attain the target at least thrice?

Solution: Given $n =$ number of patients $= 5$

probability of a person achieve a target $p = \frac{3}{4}$,

then we have $q = 1 - p = 1 - \frac{3}{4} = \frac{1}{4}$

Using binomial distribution formula,

$$P(X = x) = {}^n C_x p^x q^{n-x}$$

Thus, the required probability is: $P(X = 3) + P(X = 4) + P(X = 5)$

$$\begin{aligned} &= 5C_3 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^3 + 5C_4 \left(\frac{3}{4}\right)^1 \left(\frac{1}{4}\right)^4 + 5C_5 \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^5 \\ &= \frac{459}{512}. \end{aligned}$$

Therefore, the probability that the person will attain the target atleast thrice is $\frac{459}{512}$.

Problem 3: If from six to seven in the evening one telephone line in every five is engaged in a conversation: what is the probability that when 10 telephone numbers are chosen at random, only two are in use?

Solution: Given $n =$ number of telephone numbers $= 5$

probability of a telephone line being in use $p = \frac{1}{5}$,

then probability of a telephone line not being in use $q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$

Now, we can use the binomial probability formula

$$P(X = x) = {}^n C_x p^x q^{n-x}$$

$$\begin{aligned} P(X = 2) &= {}^{10}C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^8 \\ &= 45 \times \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^8 = 0.0897. \end{aligned}$$

So, the probability that when 10 telephone numbers are chosen at random, only two are in use is 0.0897

Problem 3: A pharmaceutical lab states that a drug causes negative side effects in 3 of every 100 patients. To confirm this affirmation, another laboratory chooses 5 people at random who have consumed the drug. What is the probability of the following events?

- (i). None of the five patients experience side effects.
- (ii). At least two experience side effects.

(iii). What is the average number of patients that the laboratory should expect to experience side effects if they choose 100 patients at random?

Solution: Given $n =$ number of patients $= 5$

$$\text{Let } p = \text{Probability of negative side effect} = \frac{3}{100} = 0.03$$

$$\text{and hence the probability o, } q = 1 - 0.03 = 0.97$$

(i) Probability of at least two experience side effects:

$$\begin{aligned} P(X = 0) &= {}^5C_0 p^0 q^{5-0} \\ &= 1 \times (0.97)^5 \\ &= 0.8687 \end{aligned}$$

(ii) Probability of at least two experience side effects.

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - {}^5C_0 p^0 q^{5-0} + {}^5C_1 p^1 q^{5-1} + {}^5C_2 p^2 q^{5-2} \\ &= 1 - 1 \times (0.03)^0 (0.97)^5 - 5 \times (0.03)^1 (0.97)^4 \\ &= 1 - 0.8587 - 0.1327 = 0.0086 \end{aligned}$$

(iii) What is the average number of patients that the laboratory should expect to experience side effects if they choose 100 patients at random?

$$\mu = np = 100 \times 0.03 = 3$$

Problem 4: The mean of the binomial distribution is 3 and variance is $\frac{9}{4}$.

Find (i) the value of n (ii) $P(X \geq 9)$

(iii) $P(1 < X < 4)$ (iv) $P(1 \leq X \leq 3)$

Solution: (a) Given mean $(\mu) = np = 3$

$$\text{Variance } (\sigma^2) = npq = \frac{9}{4}$$

$$\frac{npq}{np} = \frac{9/4}{3} \Rightarrow q = \frac{3}{4}$$

$$\therefore p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\text{and } np = 3 \Rightarrow n \cdot \frac{1}{4} = 3 \Rightarrow n = 12$$

The probability of the binomial distribution is

$$P(X = x) = {}^{12}C_x p^x q^{12-x} = {}^{12}C_x \times \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{12-x}$$

(a) Probability of getting exactly two heads:

$$\begin{aligned}
P(X \geq 9) &= P(X = 9) + P(X = 10) + P(X = 11) + P(X = 12) \\
&= 12c_9 p^9 q^{12-9} + 12c_{10} p^{10} q^{12-10} + 12c_{11} p^{11} q^{12-11} + 12c_{12} p^{12} q^{12-12} \\
&= 10 \times \left(\frac{1}{4}\right)^9 \left(\frac{3}{4}\right)^3 + 10 \times \left(\frac{1}{4}\right)^{10} \left(\frac{3}{4}\right)^2 + 10 \times \left(\frac{1}{4}\right)^{11} \left(\frac{3}{4}\right)^1 + 10 \times \left(\frac{1}{4}\right)^{12} \left(\frac{3}{4}\right)^0 \\
&= 120 \times \frac{3^3}{4^{12}} + 66 \times \frac{3^2}{4^{12}} + 12 \times \frac{3^1}{4^{12}} + 1 \times \frac{1}{4^{12}} =
\end{aligned}$$

$$\begin{aligned}
\text{(b) } P(1 < X < 4) &= P(X = 2) + P(X = 3) \\
&= P(X = 2) + P(X = 3) \\
&= 12c_2 p^2 q^{12-2} + 12c_3 p^3 q^{12-3} \\
&= 66 \times \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{10} + 120 \times \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^9
\end{aligned}$$

$$\begin{aligned}
\text{(c) } P(1 \leq X \leq 3) &= P(X = 1) + P(X = 2) + P(X = 3) \\
&= 12c_1 p^1 q^{12-1} + 12c_2 p^2 q^{12-2} + 12c_3 p^3 q^{12-3} \\
&= 12 \times \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{11} + 66 \times \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{10} + 120 \times \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^9
\end{aligned}$$

Home work:

Problem 1: If a coin is tossed 5 times, find the probability of:

(a) Exactly 2 heads (b) At least 4 heads, and (c) At most 2 heads

Problem 2: There are four fused bulbs in a lot of 10 good bulbs. If three bulbs are drawn at random, find the probability of distribution of the number of fused bulbs drawn.

Problem 3: The probability that a person can hit a target is $\frac{3}{4}$. The count of tries is 5. What is the probability that he will attain the target at least thrice?

Poisson distribution:

Definition: A discrete [random variable](#) X is said to have a Poisson distribution with parameter $\lambda > 0$ if it has a [probability mass function](#) given by:

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \dots, \infty.$$

Constants of the Poisson distribution:

(i) Mean of the Poisson distribution:

$$\begin{aligned} \text{Mean}(\mu) = E(x) &= \sum_{x=0}^{\infty} xP(x) \\ &= \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!} = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!} \\ &= \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!} \quad \text{put } y = x-1 \Rightarrow x = y+1 \\ &= \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^{y+1}}{y!} = e^{-\lambda} \lambda \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \\ &= e^{-\lambda} \lambda e^{\lambda} = \lambda \end{aligned}$$

$$\text{Mean}(\mu) = E(x) = \lambda$$

(ii) Variance of the Poisson distribution:

$$\begin{aligned} \text{Variance}(\sigma^2) &= E(x^2) - E(x) \\ &= \sum_{x=0}^{\infty} x^2 P(x) - \sum_{x=0}^{\infty} xP(x) \\ &= \sum_{x=0}^{\infty} x^2 P(x) - \sum_{x=0}^{\infty} xP(x) \\ &= \sum_{x=0}^{\infty} [x(x-1) + x] P(x) \frac{e^{-\lambda} \lambda^x}{x!} - \mu^2 \\ &= \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-2)!} + \sum_{x=0}^{\infty} xP(x) - \mu^2 \\ &= \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-2)!} + \mu - \mu^2 \quad \text{put } y = x-2 \Rightarrow x = y+2 \\ &= \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^{y+2}}{y!} + \mu - \mu^2 \\ &= e^{-\lambda} \lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} + \mu - \mu^2 \\ &= e^{-\lambda} \lambda^2 e^{\lambda} + \lambda - \lambda^2 = \lambda \end{aligned}$$

$$\text{Variance}(\sigma^2) = \lambda$$

(iii) Recurrence Relation for the Poisson distribution:

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!},$$

$$P(x+1) = \frac{e^{-\lambda} \lambda^{x+1}}{(x+1)!}$$

$$\begin{aligned} \text{Now } \frac{P(x+1)}{P(x)} &= \frac{e^{-\lambda} \lambda^{x+1} / (x+1)!}{e^{-\lambda} \lambda^x / x!} \\ &= \frac{x!}{(x+1)!} \frac{\lambda^{x+1}}{\lambda^x} \end{aligned}$$

$\therefore P(x+1) = \frac{\lambda}{x+1} P(x)$, which is the required recurrence relation.

Note: Mean and Variance of the Poisson distribution are equal to λ

Problem 1: A random variable X has a Poisson distribution such that

$$P(x=1) = P(x=2)$$

- Find** (i) Mean (ii) $P(x=4)$
 (iii) $P(x \geq 1)$ (iv) $P(1 < X < 4)$

Solution: Since a random variable X has a Poisson distribution

$$\text{i.e., } P(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \dots, \infty$$

(i) Given $P(x=1) = P(x=2)$

$$\frac{e^{-\lambda} \lambda^1}{1!} = \frac{e^{-\lambda} \lambda^2}{2!}$$

$$\Rightarrow 1 = \frac{\lambda}{2} \quad \therefore \lambda = 2$$

(ii) $P(X=4) = \frac{e^{-\lambda} \lambda^4}{4!} = \frac{e^{-2} 2^4}{4!}$
 $= \frac{16}{24e^2} = \frac{2}{3} \cdot \frac{1}{e^2} = 0.0902$

(iii) $P(X \geq 1) = 1 - P(X=0)$
 $= 1 - P(X=0)$
 $= 1 - \frac{e^{-\lambda} \lambda^0}{0!}$
 $= 1 - e^{-2} = 0.8647.$

$$\begin{aligned}
 \text{(iv) } P(1 < X < 4) &= P(X = 2) + P(X = 3) \\
 &= \frac{e^{-\lambda} \lambda^2}{2!} + \frac{e^{-\lambda} \lambda^3}{3!} \\
 &= 4.e^{-2} \left(\frac{1}{2} + \frac{2}{6} \right) \\
 &= \frac{10}{3} e^{-2} = 0.4511.
 \end{aligned}$$

Problem 2: At a small walk-in clinic, an average of five patients arrive at the clinic per hour during the opening hours. What is the probability that the patients will arrive in the next hours (i) exactly three patients (ii) at least one patient

Solution: an average no. of patients arrive at the clinic per hour during opening hours is 5.
i.e., $\lambda = 5$

(i) The probability that exactly three patients will arrive at the next hour

$$\begin{aligned}
 P(X = 3) &= \frac{e^{-\lambda} \lambda^3}{3!} \\
 &= \frac{e^{-5} 5^3}{3!}
 \end{aligned}$$

(ii) The probability that exactly three patients will arrive at the next hour

$$\begin{aligned}
 P(X \geq 1) &= 1 - P(X < 1) \\
 &= 1 - P(X = 0) \\
 &= 1 - \frac{e^{-\lambda} \lambda^0}{0!} = 1 - \frac{e^{-5} 5^0}{1} \\
 &= 1 - \frac{1}{e^5} = 0.9932
 \end{aligned}$$

Problem 3: The average number of major storms in your city is 2 per year. What is the probability that exactly 3 storms will hit your city next year?

Solution: an average number of major storms in your city per year is 2
i.e., $\lambda = 2$

The probability that exactly three storms will hit your city next year

$$\begin{aligned}
 P(X = 3) &= \frac{e^{-\lambda} \lambda^3}{3!} = \frac{e^{-2} 2^3}{3!} \\
 &= \frac{4}{3e^2} = 0.1799
 \end{aligned}$$

Problem 4: If the variance of the Poisson distribution is 3, then find the probability that

(i) $X = 3$ (ii) $0 < X \leq 3$ and (iii) $1 \leq X \leq 4$

Solution: Given the variance of the Poisson distribution is 3

For a Poisson distribution, we have

Mean = Variance = $\lambda = 3$

$$\text{(i) } P(X = 0) = \frac{e^{-\lambda} \lambda^0}{0!}$$

$$= \frac{1}{e^3} = 0.0497.$$

$$(ii) \quad P(0 < X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$$

$$= \frac{e^{-\lambda} \lambda^1}{1!} + \frac{e^{-\lambda} \lambda^2}{2!} + \frac{e^{-\lambda} \lambda^3}{3!}$$

$$= \lambda e^{-\lambda} \left(1 + \frac{\lambda}{2} + \frac{\lambda^2}{6} \right)$$

$$= 3.e^{-3} \left(1 + \frac{3}{2} + \frac{9}{6} \right) = 12e^{-3} = 0.5974.$$

$$(iii) \quad P(1 \leq X \leq 4) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

$$= \frac{e^{-\lambda} \lambda^1}{1!} + \frac{e^{-\lambda} \lambda^2}{2!} + \frac{e^{-\lambda} \lambda^3}{3!} + \frac{e^{-\lambda} \lambda^4}{4!}$$

$$= \lambda e^{-\lambda} \left(1 + \frac{\lambda}{2} + \frac{\lambda^2}{6} + \frac{\lambda^3}{24} \right)$$

$$= 3.e^{-3} \left(1 + \frac{3}{2} + \frac{9}{6} + \frac{27}{8} \right) = 0.7654$$

Homework:

Problem 5: Probability of getting no misprints in a page of book is e^{-4} . what is the probability that a page contains more than two misprints?

Problem 6: An average number of accidents on any national highway between any two cities in India is 1.8. Find the probability that the number of accidents there will be (i) at least one and (ii) at most one.

Problem 7: A bold factory manufactured 2% of the items are defective. The items are packed in the boxes. What is the probability that there will be (i) exactly 2 defective items and (ii) at least 3 defective items out 100 items.

Normal Distribution:

Definition: A continuous random variable X is said to has Normal distribution if its probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$$

Where μ = mean of the random variable x .

σ = Standard deviation of the random variable

Constants of the Normal distribution:

(i) Mean of the normal distribution

By normal distribution, we have

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$$

$$\begin{aligned} \text{Mean} &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2}z^2} \sigma dz \quad \text{put } z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2}z^2} dz \quad dx = \sigma dz, \text{ the limits are same} \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} \mu e^{-\frac{1}{2}z^2} dz + \sigma \int_{-\infty}^{\infty} ze^{-\frac{1}{2}z^2} dz \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} \mu e^{-\frac{1}{2}z^2} dz + 0 \right) \\ &= \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}z^2} dz \\ &= \frac{2\mu}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} = \mu. \end{aligned}$$

(ii) **Variance of the normal distribution**

By normal distribution, we have

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$$

$$\begin{aligned} \text{Variance} &= E(x-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^2 e^{-\frac{1}{2}z^2} \sigma dz && \text{put } z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z \\
&= \frac{2\sigma^2}{\sqrt{2\pi}} \left(\int_0^{\infty} z^2 e^{-\frac{1}{2}z^2} dz \right) && dx = \sigma dz, \text{ the limits are same} \\
&= \frac{\sqrt{2}\sigma^2}{\sqrt{\pi}} \int_0^{\infty} 2te^{-t} \frac{dt}{\sqrt{2t}} && \text{put } \frac{z^2}{2} = t \Rightarrow z^2 = 2t \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \sqrt{\pi} = \sigma^2 && 2zdz = 2dt \\
& && \text{and the limits are same}
\end{aligned}$$

(iii) Mode of the normal distribution

Mode is the value of x for which $f(x)$ is maximum

Mode is the value of $f'(x) = 0$ and $f''(x) < 0$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot -\frac{1}{2} \cdot 2 \cdot \frac{x-\mu}{\sigma} \cdot \sigma$$

$$f'(x) = -(x-\mu) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$f'(x) = -(x-\mu) f(x)$$

$$f'(x) = 0 \Rightarrow x = \mu$$

$$\begin{aligned}
\text{and } f''(x) &= -[(x-\mu) f'(x) + f(x)] \\
&= -f(x) [1 - (x-\mu)^2]
\end{aligned}$$

$$\text{At } x = \mu, f''(x) = -\frac{1}{\sigma\sqrt{2\pi}} < 0$$

Hence $f(x)$ is maximum at $x = \mu$.

Therefore, mode is the value of $x = \mu$.

(iv) Median of the normal distribution

$$\int_{-\infty}^M f(x) dx = \frac{1}{2}$$

$$\text{We have } \Rightarrow \left(\int_{-\infty}^{\mu} + \int_{\mu}^M \right) f(x) dx = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{\mu} f(x) dx + \int_{\mu}^M f(x) dx = \frac{1}{2}$$

Consider the first integral

$$\int_{-\infty}^{\mu} f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} = \sigma \int_{-\infty}^0 e^{-\frac{1}{2}z^2} dz$$

put $z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z$

$dx = \sigma dz,$

the limits are

at $x = \mu \Rightarrow z = 0$

as $x \rightarrow -\infty, z \rightarrow -\infty$

$$\therefore \int_{-\infty}^{\mu} f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^0 e^{-\frac{1}{2}z^2} \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{\pi}{2}} = \frac{1}{2}$$

$$\frac{1}{2} + \int_{-\infty}^{\mu} f(x) dx = \frac{1}{2} \Rightarrow \int_{-\infty}^{\mu} f(x) dx = 0$$

Hence median of the normal distribution is $x = \mu$

(v) **Mean deviation of the about mean of the normal distribution**

By normal distribution, we have

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$$

Mean deviation of the about mean = $\int_{-\infty}^{\infty} |x-\mu| f(x) dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x-\mu| e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sigma z| e^{-\frac{1}{2}z^2} \sigma dz$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} |z| e^{-\frac{1}{2}z^2} \sigma dz$$

$$= \sigma \sqrt{\frac{2}{\pi}} \int_0^{\infty} z e^{-\frac{1}{2}z^2} \sigma dz$$

put $z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z$

$dx = \sigma dz,$ the limits are same

$$\begin{aligned}
&= \sigma \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t} dt && \text{put } \frac{z^2}{2} = t \Rightarrow z^2 = 2t \\
&= \sigma \sqrt{\frac{2}{\pi}} \left[\frac{e^{-t}}{-1} \right]_{t=0}^{\infty} = \sqrt{\frac{2}{\pi}} \sigma && 2z dz = 2 dt \\
&&& \text{and the limits are same} \\
&= 0.8\sigma = \frac{4}{5}\sigma
\end{aligned}$$

Mean deviation of the about mean of the normal distribution = $\frac{4}{5}$ S.D

Problem 1: If X is a normal variate with mean 30 and S.D 5, Find (i) $P(26 \leq X \leq 40)$ and (ii) $P(X > 45)$

Solution: Since X follows normal distribution

Given mean (μ) = 30 and S.D (σ) = 5

$$\text{Normal variate } z = \frac{X - \mu}{\sigma} = \frac{X - 30}{5}$$

$$\text{When } X = 26, z = \frac{26 - 30}{5} = \frac{-4}{5} = -0.8 (= z_1)$$

$$\text{When } X = 40, z = \frac{40 - 30}{5} = 2 (= z_2)$$

(i) Probability that between X lies between 26 and 40 is

$$\begin{aligned}
P(26 \leq X \leq 40) &= P(-0.8 \leq Z \leq 2.0) \\
&= P(Z = 2.0) - P(Z = -0.8) \\
&= A(2.0) - A(-0.8) \\
&= 0.9772 - 0.2119 \\
&= 0.7653
\end{aligned}$$

(ii) Probability that X is greater than 55 is $P(X > 55)$

$$\text{When } X = 55, z = \frac{55 - 30}{5} = 3 (= z_1 \text{ say})$$

$$\begin{aligned}
\therefore P(X > 55) &= 1 - P(X \leq 55) \\
&= 1 - A(3) \\
&= 1 - 0.9987 \\
&= 0.0013.
\end{aligned}$$

Problem 2: The mean and S.D of the normal variate are 8 and 4 respectively. Find

(i) $P(5 \leq X \leq 10)$ and (ii) $P(X > 5)$

Solution: Since X follows normal distribution

Given mean (μ) = 8 and S.D (σ) = 4

$$\text{Normal variate } z = \frac{X - \mu}{\sigma} = \frac{X - 8}{4}$$

(i) Probability that between X lies between 26 and 40 is $P(5 \leq X \leq 10)$

$$\text{When } X = 5, z = \frac{5 - 8}{4} = \frac{-3}{4} = -0.75 (= z_1)$$

$$\text{When } X = 10, z = \frac{10 - 8}{4} = 0.5 (= z_2)$$

$$\begin{aligned}
P(26 \leq X \leq 40) &= P(-0.75 \leq Z \leq 0.5) \\
&= P(Z = 0.5) - P(Z = -0.75)
\end{aligned}$$

$$\begin{aligned}
&= A(0.5) - A(-0.75) \\
&= 0.6915 - 0.2266 \\
&= 0.4649
\end{aligned}$$

(ii) Probability that X is greater than 5 is $P(X > 5)$

$$\text{When } X = 5, z = \frac{5 - 8}{4} = \frac{-3}{4} = -0.75 (= z_1)$$

$$\begin{aligned}
\therefore P(X > 5) &= 1 - P(X \leq 5) \\
&= 1 - A(2.0) \\
&= 1 - 0.2266 \\
&= 0.7734
\end{aligned}$$

Assuming normality

Problem 3: The length of human pregnancies from conception to birth approximates a normal distribution with a mean of 266 days and a standard deviation of 16 days. What proportion of all pregnancies will last between 240 and 270 days (roughly between 8 and 9 months)?

Solution: Normal Distribution

Given $\mu = 266$ and $\sigma = 16$

$$\text{The standard normal variate } z = \frac{X - \mu}{\sigma} = \frac{X - 266}{16}$$

To find, $P(240 < X < 270)$

$$\text{When } X = 240, z = \frac{240 - 266}{16} = \frac{-26}{16} = -1.63 (= z_1)$$

$$\text{When } X = 270, z = \frac{270 - 266}{16} = \frac{4}{16} = 0.25 (= z_2)$$

$$\begin{aligned}
P(240 < X < 270) &= P(-1.63 < Z < 0.25) \\
&= P(Z < 0.25) - P(Z < -1.63) \\
&= 0.5987 - 0.0516 \\
&= 0.5471.
\end{aligned}$$

Problem 4: The average number of acres burned by puddy grace fires in New Delhi is 4,300 acres per year, with a standard deviation of 750 acres. The distribution of the number of acres burned is normal. What is the probability that between 2,500 and 4,200 acres will be burned in any given year?

Solution: Given mean (μ) = 4300 and S.D (σ) = 750

$$\text{Normal variate } z = \frac{X - \mu}{\sigma} = \frac{X - 4300}{750}$$

$$\text{When } X = 2500, z = \frac{2500 - 4300}{750} = \frac{-1800}{750} = -2.40$$

$$\text{When } X = 4200, z = \frac{4200 - 4300}{750} = \frac{-100}{750} = -0.13$$

probability that between 2,500 and 4,200 acres will be burned in any given year

$$\begin{aligned}
P(2500 < X < 4200) &= P(-2.40 < Z < -0.13) \\
&= P(Z < -0.13) - P(Z < -2.40) \\
&= 0.4483 - 0.0082 \\
&= 0.4401
\end{aligned}$$

Assuming normality

Problem 5: An Average IQ of a group of 800 children is 98 and the S.D is 8. Assuming normality find the expected number of children having IQ between 100 and 120.

Solution: Given mean (μ) = 98 and S.D (σ) = 8

$$\text{Normal variate } z = \frac{X - \mu}{\sigma} = \frac{X - 98}{8}$$

$$\text{When } X = 100, z = \frac{100 - 98}{8} = \frac{2}{8} = 0.25$$

$$\text{When } X = 120, z = \frac{120 - 98}{8} = \frac{22}{8} = 2.75$$

Probability of IQ of a child is between 100 and 120

$$\begin{aligned} P(100 < X < 120) &= P(0.25 < Z < 2.75) \\ &= P(Z < 2.75) - P(Z < 0.25) \\ &= 0.4970 - 0.0987 \\ &= 0.3983 \end{aligned}$$

Assuming normality

$$\begin{aligned} \text{Among 800 children, expected number of children having IQ between 100 and 120 is} \\ &= N \times P(100 < X < 120) \\ &= 800 \times 0.3983 \\ &= 318.64 = 319. \end{aligned}$$

Home work:

Problem 6: Given that the mean of the students in a class is 158 cm, with S.D of 20 cm. Find how many students height lie between 150 cm and 170 cm if there are 100 students in the class.