ANNAMACHARYA INSTITUTE OF TECHNOLOGY & SCIENCES
(AUTONOMOUS)
UTUKUR, C. K. DINNE (V & M), KADAPA, YSR DIST.
Approved by AICTE, New Delhi & Affiliation to JNTUA, Anantapuramu.
Accredited by NAAC with 'A' Grade, Bangalore & NBA (EEE, ECE & CSE)

DEPARTMENT OF
ELECTRONICS AND COMMUNICATION ENGINEERING

LINEAR CONTROL SYSTEMS

(23HES0204)

COURSE MATERIAL

B. TECH III YEAR I SEM

II Year B.Tech. – II Semester

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(23HES0204) LINEAR CONTROL SYSTEMS

Course Objectives:

- Introduce the basic principles and applications of control systems.
- Learn the time response and steady state response of the systems.
- Know the time domain analysis and solutions to time invariant systems.
- Understand different aspects of stability analysis of systems in frequency domain.
- Understand the concept of state space, controllability and observability.

Course Outcomes: After completing the course, the student should be able to:

- Summarize the basic principles and applications of control systems. (L2)
- Understand the time response and steady state response of the systems. (L2)
- Understand the concept of state space, controllability and observability. (L2)
- Apply time domain analysis to find solutions to time invariant systems. (L3)
- Analyze different aspects of stability analysis of systems in frequency domain. (L4)

UNIT I

Control Systems Concepts: Open loop and closed loop control systems and their differences- Examples of control systems- Classification of control systems, Feedback characteristics, Effects of positive and negative feedback, Mathematical models – Differential equations of translational and rotational mechanical systems and electrical systems, Analogous Systems, Block diagram reduction methods – Signal flow graphs - Reduction using Mason's gain formula.

UNIT II

Time Response Analysis: Step Response - Impulse Response - Time response of first order systems – Characteristic Equation of Feedback control systems, Transient response of second order systems - Time domain specifications – Steady state response - Steady state errors and error constants.

UNIT III

Stability Analysis in Time Domain: The concept of stability – Routh's stability criterion – Stability and conditional stability - limitations of Routh's stability. The Root locus concept - construction of root loci-effects of adding poles and zeros to G(s) H(s) on the root loci.

UNIT IV

Frequency Response Analysis: Introduction, Frequency domain specifications-Bode diagrams-Determination of Frequency domain specifications and transfer function from the Bode Diagram - Stability Analysis from Bode Plots. Polar Plots- Nyquist Plots- Phase margin and Gain margin-Stability Analysis.

Compensation techniques – Basic introduction of Lag, Lead, Lag-Lead Compensator.

UNIT V

State Space Analysis of Continuous Systems: Concepts of state, state variables and state model - differential equations & Transfer function models - Block diagrams. Transfer function from state model, solving the Time invariant state Equations- State Transition Matrix and its Properties. System response through State Space models. The concepts of controllability and observability,

Textbooks:

1. Modern Control Engineering by Katsuhiko Ogata, Prentice Hall of India Pvt. Ltd., 5th edition, 2010.
2. Control Systems Engineering by I. J. Nagrath and M. Gopal, New Age International (P) Limited Publishers, 5th edition, 2007.

References:

1. Control Systems Principles & Design by M.Gopal, 4th Edition, McGraw Hill Education,2012.
2. Automatic Control Systems by B. C. Kuo and Farid Golnaraghi, John wiley and sons, 8th edition,2003.
3. Feedback and Control Systems, Joseph J Distefano III, Allen R Stubberud & Ivan J Williams, 2nd Edition, Schaum's outlines, McGraw Hill Education,2013.
4. Control System Design by Graham C. Goodwin, Stefan F. Graebe and Mario E. Salgado, Pearson, 2000.
5. Feedback Control of Dynamic Systems by Gene F. Franklin, J.D. Powell and Abbas Emami- Naeini, 6th Edition, Pearson,2010.

UNIT-1
CONTROL SYSTEM CONCEPTS

control system:-

The control system evolved as an engineering discipline and due to universality of the principles involved, it is extended to various fields like economy, sociology, biology and medicine etc.,

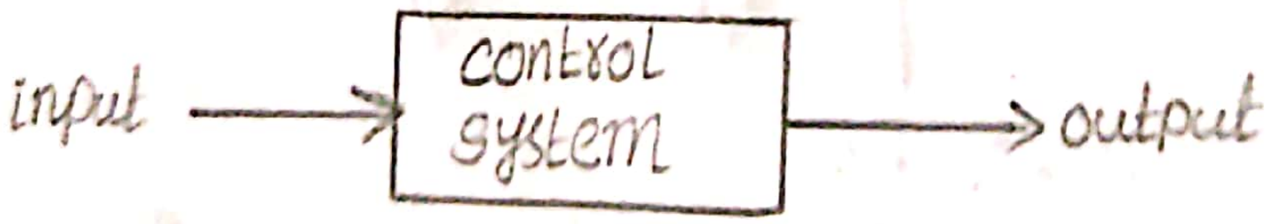
system:-

When a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a system.

EX:- Every physical object is actually a system. A classroom is a good example of physical system. A room along with the combination of benches, blackboard, lights, fans etc.,

control system (Defination):-

In a system when the output quantity is controlled by varying the input quantity, the system is called as control system.



i.e., output quantity is called controlled variable or response. and input quantity is called command signal or Excitation.

classification of control system:-

The control system are broadly classified into following types:

1. Natural control system:-

The biological system inside human being or of nature type.

EX:- The perspiration system inside the human being is a good example of natural control system.

2. Manmade control system:-

The various system, we are using in our day to day life or designs and manufactured by human beings. such systems like vehicles, switches, various controllers etc.,

EX:- An automobile system with gears, accelerator, breaking system is a perfect example for the manmade control system

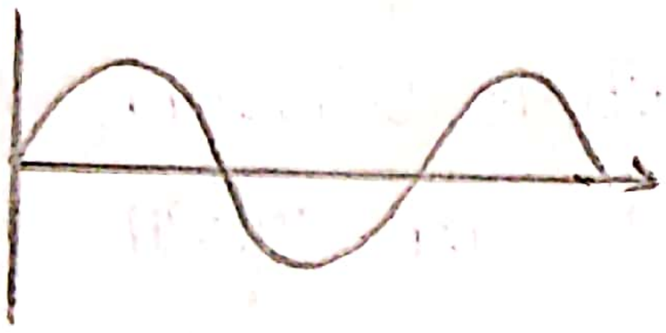
3. Combinational control system:-

This system is the one having both combination of natural and manmade together.

EX:- Driver driving a vehicle.

4. Continous and Discrete time control system:-

In continous time control system all the signals are continous in time but discrete time control system there exist one or more discrete time signal.





5. Linear and Nonlinear control system:-

A control system is said to be linear if it satisfies the following properties:

(a) The principle of superposition is applicable to the system. This means the response of the several inputs can be obtained by considering one input at a time and then algebraically adding the individual results.

Mathematically, the principle of superposition is expressed in two ways:-

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(i) Additive property, $f(x+y)=f(x)+f(y)$

(ii) Homogeneous property, $f(\alpha x)=\alpha f(x)$

(b) The differential equation describing the system is linear having its coefficients as constant.

(c) practically, the output varies linearly as the input varies.

6. SISO and MIMO control system:-

SISO (single input single output) control system have one input and one output.

MIMO (Multiple input and Multiple outputs) control system have more than one inputs and more than one output.

7. Time variant and Time Invariant control system:-

Time variant control system are those in which parameters of the system varying with time.

Time invariant systems are those the input and output are functions of time, but the parameters of the system are independent of time, which are not varied with time & constants.

8. open loop and closed loop control systems:-

A system which does not have a feedback element to get the desired output is known as open loop control system.

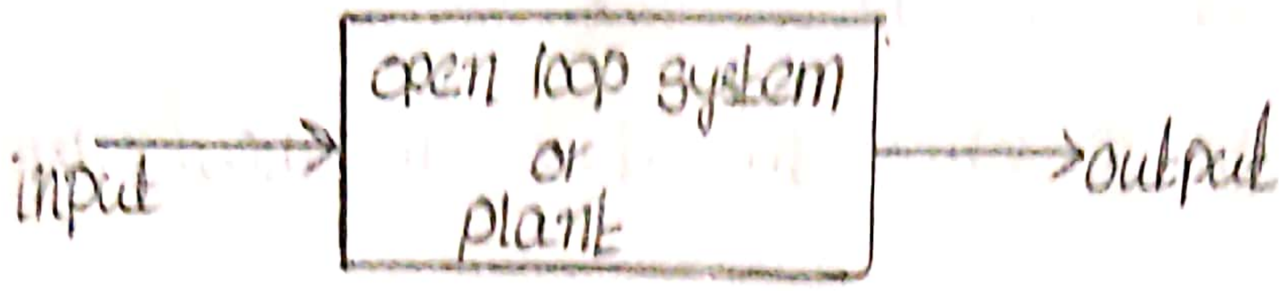
A system having a feedback element is known as closed loop control system.

*. open loop control system:-

Any physical system which does not automatically correct the variations in its output is called open loop system.

(or)

A control system in which the output quantity has no effect upon the input quantity are called open loop control system. This means the output is not feedback to the input for correction.



* In open loop control system, the output is varied by varying the input, but due to external disturbtions the system output may change. when the output changes due to external disturbances, it is not followed by changes in input to correct the output. In open loop control system, the changes in output are corrected by changing the input manually.

* Advantages:-

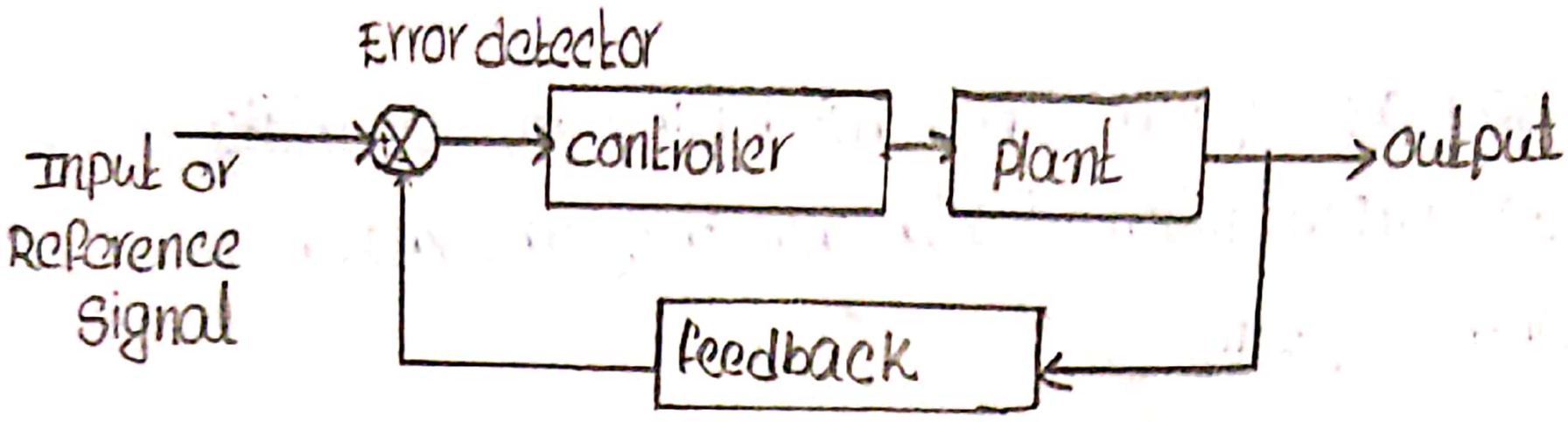
1. The loop systems are simple and economical.
2. The open loop systems are easier to construct.
3. Generally, open loop systems are stable.

* Disadvantages:-

1. The open loop systems are inaccurate and unreliable.
2. The changes in the output due to external disturbances are not corrected automatically.

closed ~~open~~ loop control system:-

- control system in which the output has an effect upon the input quantity in such a manner. so, it has to maintain the desired output value are called closed loop control system.



The open loop system can be modified as closed loop system by providing a feedback. The addition of feedback element automatically connects the changes in output due to disturbances. Hence, the closed loop systems are also called as automatic control system.

The reference or input signal corresponds to desired output. The feedback path samples the output and converts it to a signal of same type as that of reference signal. The feedback signal is proportional to output signal and is fed to the error detector. The error detector compares the difference between the reference and feedback signal to produce better action. The modified error signal is fed to the plant to correct as output.

*. Advantages:-

1. The closed loop systems are accurate and reliable.
2. these are less effected by noise
3. these systems are accurate even in the presence of non-linearity.

* Disadvantages:-

1. The closed loop systems are complex and costlier.
2. The feedback reduces the overall gain of the system.
3. The feedback is closed and it may lead to the oscillatory response.
4. The closed loop system has major problem of stability.

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Note:-

1. since the openloop systems does not having the capability of correcting its output automatically, thus they are known as "Manually control system".

2. the closed loop control system are having the capable of correcting its output for various variations (external disturbances). so thus are known as "Automatic control system".

Examples of open & closed loop system:-

(a) Traffic signals

Open loop control system:- Traffic controlled by means of traffic signals operated on a time basis constitutes an open loop control system. the sequence of control signals are based on a time slot given for each signal. the time slots are decided based on a traffic study. The system will not measure the density of traffic. since the time slots does not change according to the traffic density.

closed loop control system:- Traffic control system can be made as a closed loop system if the time slots are decided based on the traffic density. In these type of system, the traffic density will be measured from all the sides and the information is fed to the computer. the timing of signals are decided by the computer based on the traffic density. since, the closed loop system dynamically changes the timings of traffic signals will be better than open loop control systems.

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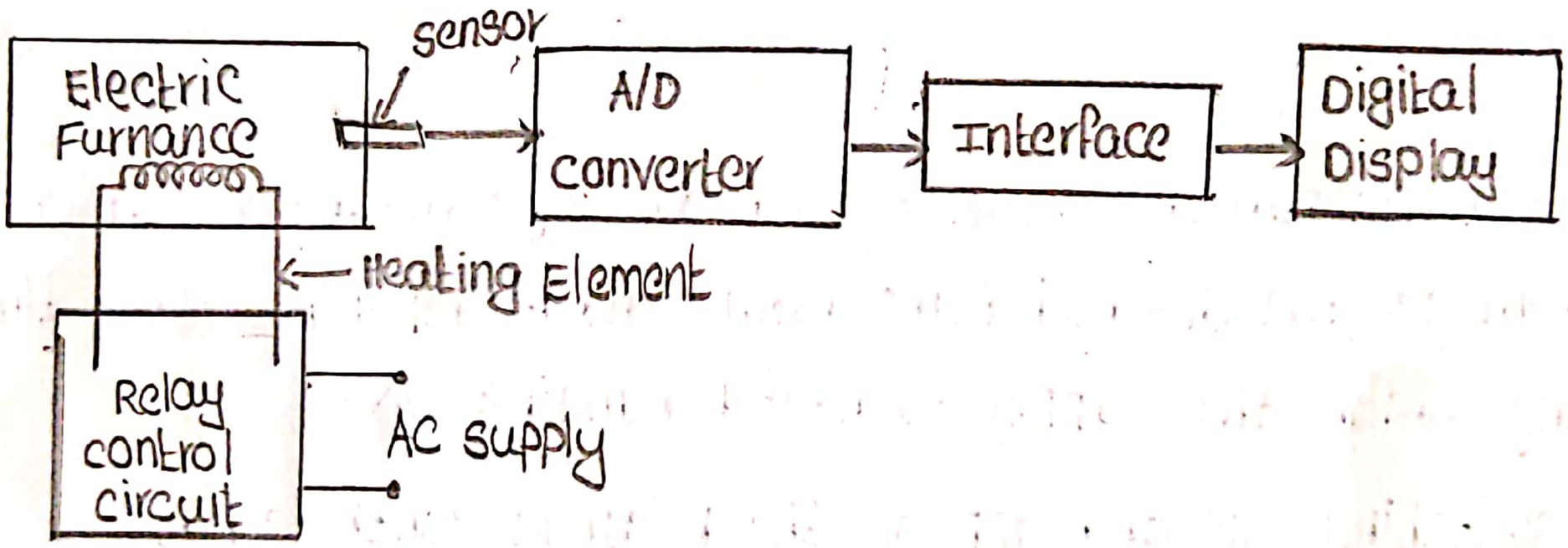
(b) Washing Machine:

A washing machine without any cleanliness measuring system is an example for open loop system. Here washing, rinshing in the washer will be operates on a time basis. The machine does not measure the output that is the cleanliness of the cbthes. once the set on time is over, the machine will automatically stops, whatever may be the level of cleanliness.

This can be a closed loop control system, if the level of cleanliness can be measured and compared with the desired cleanliness.

(c). Temperature control system:

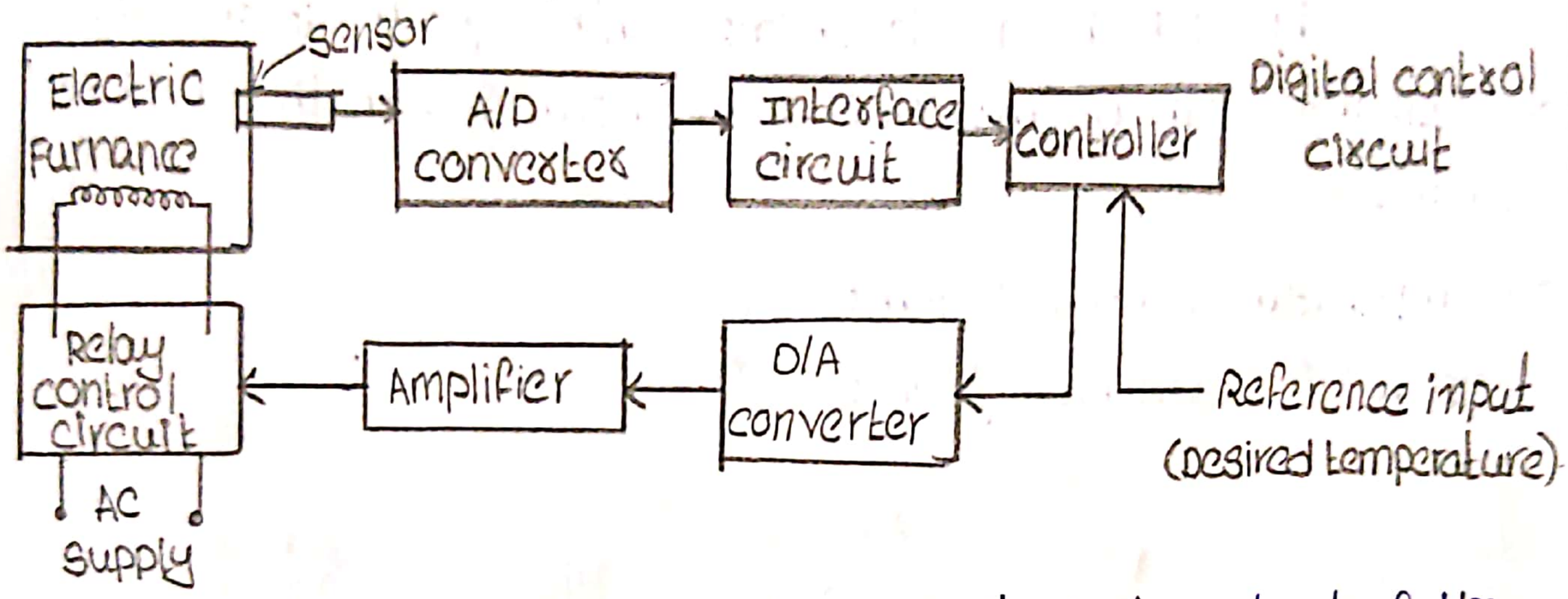
open loop control system:-



- The electric Furnance is an open loop system. The output in the system is the desired temperature.
- The temperature of the system is raised by heat generated by the heating element. The output temperature depends on the time during which the supply to heat remains ON.
- The ON and OFF of the supply is governed by the time setting of the relay. The temperature is measured by a sensor, which gives an analog voltage corresponding to the temperature of the furnance.

- The analog signal is converted to digital signal by an Analog-to-Digital converter. (A/D converter).
- the digital signal is given to the digital display device to display the temperature. In this system if there is any change in output temperature then the time setting of the relay is not altered automatically.

closed loop control system:-



- The electric furnance is a closed loop system. The output of the system is the desired temperature and it depends on the time during which the supply to heater remains ON.
- The switching ON and OFF of the relay is controlled by a controller which is a digital system or computer.
- The desired temperature is input to the system through keyboard or as a signal corresponding to desired temperature via ports.
- The actual temperature is sensed by sensor and converted to digital signal by using the A/D converter.
- The computer reads the actual temperature and compares with desired temperature.

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• If it finds any difference then it sends signal to switch ON or OFF the relay through D/A converter and amplifier. Thus, the system automatically corrects any charge in output. Hence it is a closed loop system.

⇒ Differences between open & closed loop control system:-

open loop control system	closed loop control system
1. It is inaccurate and unreliable	1. It is accurate and reliable
2. It is simple and conventional.	2. It is complex and costlier.
3. Generally, it is a stable system.	3. Great effects are need to design a stable system.
4. External disturbances are not corrected automatically	4. External disturbances are corrected automatically.

Transfer Function:-

The mathematical representation of the transfer function is defined as the ratio of laplace transform of the output to the laplace transform of the input with zero initial conditions.

$$T.F = \frac{L[\text{output}]}{L[\text{input}]} \Big|_{\text{zero initial conditions}}$$

• It is also defined as the laplace transform of impulse response of a system with zero initial conditions.

• The transfer function technique is not applicable for the non-linear system.

• Also, the transfer function of a system is independent of input and it is dependent only on system parameters.

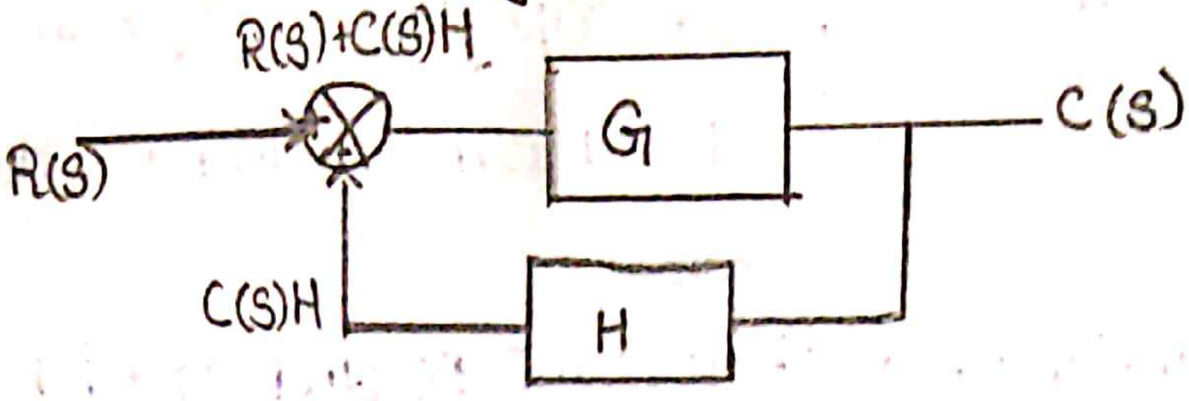
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Types of Feedback:-

There are two types of feedback system:

1. positive feedback system
2. Negative feedback system

1. positive feedback system:-



$C(S) = G_1 [R(S) + C(S)H]$

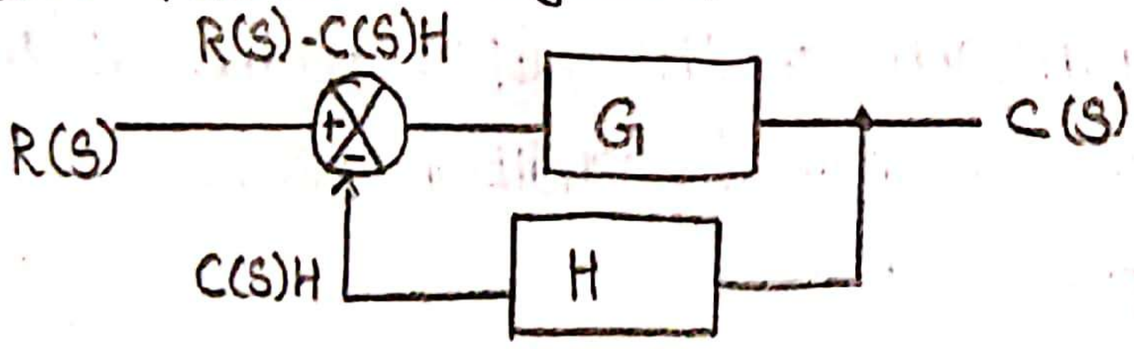
$C(S) = G_1 \cdot R(S) + G_1 \cdot H \cdot C(S)$

$C(S) - G_1 \cdot H \cdot C(S) = G_1 R(S)$

$C(S)[1 - GH] = G_1 R(S)$

$\therefore$ Transfer Function $= \frac{C(S)}{R(S)} = \frac{G_1}{1 - GH}$.

2. Negative feedback system:-



$C(S) = G_1 [R(S) - C(S)H]$

$C(S) = G_1 \cdot R(S) - G_1 \cdot H \cdot C(S)$

$C(S) + G_1 \cdot H \cdot C(S) = G_1 R(S)$

$C(S)[1 + G_1 H] = G_1 R(S)$

$\therefore$ Transfer Function $= \frac{C(S)}{R(S)} = \frac{G_1}{1 + GH}$.

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Effects of positive Feedback:-

- The positive feedback increases the error signal and dries the output to instability. Sometimes, the positive feedback signal also uses in control systems to amplify certain internal signals or system parameters.

Effects of Negative Feedback:-

- The negative feedback results in better stability and it rejects any disturbance signals. Therefore, negative feedback is preferred in closedloop systems.

Mathematical Models of physical system:-

1. The input and output relations of various physical components of a system are governed by differential equations.
2. A Mathematical model will be linear, if the differential equations described by the system having constant coefficients.

Mechanical Translational system:-

List of symbol used in Mechanical Translational system:-

- X = Displacement (meters)
- Y = velocity = $\frac{dx}{dt}$ (metre/sec)
- A = Acceleration = $\frac{dV}{dt} = \frac{d^2x}{dt^2}$ (m/sec^2)
- Fm = (Applied force). opposing force offered by mass of the body
- F_b = opposing force offered by Friction of the body
- F_k = opposing force offered by velocity of the body
- M = Mass (Kg)
- B = viscous Friction coefficient - Dashpot (N-sec/m)
- K = stiffness of a spring (N/m).

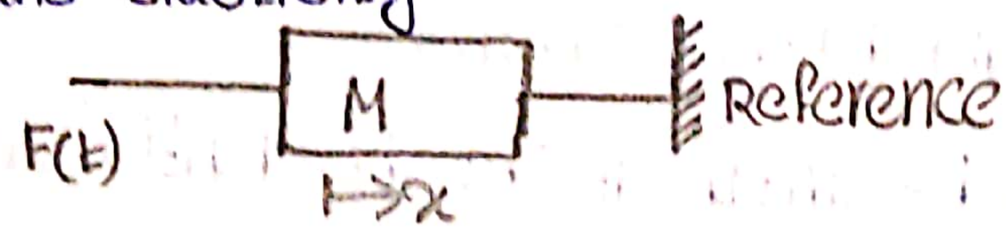
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Force Balance Equation for Ideal Elements:-

- the Mechanical Translational system can be obtained by using 3 basic Elements namely Mass, Dashpot and spring.
- the weight of the mechanical is represented by the elements (m) and it is assumed to be concentrated that the centre of the body.
- the Friction existing in Mechanical system can be represented by the element (Dashpot as B). the elastic deformation of the body can be represented by the spring(K).

1. Mass:-

consider, a ideal mass element as shown below, which has negliable Friction and elasticity.



let be Force applied on it, offers an opposing force which is proportional to acceleration of the body.

where, F = applied Force

x = Displacement of Body

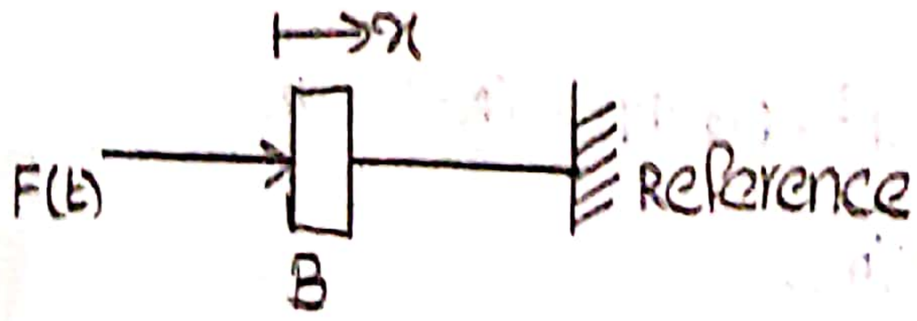
F_m = opposing Force applied to body

According to Newtons 2nd law,

$$F = ma = m\frac{d^2x}{dt^2} = m\frac{dv}{dt}.$$

2. Dashpot :-

similarly, consider an ideal element dashpot as shown in figure which is negliable from mass and spring.



The dashpot will offer an opposing force which is proportional to velocity of body.

$$F_B \propto V$$

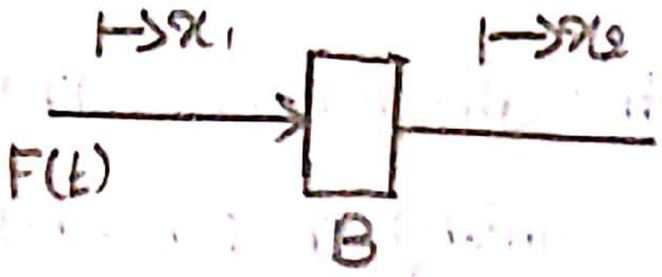
$$F_B = B \cdot V$$

$$F_B = B \frac{dx}{dt}$$

According to Newton's 2nd law,

$$\therefore F = F_B = B \frac{dx}{dt}.$$

When a dashpot has displacement at both ends shown in figure



$$F_B \propto V$$

$$F_B = B \cdot V$$

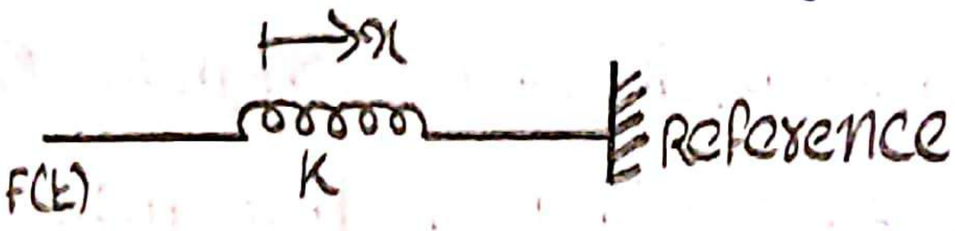
$$F_B = B(x_2 - x_1)$$

According to Newton's 2nd law,

$$\therefore F = F_B = B \frac{d}{dt}(x_1 - x_2).$$

3. Spring:-

consider an ideal elastic element spring as shown in figure



The spring will offer the displacement of the body. The spring will offer an opposing force which is proportional to displacement of the body.

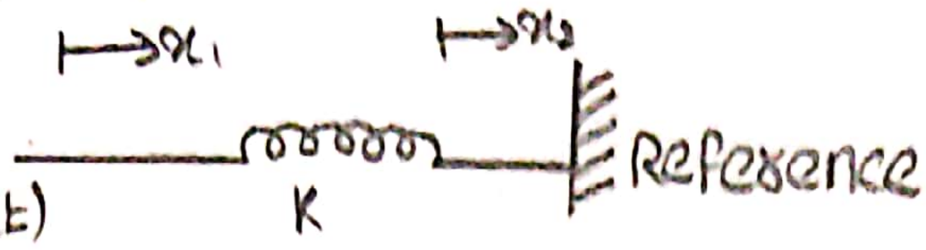
$$F_K \propto x$$

$$F_K = K \cdot x$$

According to Newton's 2nd law,

$$\therefore F = F_K = Kx.$$

When the spring has displacement at both ends as shown in Figure



The opposing force is proportional to two different displacement

$$F = F_K \quad k \cdot x$$

$$\therefore F = F_K = k(x_1 - x_2).$$

Guidelines to determine the Mechanical Translational system [To determine Transfer Function]:-

1. In mechanical Translation system, the differential equation governing the system are obtained by writing in force balance equations. Set the knots at meeting of elements.
2. The linear displacement of the masses are assumed as x_1, x_2, x_3 and so,on are assigned a displacement to each mass knot.
3. The first derivative of the displacement is velocity and second derivative of displacement is acceleration.
4. Then draw free body diagram of the system, the free body diagram is obtained by diagram drawing each mass separately and then marking all the force acting on that mass, always the opposing force exists on a direction opposite to applied force.
5. The mass has to move in the direction of applied force for each free body diagram, write one differential equation by equating the sum of applied forces to the sum of opposing forces.
6. Take laplace transform of differential equation to convert them to algebraic equation then rearrange the s-domain

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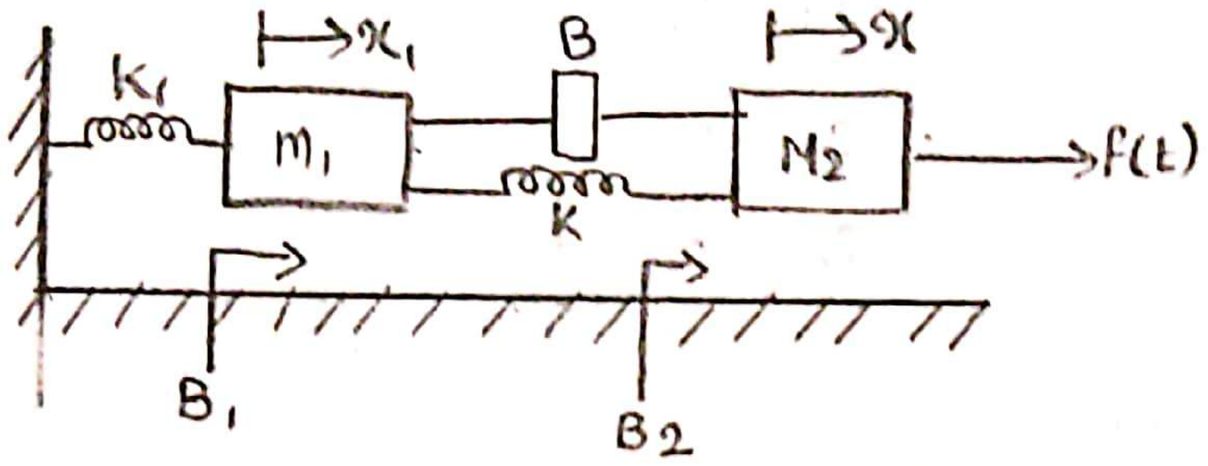
equations to eliminate the wanted variable and obtain the ratio between output and input variable.

Note:-

- The laplace transform of $L[x(t)] = X(s)$
- The laplace transform of $\dot{x}(t) = L\left[\frac{d}{dt}x(t)\right] = s\,X(s)$
- The laplace transform of $\ddot{x}(t) = L\left[\frac{d^2}{dt^2}x(t)\right] = s^2\,X(s)$.

Problems:-

1. Write the differential equation governing the Mechanical system shown in Figure. Determine the Transfer function.



From the Mechanical system f(t) is input and displacement x is output. Hence, there are two mass and so the two free Body diagram to be needed

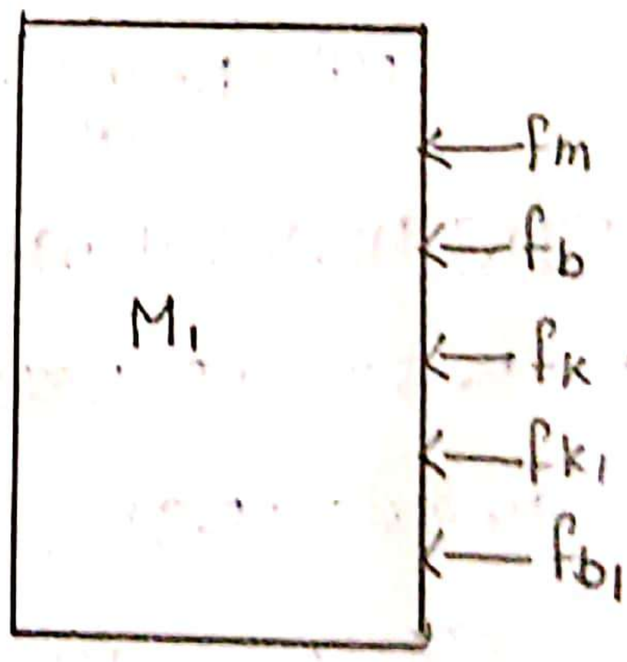
$$f_{m1} = M_1 \frac{d^2x_1}{dt^2}$$

$$f_b = B \cdot \frac{d}{dt}(x_1 - x)$$

$$f_k = k(x_1 - x)$$

$$f_{k_1} = k_1 \cdot x_1$$

$$f_{b1} = B_1 \frac{d}{dt}x_1$$



According to newton's 2nd law

sum of applied forces = sum of opposing forces

$$0 = f_{m_1} + f_b + f_k + f_{k_1} + f_{b_1}$$

$$0 = M \cdot \frac{d^2x_1}{dt^2} + B \cdot \frac{d}{dt}(x_1 - x) + B_1 \frac{d}{dt}x_1 + k(x_1 - x) + k_1 x_1.$$

Apply laplace transform on both sides with zero initial conditions

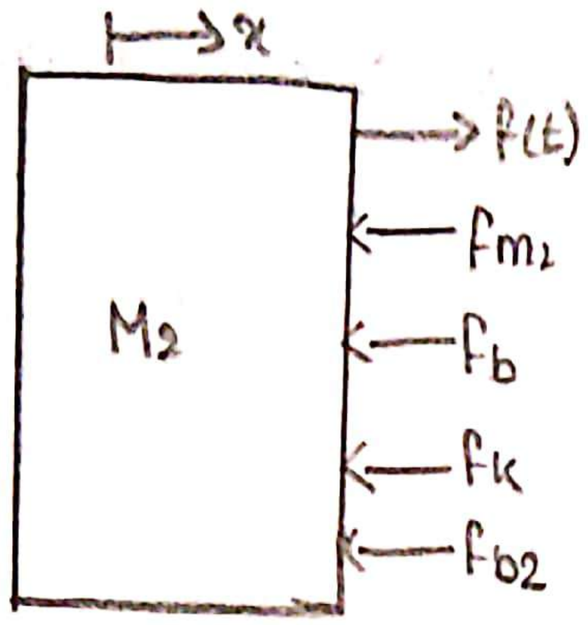
$$0 = M_1S^2X_1(S) + B\cdot S[X_1(S) - X(S)] + B_1S X_1(S) + K[X_1(S) - X(S)] + K_1X_1(S)$$

$$M_1S^2X_1(S) + B\cdot S X_1(S) - B\cdot S X(S) + B_1S X_1(S) + K\cdot X_1(S) - K\cdot X(S) + K_1\cdot X_1(S) = 0$$

$$X_1(S)\cdot[M_1S^2 + B\cdot S + B_1\cdot S + K_1 + K] - X(S)[B\cdot S + K] = 0$$

$$X_1(S)[M_1S^2 + (B+B_1)S + (K+K_1)] = X(S)[B\cdot S + K]$$

$$X_1(S) = \frac{X(S)[BS+K]}{[M_1S^2 + S(B+B_1) + (K+K_1)]} \quad \text{——(1)}$$



$$Fm_2 = M_2\frac{d^2x}{dt^2}$$

$$f_b = B\frac{d}{dt}(x - x_1)$$

$$f_K = K(x - x_1)$$

$$f_{b2} = B_2\frac{d}{dt}x$$

According to Newton's 2nd law,

Sum of applied forces = sum of opposing forces

$$F(t) = f_{m2} + f_b + f_k + f_{b2}$$

$$F(t) = M_2\frac{d^2x}{dt^2} + B\frac{d}{dt}(x - x_1) + K(x - x_1) + B_2\frac{d}{dt}x$$

Applying laplace transform on both sides with zero initial conditions

$$F(S) = M_2S^2X(S) + B\cdot S[X(S) - X_1(S)] + K[X(S) - X_1(S)] + B_2SX(S)$$

$$F(S) = M_2S^2X(S) + B\cdot S X(S) - B\cdot S X_1(S) + K\cdot X(S) - K\cdot X_1(S) + B_2SX(S)$$

$$F(S) = X(S)[M_2S^2 + S(B+B_2) + K] - X_1(S)[B\cdot S + K] \quad \text{——(2)}$$

Substitute eq-(1) in eq(2), we get

$$F(S) = X(S)[M_2S^2 + S(B+B_2) + K] - \left[\frac{X(S)(B\cdot S + K)}{M_1S^2 + S(B+B_1) + (K+K_1)}\right][B\cdot S + K]$$

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$$F(S) = \frac{X(S)\left[M_2S^2+S(B+B_2)+K\right]\left[M_1S^2+S(B+B_1)+(K+K_1)\right] - X(S)\left[B\cdot S+K\right]^2}{M_1S^2+S(B+B_1)+(K+K_1)}$$

$$F(S) = \frac{X(S)\left[(M_2S^2+S(B+B_2)+K)(M_1S^2+S(B+B_1)+(K+K_1) - (B\cdot S+K)^2\right]}{M_1S^2+S(B+B_1)+(K+K_1)}$$

$$\text{Transfer function} = \frac{X(S)}{F(S)} = \frac{M_1S^2+S(B+B_1)+(K+K_1)}{\left[(M_2S^2+S(B+B_2)+K)(M_1S^2+S(B+B_1)+(K+K_1)-(BS+K)^2)\right]}$$

Result :- The differential equations governing system

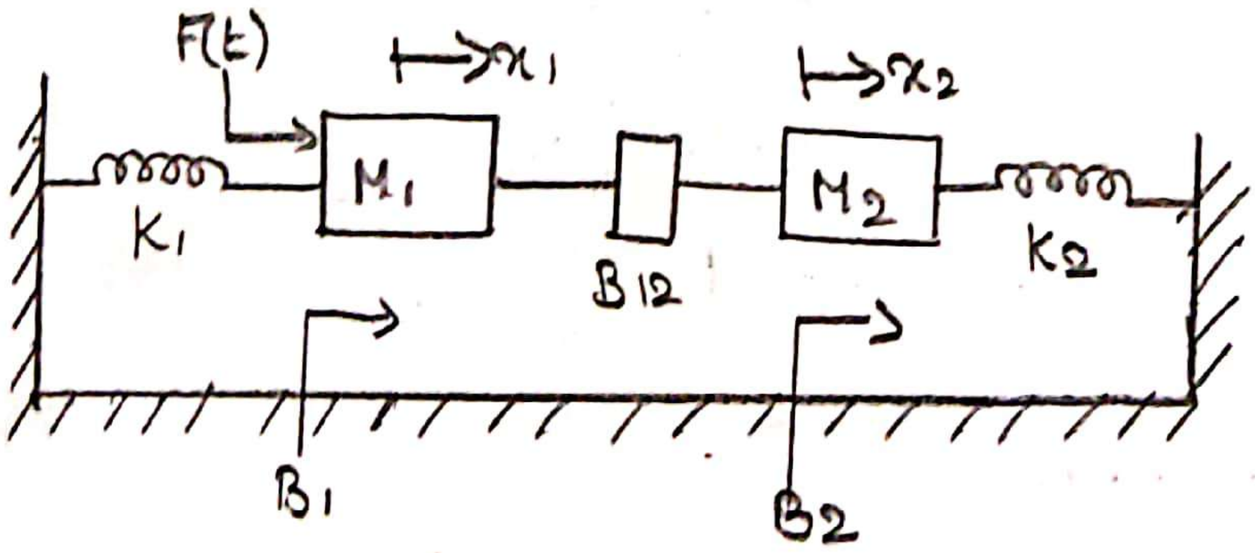
(i) $M_1\frac{d^2x}{dt^2} + B\frac{d}{dt}(x_1-x) + B_1\frac{d}{dt}x_1 + k(x_1-x) + K_1\cdot x_1 = 0$

(ii) $M_2\frac{d^2x}{dt^2} + B\frac{d}{dt}(x-x_1) + B_2\frac{d}{dt}x + k(x-x_1) = F(t)$

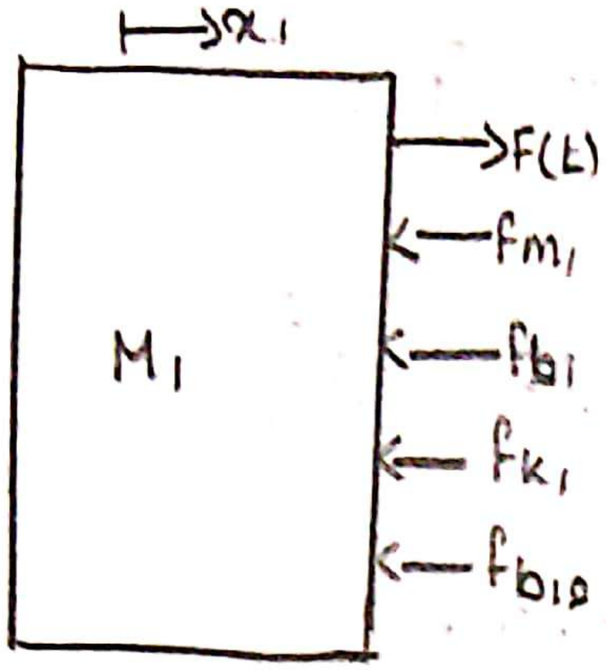
The transfer function of the system is

$$= \frac{M_1S^2+S(B+B_1)+(K+K_1)}{\left[M_2S^2+S(B+B_2)+K\right]\left[M_1S^2+S(B+B_1)+(K+K_1)-(B\cdot S+K)^2\right]}.$$

2. Determine the transfer function $\frac{X_1(S)}{F(S)}$ and $\frac{X_2(S)}{F(S)}$ for the system given below.



Here, there are two masses and so the two free body diagram to be needed is given as



$$f_{m1} = M_1 \frac{d^2x_1}{dt^2}$$

$$f_{b1} = B_1 \frac{d}{dt} x_1$$

$$f_{k1} = K_1 \cdot x_1$$

$$f_{b12} = B_{12} \frac{d}{dt}(x_1 - x_2)$$

According to Newton's 2nd law;

Sum of applied forces = sum of opposing forces

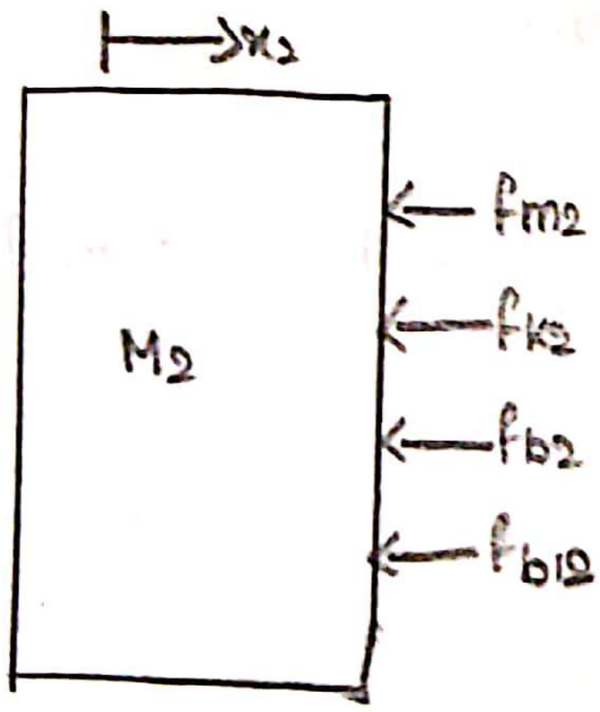
$$F(t) = f_{m1} + f_{b1} + f_{k1} + f_{B12}$$

$$F(t) = M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{d}{dt} x_1 + K_1 x_1 + B_{12} \frac{d}{dt}(x_1 - x_2)$$

Apply laplace transform on both sides with zero initial conditions

$$F(S) = M_1 S^2 X_1(S) + B_1 S X_1(S) + K_1 X_1(S) + B_{12} S[X_1(S) - X_2(S)]$$

$$F(S) = X_1(S)\left[M_1 S^2 + B \cdot S + K_1\right] + B_{12} S] - X_2(S)[B_{12} S] \quad \text{---(1)}$$



$$f_{m2} = M_2 \frac{d^2x_2}{dt^2}$$

$$f_{k2} = K_2 x_2$$

$$f_{b2} = B_2 \frac{d}{dt} x_2$$

$$f_{b12} = B_{12} \frac{d}{dt}(x_2 - x_1)$$

According to Newton's 2nd law,

Sum of applied forces = sum of opposing forces

$$0 = F_{m2} + f_{k2} + f_{b2} + F_{b12}$$

$$0 = M_2 \frac{d^2x_2}{dt^2} + K_2 x_2 + B_2 \frac{d}{dt} x_2 + B_{12} \frac{d}{dt}(x_2 - x_1)$$

Apply laplace transform on both sides with zero initial conditions

$$0 = M_2 S^2 X_2(S) + k_2 X_2(S) + B_2 \cdot S X_2(S) + B_{12} S [X_2(S) - X_1(S)]$$

$$0 = X_2(S)[M_2 S^2 + k_2 + S(B_2 + B_{12})] - [B_{12} S] X_1(S)$$

$$X_2(S)[M_2 S^2 + k_2 + S(B_2 + B_{12})] = [B_{12} S] X_1(S)$$

$$X_2(S) = \frac{(B_{12} S) X_1(S)}{M_2 S^2 + k_2 + S(B_2 + B_{12})} \quad \text{——(2)}$$

From eq-(2),

$$X_1(S) = \frac{X_2(S)[M_2 S^2 + k_2 + S(B_2 + B_{12})]}{B_{12} \cdot S} \quad \text{——(3)}$$

Substitute eq-(2) in eq-(1), we get

$$F(S) = X_1(S)[M_1 S^2 + (B + B_{12})S + k_1] - \left[\frac{B_{12} \cdot S X_1(S)}{M_2 S^2 + S(B_2 + B_{12}) + k_2}\right](B_{12} \cdot S)$$

$$F(S) = \frac{X_1(S)[M_1 S^2 + S(B + B_{12}) + k_1][M_2 S^2 + S(B_2 + B_{12}) + k_2] - [B_{12} S]^2 X_1(S)}{M_2 S^2 + k_2 + S(B_2 + B_{12})}$$

$$\therefore \frac{X_1(S)}{F(S)} = \frac{M_2 S^2 + k_2 + S(B_2 + B_{12}) \cdot}{[M_1 S^2 + S(B_2 + B_{12}) + k_1][M_2 S^2 + S(B_2 + B_{12}) + k_2] - [B_{12} S]^2}$$

Substitute eq(3) in eq(1), we get

$$F(S) = \left[\frac{X_2(S)[M_2 S^2 + k_2 + S(B_2 + B_{12})]}{B_{12} S}\right][M_1 S^2 + S(B + B_{12}) + k_1] - B_{12} S X_2(S)$$

$$F(S) = X_2(S)\left[\frac{[M_2 S^2 + S(B_2 + B_{12}) + k_2)][M_1 S^2 + S(B + B_{12}) + k_1] - [B_{12} S]^2}{B_{12} S}\right]$$

$$\therefore \frac{X_2(S)}{F(S)} = \frac{B_{12}\,S}{\left[[M_2S^2+K_2+S(B_2+B_{12})][M_1S^2+S(B+B_{12})+K_1]-[B_{12}S]^2\right]}.$$

Result :-

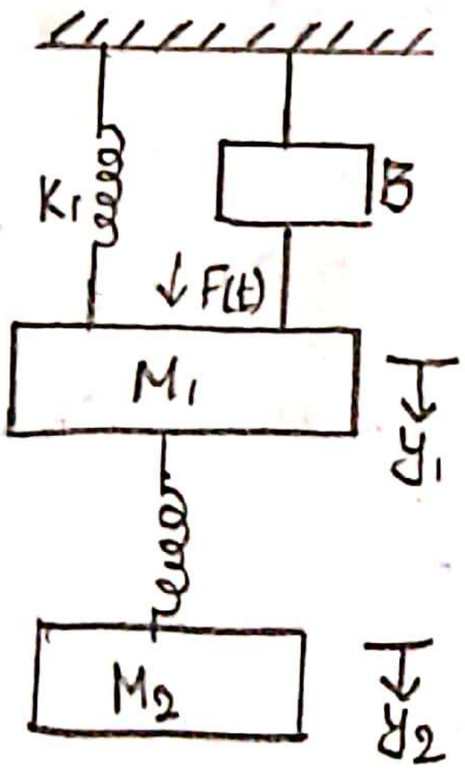
The transfer functions of the system are

(i) $$\frac{X_1(S)}{F(S)} = \frac{M_2S^2+S(B_2+B_{12})+K_2}{[M_1S^2+S(B+B_{12})+K_1][M_2S^2+K_2+S(B_2+B_{12})]-[B_{12}S]^2}.$$

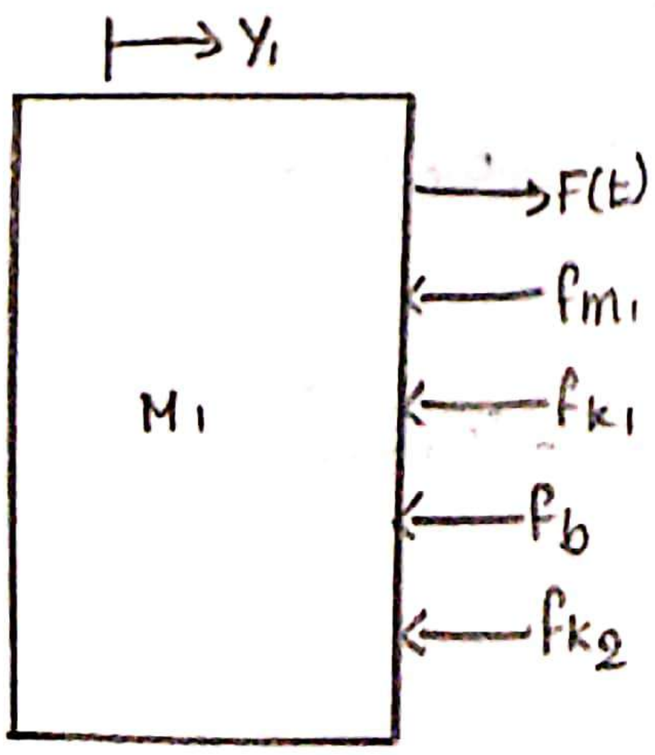
(ii) $$\frac{X_2(S)}{F(S)} = \frac{B_{12}\,S}{\left[[M_2S^2+K_2+S(B_2+B_{12})][M_1S^2+S(B+B_{12})+K_1]-[B_{12}S]^2\right]}.$$

---

5. Determine the transfer function $\frac{Y_2(s)}{F(s)}$ of the system as shown in figure:



From the Mechanical system F(t) is the input and y_2 is the output. Hence, there are two mass and so the free body diagram is needed.



$$F_{m_1} = M_1\frac{d^2y_1}{dt^2}$$

$$F_{K1} = K_1y_1$$

$$F_b = B\frac{dy_1}{dt}$$

$$f_{k2} = K_2(y_1-y_2)$$

According to Newton's 2nd law,

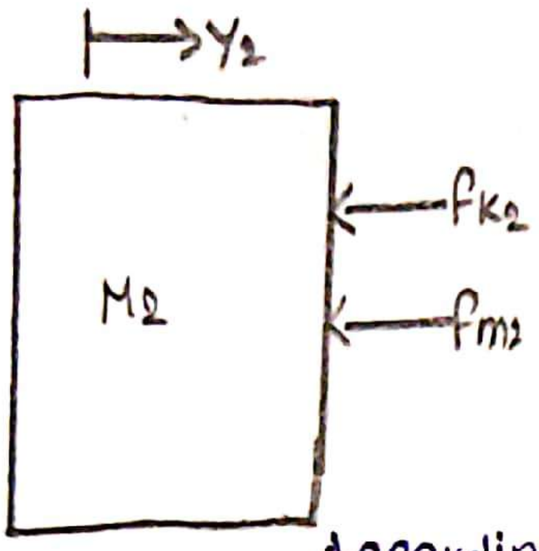
$$F(t) = f_{m1} + f_b + f_{k1} + f_{k2}$$

$$F(t) = M_1 \frac{d^2y_1}{dt^2} + B \frac{dy_1}{dt} + k_1 y_1 + k_2 (y_1 - y_2)$$

Apply laplace transform on both sides with zero initial conditions

$$F(S) = M_1 S^2 Y_1(S) + B \cdot S\, Y_1(S) + k_1 Y_1(S) + k_2 [Y_1(S) - Y_2(S)]$$

$$F(S) = Y_1(S) [M_1 S^2 + B \cdot S + k_1 + k_2] - k_2 Y_2(S) \quad \text{——— (1)}$$



$$F_{m2} = M_2 \frac{d^2y_2}{dt^2}$$

$$F_{k2} = k_2 (y_2 - y_1)$$

According to Newton's 2nd law,

$$0 = f_{m2} + f_{k2}$$

$$0 = M_2 \frac{d^2y_2}{dt^2} + k_2 (y_2 - y_1)$$

Apply laplace transform on both side with zero initial conditions

$$0 = M_2 S^2 Y_2(S) + k_2 [Y_2(S) - Y_1(S)]$$

$$Y_2(S) [M_2 S^2 + k_2] - k_2 Y_1(S) = 0$$

$$Y_2(S) [M_2 S^2 + k_2] = k_2 Y_1(S)$$

$$Y_1(S) = \frac{Y_2(S) [M_2 S^2 + k_2]}{k_2} \quad \text{——— (2)}$$

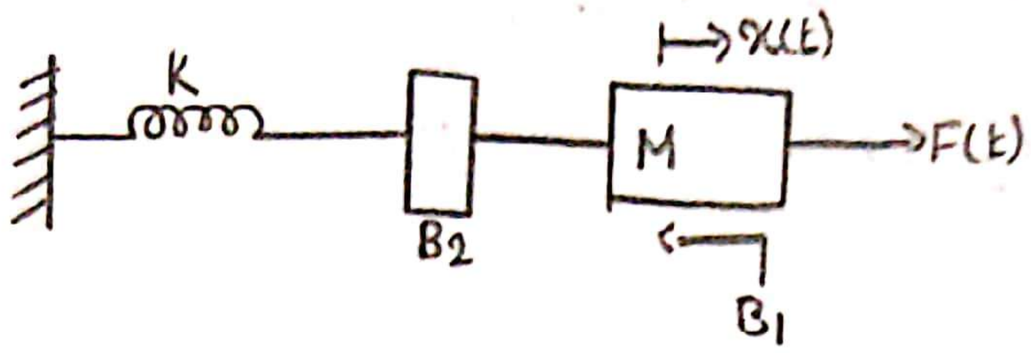
substitute eq-(2) in eq-(1), we get

$$F(S) = \frac{Y_2(S) [M_2 S^2 + k_2]}{k_2} [M_1 S^2 + B \cdot S + k_1 + k_2] - k_2 Y_2(S)$$

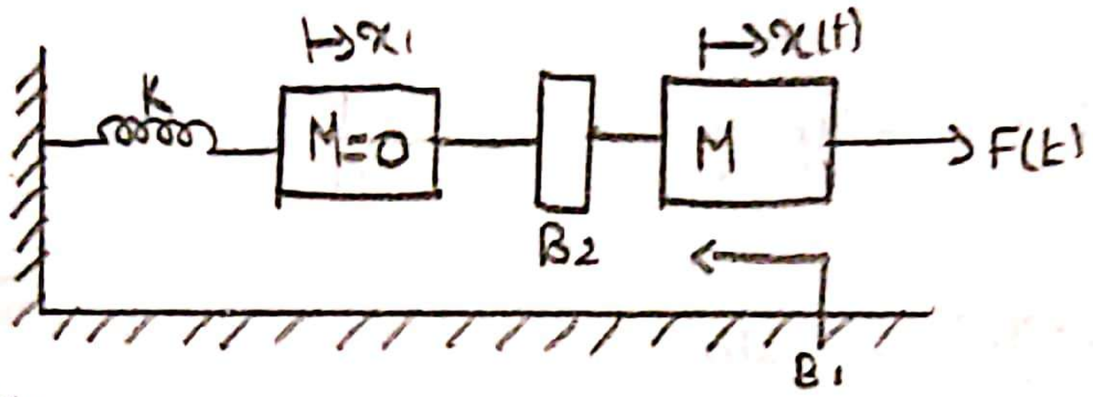
$$= Y_2(S) \left[\frac{(M_2 S^2 + k_2)(M_1 S^2 + B \cdot S + k_1 + k_2) - k_2^2}{k_2} \right]$$

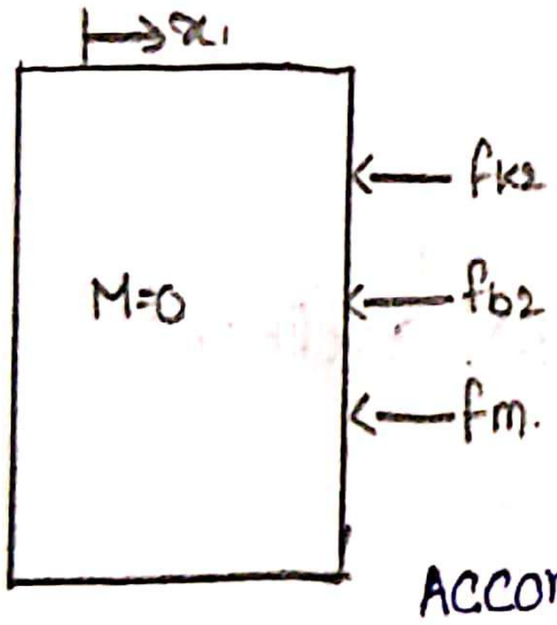
$$\therefore \frac{Y_2(S)}{F(S)} = \frac{K_2}{(M_2S^2+K_2)[M_1S^2+B\cdot S+(K_1+K_2)]-K_2^2}$$

4. Write the equation of motion in s-domain for the system shown in figure. Determine the transfer function of the system



Sol:- Assume another mass between k and B_2 as M=0. and it is given as





$$Fm = 0 \cdot \frac{d^2x}{dt^2} = 0$$

$$Fb_2 = B_2 \frac{d}{dt}[x_1 - x]$$

$$f_k = k x_1(t)$$

According to Newton's 2nd law,

$$0 = f_m + f_{b2} + f_k$$

$$0 = 0 + B_2 \frac{d}{dt}(x_1 - x) + k \cdot x_1$$

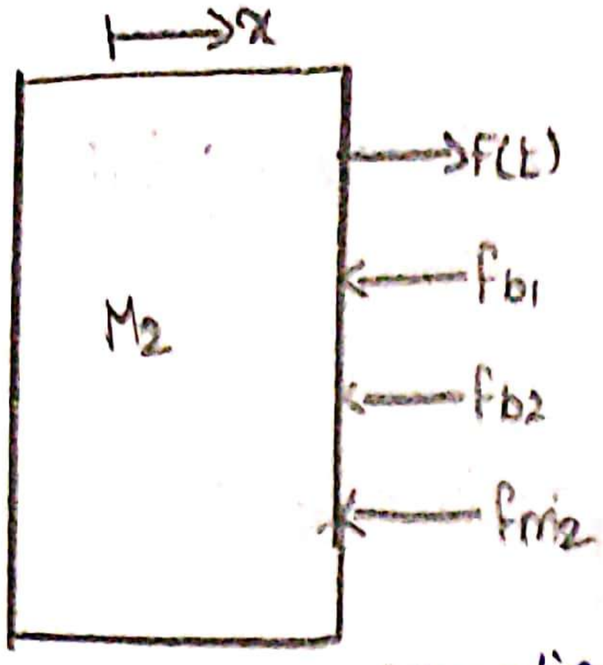
Apply laplace transform on both sides with zero initial conditions

$$0 = B_2 S [X_1(S) - X(S)] + K X_1(S)$$

$$0 = X_1(S)[B_2S + k] - B_2S X(S)$$

$$X_1(S)[B_2S + K] = B_2S X(S)$$

$$X_1(S) = \frac{B_2S X(S)}{B_2S + K} \quad \text{——(1)}$$



$$F_{m2} = M_2 \frac{d^2x}{dt^2}$$

$$F_{b1} = B_1 \frac{d}{dt}(x)$$

$$F_{b2} = B_2 \frac{d}{dt}[x - x_1]$$

According to newton's 2nd law

$$F(t) = f_{m_2} + f_{b1} + f_{b2}$$

$$F(t) = M_2 \frac{d^2x}{dt^2} + B_1 \frac{d}{dt}x + B_2 \frac{d}{dt}(x - x_1)$$

Apply laplace transform on both sides with zero initial conditions

$$F(S) = M_2 S^2 X(S) + B_1 S X(S) + B_2 \cdot S[X(S) - X_1(S)]$$

$$F(S) = X(S)[M_2 S^2 + B_1 S + B_2 S] - B_2 S X_1(S) \quad \text{——(2)}$$

substitute eq-(1) in eq-(2), we get

$$F(S) = X(S)[M_2 S^2 + S(B_1 + B_2)] - B_2 \cdot S\left[\frac{B_2 S \; X(S)}{B_2 S + k}\right]$$

$$= X(S) \frac{[M_2 S^2 + S(B_1 + B_2)][B_2 S + k] - [B_2 \cdot S]^2}{B_2 S + k}$$

$$\therefore \frac{X(S)}{F(S)} = \frac{B_2 S + k}{[M_2 S^2 + S(B_1 + B_2)][B_2 S + k] - [B_2 S]^2}$$

Result :-

The transfer function of the system is

$$\frac{X(S)}{F(S)} = \frac{B_2 S + k}{[M_2 S^2 + S(B_1 + B_2)][B_2 S + k] - [B_2 S]^2}$$

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Mechanical Rotational system:-

The model of rotational system can be obtained by using three elements

(i) Moment of Inertia (J)

(ii) Dashpot with rotational friction of coefficient (B)

(iii) Torsional spring with stiffness (k).

- When a torque is applied to a rotational mechanical system it is opposed by opposing torque due to moment of inertia, friction and elasticity of system.
- The torque acting on a rotational mechanical body is governed by Newton's 2nd of law of motion.

List of symbols used in Mechanical rotational system:-

θ = Angular Displacement (radian)

$v = \frac{d\theta}{dt}$ = Angular velocity (rad/sec)

$a = \frac{d^2\theta}{dt^2}$ = Angular acceleration (rad/sec^2)

T = applied Torque (N-m)

J = Moment of Inertia (kg-m^2/rad)

B = Friction coefficient (N-m)

K = Stiffness of spring (N-m/rad).

Torque Balance Equations:-

1. Moment of Inertia (J):-

consider an ideal mass element, the opposing torque due to moment of inertia is proportional to the angular acceleration.

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$T_J \propto a$

$T_J = J \cdot a$ [$\because T_J$ = opposing torque due to moment of inertia]

Acc. to newton's 2nd law,

sum of applied torque = sum of opposing torques

$$\therefore T = T_J = J \cdot \frac{d^2\theta}{dt^2}.$$

2. Dashpot (B):-

- consider an ideal element dashpot. the dashpot will offer an opposing torque which is proportional to angular velocity of body.



$T_B \propto v$

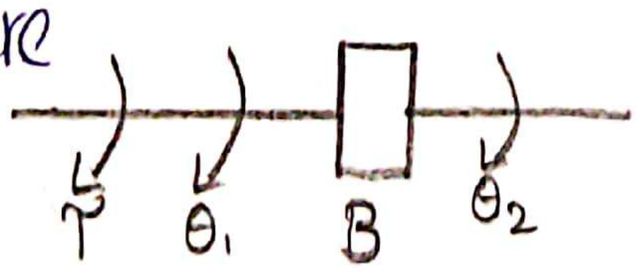
$T_B = B \cdot v$

$T_B = B \cdot \frac{d\theta}{dt}$

Acc. to newton's 2nd law

$$T = T_B = B \cdot \frac{d\theta}{dt}.$$

- when consider dashpot has angular displacement at both ends as shown in figure



$T_B \propto v$

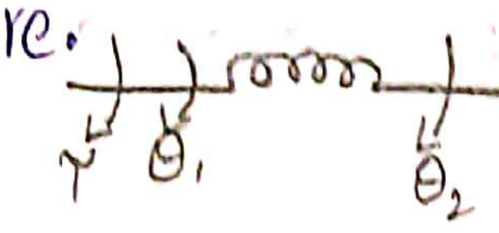
$T_B = B \frac{d}{dt}(\theta_1 - \theta_2)$

Acc. to newton's 2nd law,

$$T = T_B = B \frac{d}{dt}(\theta_1 - \theta_2).$$

3. spring:-

- consider an ideal element spring. The spring will offer an opposing force which is proportional to displacement of body.



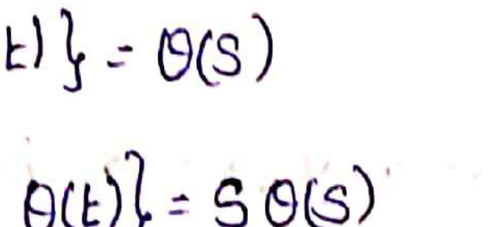
$T_k \propto \theta$

$T_k = K\cdot\theta$

Acc. to newton's 2nd law,

$T = T_k = K\cdot\theta$.

- when consider spring has angular displacement at both ends as shown in Figure.



$T_k \propto (\theta_1 - \theta_2)$

$T_k = K(\theta_1 - \theta_2)$

Acc. to newton's 2nd law

$\therefore \quad T = T_k = K(\theta_1 - \theta_2)$

Note:-

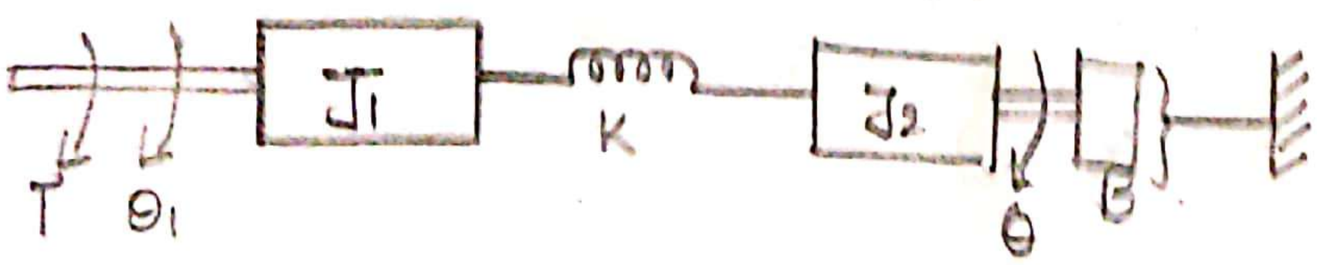
(i) $L\{\theta(t)\} = \theta(s)$

(ii) $L\left\{\frac{d}{dt}\theta(t)\right\} = s\,\theta(s)$

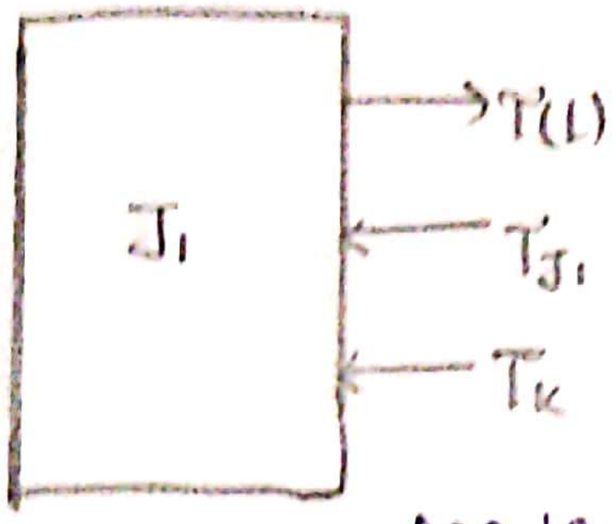
(iii) $L\left\{\frac{d^2}{dt^2}\theta(t)\right\} = s^2\theta(s)$.

problems:-

1. write the differential equation governing the mechanical rotational system shown in figure. Determine the transfer function of the system.



Sol:-



$T_{J1} = J_1 \frac{d^2\theta}{dt^2}$

$T_k = K(\theta_1 - \theta)$

Acc. to newton's 2nd law,

Sum of applied torque = sum of opposing torque

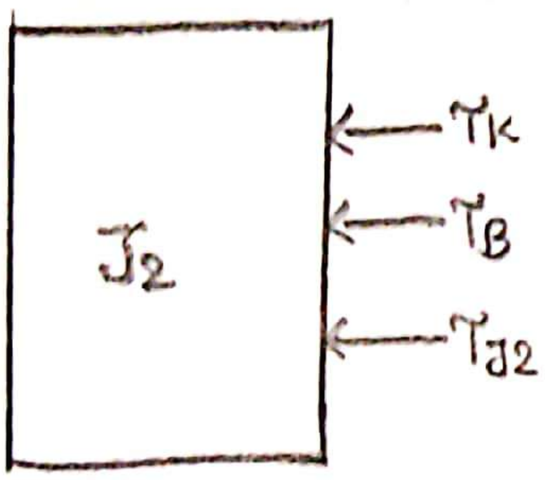
$T(t) = T_{J1} + T_k$

$T(t) = J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta)$

Apply laplace transform on both sides with zero initial conditions

$T(s) = Js^2\theta_1(s) + k[\theta_1(s) - \theta(s)]$

$T(s) = \theta_1(s)[J_1 s^2 + K] - K\theta(s)$ ——— (1)



$T_{J2} = J_2 \frac{d^2\theta}{dt^2}$

$T_B = B \cdot \frac{d\theta}{dt}$

$T_k = k(\theta - \theta_1)$

Acc. to newton's 2nd law,

$0 = T_k + T_B + T_{J2}$

$0 = k(\theta - \theta_1) + B\frac{d\theta}{dt} + J_2\frac{d^2\theta}{dt^2}$

Apply laplace transform on both sides with zero initial conditions

$0 = K[\theta(s) - \theta_1(s)] + B \cdot s\,\theta(s) + J_2 s^2\theta(s)$

$0 = \theta(s)[k + B \cdot s + J_2 s^2] - k\theta_1(s)$

$\theta(s) = \frac{[k + B \cdot s + J_2 s^2]\theta(s)}{k}$ ——— (2)

Substitute eq-(2) in eq-(1)

$$T(s) = \theta_1(s)\left[J_1 s^2 + K\right] - K\,\theta(s)$$

$$T(s) = \left[\frac{\theta(s)\,(K + Bs + J_2 s^2)}{K}\right]\left[J_1 s^2 + K\right] - K\,\theta(s)$$

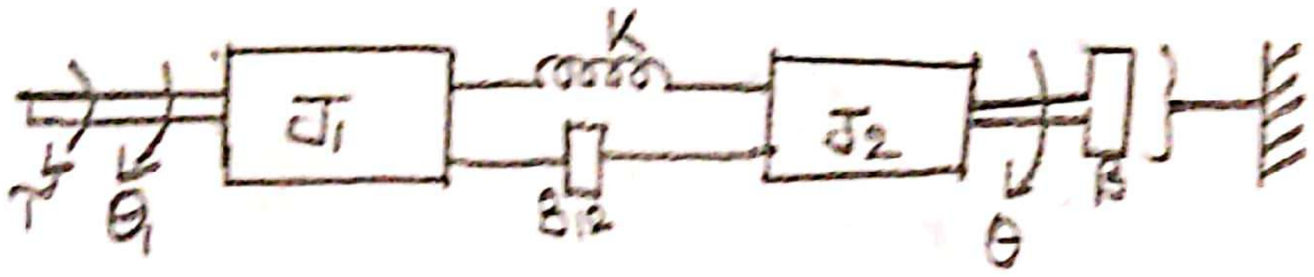
$$T(s) = \theta(s)\left[\frac{(K + Bs + J_2 s^2)(J_1 s^2 + K) - K^2}{K}\right]$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{K}{(K + Bs + J_2 s^2)(J_1 s^2 + K) - K^2}.$$

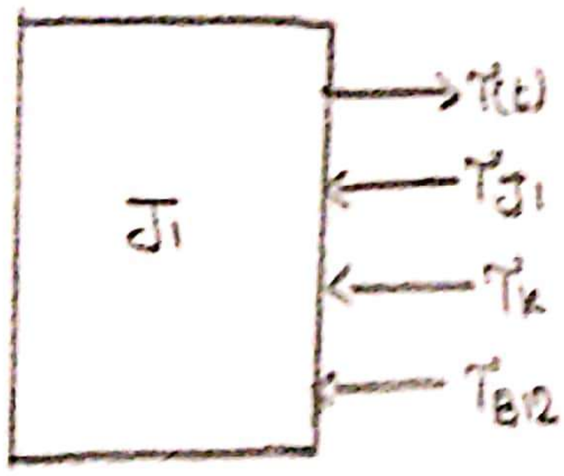
∴ The transfer function of the system is

$$\frac{\theta(s)}{T(s)} = \frac{K}{(K + B\cdot s + J_2 s^2)(J_1 s^2 + K) - K^2}.$$

2. Write the differential equation governing the mechanical rotational system as shown in Figure. Determine the transfer function of $\frac{\theta(s)}{T(s)}$.



Sol:-



$$T_{J1} = J_1 \frac{d^2\theta_1}{dt}$$

$$T_K = K(\theta_1 - \theta)$$

$$T_{B12} = B_{12} \frac{d}{dt}(\theta_1 - \theta)$$

Acc. to newton's 2nd law

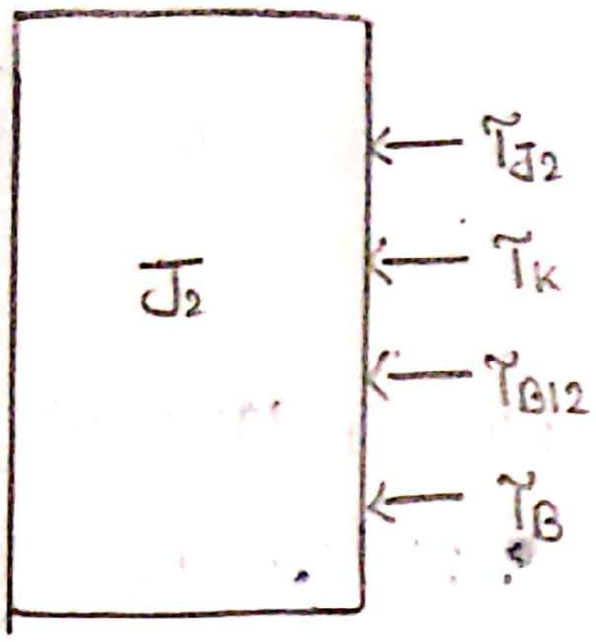
$$T(t) = T_{J1} + T_K + T_{B12}$$

$$T(t) = J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta_2) + B \frac{d}{dt}(\theta_1 - \theta_2)$$

Apply laplace Transform on both sides with zero initial conditions

$$T(s) = J_1 s^2 \theta_1(s) + K\left[\theta_1(s) - \theta(s)\right] + B\,s\left[\theta_1(s) - \theta(s)\right]$$

$$T(s) = \theta_1(s)\left[J_1 s^2 + s\cdot B + K\right] - \theta(s)\left[K + B\cdot s\right] \quad \text{——— (1)}$$



$$T_{J2} = J_2 \frac{d^2\theta}{dt^2}$$

$$T_K = K(\theta - \theta_1)$$

$$T_{B12} = B_{12} \frac{d}{dt}(\theta - \theta_1)$$

$$T_B = B \frac{d\theta}{dt}$$

Acc. to newton's 2nd law

$$0 = T_{J2} + T_K + T_{B12} + T_B$$

$$0 = J_2 \frac{d^2\theta}{dt^2} + K(\theta - \theta_1) + B_{12} \frac{d}{dt}(\theta - \theta_1) + B \frac{d\theta}{dt}$$

Apply laplace transform on both sides with zero initial conditions

$$0 = J_2 S^2\theta(S) + K[\theta(S) - \theta_1(S)] + B_{12} S[\theta(S) - \theta_1(S)] + B \cdot S\theta(S)$$

$$0 = \theta(S)[J_2S^2 + K + S(B_{12}+B)] - \theta_1(S)[K + B_{12} \cdot S]$$

$$\theta(S)[J_2S^2 + S(B_{12}+B) + K] = \theta_1(S)[K + B_{12}S]$$

$$\theta_1(S) = \frac{\theta(S)[J_2S^2 + S(B_{12}+B) + K]}{K + B_{12} \cdot S} \quad \text{——— (2)}$$

Substitute eq-(2) in eq-(1), we get

$$T(S) = \theta_1(S)[J_1S^2 + K + B \cdot S] - \theta(S)[K + B \cdot S]$$

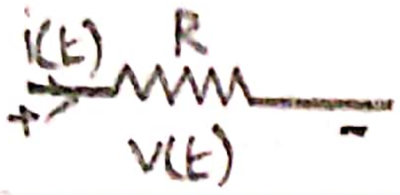
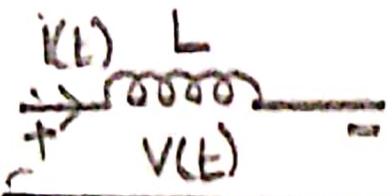
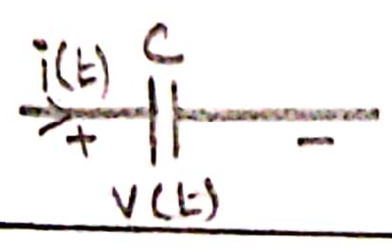
$$T(S) = \left[\frac{\theta(S)(J_2S^2 + K + S(B_{12}+B))}{K + B_{12} \cdot S}\right](J_1S^2 + K + B \cdot S) - \theta(S)[K + B \cdot S]$$

$$T(S) = \theta(S)\left[\frac{(J_2S^2 + K + S(B_{12}+B))(J_1S^2 + B \cdot S + K) - (K + B \cdot S)(K + B_{12}S)}{K + B_{12}S}\right]$$

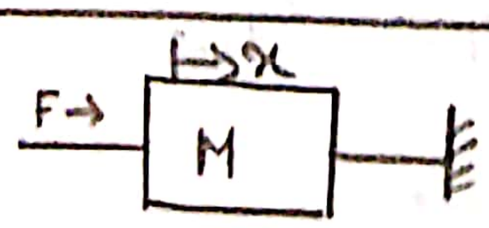
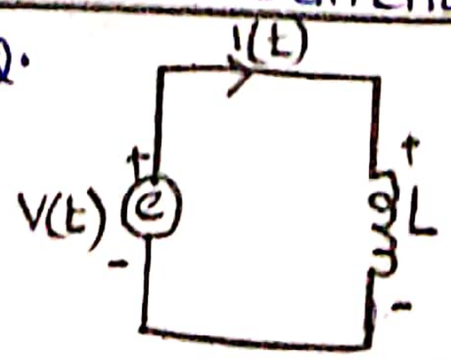
$$\therefore \frac{\theta(S)}{T(S)} = \frac{K + B_{12}S}{(J_2S^2 + K + S(B_{12}+B))(J_1S^2 + B \cdot S + K) - (K + B \cdot S)(K + B_{12}S)}.$$

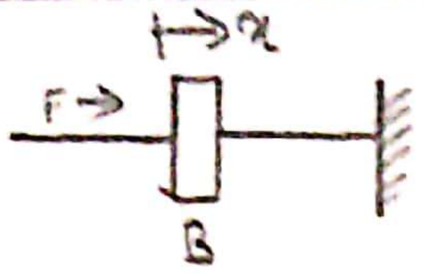
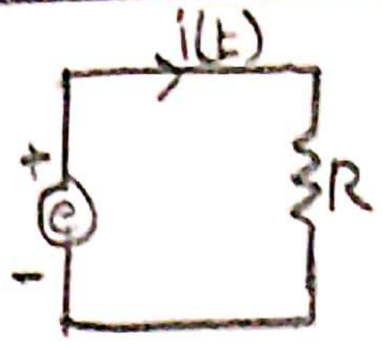
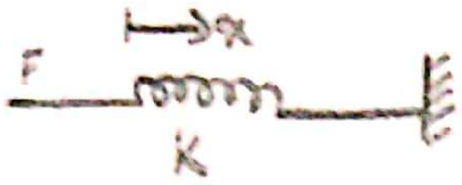
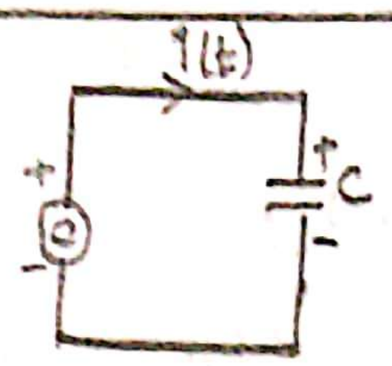
Electrical system:-

- The model of electrical systems can be obtained by using resistor, capacitor and inductor.
- The current and voltage relation of resistor, inductor and capacitor are given below. For modelling electrical system, then electrical network formed by using R, L and C, voltage source, current source.
- The differential equations governing the electrical systems can be formed by writing KCL equations by choosing various nodes in the network or KVL equations by choosing various closed paths in the network.

Element	Voltage across the element	current across the element
	$V(t) = i(t) \cdot R$	$i(t) = \frac{V(t)}{R}$
	$V(t) = L \frac{d}{dt} i(t)$	$i(t) = L \int V(t)\, dt$
	$V(t) = \frac{1}{C} \int i(t)\, dt$	$i(t) = C \frac{d}{dt} V(t)$

Force voltage Analogy:-

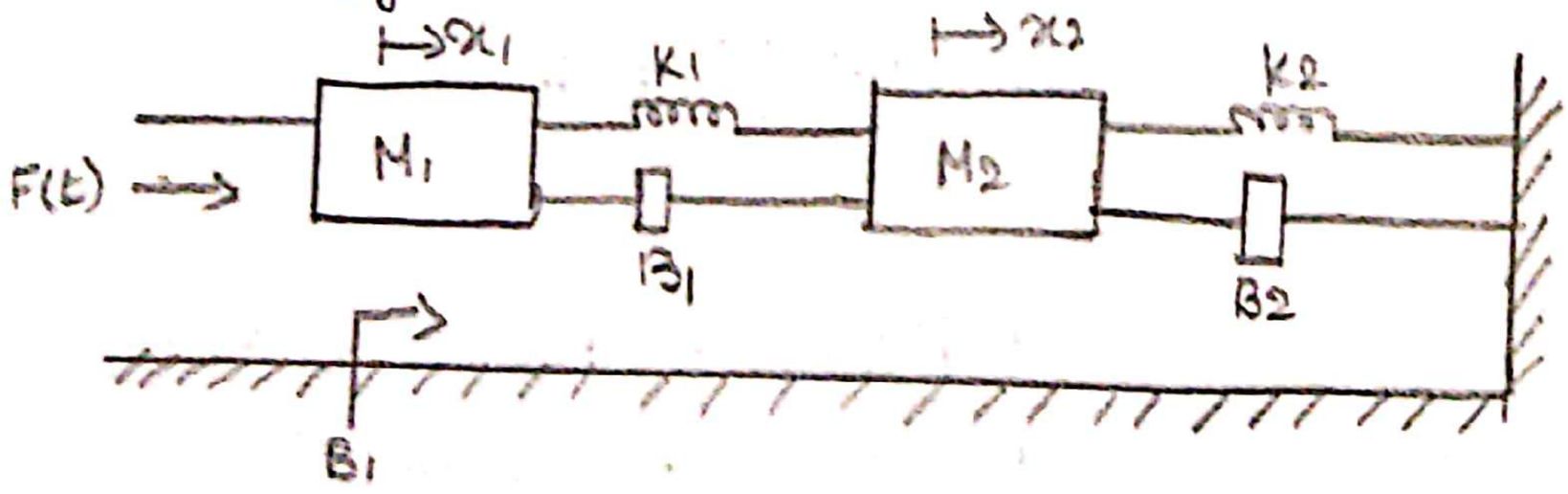
Mechanical translation system	Electrical system
1. Input : Force 2. Displacement : output	1. Input : voltage source output : current across the element
2.  $F = f_m = M \frac{d^2x}{dt^2} = M \frac{dV}{dt}$.	2.  $V(t) = e = L \cdot \frac{di(t)}{dt}$

3.  $f = f_b = B\frac{dx}{dt} = B \cdot V$	3.  $V(t) = e = i(t) \cdot R$
4.  $f = f_k = k \cdot x$	4.  $V(t) = \frac{1}{C}\int i(t)\,dt$

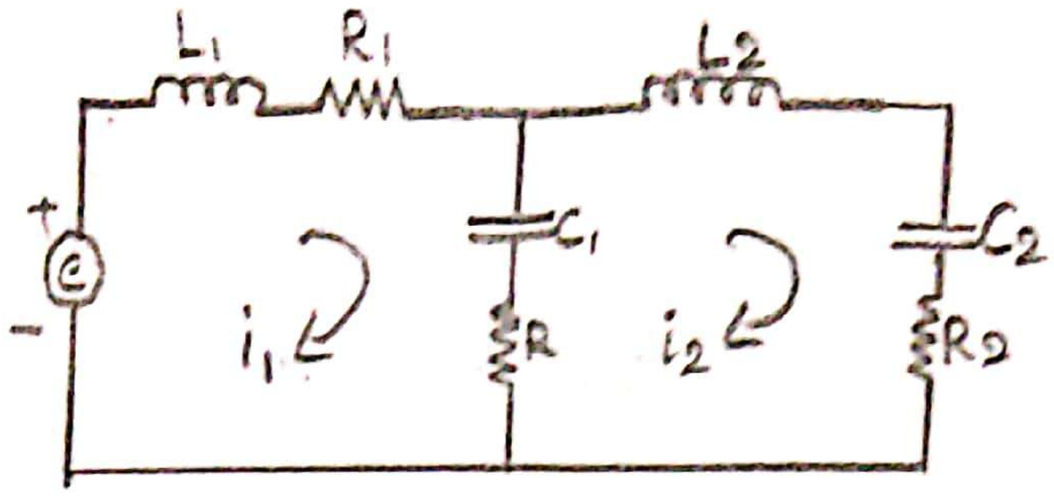
Guidelines:-

1. In mechanical system, the elements having same velocity and said to be in series. similarly, in electrical system, the elements in series will have same current.
2. Each node (mass) in the mechanical system correspond to a closed loop in an electrical system.
3. Number of mesh's in electrical system are equal to number of masses in mechanical system.
4. The element connected between two masses in mechanical system is represented as common element between two mesh's in electrical analogous system.

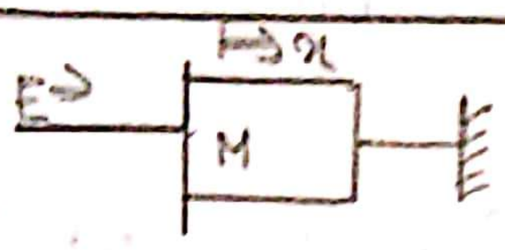
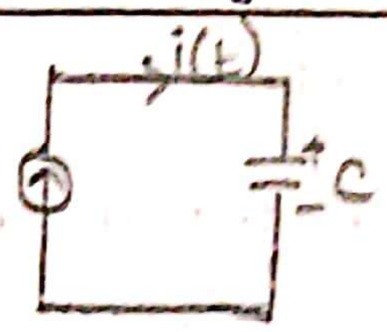
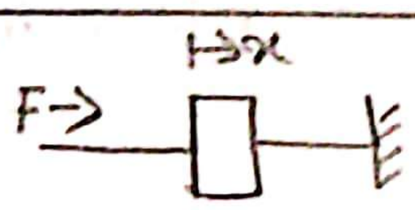
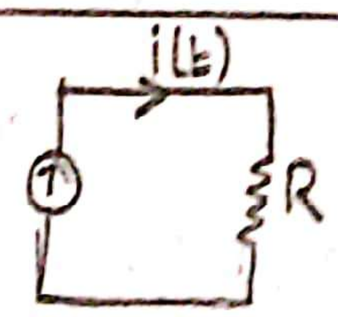
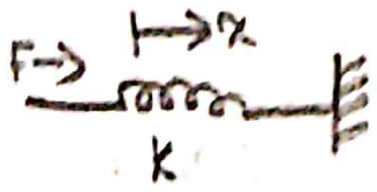
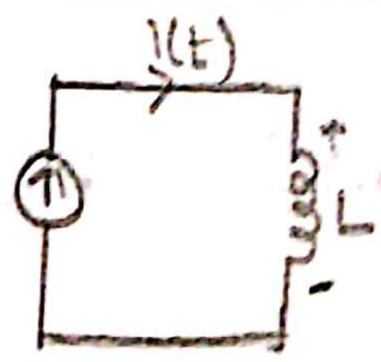
Ex:- convert the Mechanical Translational system given below into Electrical system.



Sol:-



Force-current Analogy:-

Mechanical Translational system	Electrical system
1. Input : Force output : velocity	1. Input : Current source output : voltage source through the element.
2.  $F = m\frac{d^2x}{dt^2} = m\frac{dv}{dt}$	2.  $i = C\frac{dv}{dt}$.
3.  $F = B \cdot v = B\frac{dx}{dt}$	3.  $i(t) = \frac{V(t)}{R}$
4.  $F = k \cdot x = k\int v\,dt$	4.  $i(t) = \frac{1}{L}\int v(t)\,dt$

Guidelines:-

1. The electrical system elements in parallel will have same voltage likewise in mechanical system, the elements having same Force are said to be in parallel.

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2. The elements having same velocity in mechanical system should have analogous same voltage in electrical analogous system.

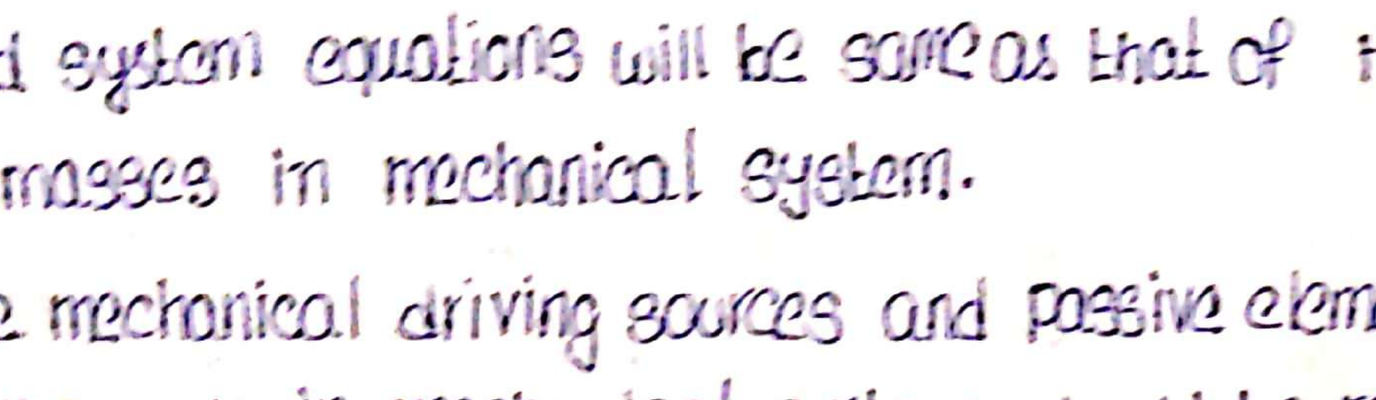
3. Each node in mechanical system corresponds to a node in electrical system. A mass is considered as a node.

4. The number of nodes in electrical system is same as that of number of nodes (masses) in mechanical system. Hence, the node voltages and system equations will be same as that of the number of velocities of masses in mechanical system.

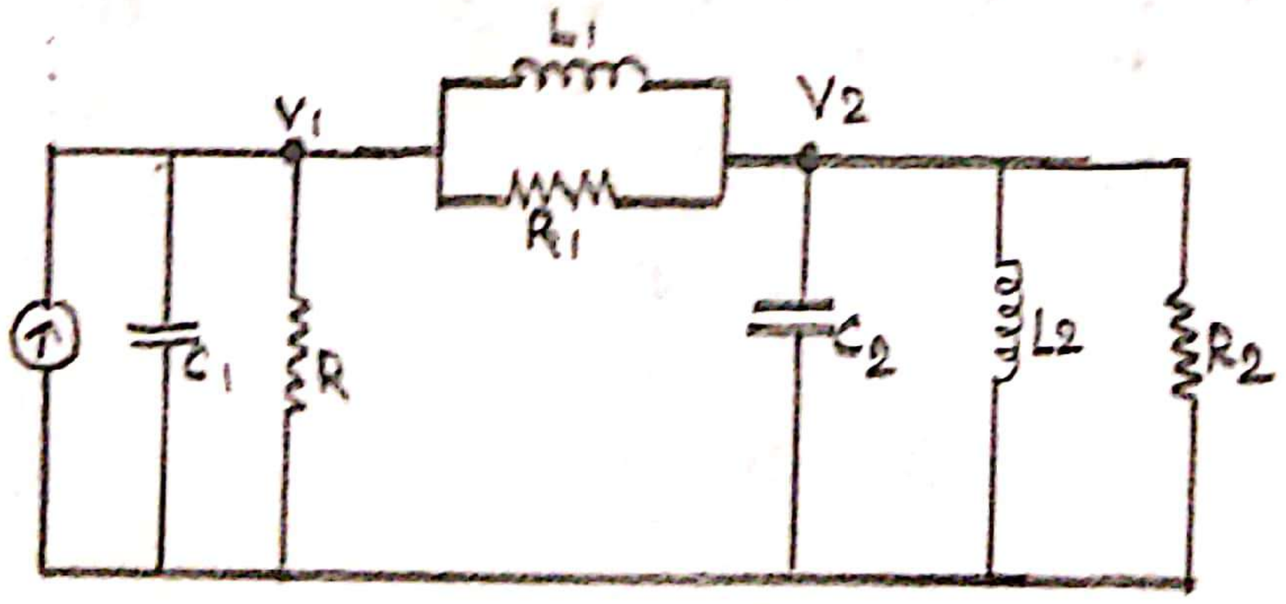
5. The mechanical driving sources and passive elements (RLC) connecting to the node in mechanical system should be represented by analogous elements connected to a node in electrical system.

6. The element connected between two nodes (mass) in mechanical system is represented as a common element between two nodes in electrical analogous system.

EX:- Convert the Mechanical Translation System given below into electrical system.

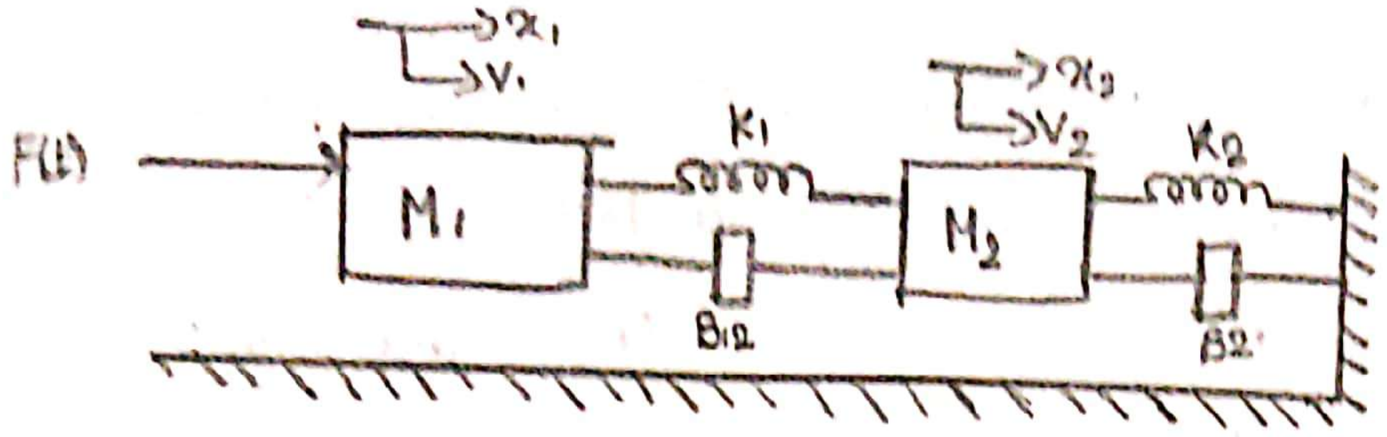


Sol:-



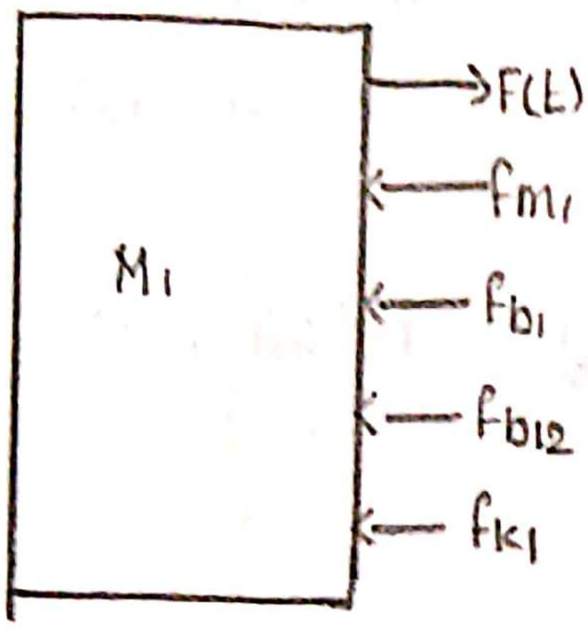
Problem:-

1. Write the differential equations governing the mechanical system shown in Figure. Draw the force voltage and force current electrical analogous circuits and verify by writing mesh and node equations.



Sol:- The given mechanical system has two nodes (masses). The differential equation governing the mechanical system are given by force balance equations at these nodes.

The Free body diagram of m_1 is given as



$$F_{m1} = M_1 \frac{d^2x}{dt^2}$$

$$F_{b1} = B_1 \frac{dx_1}{dt}$$

$$f_{b12} = B_{12} \frac{d}{dt}(x_1 - x_2)$$

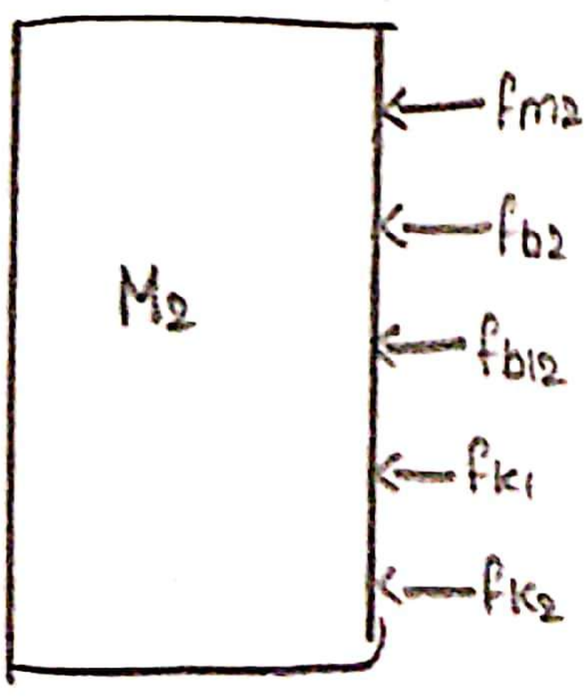
$$f_{k1} = K_1(x_1 - x_2)$$

By newton's second law,

$$f(t) = f_{m1} + f_{b1} + f_{b12} + f_{k1}$$

$$F(t) = M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + K_1(x_1 - x_2) \quad \text{——(1)}$$

The free body diagram of m_2 is shown as



$$f_{m2} = M_2 \frac{d^2x_2}{dt^2}$$

$$f_{b2} = B \cdot \frac{dx_2}{dt}$$

$$f_{b12} = B_{12} \frac{d}{dt}(x_2 - x_1)$$

$$f_{k1} = K_1(x_2 - x_1)$$

$$f_{k2} = K_2 x_2$$

By newtons 2nd law,

$$0 = f_{K1} + f_{m2} + f_{b2} + f_{K2} + f_{b12}$$

$$0 = M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + B_{12} \frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) \quad \text{——(2)}$$

on replacing the displacements by velocity in the differential equations (1) and (2) of the mechanical system,

$$\text{i.e., } \frac{d^2x}{dt^2} = \frac{dV}{dt} \; ; \frac{dx}{dt} = V \text{ and } x = \int V\,dt$$

$$M_1 \frac{dV_1}{dt} + B_1 V_1 + B_{12}(V_1 - V_2) + K_1 \int (V_1 - V_2)\,dt = f(t) \quad \text{——(3)}$$

$$M_2 \frac{dV_2}{dt} + B_2 V_2 + K_2 \int V_2\,dt + B_{12}(V_2 - V_1) + K_1 \int (V_2 - V_1)\,dt = 0 \quad \text{——(4)}$$

Force-voltage Analogous circuit:-

The given mechanical system has two nodes (masses). Hence it will have two meshes. The forces applied to mass m, is represented by a voltage source in first mesh.

The elements M_1, B_1, K_1 and B_{12} are connected to first node. Hence they are represented by analogous element in mesh-1 forming closed path.

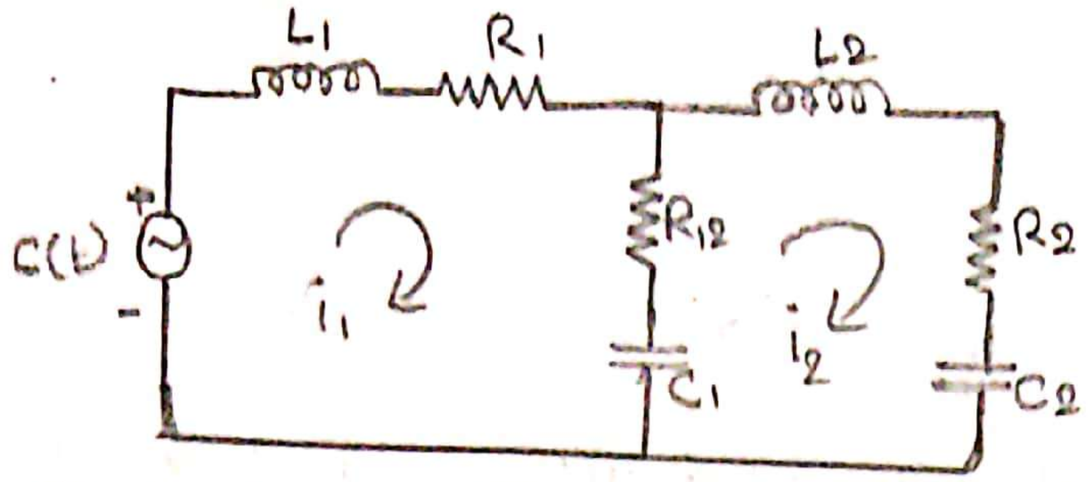
The elements M_2, B_2, K_2, B_{12} and K are second node. Hence they are represented as analogous element in mesh-2 forming a closed path. The elements K_1 and B_{12} are connected between node-1 and node-2. so they are common elements in two meshes.

The electrical analogous elements for the element of mechanical system are given below

$f(t) \rightarrow e(t)$ $\quad M_1 \rightarrow L_1$ $\quad B_1 \rightarrow R_1$ $\quad K_1 \rightarrow \frac{1}{C_1}$

$V_1 \rightarrow i_1$ $\quad M_2 \rightarrow L_2$ $\quad B_2 \rightarrow R_2$ $\quad K_2 \rightarrow \frac{1}{C_2}$

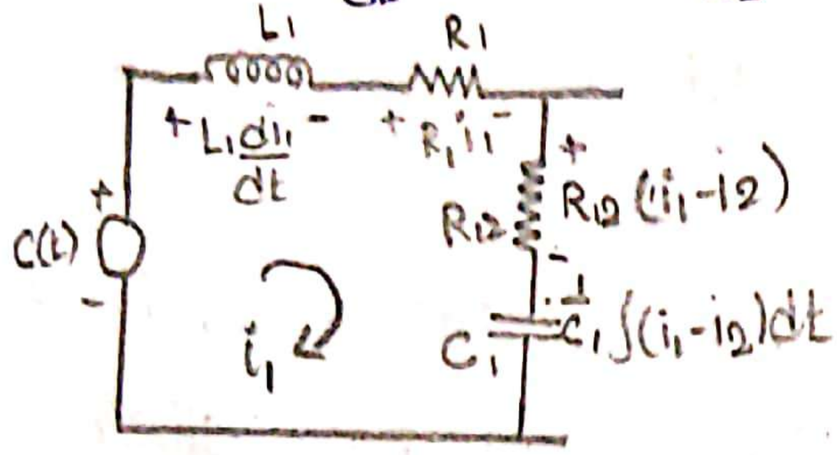
$V_2 \rightarrow i_2$ $\quad B_{12} \rightarrow R_{12}$

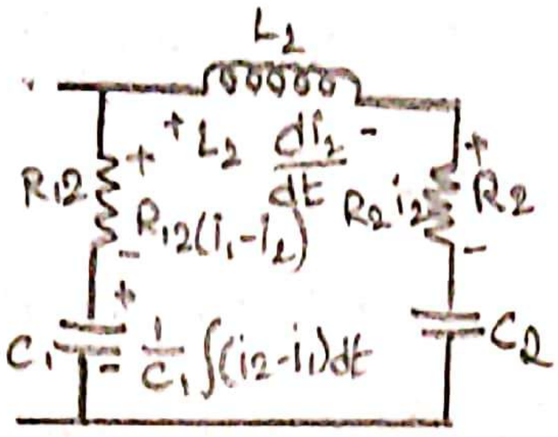


The mesh basis equations using kirchoff's voltage law for the circuit shown in above figure are

$$L_1 \frac{di_1}{dt} + R_1 i_1 + R_{12}(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2)\,dt = e(t) \quad \text{——(5)}$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt + R_{12}(i_2 - i_1) + \frac{1}{C_1}\int (i_2 - i_1)\,dt = 0 \quad \text{——(6)}$$





It is observed that the mesh basis equations (5) & (6) are similar to the differential equations (3) & (4) governing the mechanical system.

Force - current Analogous circuit:-

The given mechanical system has two nodes (masses). Hence, Force current analogous electrical circuit will have two nodes.

The force applied to mass M_1 is represented as a current source connected to node-1 in analogous electrical circuit.

Node-1 :- M_1, B_1, K_1 and B_{12}

Node-2 : K_1, B_{12}, M_2, K_2 and B_2

The elements K_1 and B_{12} are common node-1 and node-2. The force current electrical analogous circuit

$F(t) \rightarrow I(t)$

$i_1 \rightarrow V_1$

$i_2 \rightarrow V_2$

$M_1 \rightarrow C_1$

$M_2 \rightarrow C_2$

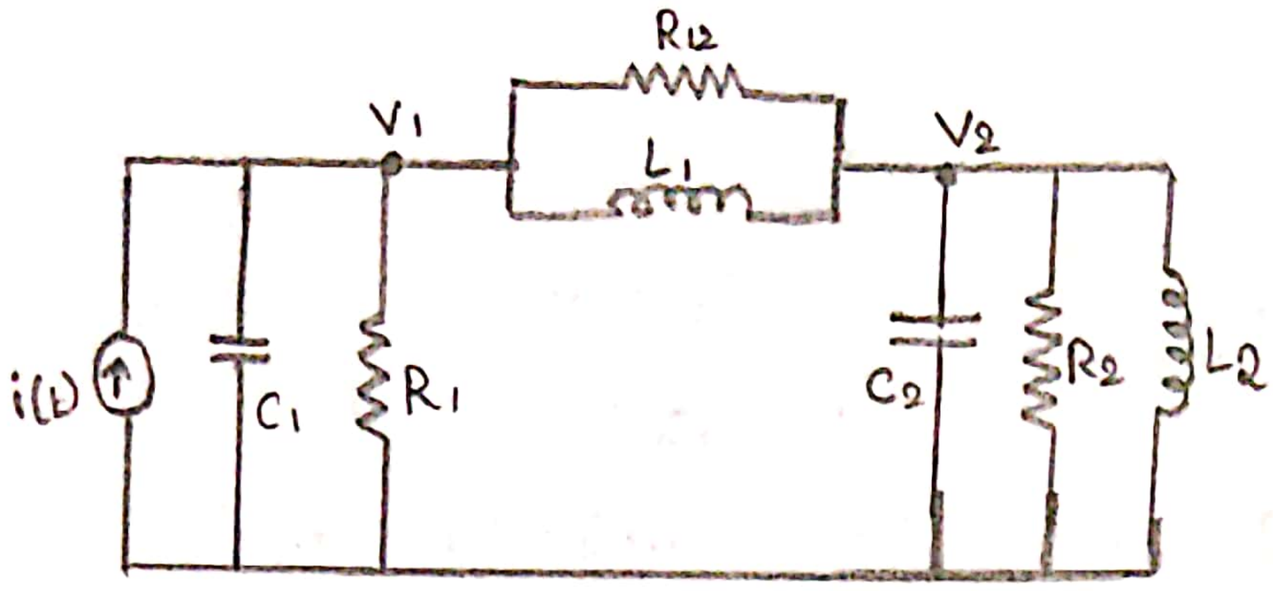
$B_1 \rightarrow \frac{1}{R_1}$

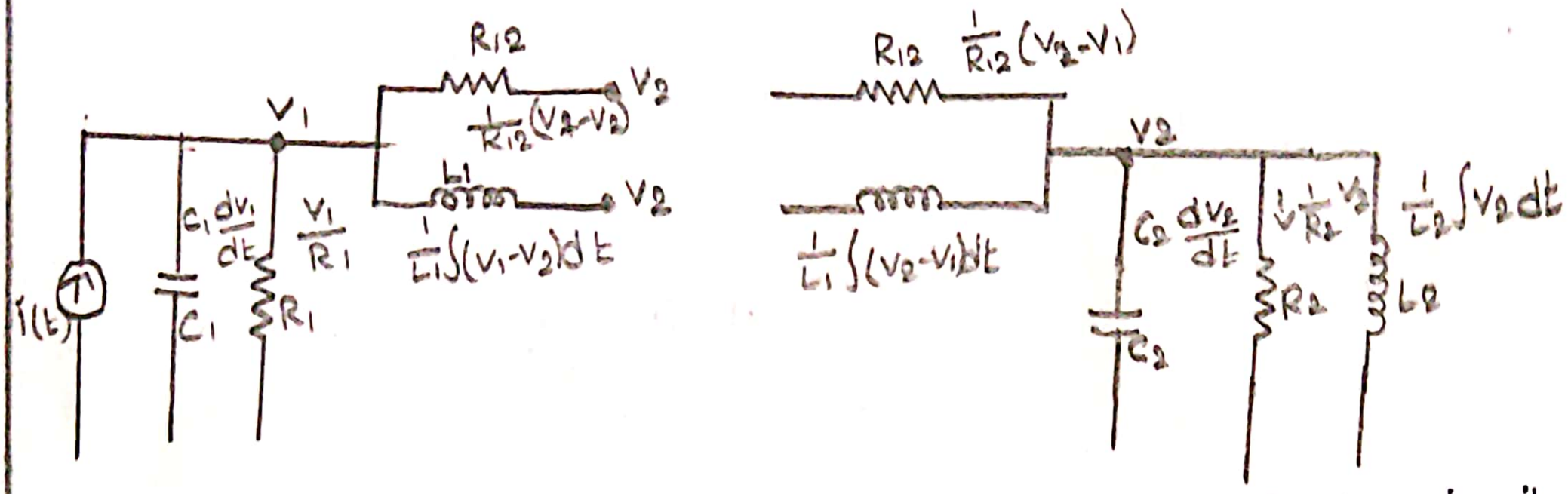
$B_2 \rightarrow \frac{1}{R_2}$

$B_{12} \rightarrow \frac{1}{R_{12}}$

$K_1 \rightarrow \frac{1}{L_1}$

$K_2 \rightarrow \frac{1}{L_2}$.





The node bias equations using kirchoff's current law for the circuit as shown in above figure

$$C_1\frac{dV_1}{dt}+\frac{1}{R_1}V_1+\frac{1}{R_{12}}(V_1-V_2)+\frac{1}{L_1}\int(V_1-V_2)dt=i(t)\quad\text{——(7)}$$

$$C_2\frac{dV_2}{dt}+\frac{1}{R_2}V_2+\frac{1}{L_2}\int V_2\,dt+\frac{1}{R_{12}}(V_2-V_1)+\frac{1}{L_1}\int(V_2-V_1)dt=0\quad\text{——(8)}$$

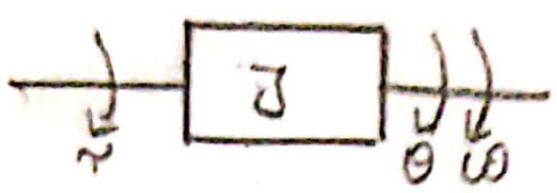
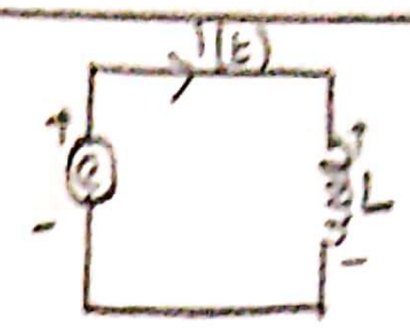
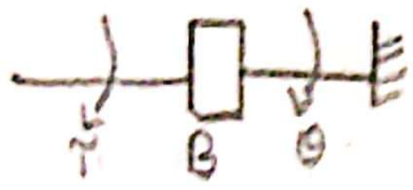
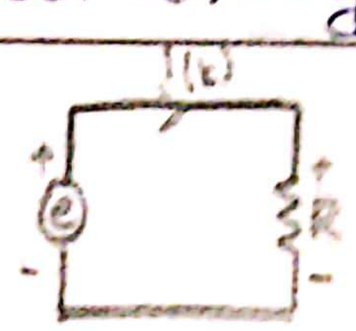
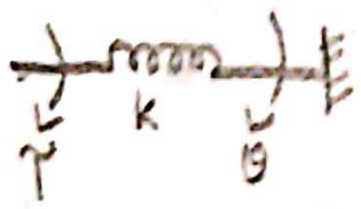
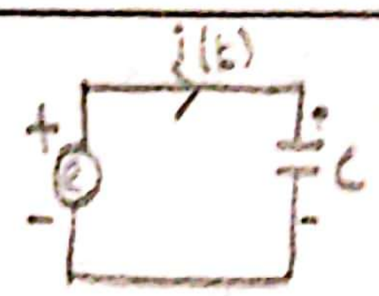
It is observed that the node bais equations (7) & (8) are similar to the differential equations (3) and (4) governing the mechanical system.

Electrical Analogous of Mechanical Rotational system:-

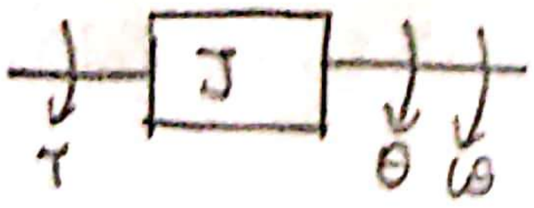
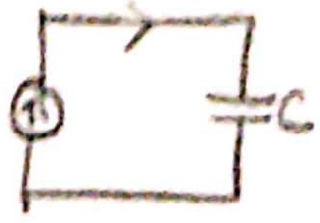
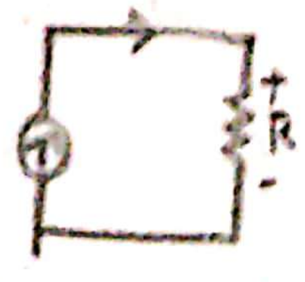
- The three basic elements are moment of Inertia(J), dashpot (B) and torsional spring (k) that are used in modelling mechanical rotational systems are analogous to resistance, inductance and capacitance of electrical system.
- since the electrical system has two types of input i.e., either voltage and current source. These are of two types

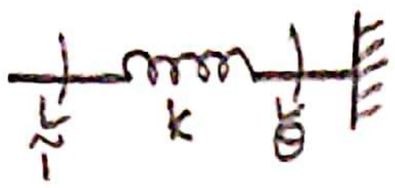
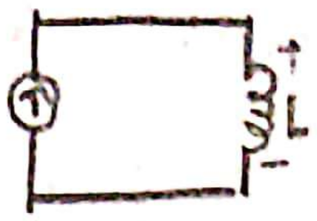
  1. Torque voltage analogy
  2. Torque current analogy

Torque - Voltage Analogy:-

Mechanical Rotational system	Electrical system
1. Input : Torque output : Angular velocity	1. Input : voltage source output : current across the element.
2.  $\tau = J\frac{d^2\theta}{dt} = J\frac{d\omega}{dt}$	2.  $e(t) = V(t) = L \cdot \frac{di}{dt}$
3.  $\tau = B\frac{d\theta}{dt} = B \cdot \omega$	3.  $V(t) = e(t) = i(t) \cdot R$
4.  $\tau = k\theta = k\int \omega\, dt$	4.  $e(t) = V(t) = \frac{1}{C}\int i(t)\, dt$

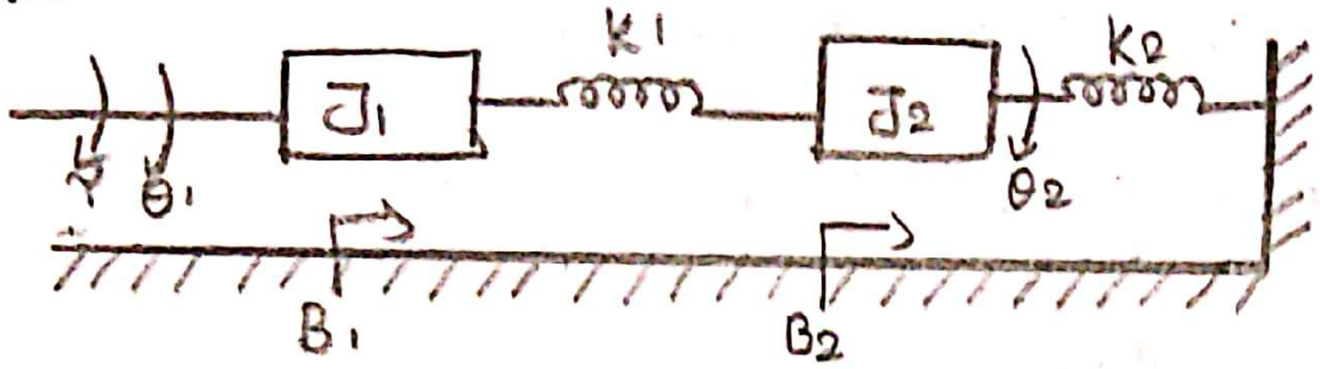
Torque - current Analogy:-

Mechanical Rotational System	Electrical system
1. Input : Torque 2. output : Angular velocity	1. input : current source output : voltage across the element
2.  $\tau = J\frac{d^2\theta}{dt} = J\frac{d\omega}{dt}$	2.  $i(t) = C\frac{dV}{dt}$
3.  $\tau = B\frac{d\theta}{dt} = B\omega$	3.  $i(t) = \frac{1}{R}V(t)$

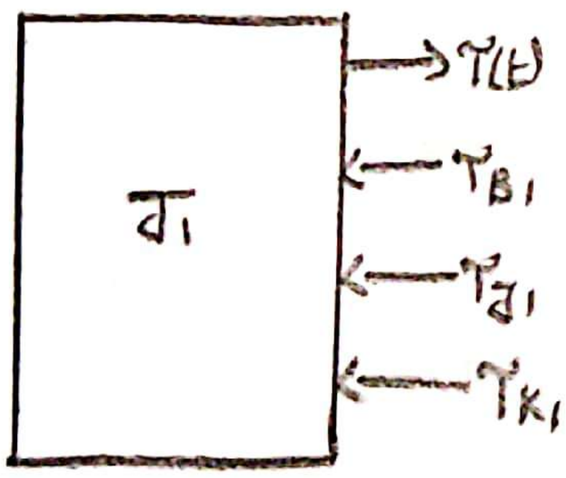
4. 	4. 
$\tau = k\theta = k\int \omega\, dt$	$i(t) = \frac{1}{L}\int v(t)\, dt$

problem :-

1. write the differential equations governing the mechanical rotational system shown in figure. Draw the torque voltage and torque current electrical analogous circuit and verify by writing mesh and node equations



sol: The given mechanical system has two nodes (masses). The differential equation governing the mechanical system are given by force balance equations at these nodes



$$T_{J1} = J_1 \frac{d^2\theta}{dt}$$

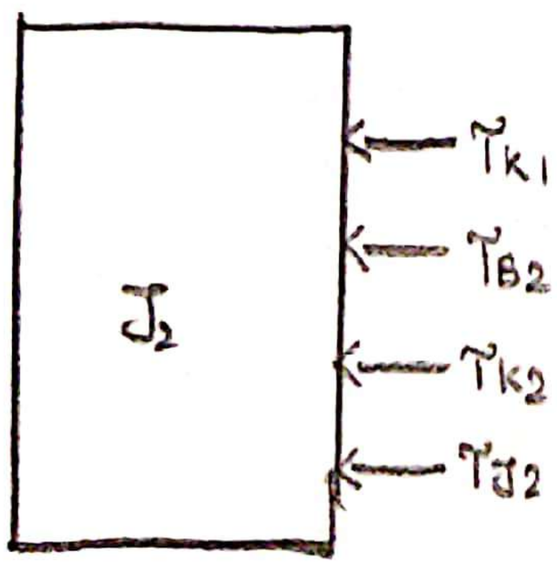
$$T_{B1} = B_1 \frac{d\theta_1}{dt}$$

$$T_{k1} = K(\theta_1 - \theta_2)$$

ACC. to newton's 2nd law,

$$T(t) = T_{J1} + T_{k1} + T_{b1}$$

$$T(t) = J_1 \frac{d^2\theta}{dt^2} + k(\theta_1 - \theta_2) + B_1 \frac{d\theta_1}{dt} \quad \text{——(1)}$$



$$T_{J2} = J_2 \frac{d^2\theta_2}{dt^2}$$

$$T_{k2} = k_2\theta_2$$

$$T_{k1} = k_1(\theta_2 - \theta_1)$$

$$T_{B2} = B_2 \frac{d\theta_2}{dt}$$

Acc. to newton's 2nd law,

$$0 = T_{a2} + T_{k1} + T_{B2} + T_{k2}$$

$$0 = J_2 \frac{d^2\theta_2}{dt^2} + K_1(\theta_2 - \theta_1) + B_2 \frac{d\theta_2}{dt} + k_2\theta_2 \quad \text{---(3)}$$

on replacing the displacements by velocity in the differential equations (1) and (2) of the mechanical system,

$$\text{i.e., } \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} \text{ & } \frac{d\theta}{dt} = \omega \quad ; \theta = \int \omega\, dt$$

$$J_1 \frac{d\omega}{dt} + K\int(\omega_1 - \omega_2) + B_1\omega_1 = T(t) \quad \text{---(3)}$$

$$J_2 \frac{d\omega}{dt} + K\int(\omega_2 - \omega_1) + K_2\int\omega_2 + B_2\omega_2 = 0 \quad \text{---(4)}$$

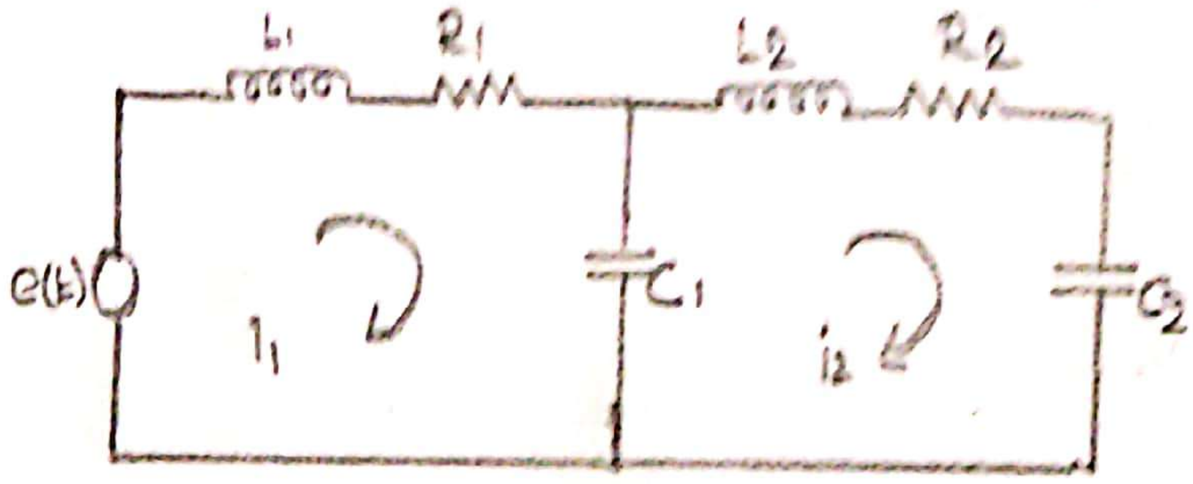
Force voltage Analogous circuit:-

The electrical analogous elements for the element of mechanical system are given below

$T(t) \rightarrow e(t)$ $\quad B_1 \rightarrow R_1$ $\quad K_1 \rightarrow \frac{1}{C_1}$

$J_1 \rightarrow L_1$ $\quad B_2 \rightarrow R_2$ $\quad K_2 \rightarrow \frac{1}{C_2}$

$J_2 \rightarrow L_2$



The mesh basis equations using kirchoff's voltage law for the circuit shown in above figure are

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int(i_1 - i_2)dt = e(t) \quad \text{---(5)}$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_1}\int i_2\, dt + \frac{1}{C_1}\int(i_2 - i_1)dt = 0 \quad \text{---(6)}$$

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Force-current Analogy:-

The electrical analogous elements for the element of mechanical system are given below

$q(t) \rightarrow i(t)$

$J_1 \rightarrow C_1$

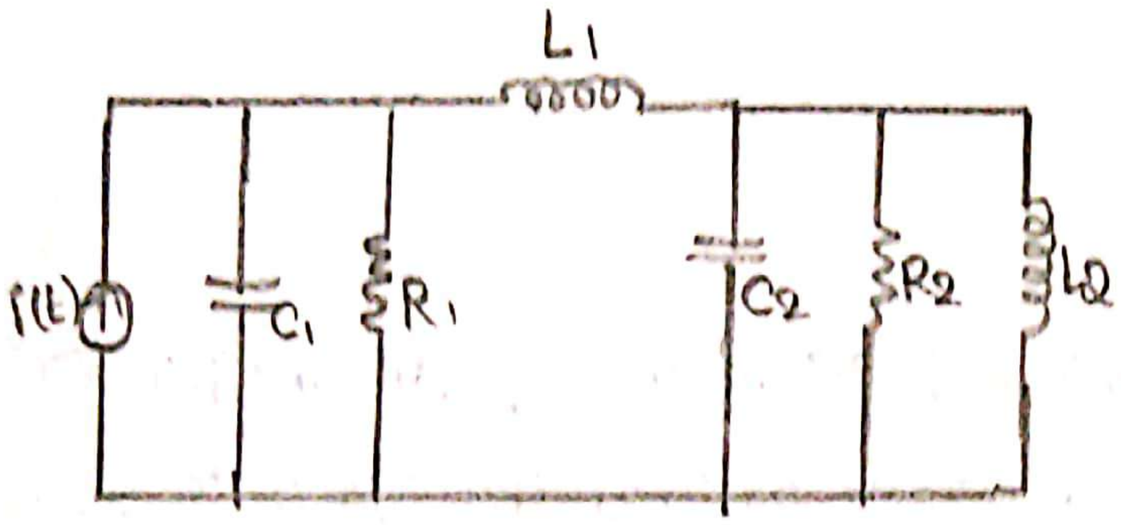
$J_2 \rightarrow C_2$

$B_1 \rightarrow \frac{1}{R_1}$

$B_2 \rightarrow \frac{1}{R_2}$

$K_1 \rightarrow \frac{1}{L_1}$

$K_2 \rightarrow \frac{1}{L_2}$



The node basis equations by using kirchhoff current law for the circuit as shown below.

$$C_1 \frac{dV_1}{dt} + \frac{1}{R_1} V_1 + \frac{1}{L_1} \int (V_1 - V_2) = i(t) \quad \text{---(7)}$$

$$C_2 \frac{dV_2}{dt} + \frac{1}{R_2} V_2 + \frac{1}{L_2} \int V_2 dt + \frac{1}{L_1} \int (V_2 - V_1) dt = 0 \quad \text{---(8)}$$

It is observed that the node bais equations (7) & (8) are similar to the differential equations (3) and (4) governing the mechanical system.

Block Diagram Reduction:

- A control system may consist of a number of components. In control system engineering to show the function performed by each component commonly used is called Block diagram.
- The basic elements of a block diagram are
  1. Block
  2. Summing point
  3. Branch point

1. Block:-

- In a block diagram all the system variables are linked to eachother with the help of fundamental blocks.
- A functional block is a simply block and is a symbol for the mathematical operation on the input signal to the block that produces an output.



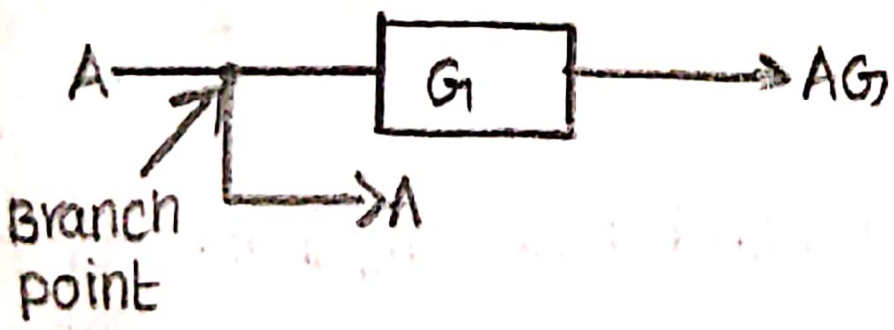
2. Summing point:-

- These are used to add two or more signals in the system. the plus (or) minus sign at each arrow have indicates whether the signal is to be added or to be subtracted.



3. Branch point :-

- Branch point is a point form which the signal from block goes concurrently to other blocks or summing points used to carry a practical signal to any other block or to any summing point.

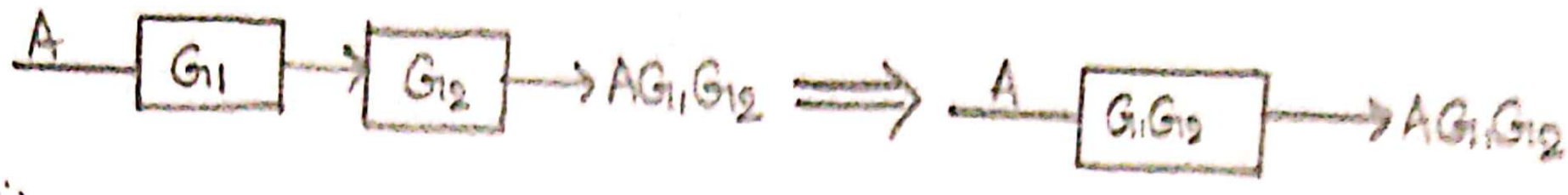


Rules:-

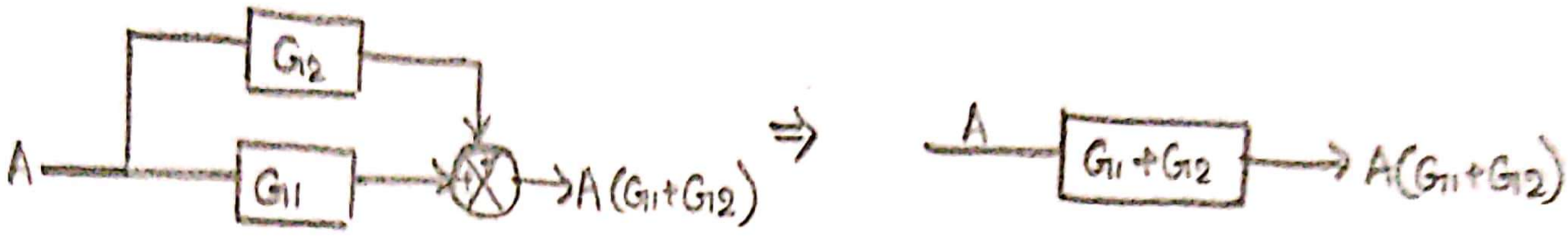
1. Block diagram can be reduced to find the overall transfer function of the system.
2. A control system can be represented diagramtically or pictoxial by block diagram
3. the differential equations governing the system are used to construct the block diagram.

4. The following rules can be used for block diagram reduction.

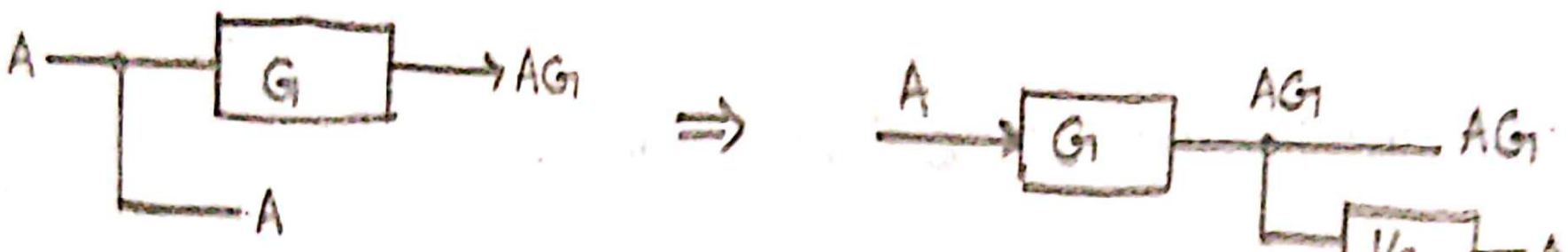
(i) Combining blocks are cascaded.



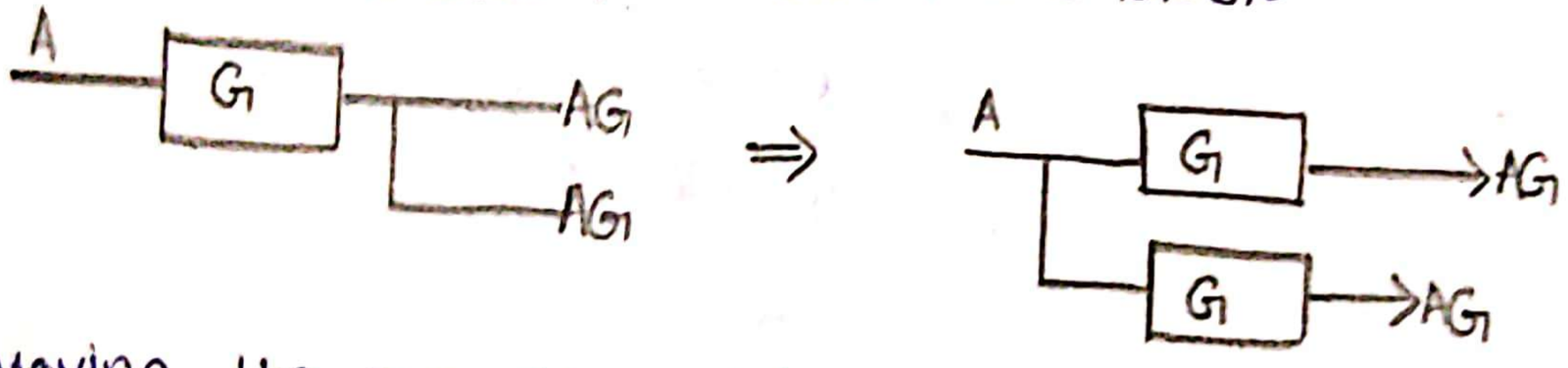
(ii) Combining blocks are parallel.



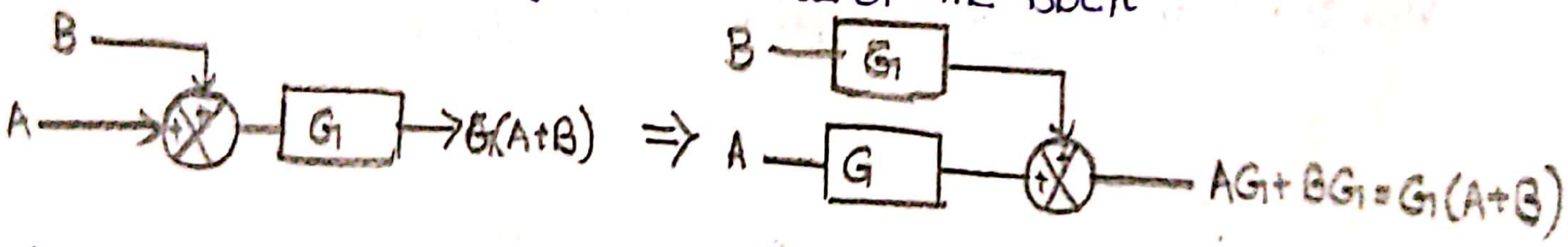
(iii) Moving the branch point ahead of block



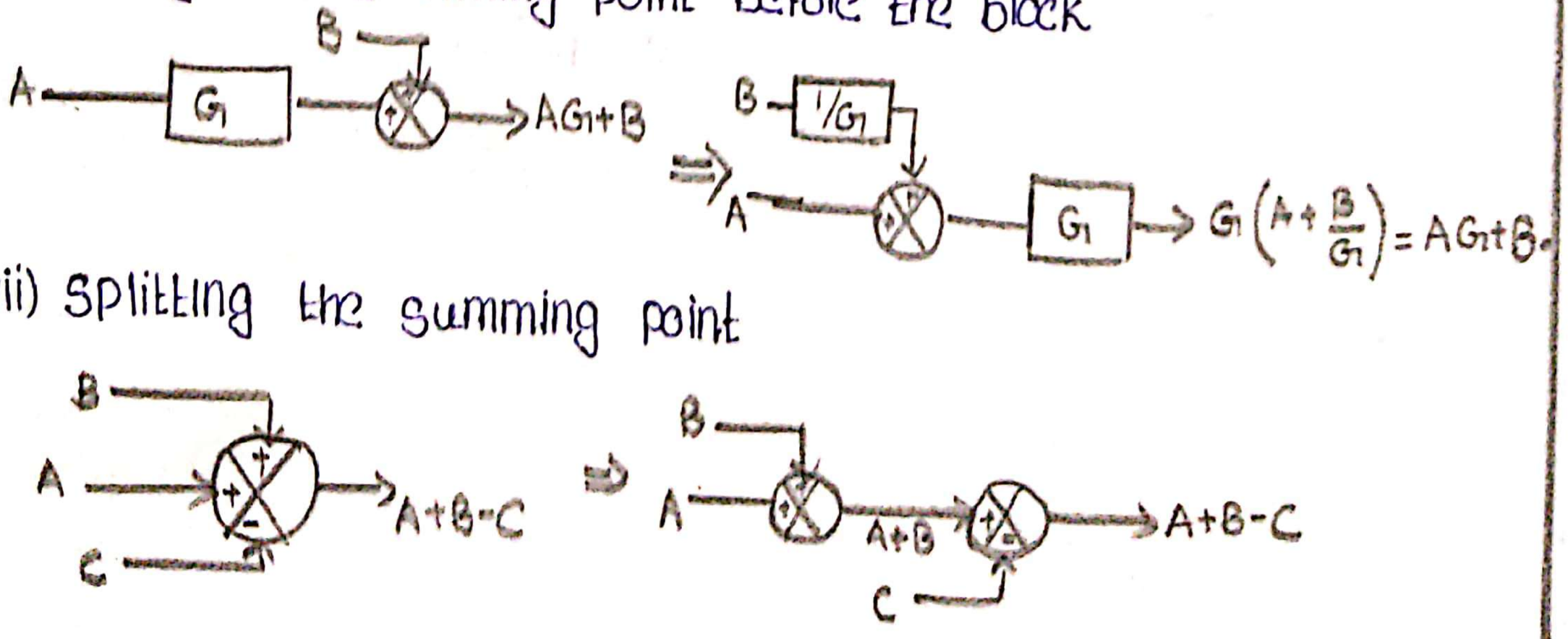
(iv) Moving the branch point before the block



(v) Moving the summing point ahead of the bbck

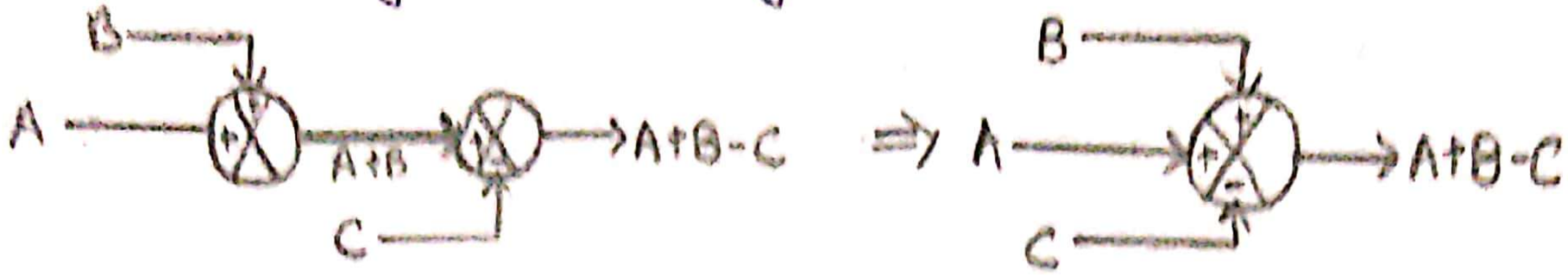


(vi) Moving the summing point before the block

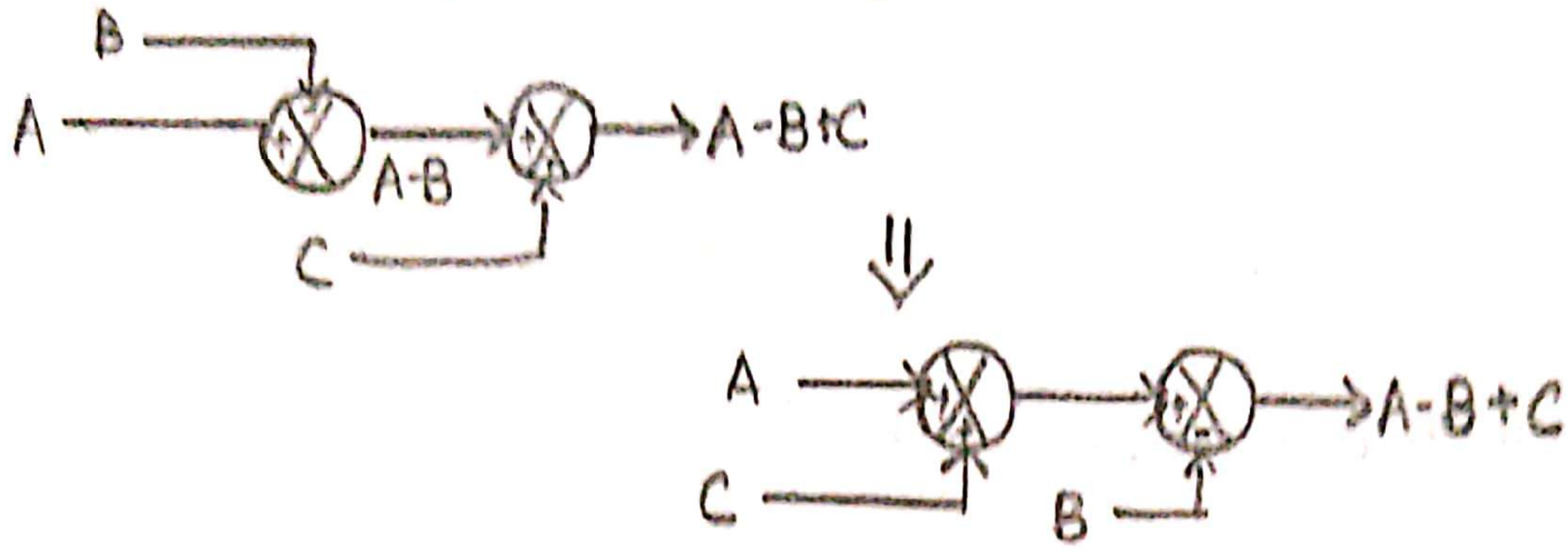


(vii) Splitting the summing point

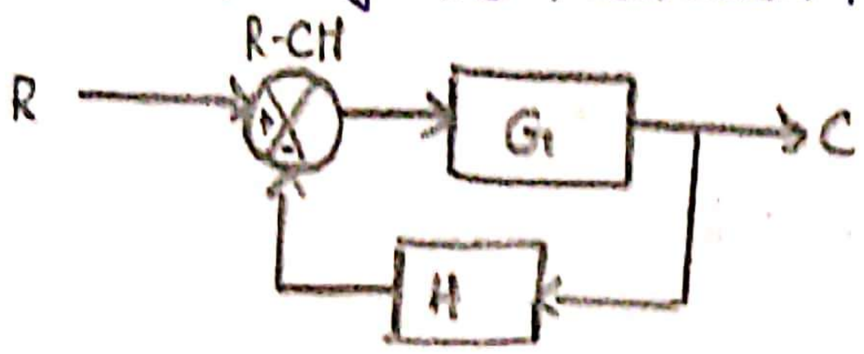
(viii) Combining the summing point



(ix) Interchanging the summing point



(x) Eliminating the feedback path (-ve feedback)



$C = (R-CH)G_1$

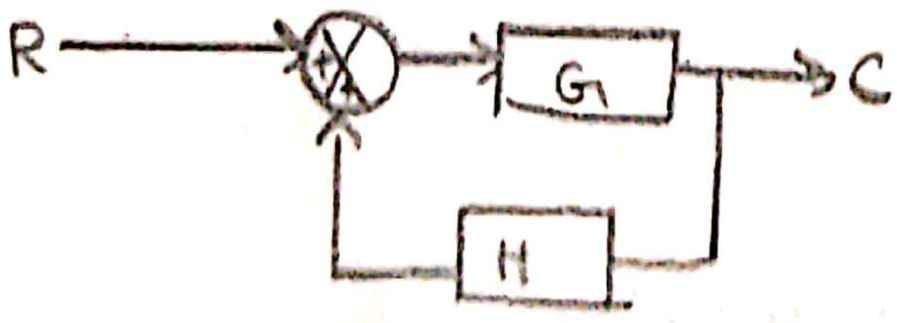
$C = RG_1 - CHG_1$

$C + CHG_1 = RG_1$

$C(1+HG_1) = RG_1$

$\frac{C}{R} = \frac{G_1}{1+G_1H}$.

(xi) Eliminating the feedback path (+ve feedback)



$C = (R+CH)G_1$

$C = RG_1 + CHG_1$

$C - CHG_1 = RG_1$

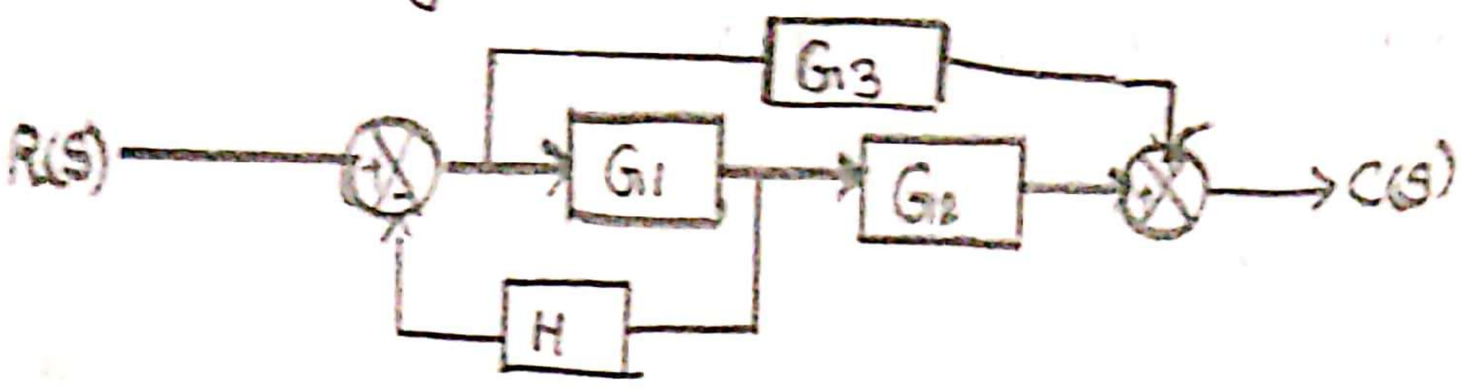
$C(1-HG_1) = RG_1$

$\frac{C}{R} = \frac{G_1}{1-G_1H}$.

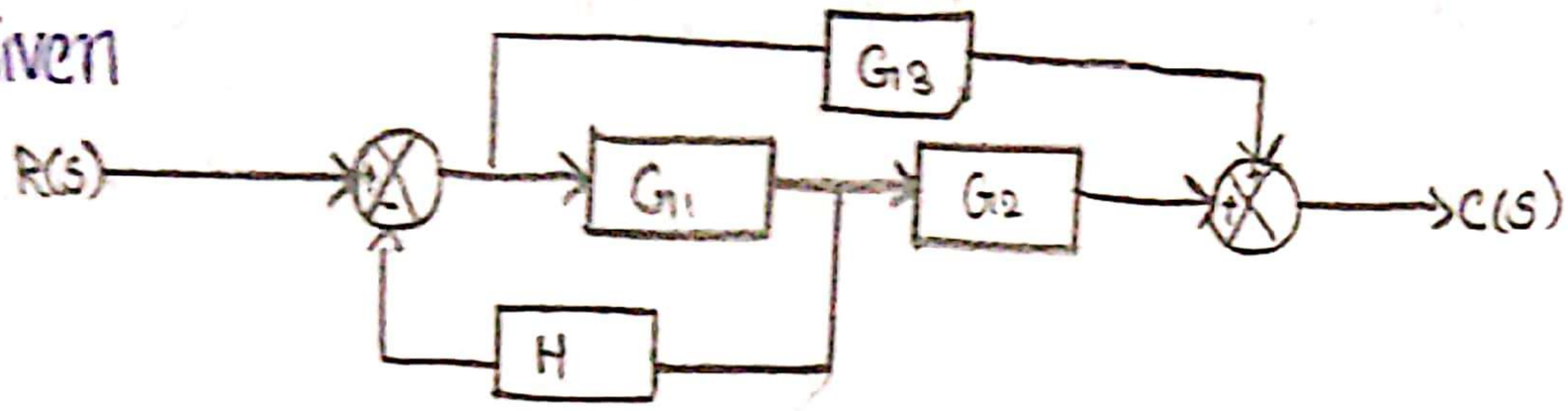
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Problems:-

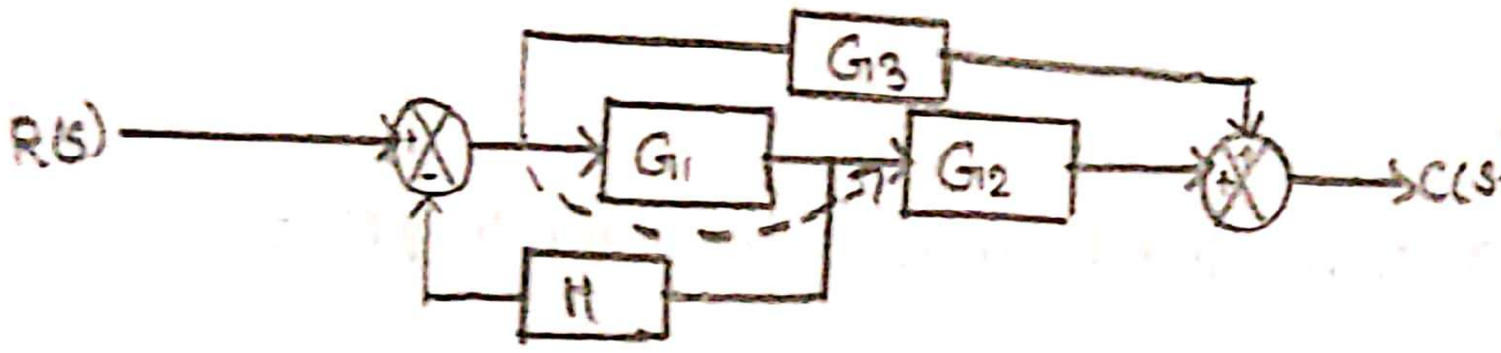
1. Reduce the block diagram shown in the below figure at point C/R.



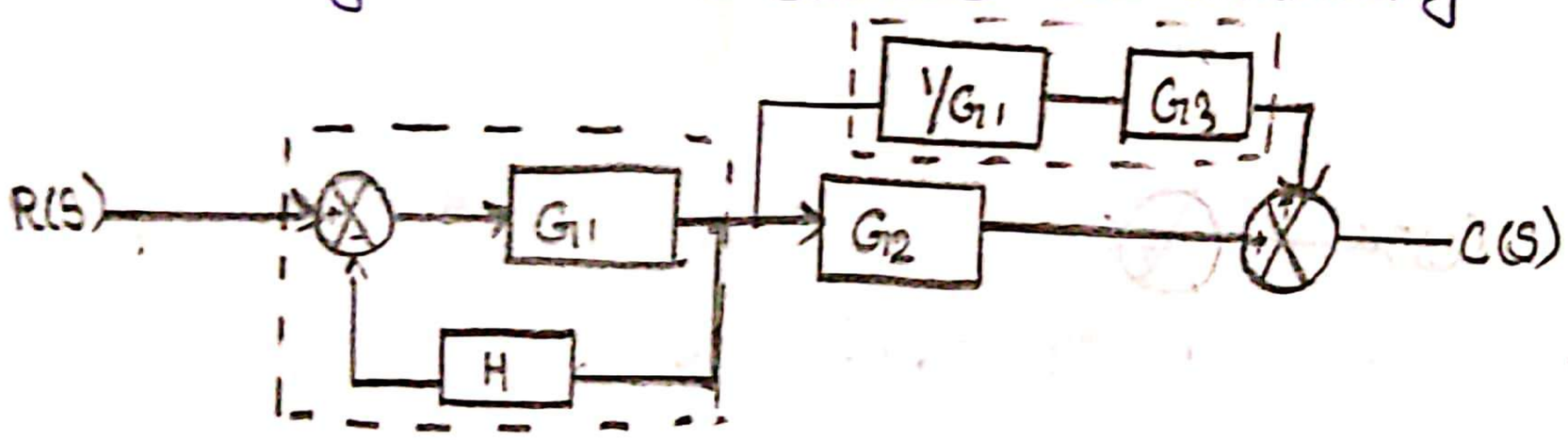
Sol: Given



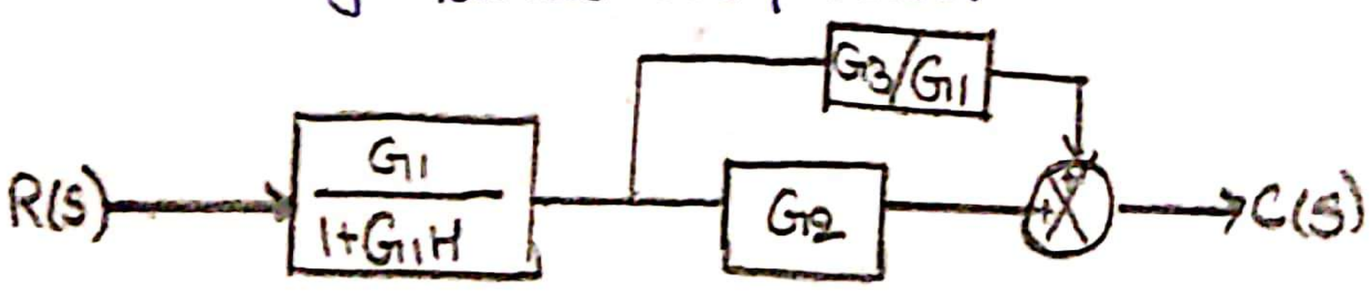
Step-1: Moving the branch point ahead of block



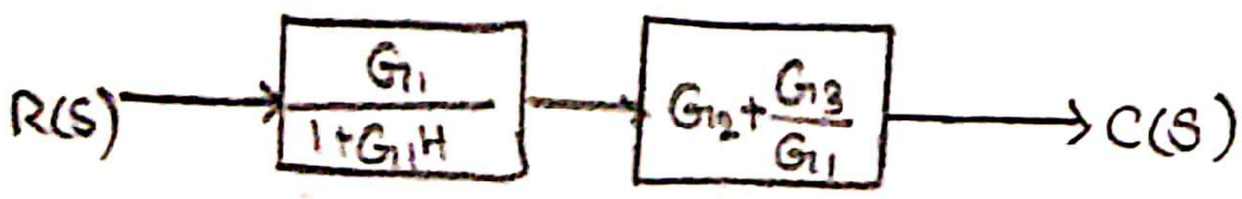
Step-2:- combining blocks are cascaded and eliminating the feedback path.



Step-3:- Combining blocks are parallel



Step-4:- combing blocks are cascaded



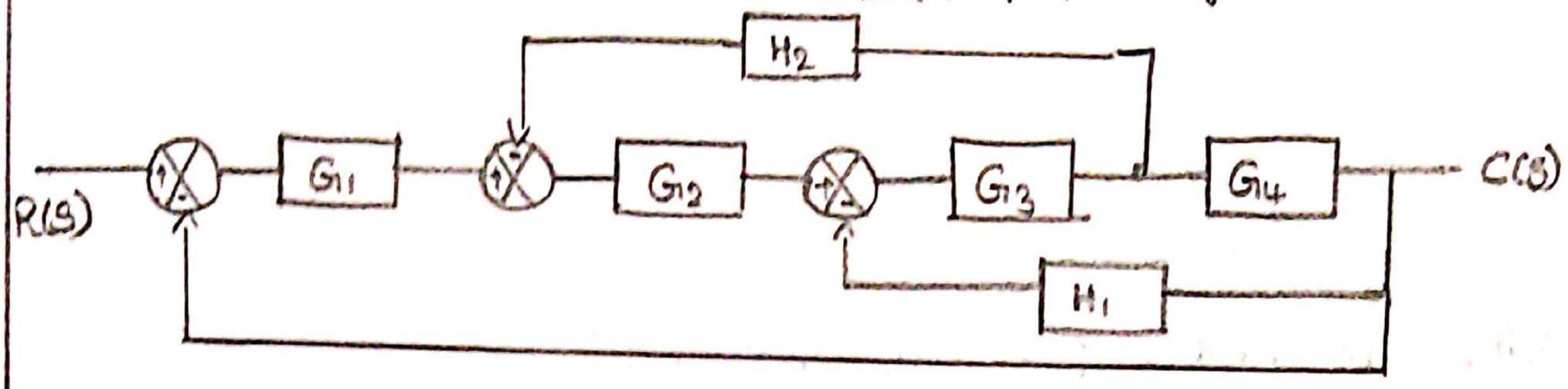
$$\therefore \quad \frac{C}{R} = \left(\frac{G_1}{1+G_1H}\right)\left(\frac{G_1G_2+G_3}{G_1}\right)$$

$$\frac{C}{R} = \frac{G_1G_2 + G_3}{1 + G_1H}$$

Result : The overall transfer function is given as

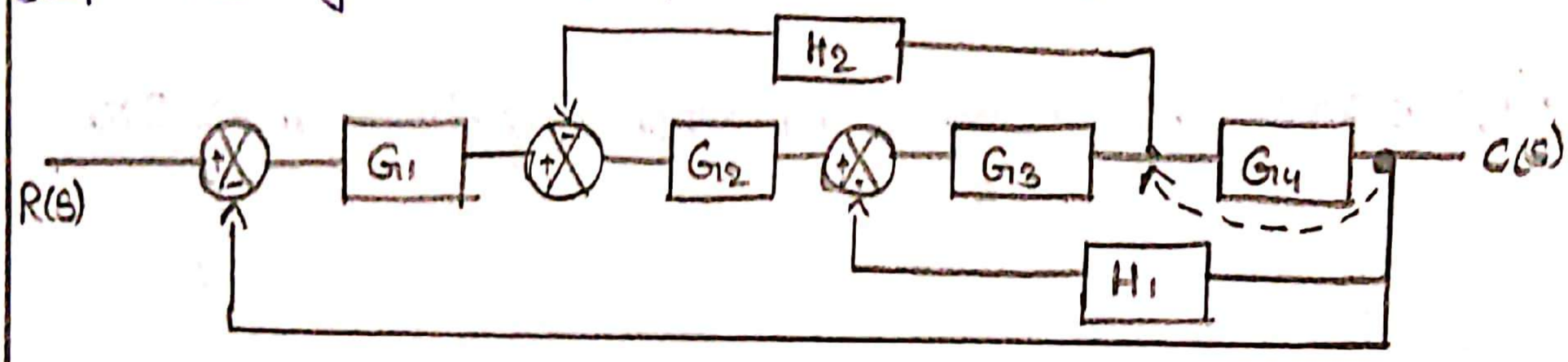
$$\frac{C(s)}{R(s)} = \frac{G_1G_2 + G_3}{1 + G_1H}.$$

2. Determine the overall function C(s)/R(s) of the system in given figure

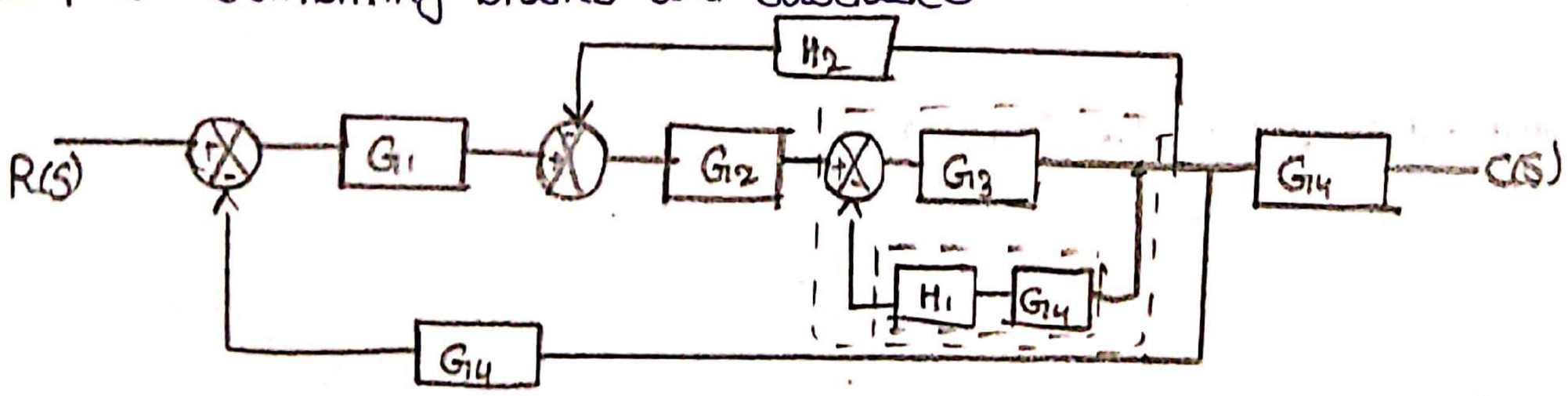


Sol:- Given, From the figure

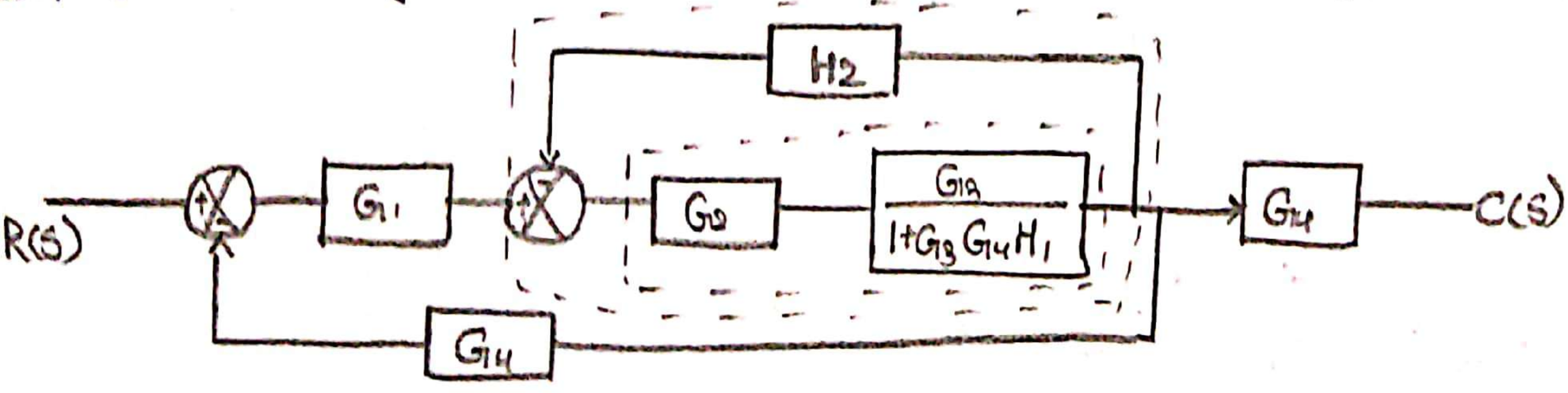
Step-1: Moving the branch point and rearrange the branch point



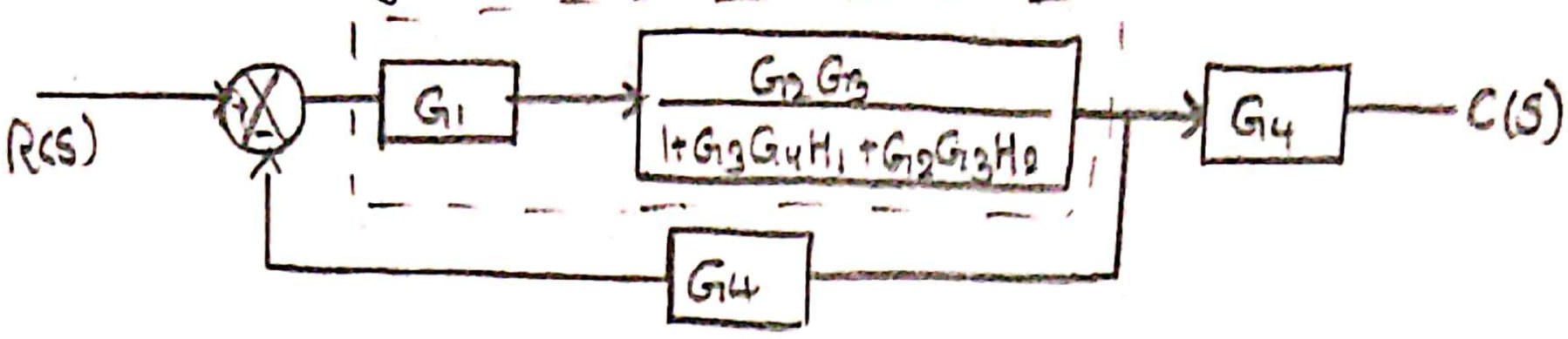
Step-2 :- combining blocks are cascaded



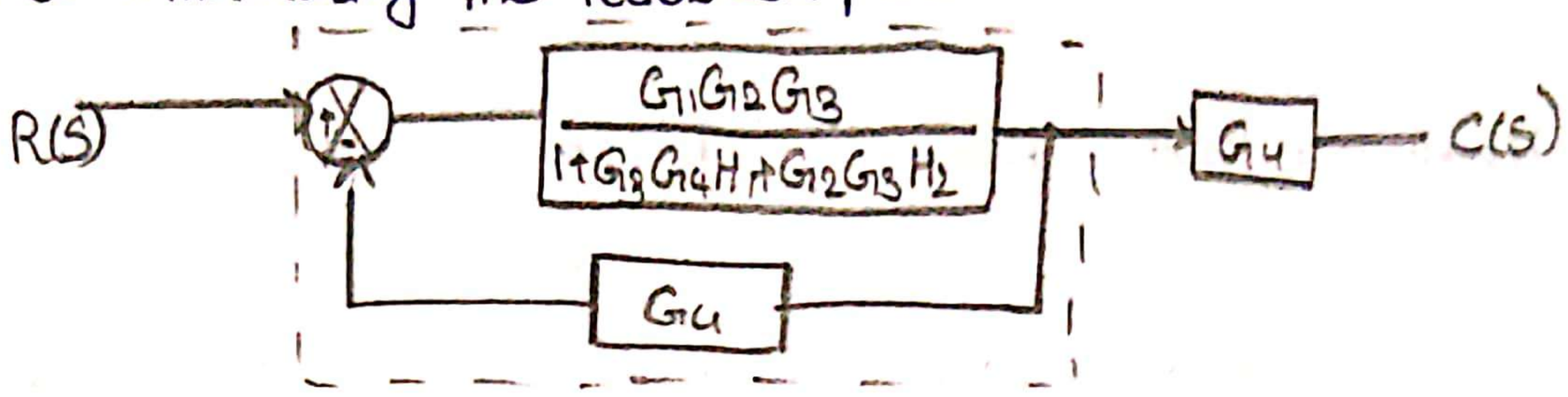
Step-3 :- combining blocks are cascaded and eliminating the feedback



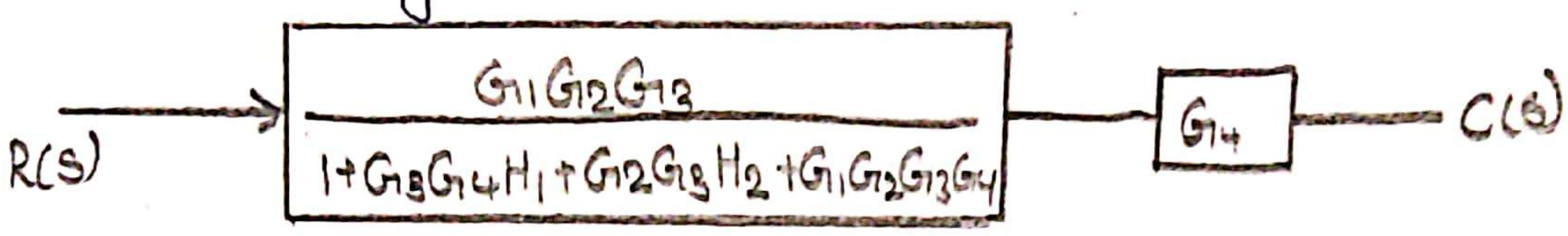
Step-4 :- combining blocks are cascaded



Step-5 :- Eliminating the feedback path



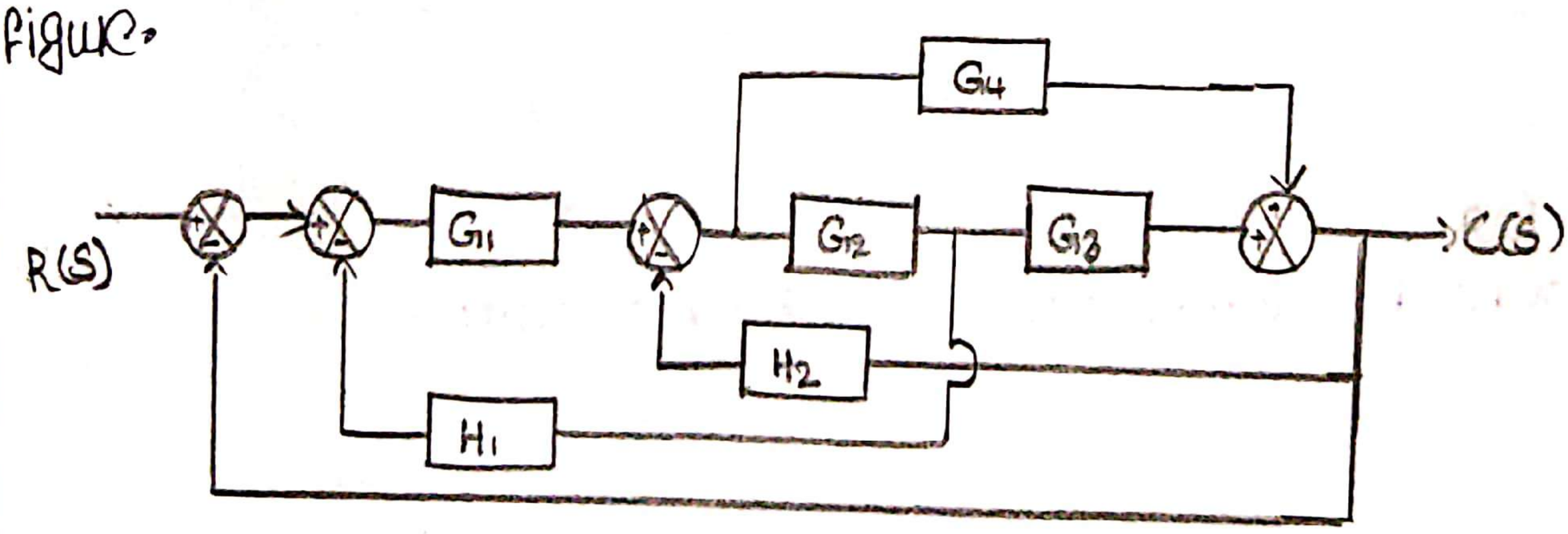
Step-6 :- combining blocks are cascaded



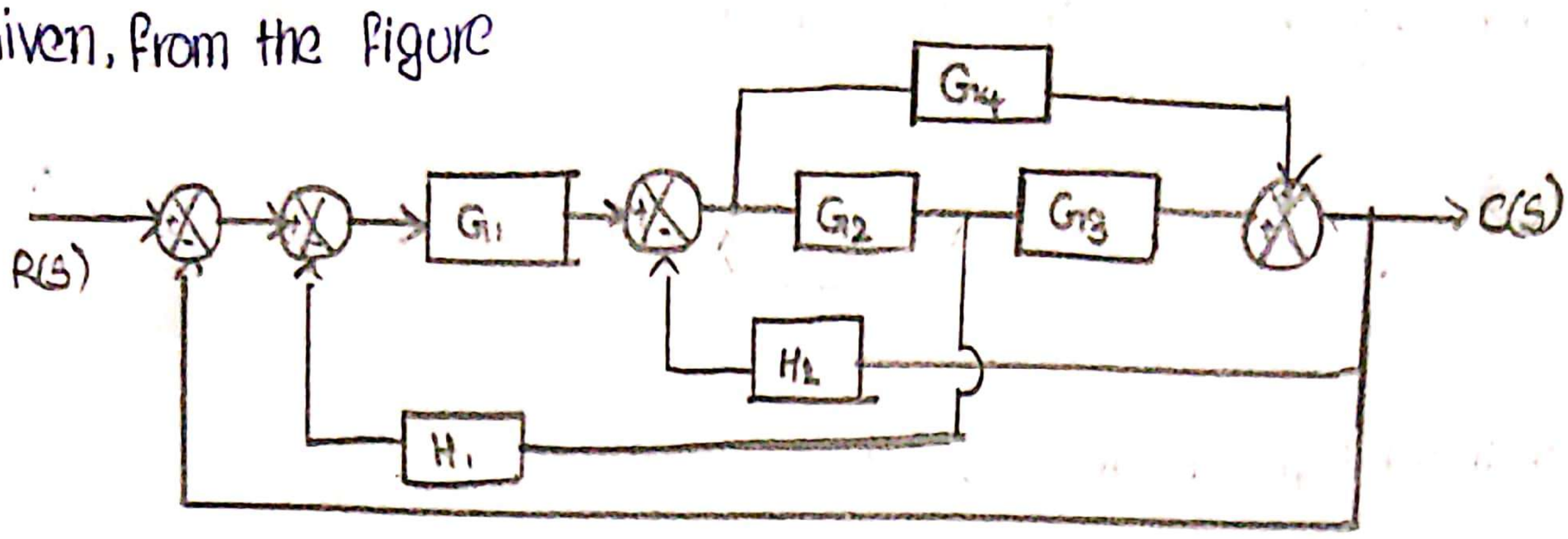
$$\frac{C(S)}{R(S)} = \left(\frac{G_1G_2G_3}{1+G_2G_3H_2+G_3G_4H_1+G_1G_2G_3G_4}\right)G_4$$

$$\therefore \frac{C(S)}{R(S)} = \frac{G_1G_2G_3G_4}{1+G_2G_3H_2+G_3G_4H_1+G_1G_2G_3G_4}.$$

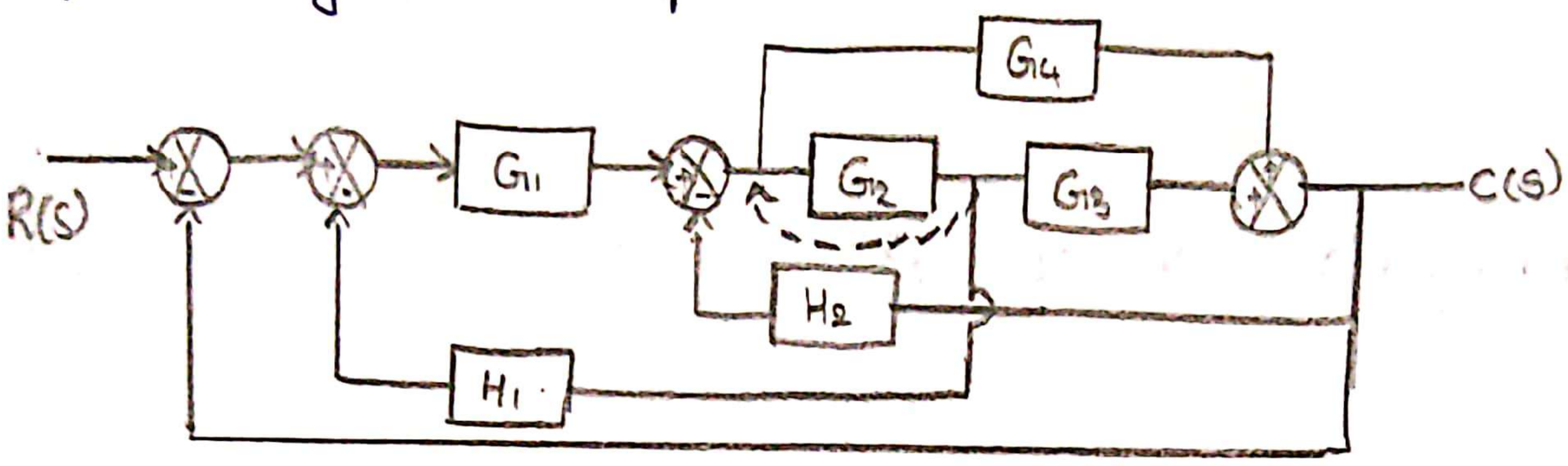
3. Using the block diagram reduction technique, Find the closed loop transfer function of the system whose block diagram is shown in figure.



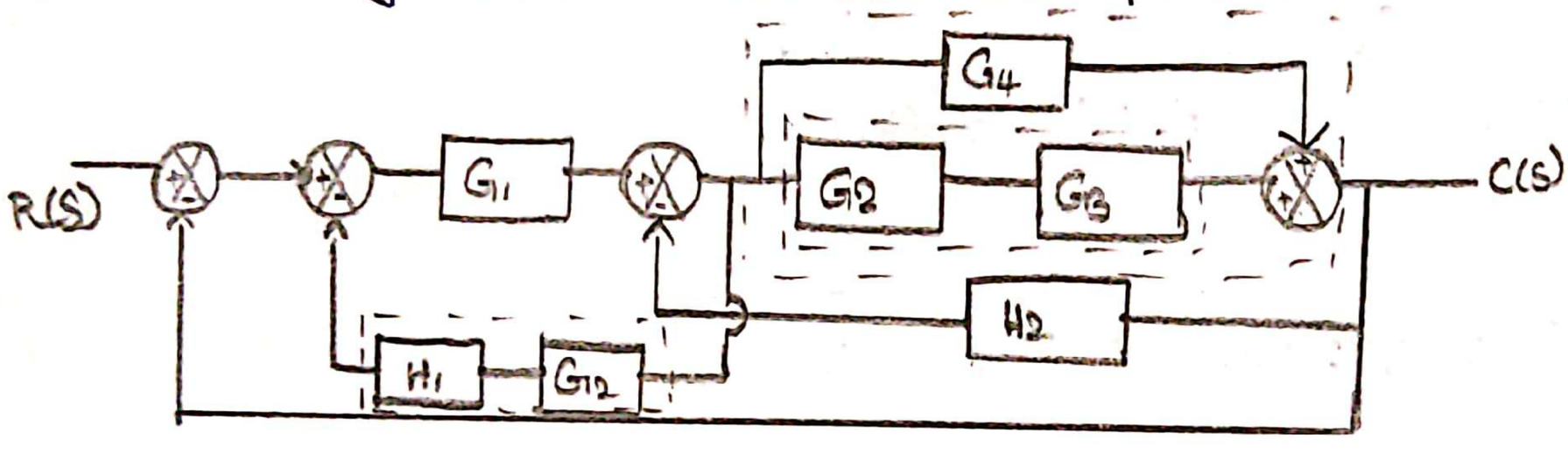
Sol:- Given, from the figure



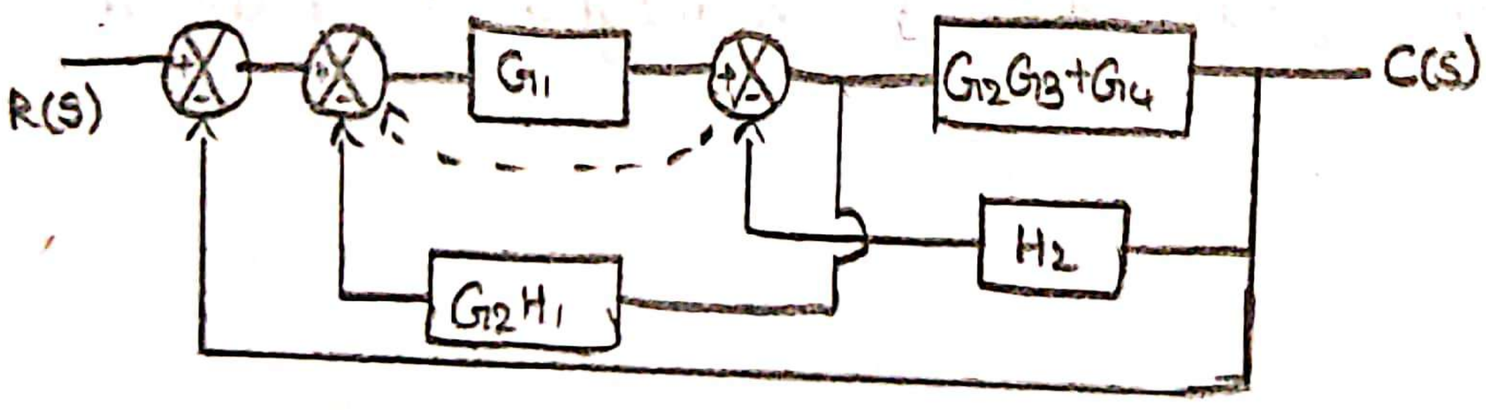
Step-1:- Moving the branch point before the block



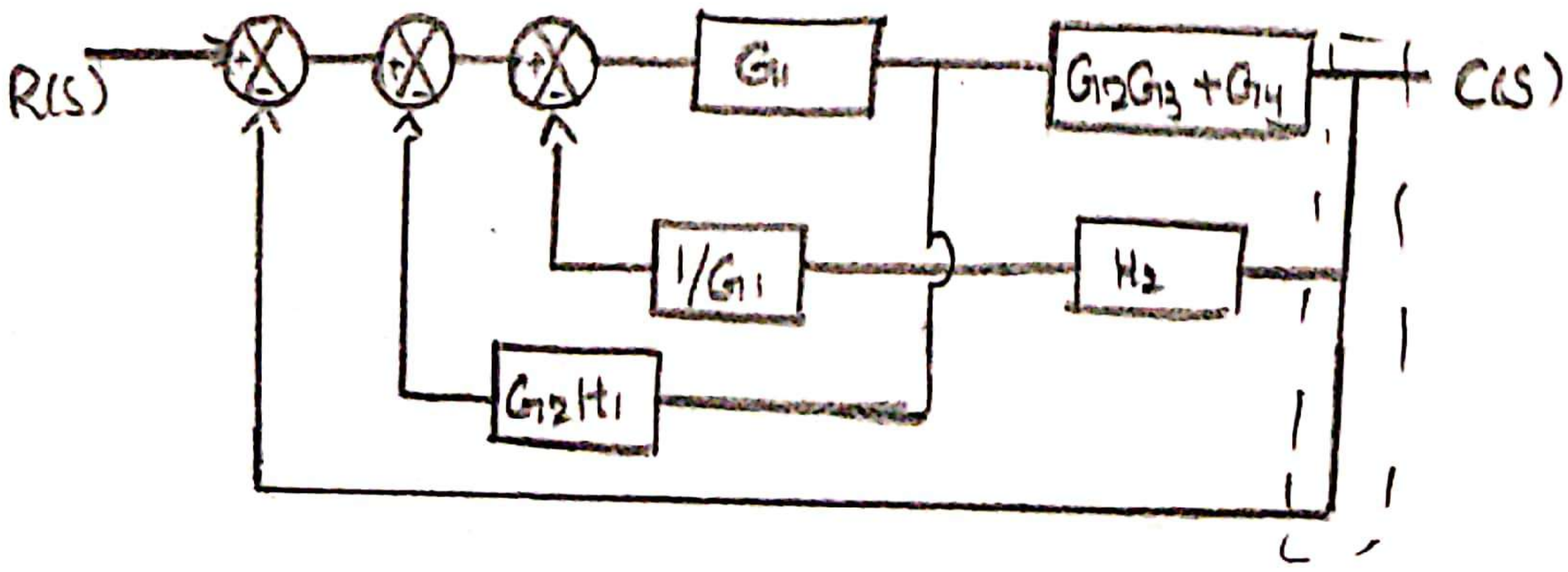
Step-2:- combining blocks are cascaded and parallel



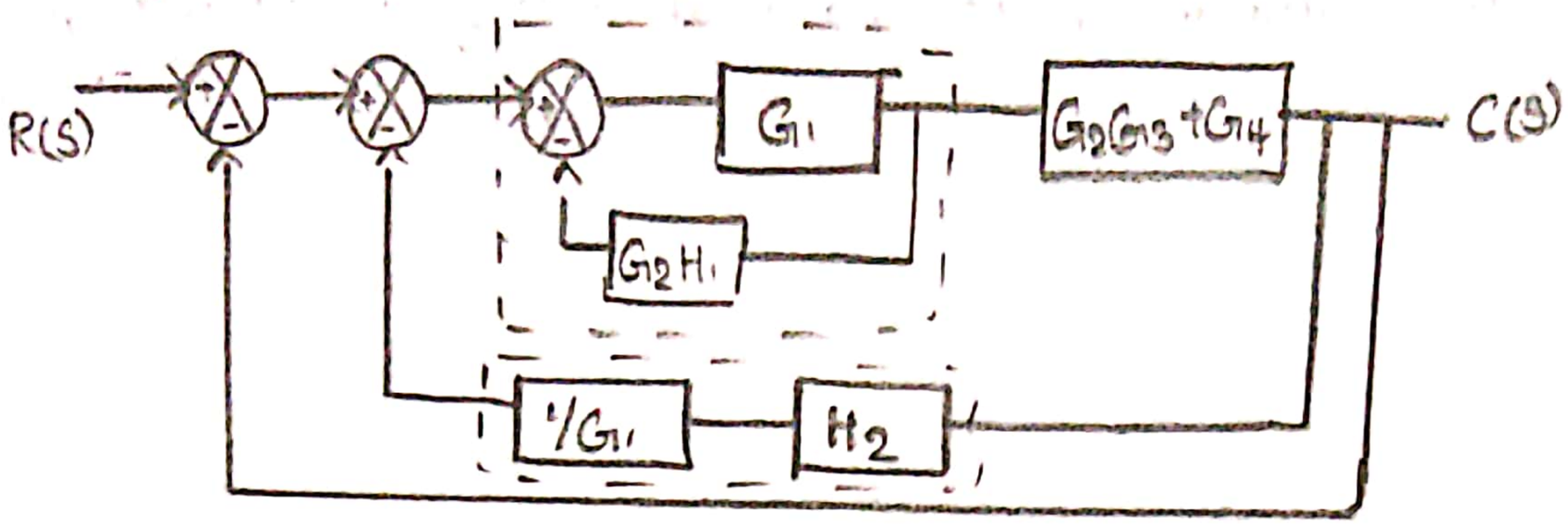
Step-3:- combining blocks are cascaded and parallel



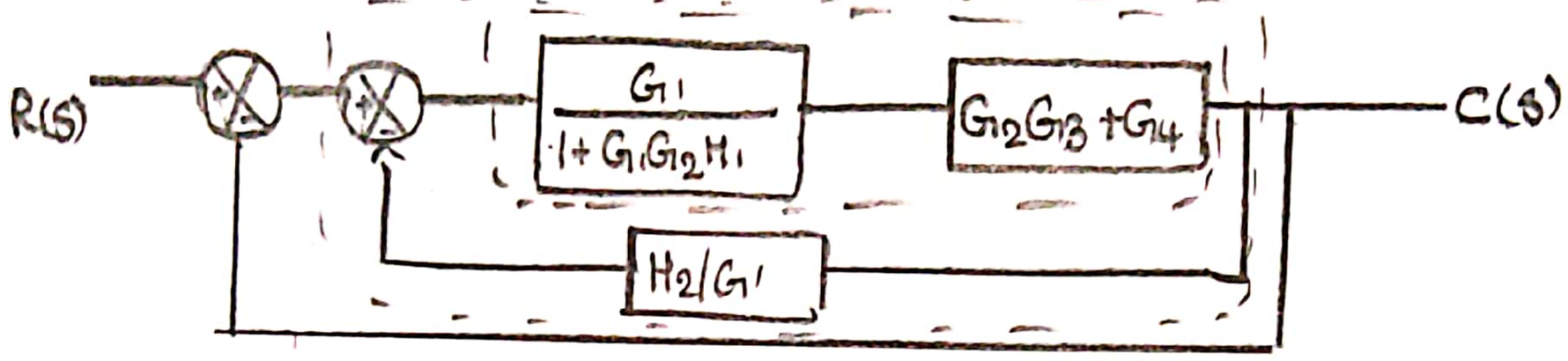
Step-4:- Moving the summing point before the block and



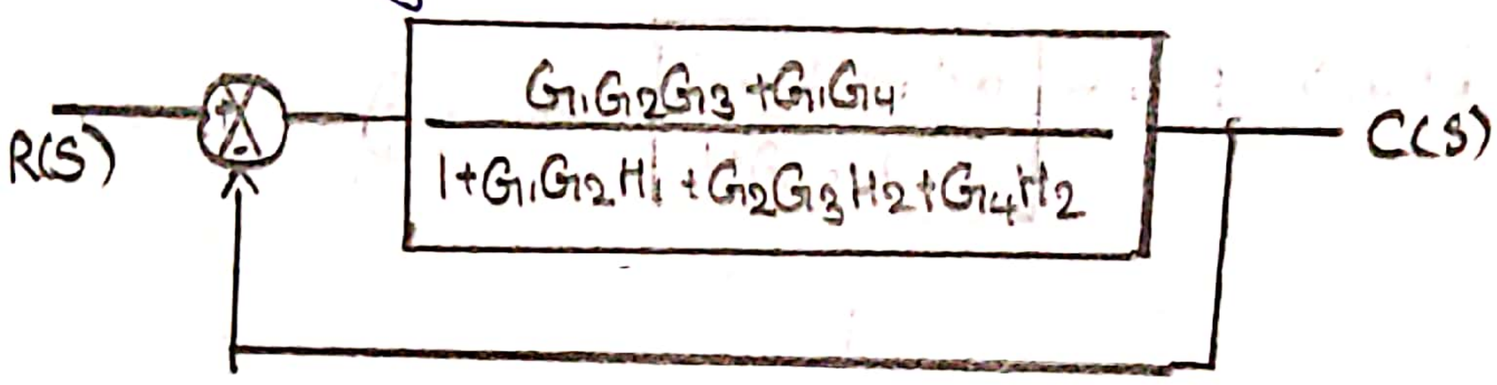
Step-5:- Rearranging the branch point and interchanging the summing point



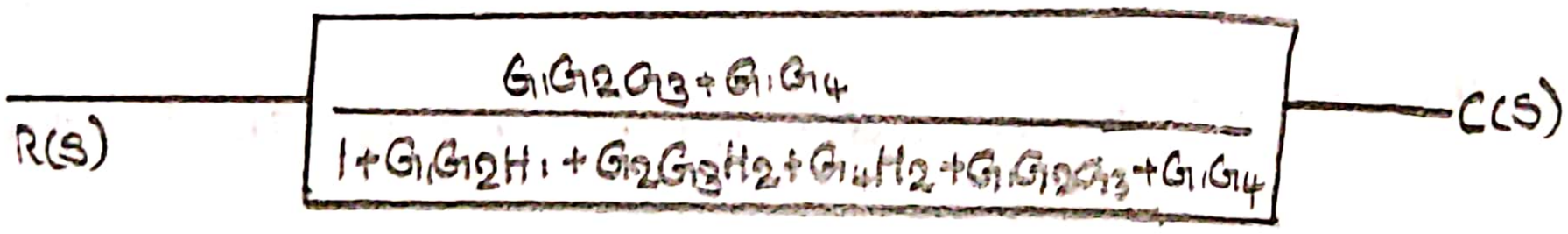
Step-6:- Eliminating the feedback and combining blocks are cascaded



Step-7:- combining blocks are cascaded and eliminating the feedback

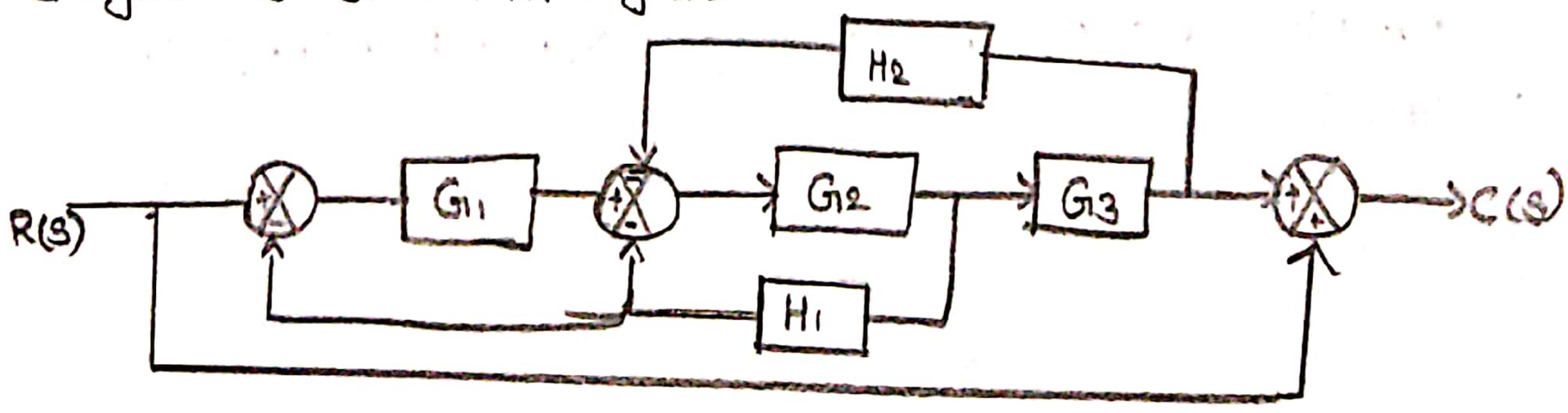


Step-8:- Eliminating the feedback



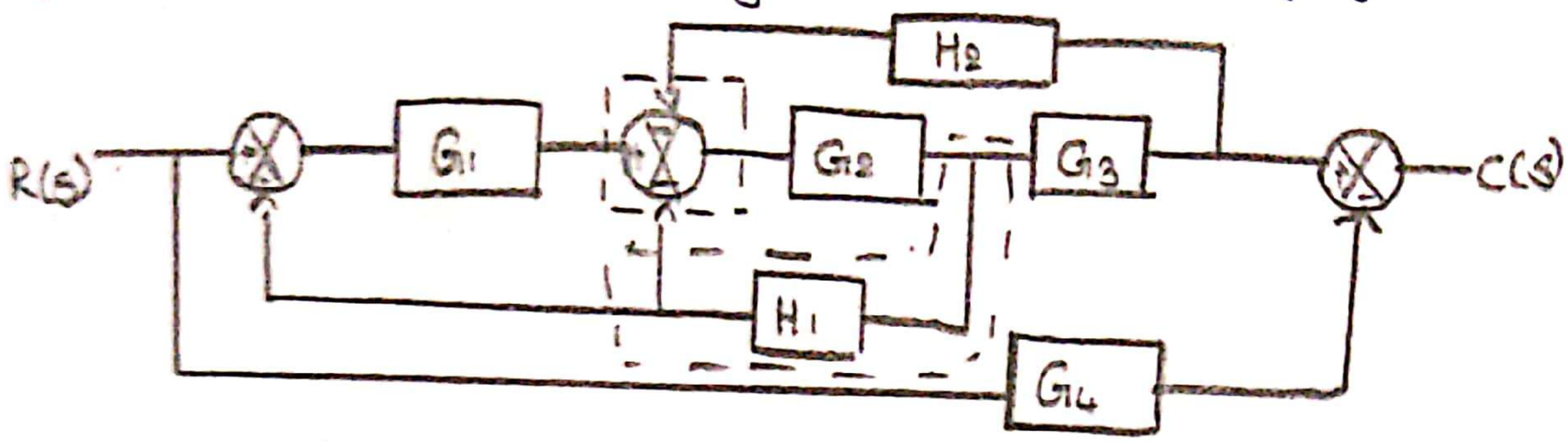
$$\therefore \frac{C(S)}{R(S)} = \frac{G_1G_2G_3 + G_1G_4}{1+G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_2G_3 + G_1G_4}$$

4. obtain the closed loop transfer function C(S)/R(S) of the system whose block diagram is shown in figure.

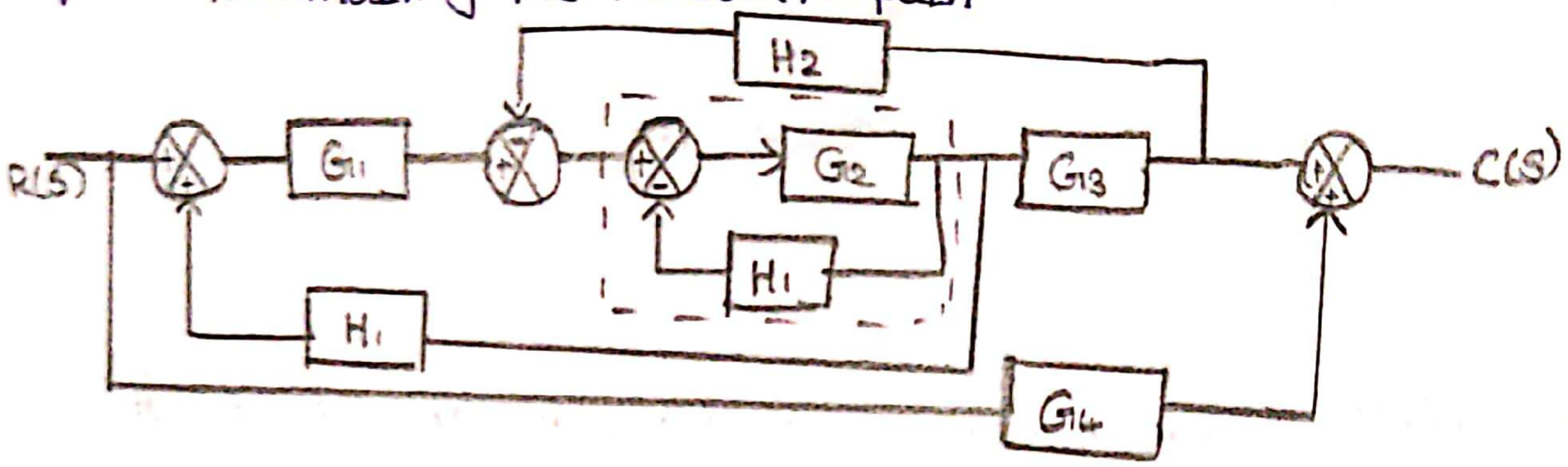


Sol: Given ; From the figure

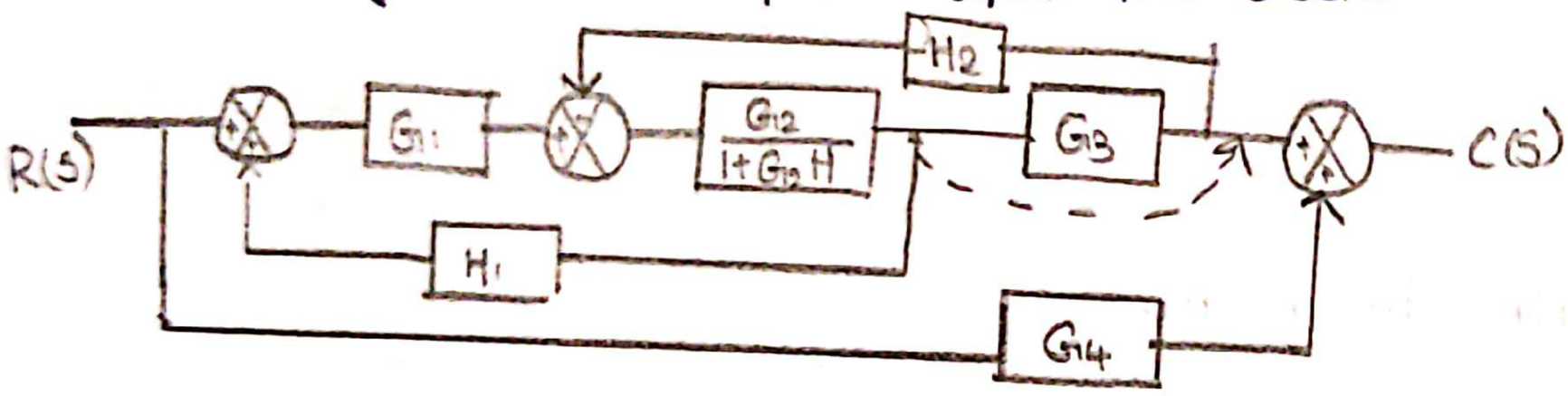
Step-1 :- splitting the summing point and rearranging the branch point



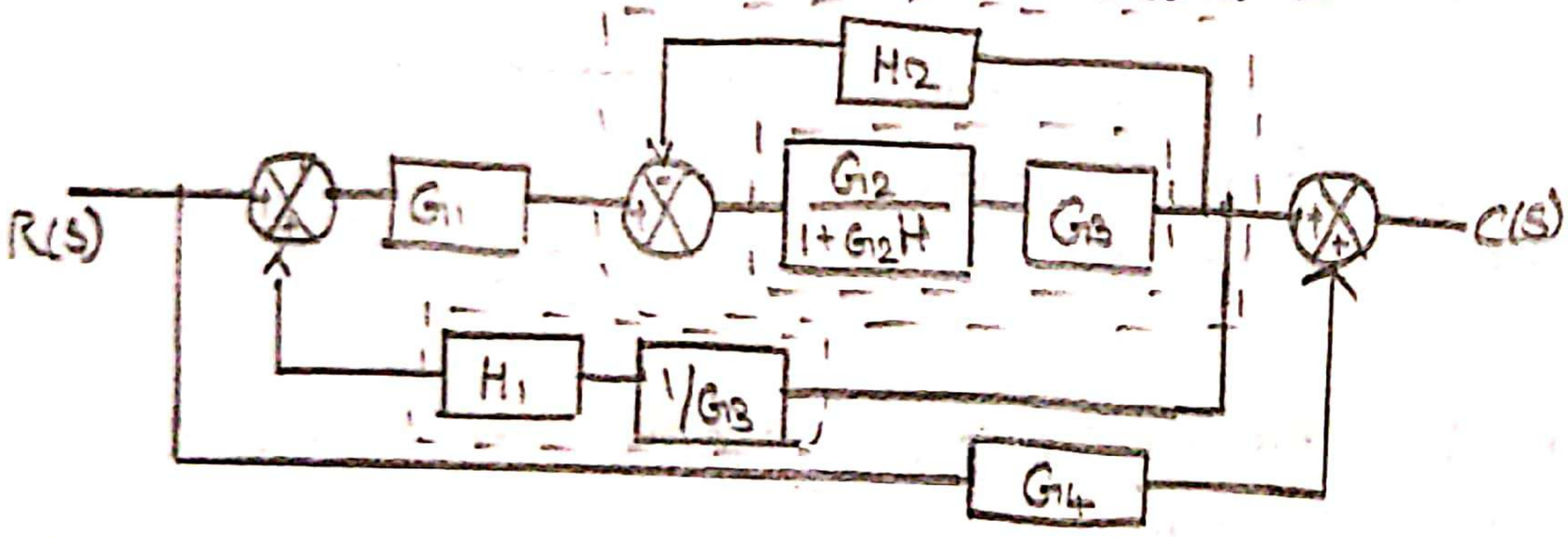
Step-2 :- Eliminating the Feedback path



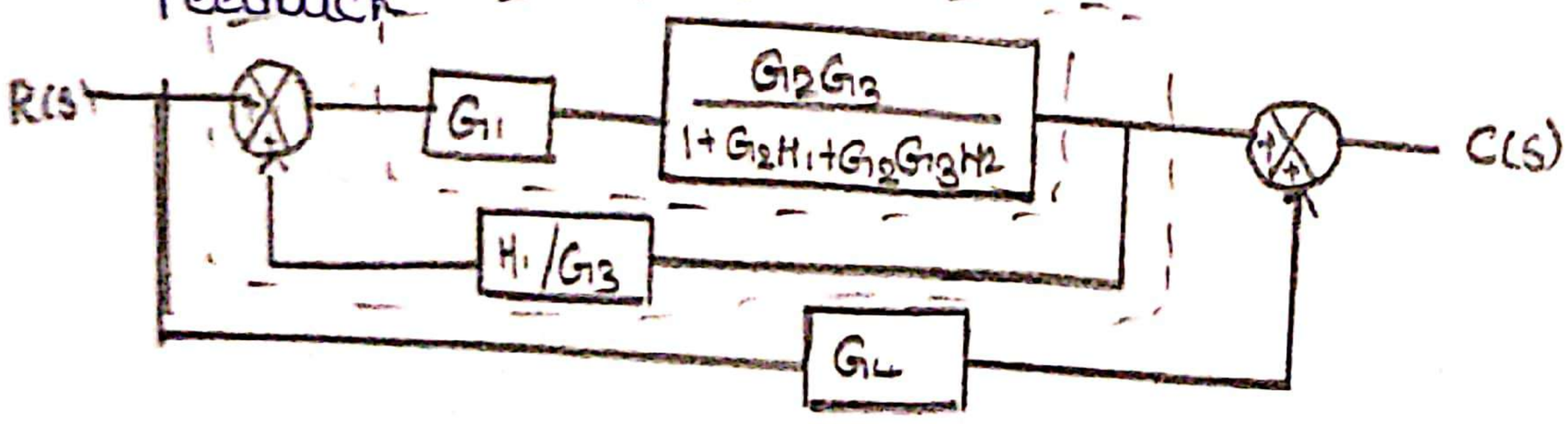
Step-3 : Moving the branch point after the block



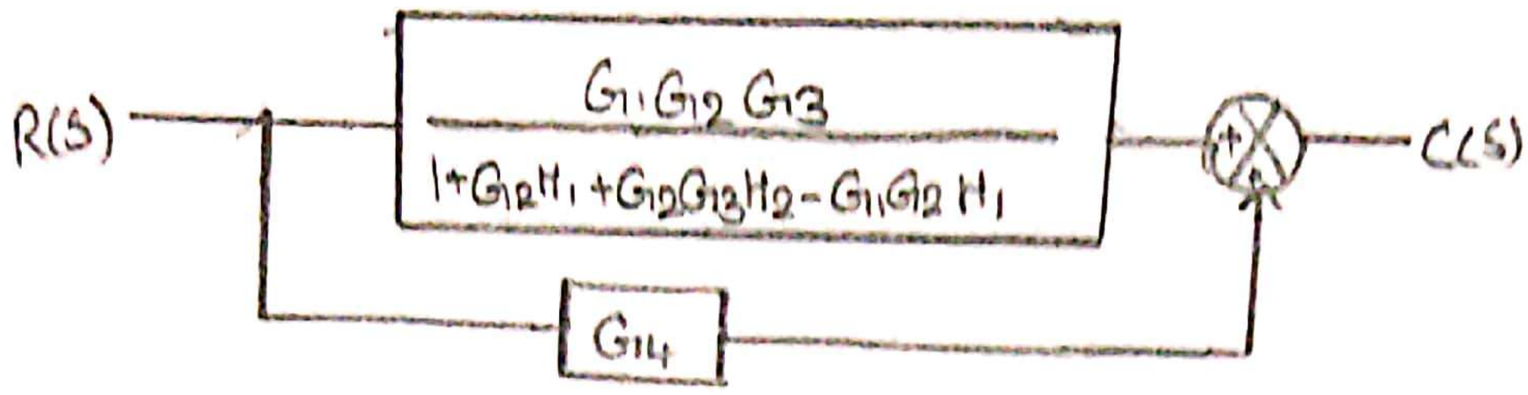
Step-4 :- combining blocks are cascaded and Eliminating the Feedback



Step-5 : combining blocks are cascaded and eliminating the Feedback

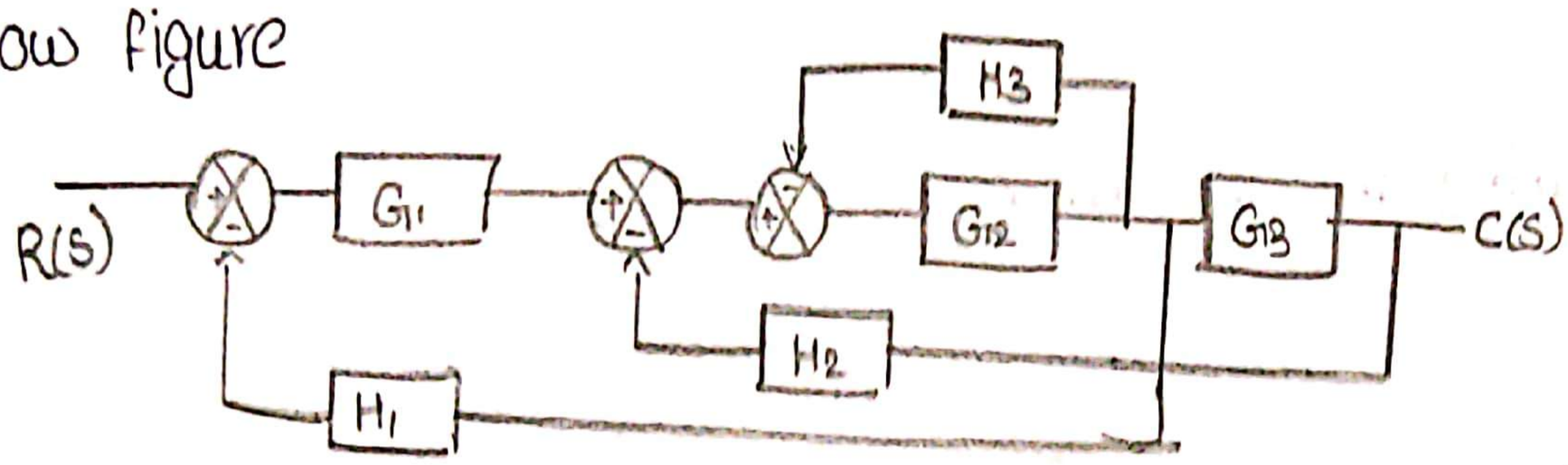


Step-6:- Eliminating the feedback path



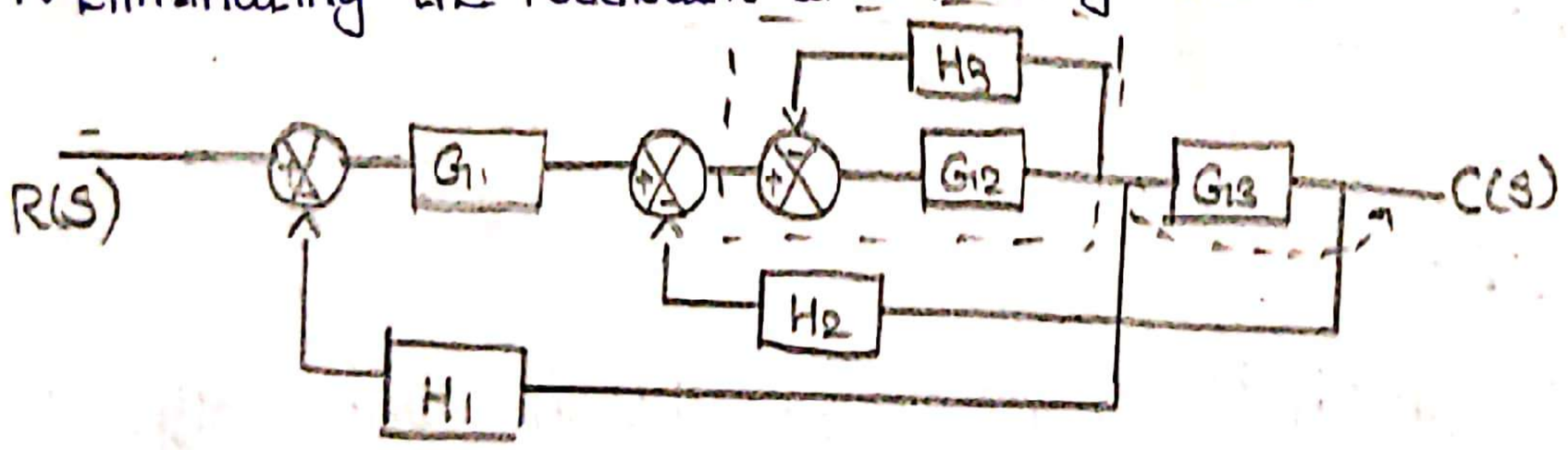
$$\therefore \frac{C(S)}{R(S)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1} + G_4 .$$

5. Determine the overall function C(S)/R(S) of the system given in below figure

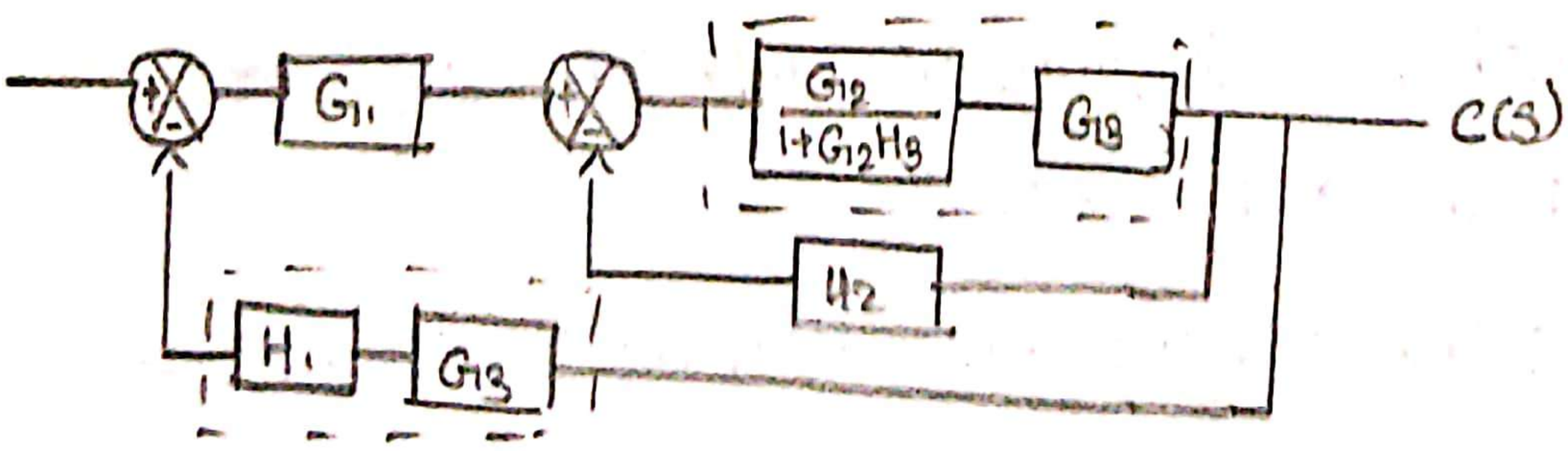


Sol: Given, From the figure

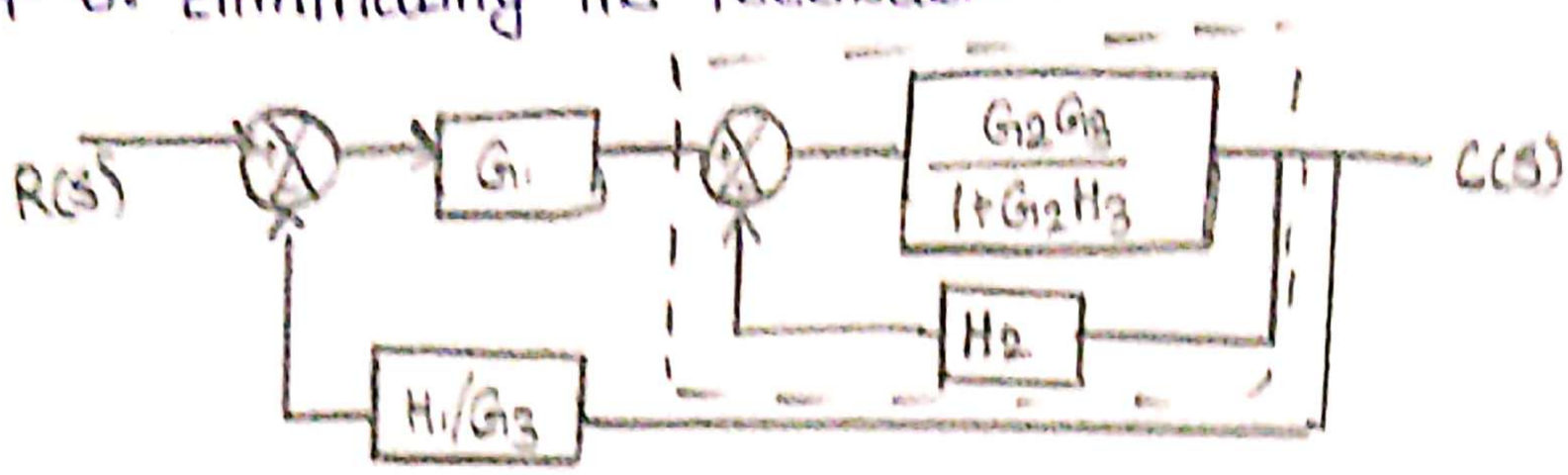
Step-1: Eliminating the feedback and moving the branch point after block



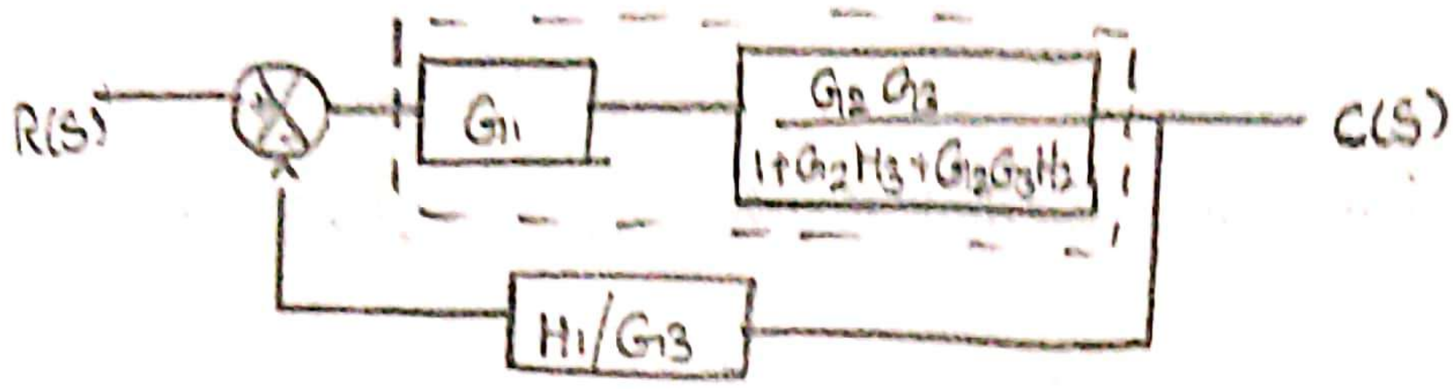
Step-2:- ~~case~~ combining blocks are cascaded



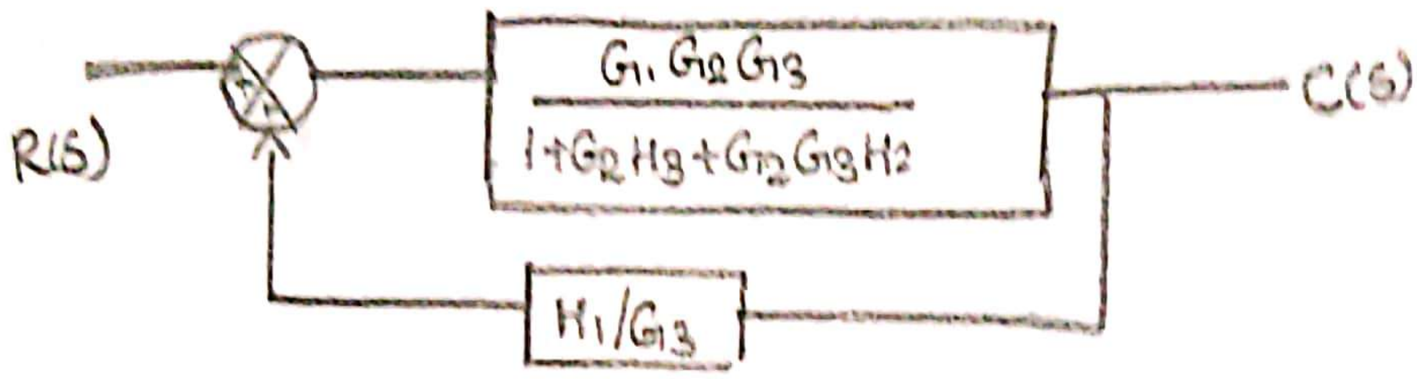
Step-3: Eliminating the feedback



Step-4 :- combining blocks are cascaded



step-5:- Eliminating the feedback



$$\therefore \frac{C(S)}{R(S)} = \frac{G_1 G_2 G_3}{1+G_2H_3+G_2G_3H_2+G_1G_2H_1}$$

Signal Flow Graph:-

- The signal Flow graph is used to represent the control system graphically and it is developed by S.J. Mason.
- A signal Flow graph is a diagram that represents a set of simultaneous linear algebraic equations.
- By taking laplace transform, the time domain differential equations governing a control system to a set of algebraic equations in s-domain.

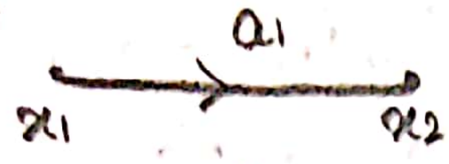
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Terms used in signal flow graph.

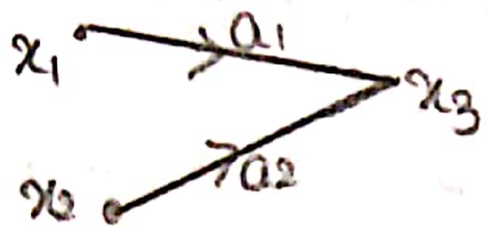
1. Node :- It is a point representating a variable or a signal.
2. Branch :- A branch is directed line segment joining two nodes the arrow indicates the branch of direction of signal flow.
3. Transmittance :- The gain acquired by the signal when it travels from one node to another node is called transmittance. It can be real or complex.
4. Input node(source) :- It is a node that has only outgoing branches.
5. Output node(sink) :- It is a node that has only incoming branches.
6. Mixed node :- It is a node that has both incoming and outgoing branches.
7. path :- A path is a traverse of connected branches in the direction of the branch arrows, the path should not cross a node more than once.
   (a) open path :- A open path starts at a node and ends at another node.
   (b) closed path :- A closed path starts and ends at the same node.
   (c) forward path :- It is a path from input node to output node that does not cross anymore than once.
8. Forward path gains :- It is the product of branch transmittances (gain) of a forward path.
9. Loop gain :- It is the product of transmittance gains of the loop.
10. Individual loop :- It is a closed path starting from node after passing certain part and that does not cross anymore more than once.

11. Non-touching loops:- If the loop doesn't have a common point then they are said to be non touching loops.

Rule-1:-

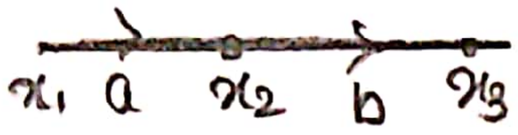


$\Rightarrow x_2 = a_1 x_1$



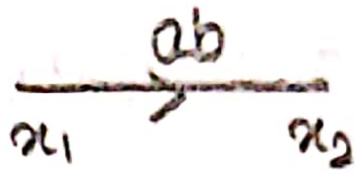
$\Rightarrow x_3 = a_1 x_1 + a_2 x_2$

Rule-2:-



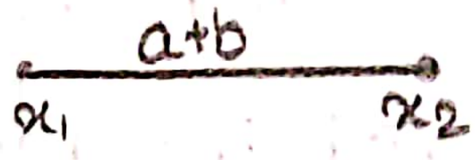
$\Rightarrow x_2 = ax_1$

$x_3 = bx_2 = abx_1$

$\Rightarrow$ 

Rule-3:-



$\Rightarrow$ 

Rule-4:-



$\Rightarrow x_4 = acx_1 + bcx_2$.

properties of signal Flow graph:-

The basic properties of signal Flowgraph are the following:

(i) The algebraic equations which are used to construct signal flow graph must be in the form of cause and effect.

(ii) signal flow graph is applicable to linear systems only.

(iii) A node in the signal flow graph represents a variable or a signal.

(iv) The signal flow graph of the system is not unique, by arranging the system equations different types of signal flow graph can be drawn for a given system.

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Signal Flow graph reduction:-

- The signal flow graph of a system can be reduced either by using the rules of a signal flow graph algebra (or) by using mason's gain formula.
- write equations at every node and then rearrange these equations to get the ratio of output to input.
- The signal flow graph reduction by above method will be time consuming and tedious.
- It has developed a formula called by his name mason's gain formula, which can be directly used to find transfer function of the system.

Mason's gain formula :-

- The mason's gain formula used to determine the transfer function of the system from the signal flow graph.
- Let R(s) be the input to the system and C(s) be the output of the system and Transfer function is

$$T(s) = \frac{C(s)}{R(s)}$$

- Mason's gain formula states that overall gain of the system (Transfer function) as follows :-

$$\text{overall gain } (T) = \frac{1}{\Delta} \sum_k P_k \Delta_k$$

where,

$T = T(s)$ = Transfer function of the system

P_k = Forward path gain of the kth forward path

k = No. of Forward path in signal flow graph

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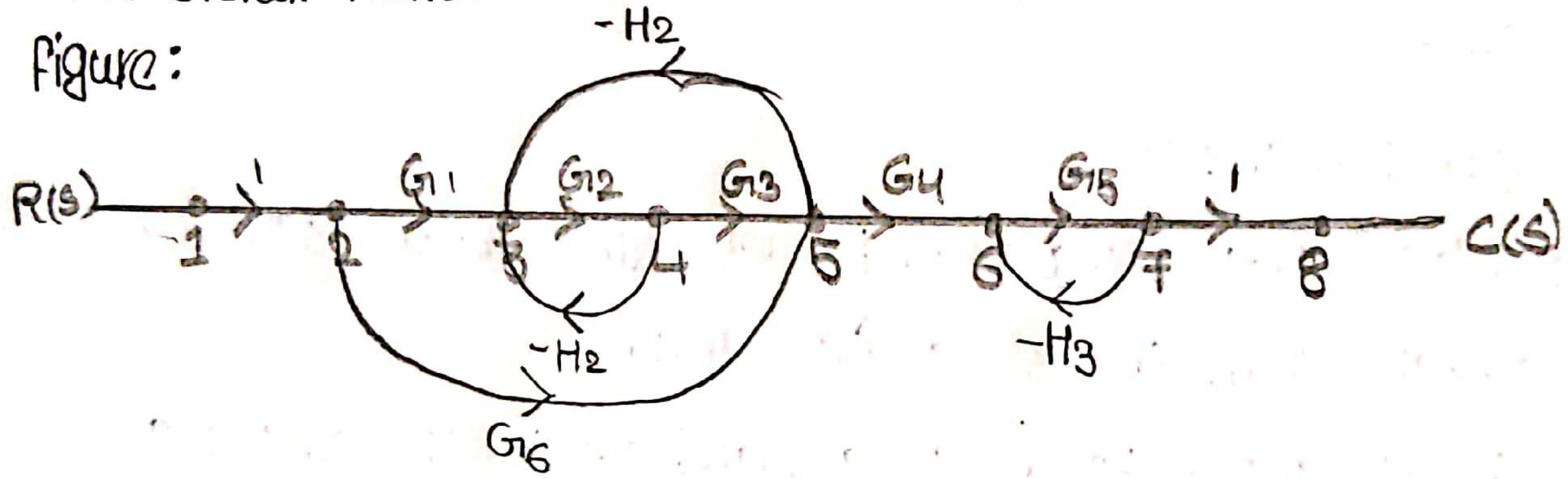
$$\Delta = 1 - \begin{bmatrix}\text{sum of individual}\\ \text{loop gains}\end{bmatrix} + \begin{bmatrix}\text{sum of gain products of}\\ \text{all possible combinations}\\ \text{of two non-touching loops}\end{bmatrix} -$$

$$\begin{bmatrix}\text{sum of gain products of}\\ \text{all possible combinations}\\ \text{of three non-touching loops}\end{bmatrix} + \ldots$$

Δ_K = Δ for that part of the graph which is not touching K^{th} forward path.

Problems :-

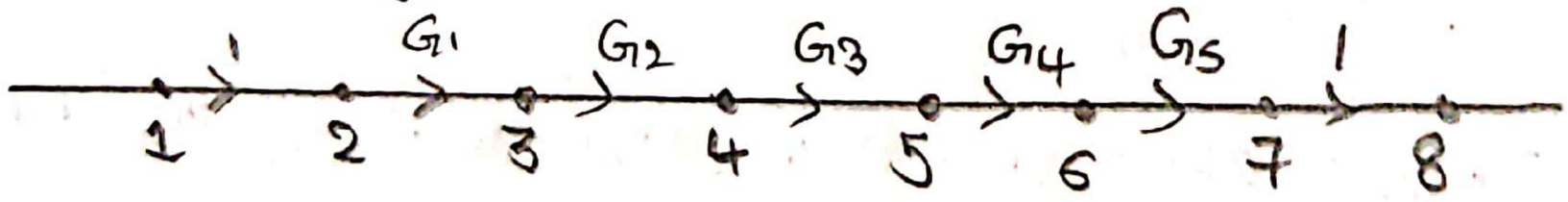
1. Find the overall transfer function whose signal flow graph is shown in figure:



Sol: No. of forward paths (K) = 2

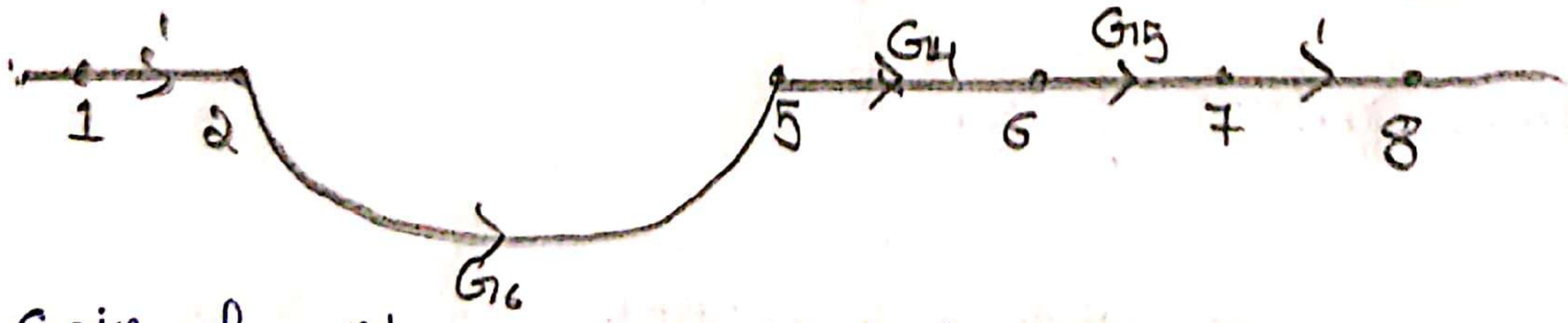
Let forward path gains P_1 and P_2

(i)



Gain of 1st forward path $(P_1) = G_1G_2G_3G_4G_5$

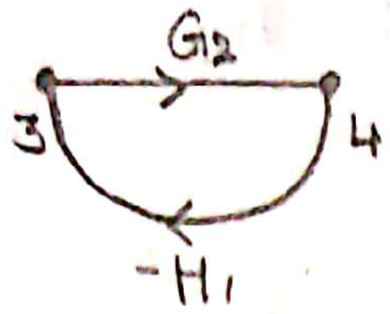
(ii)



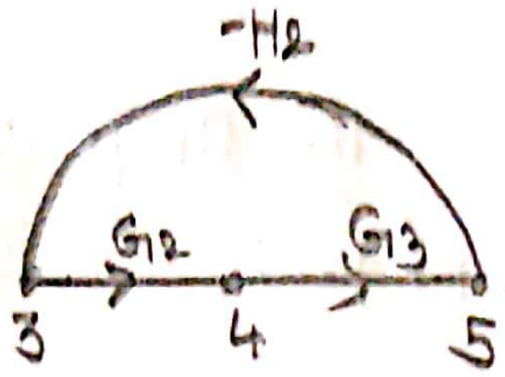
Gain of 2nd forward path $(P_2) = G_4G_5G_6$

Individual loop gains:-

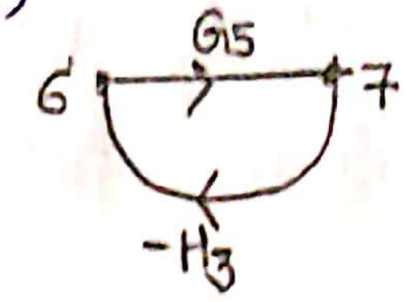
(i)



(ii)



(iii)



(i) Loop of 1st individual loop gain $P_{11} = -G_2H_1$

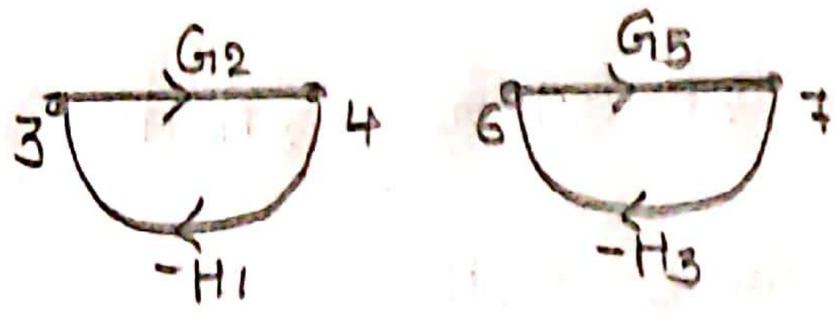
(ii) Loop gain of 2nd individual loop $P_{12} = -G_2G_3H_2$

(iii) Loop gain of 3rd individual loop $P_{13} = -G_5H_3$.

Two non-touching loops:-

There are two combinations of two non-touching loops. Let the gain products of two non-touching loops be P_{12} and P_{22}.

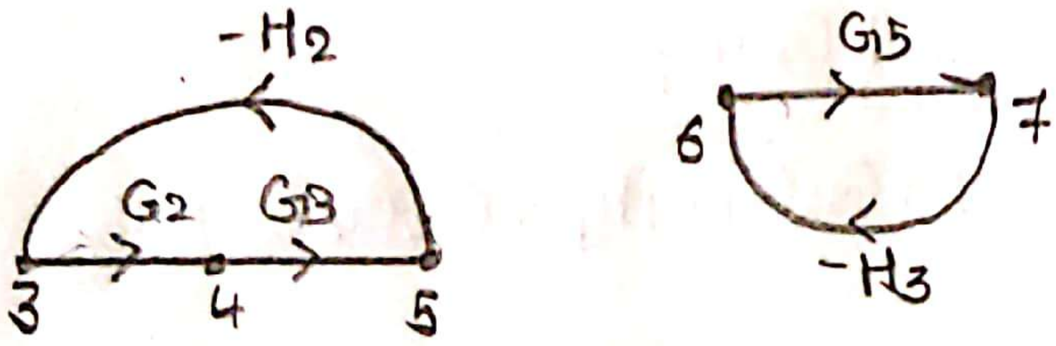
(i)



Gain of two non touching loops $(P_{12}) = (-G_2H_1)(-G_5H_3)$

$= G_2G_5H_1H_3$

(ii)



2nd combination of non touching loops $(P_{22}) = (-G_2G_3H_2)(-G_5H_3)$

$= G_2G_3G_5H_2H_3$.

Calculation of Δ and Δk:-

$$\Delta = 1 - \left[\text{sum of individual loop gain}\right] + \left[\text{sum of two non touching loops}\right] -$$

$$= 1 - [(P_{11} + P_{21} + P_{31})] + [(P_{12} + P_{22})] -$$

$$= 1 - [(-G_2H_1) + (-G_2G_3H_2) + (-G_5H_3)] + [G_2G_5H_1H_3 + G_2G_3G_5H_2H_3].$$

$$\Delta = 1 + G_{12}H_1 + G_{12}G_{13}H_2 + G_{15}H_3 + G_{12}G_{15}H_1H_3 + G_{12}G_{13}G_{15}H_2H_3$$

$\Rightarrow \Delta_1 = 1$, Since there is no part of the graph which is not touching with first forward path.

$\Delta_2 = 1 - (G_{12}H_1)$; The path of the graph which is non touching with 2nd forward path.

$= 1 + G_{12}H_1$

then the transfer function in by using mason's gain formula:

$$T = T(S) = \frac{1}{\Delta}\sum_K P_K \Delta_K$$

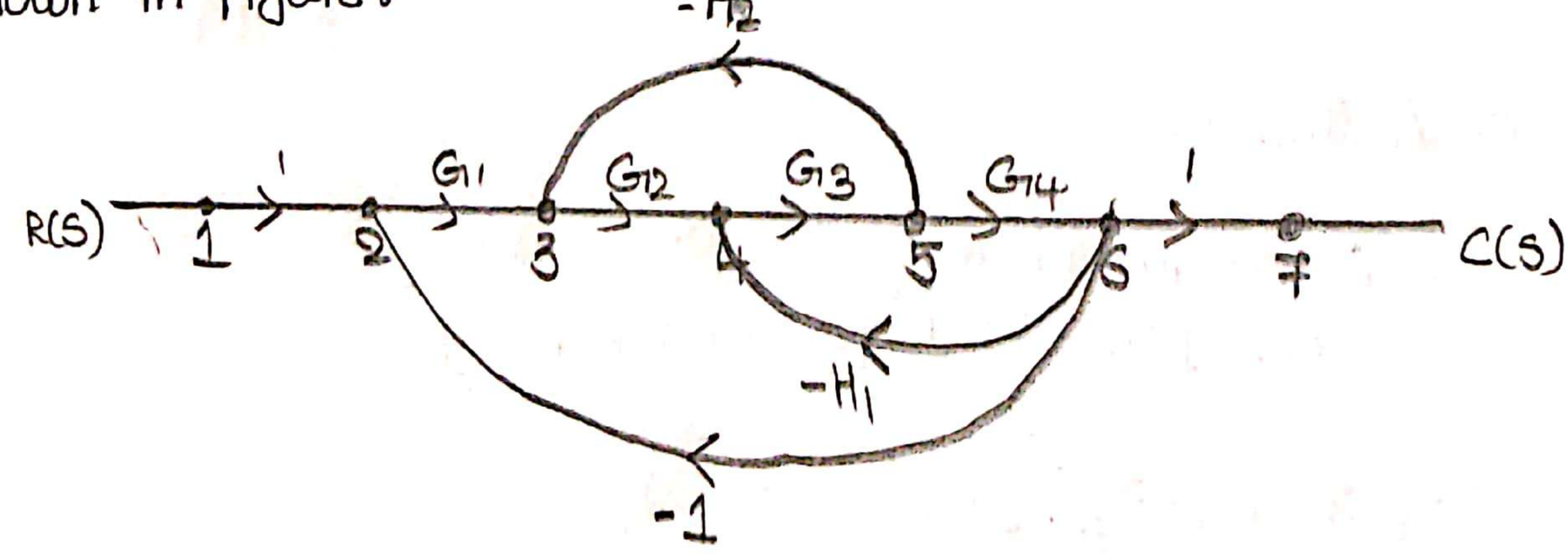
$$= \frac{1}{\Delta}\left[P_1\Delta_1 + P_2\Delta_2\right]$$

$$T(S) = \frac{G_1G_2G_3G_4G_5(1) + G_4G_5G_6(1+G_{12}H_1)}{1+G_{12}H_1 + G_{12}G_{13}H_2 + G_{15}H_3 + G_{12}G_{15}H_1H_3 + G_{12}G_{13}G_{15}H_2H_3}$$

$$T(S) = \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6 + G_4G_5G_6G_{12}H_1}{1+G_{12}H_1 + G_{12}G_{13}H_2 + G_{15}H_3 + G_{12}G_{15}H_1H_3 + G_{12}G_{13}G_{15}H_2H_3}$$

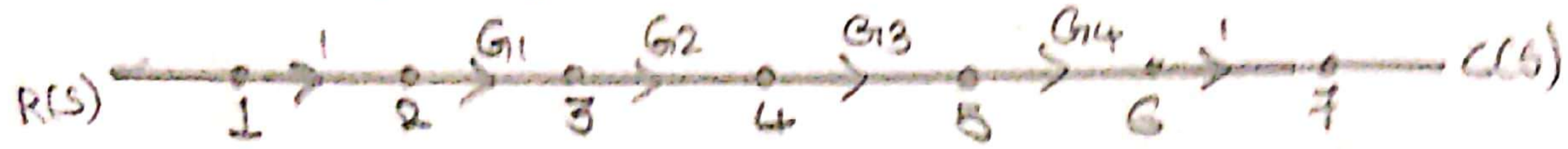
$$\therefore T(S) = \frac{G_{12}G_4G_{15}\left[G_1G_3 + G_6/G_{12} + G_6H_1\right]}{1+G_{12}H_1 + G_{12}G_{13}H_2 + G_{15}H_3 + G_{12}G_{15}H_1H_3 + G_{12}G_{13}G_{15}H_2H_3}.$$

2. Find the overall transfer function whose signal flow graph is shown in figure:-



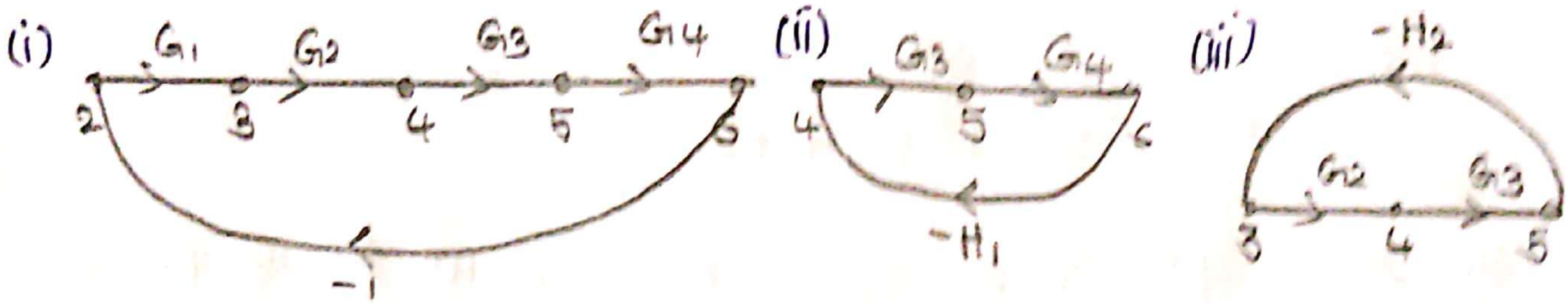
Sol: No. of forward paths (K) = 1

Let the forward path gain is P_1.



Gain of 1st forward path (P_1) = $G_1G_2G_3G_4$.

Individual loop gain:-



(i) loop gain of 1st individual loop (P_{11}) = $-G_1G_2G_3G_4$

(ii) loop gain of 2nd individual loop (P_{12}) = $-G_3G_4H_1$

(iii) loop gain of 3rd individual loop (P_{13}) = $-G_2G_3H_2$

Two non-touching loops:-

There are no two non-touching loops, since no loops are touching loops

$\therefore$ Two non-touching loops = 0

calculation of Δ and Δ_k:-

$$\Delta = 1 - \begin{bmatrix}\text{sum of individual}\\ \text{loop gain}\end{bmatrix} + \begin{bmatrix}\text{sum of two non}\\ \text{touching loops}\end{bmatrix} -$$

$$= 1 - [P_{11} + P_{12} + P_{13}] + 0$$

$$= 1 - [(-G_1G_2G_3G_4) + (-G_3G_4H_1) + (-G_2G_3H_2)]$$

$$= 1 + G_1G_2G_3G_4 + G_3G_4H_1 + G_2G_3H_2.$$

$\Rightarrow \Delta_1 = 1$; since there is no part of the graph which is not touching with first forward path.

Then the transfer function.

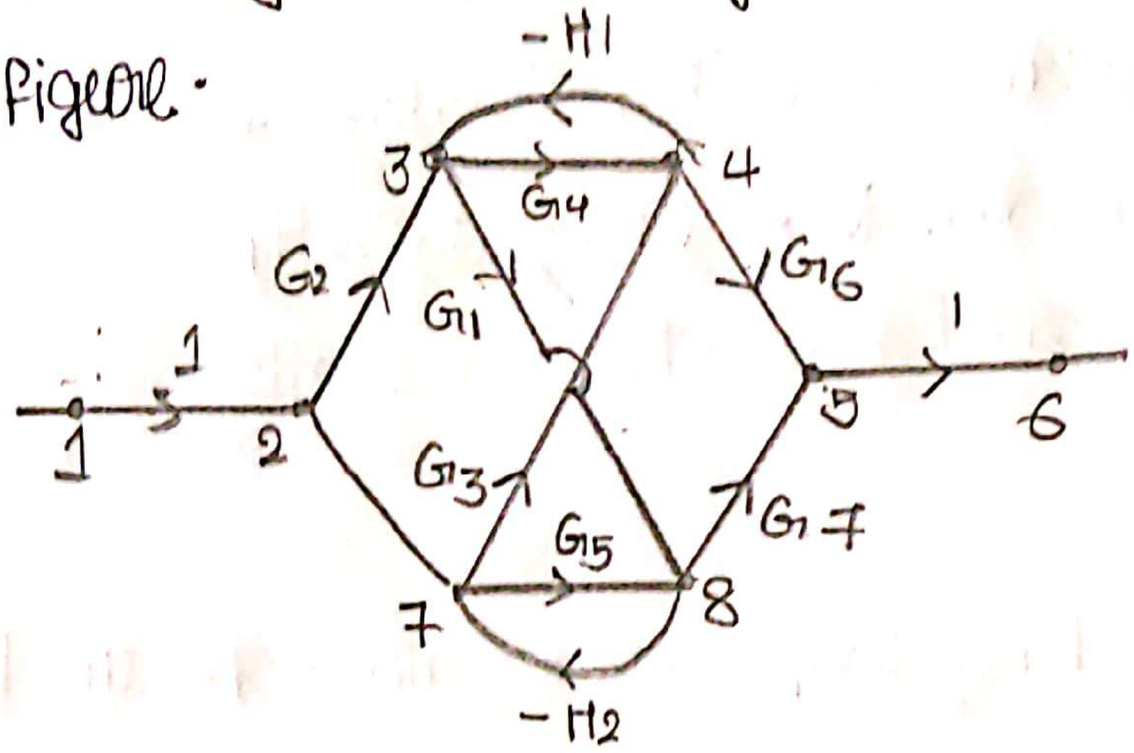
$$T = T(s) = \frac{1}{\Delta} \sum_k P_k \Delta_k$$

$$= \frac{1}{\Delta} [P_1 \Delta_1]$$

$$= \frac{G_1 G_2 G_3 G_4 (1)}{1 + G_1 G_2 G_3 G_4 + G_3 G_4 H_3 + G_2 G_3 H_2}$$

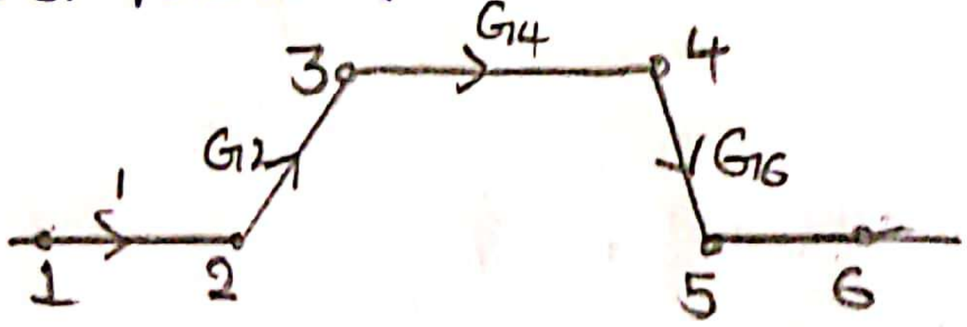
$$\therefore T(s) = \frac{G_1 G_2 G_3 G_4}{1 + G_1 G_2 G_3 G_4 + G_3 G_4 H_3 + G_2 G_3 H_2} .$$

3. Find the overall gain of the system whose signal flowgraph is shown in figure.



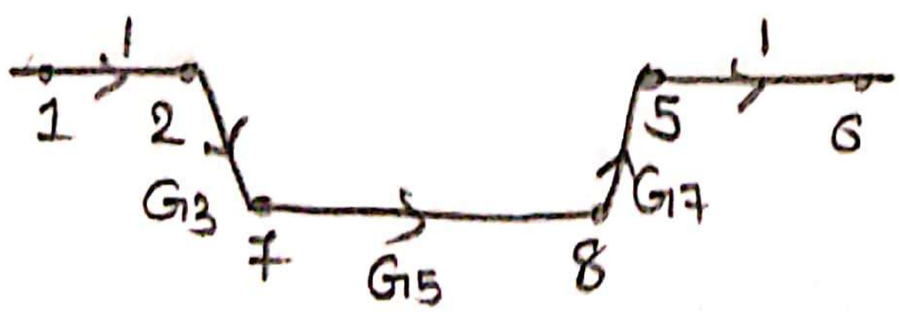
(i) No. of Forward path (k) = 6

(1)



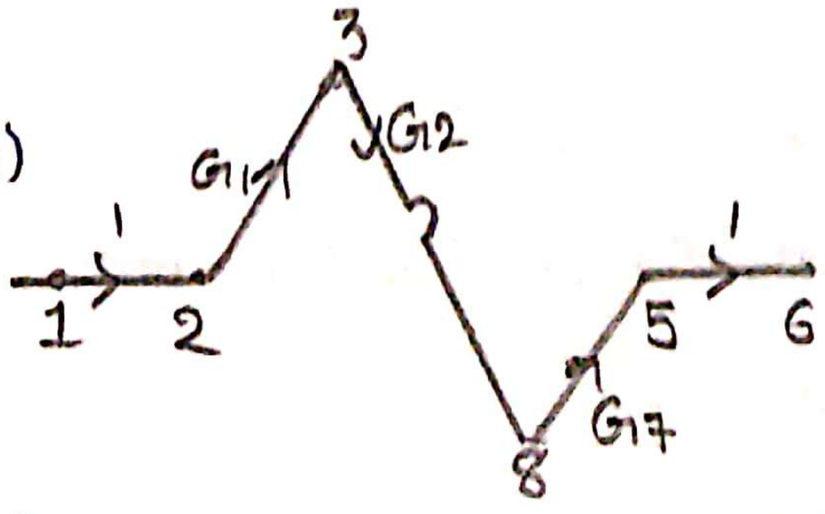
Gain of 1st Forward path (P_1) = $G_2 G_4 G_6$

(ii)



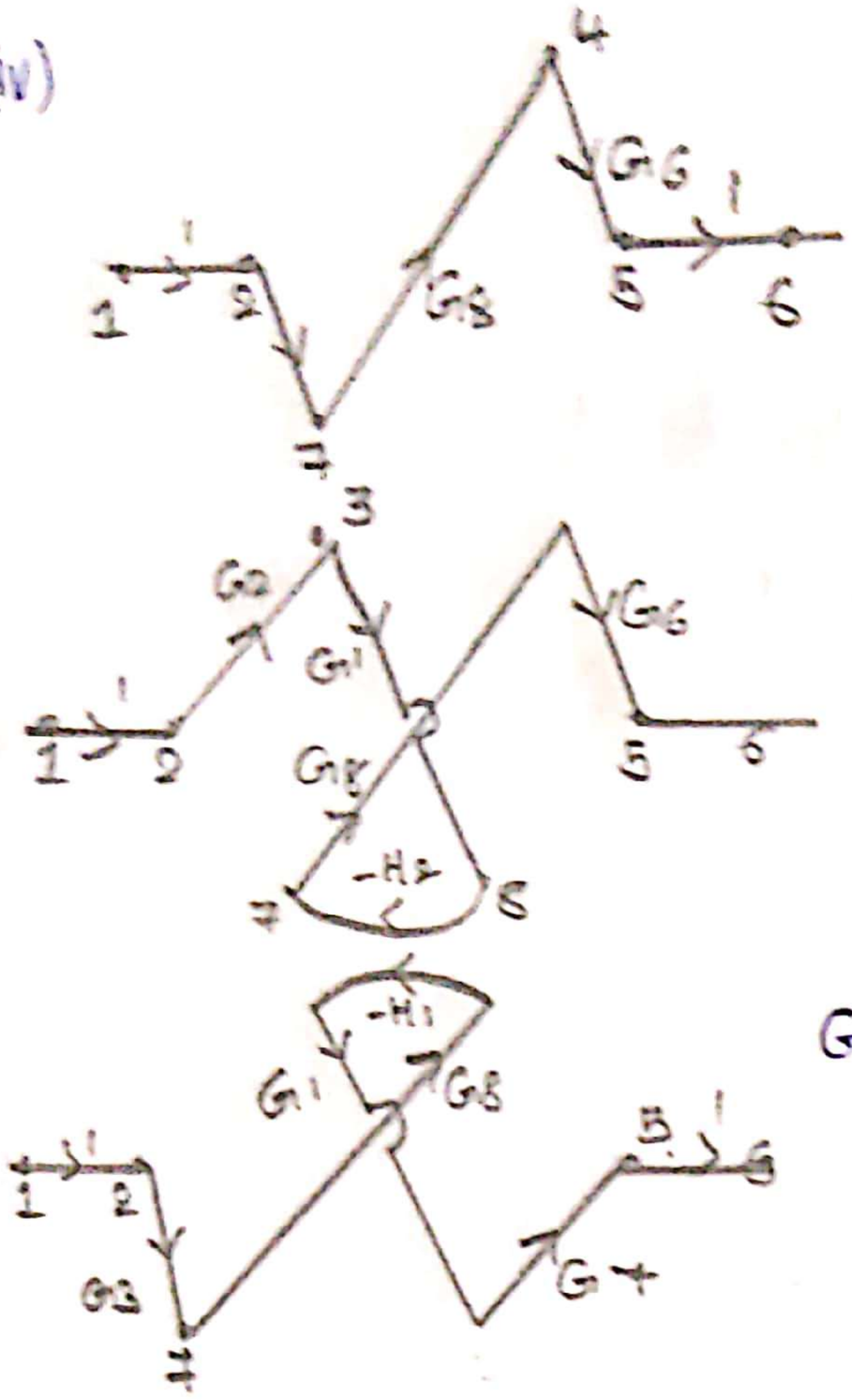
Gain of 2nd Forward path (P_2) = $G_3 G_5 G_7$

(iii)



Gain of 3rd forward path (P_3) = $G_1 G_2 G_7$

(iv)



Gain of 4th forward path $(P_4) = G_{13}G_{16}G_{85}$

Gain of 5th forward path $(P_5) = -G_{12}G_{11}G_{16}G_{18}H_2$

Gain of 6th forward path $(P_6) = -G_1G_3G_{17}G_{18}H_1$

Individual loop gain :-

(i)



(ii)



(iii)



(i) Loop gain of 1st individual loop $(P_{11}) = -G_{14}H_1$

(ii) Loop gain of 2nd individual loop $(P_{21}) = -G_{15}H_2$

(iii) Loop gain of 3rd individual loop $(P_{31}) = G_1G_{18}H_1H_2$

Two non-touching loops:-



(i) Gain of two nontouching loops $(P_{21}) = (-G_{14}H_1)(-G_{13}H_2)$

$= G_4G_5H_1H_2$.

calculation of Δ and Δ_K :-

$$\Delta = 1 - [P_{11} + P_{21} + P_{31}] + [P_{12}]$$

$$= 1 - [(G_4H_1) + (-G_5H_2) + G_1G_8H_1H_2] + [G_4G_5H_1H_2]$$

$$= 1 + G_4H_1 + G_5H_2 - G_1G_8H_1H_2 + G_4G_5H_1H_2$$

$\Rightarrow \Delta_1 = 1 - (-G_5H_2) = 1 + G_5H_2$

$\Delta_2 = 1 - (-G_4H_1) = 1 + G_4H_1$

$\Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$.

Then the transfer function,

$$T = T(s) = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

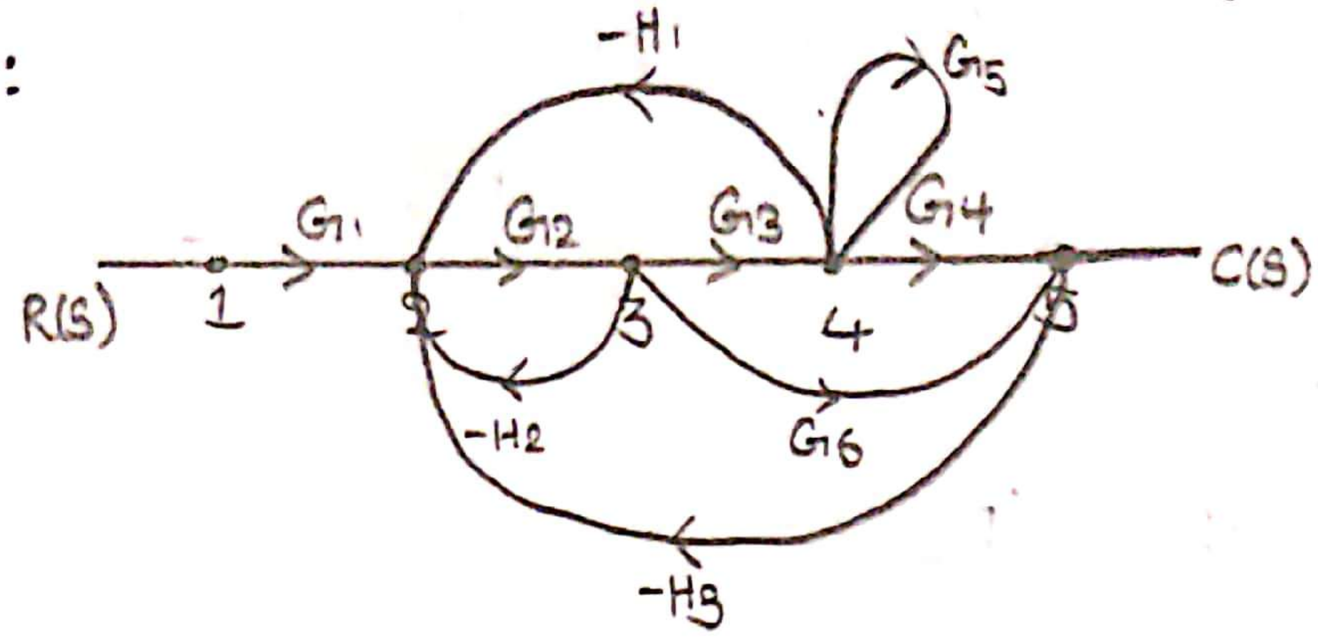
$$= \frac{1}{\Delta} [P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6]$$

$$= \frac{G_2G_4G_6(1 + G_5H_2) + G_3G_5G_7(1 + G_4H_1) + G_1G_2G_7 + G_3G_6G_8 - G_1G_2G_6G_8H_2 - G_1G_3G_7G_8H_1}{1 + G_4H_1 + G_5H_2 - G_1G_8H_1H_2 + G_4G_5H_1H_2}$$

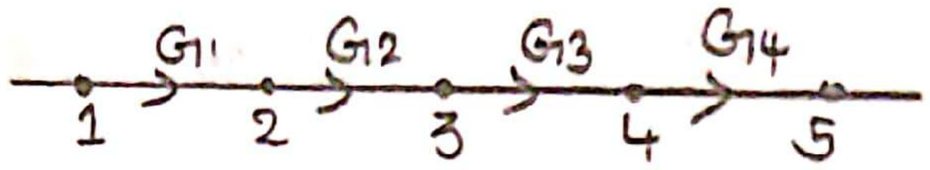
$$= \frac{G_2G_4G_6 + G_2G_4G_5G_6H_2 + G_3G_5G_7 + G_3G_4G_5G_7H_1 + G_1G_2G_7 + G_3G_6G_8 - G_1G_2G_6G_8H_2 - G_1G_3G_7G_8H_1}{1 + G_4H_1 + G_5H_2 - G_1G_8H_1H_2 + G_4G_5H_1H_2}.$$

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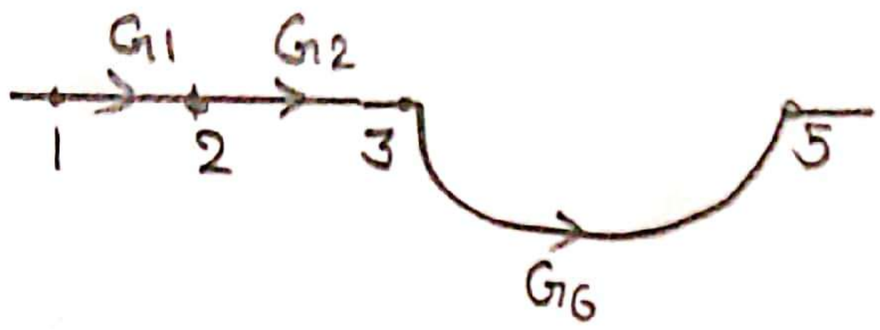
4. Find the overall transfer function whose signal flow graph is shown in figure:



Sol:- No. of forward path (K) = 2

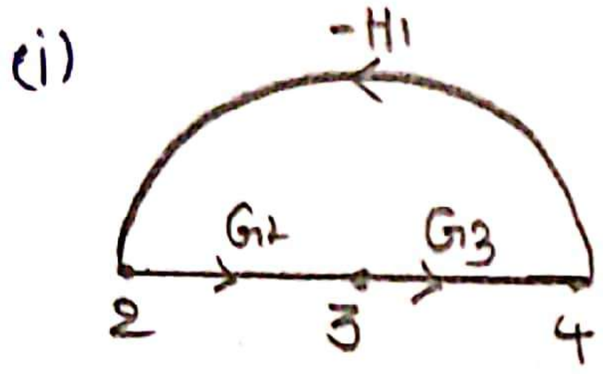
(i) 

Gain of 1st forward path (P_1) = $G_1G_2G_3G_4$

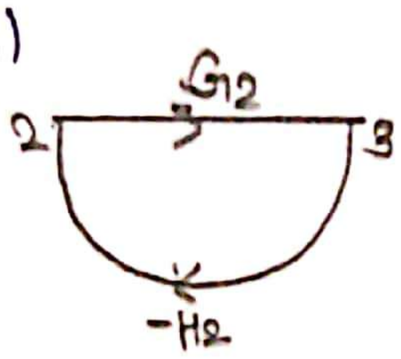
(ii) 

Gain of 2nd forward path (P_2) = $G_1G_2G_6$

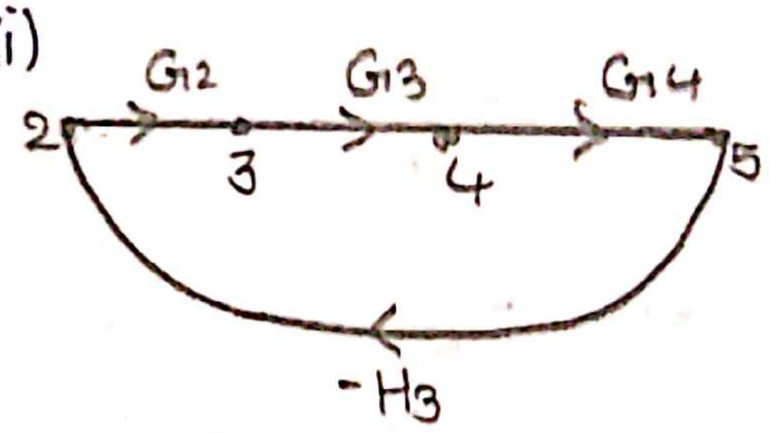
Individual loop gain:

(i) 

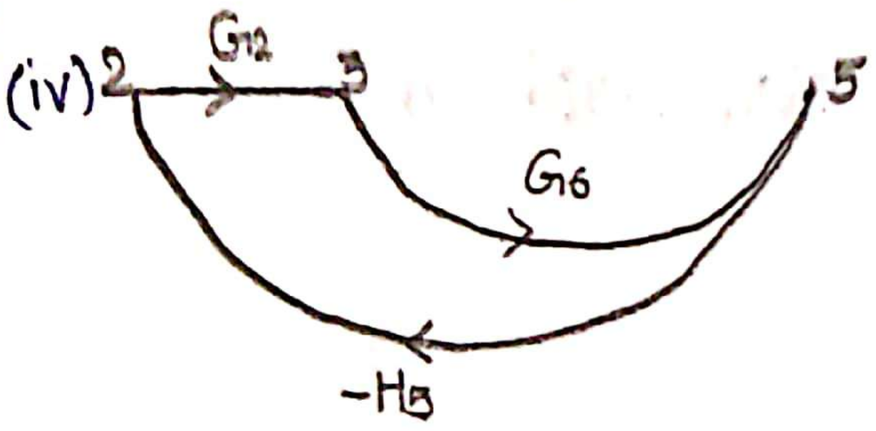
$P_{11} = -G_{12}G_{13}H_1$

(ii) 

$P_{21} = -G_{12}H_2$

(iii) 

$P_{31} = -G_{12}G_{13}G_{14}.H_3$

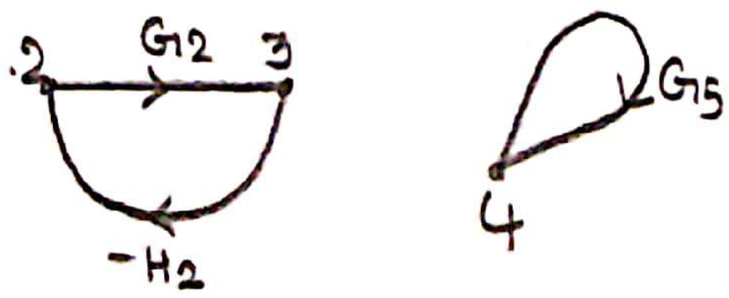
(iv) 

$P_{41} = -G_{12}G_{16}H_3$

(v) 

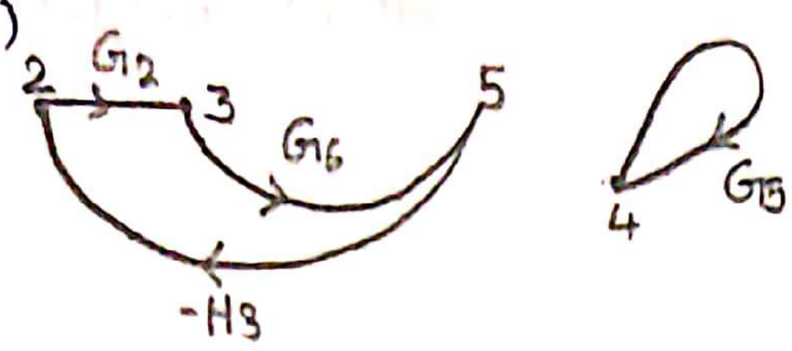
$P_{51} = G_{15}$

Non Touching Loops:-

(i) 

$P_{12} = (-G_{12}H_2)(G_{15}) = -G_{12}G_{15}H_2$.

(ii)



$P_{22} = (-G_{12}G_6H_3)(G_5)$

$= -G_{12}G_6G_5H_3$.

calculation of Δ and Δ_K :-

$$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41} + P_{51}) + (P_{12} + P_{22})$$

$$= 1 - (-G_2G_3H_1 - G_{12}H_2 - G_{12}G_3G_4H_3 + G_5 - G_{12}G_6H_3) + (-G_2G_5H_2 - G_{12}G_5G_6H_3)$$

$$= 1 + G_{12}G_3H_1 + G_{12}H_2 + G_{12}G_3G_4H_3 - G_5 + G_{12}G_6H_3 - G_{12}G_5H_2 - G_{12}G_5G_6H_3.$$

$\Rightarrow \Delta_1 = 1$

$\Delta_2 = 1 - G_5$

Then the transfer function,

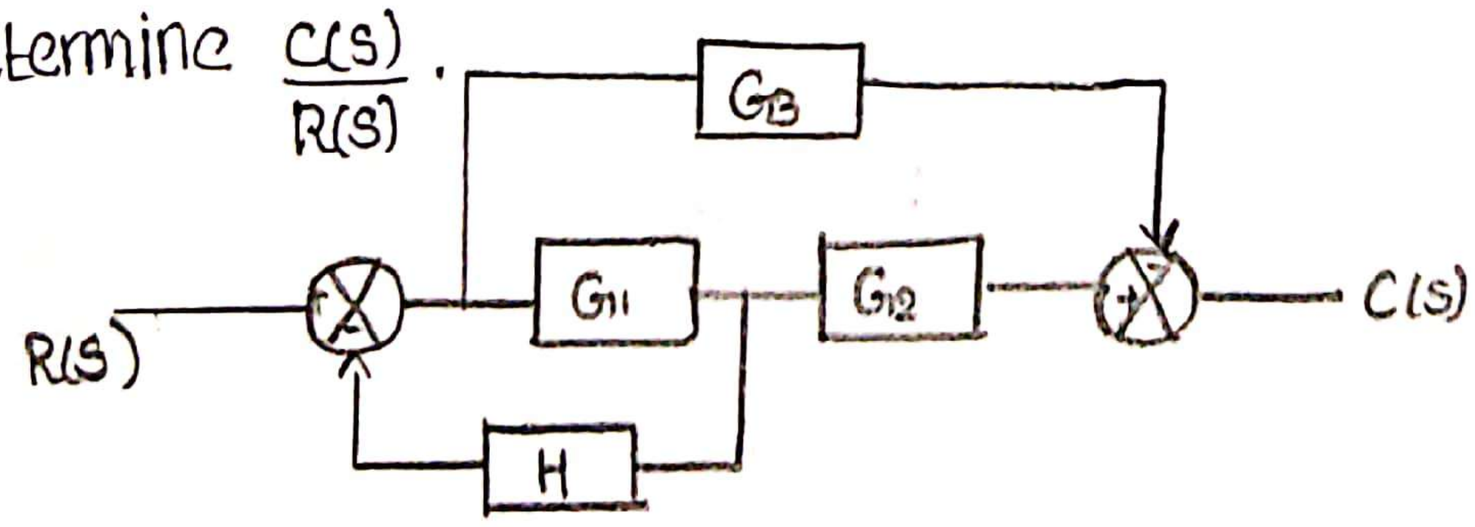
$$T = T(S) = \frac{1}{\Delta} \Sigma_K P_K \Delta_K$$

$$= \frac{1}{\Delta} [P_1\Delta_1 + P_2\Delta_2]$$

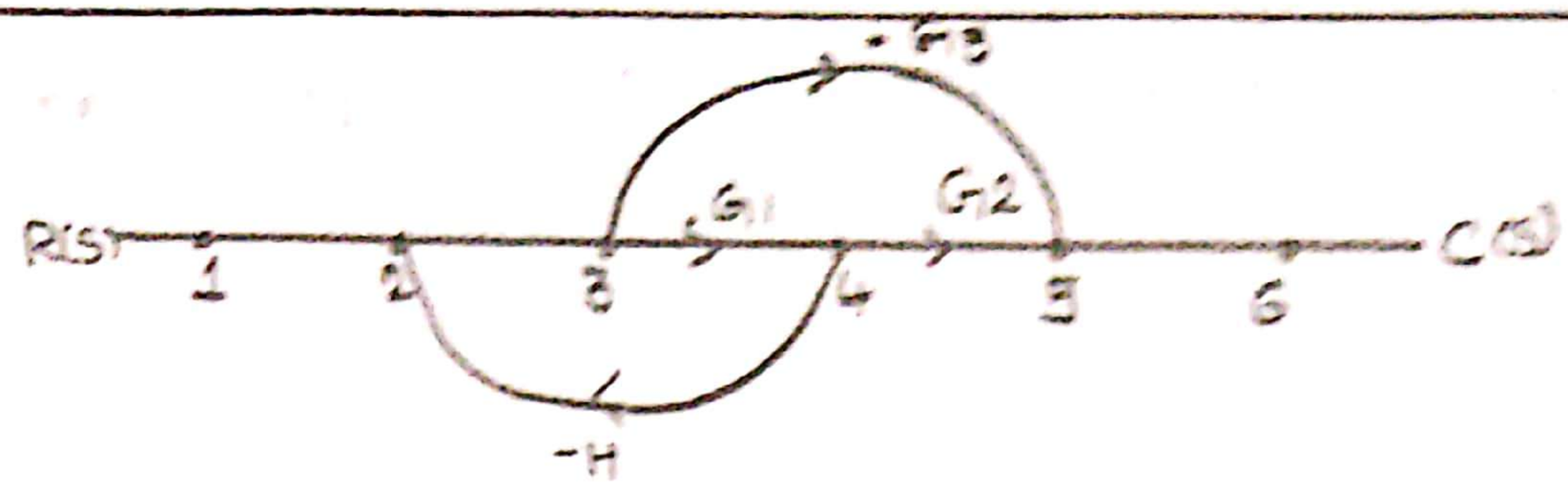
$$= \frac{1}{\Delta} [G_1G_2G_3G_4(1) + G_1G_2G_6(1 - G_5)]$$

$$\therefore T(S) = \frac{G_1G_2G_3G_4 + G_1G_2G_6 - G_1G_2G_5G_6}{1 + G_2G_3H_1 + G_{12}H_2 + G_{12}G_3G_4H_3 - G_5 + G_{12}G_6H_3 - G_{12}G_5H_2 - G_{12}G_5G_6H_3}.$$

5. convert the given block diagram to signal flow graph and determine $\frac{C(S)}{R(S)}$.

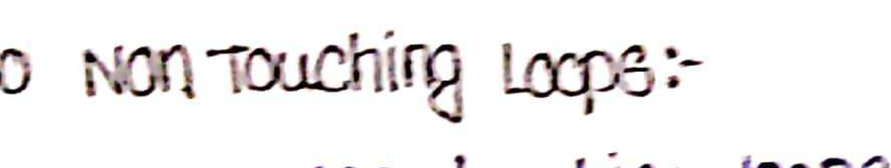


Sol:



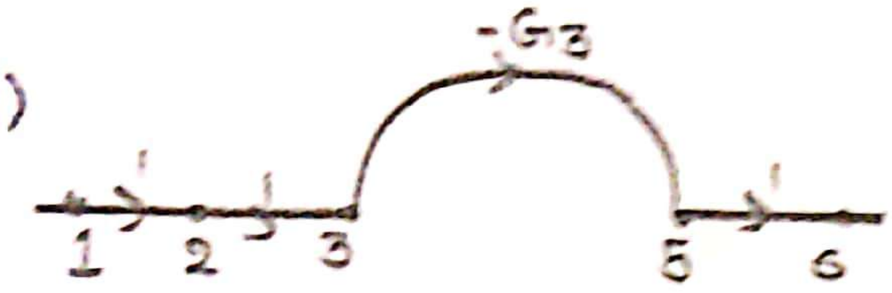
No. of Forward paths (k)=2

(i)



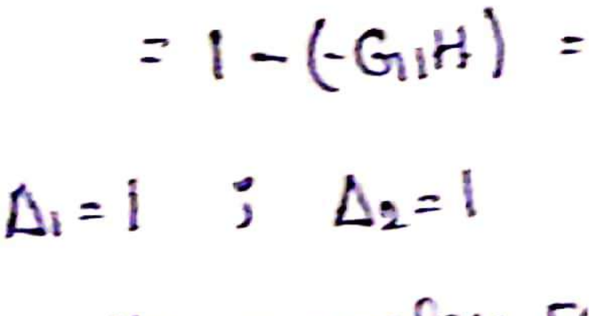
Gain of 1st Forward path (P_1)=G_1G_2

(ii)



Gain of 2nd forword path (P_2)= $-G_3$

Individual Loops:-



$P_{11} = -G_1H$

Two Non Touching Loops:-

·No non touching loops

calculation of Δ and Δ_K :-

$$\Delta = 1 - (P_{11})$$

$$= 1 - (-G_1H) = 1 + G_1H$$

$\Rightarrow \Delta_1 = 1 \quad ; \quad \Delta_2 = 1$

Then the Transfer Function

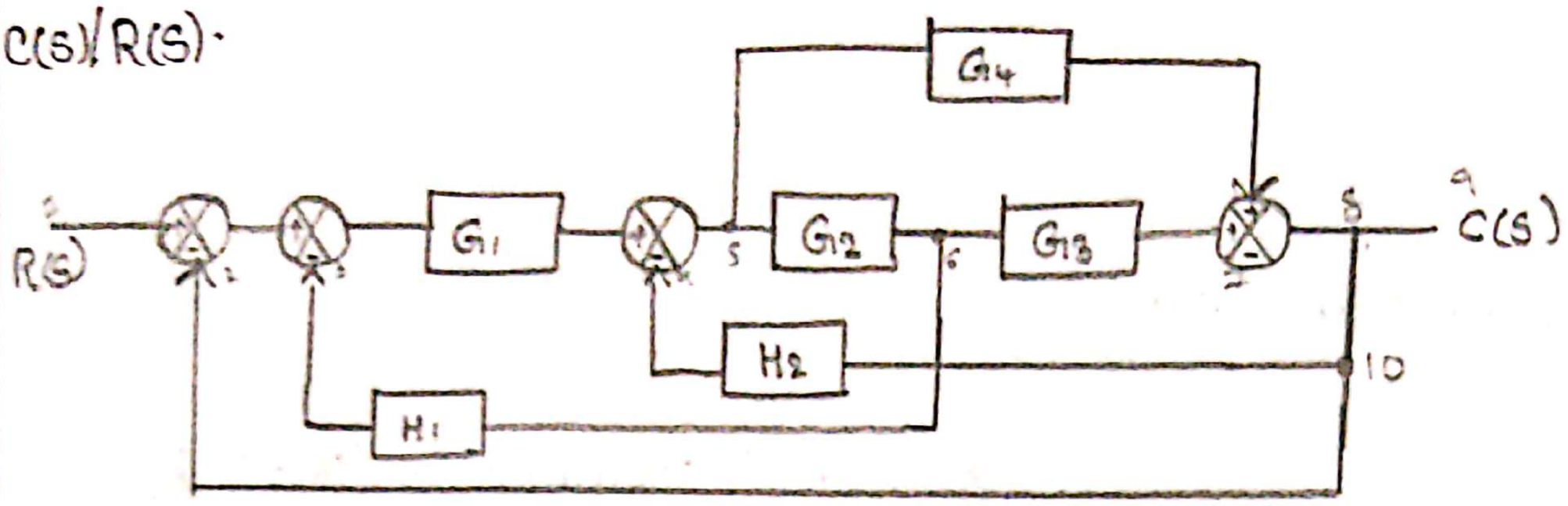
$$T = T(s) = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

$$= \frac{1}{\Delta} [P_1\Delta_1 + P_2\Delta_2]$$

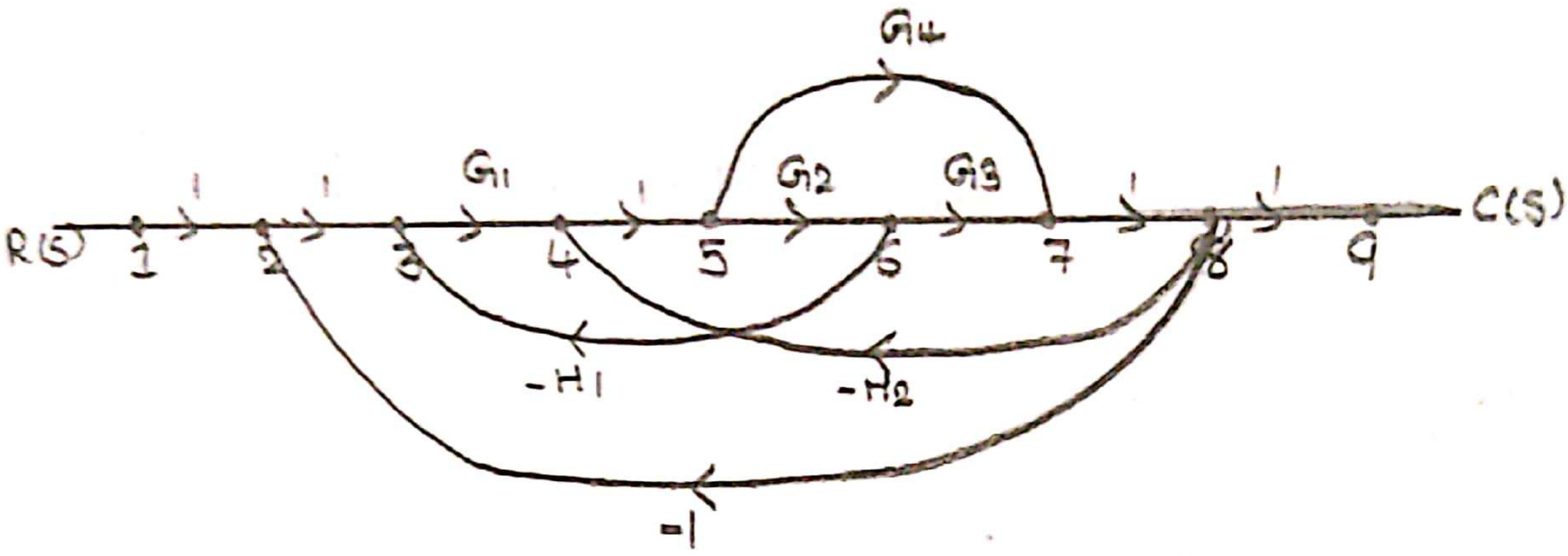
$$= \frac{G_1G_2(1) - G_3(1)}{1+G_1H} = \frac{G_1G_2 - G_3}{1+G_1H}.$$

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6. convert the block diagram to signal flow graph and determine C(s)/R(s).

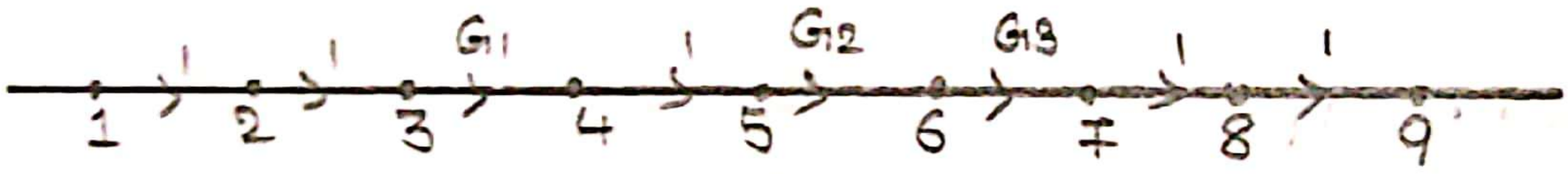


Sol:-



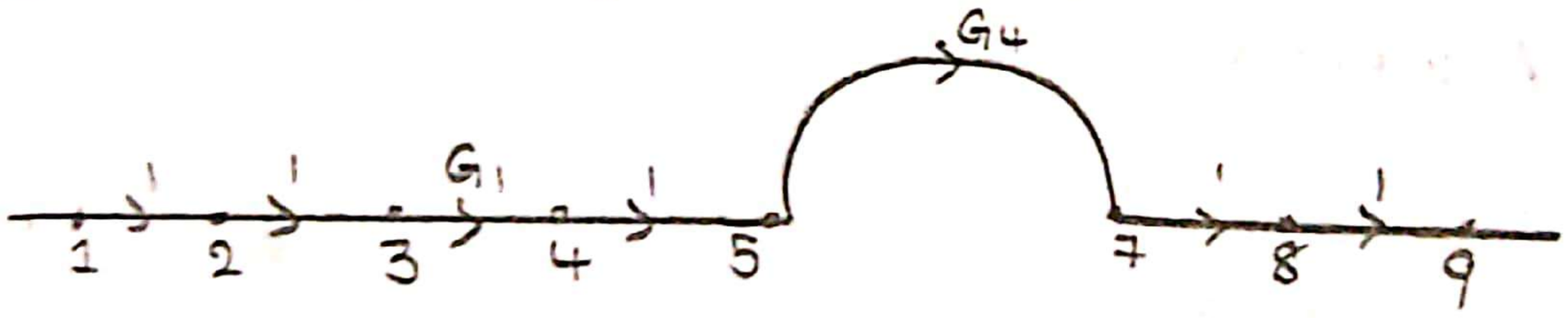
No of Forward path(k) = 2

(i)



Gain of 1st Forward gain (P_1) = $G_1G_2G_3$

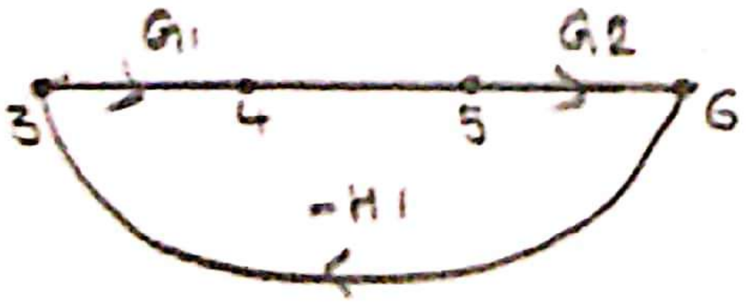
(ii)



Gain of 2nd Forward gain (P_2) = G_1G_4.

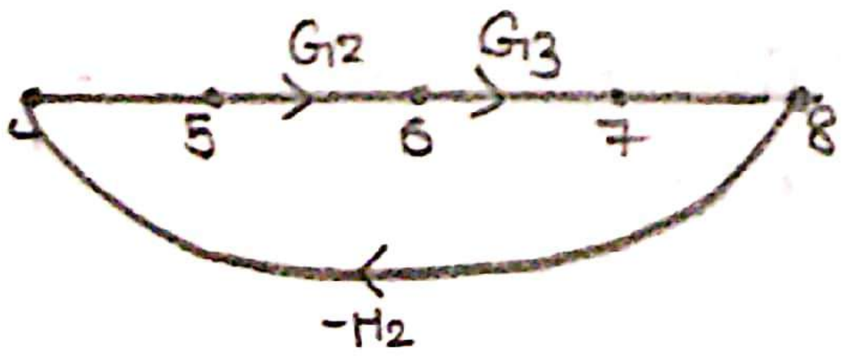
Individual loops:-

(i)



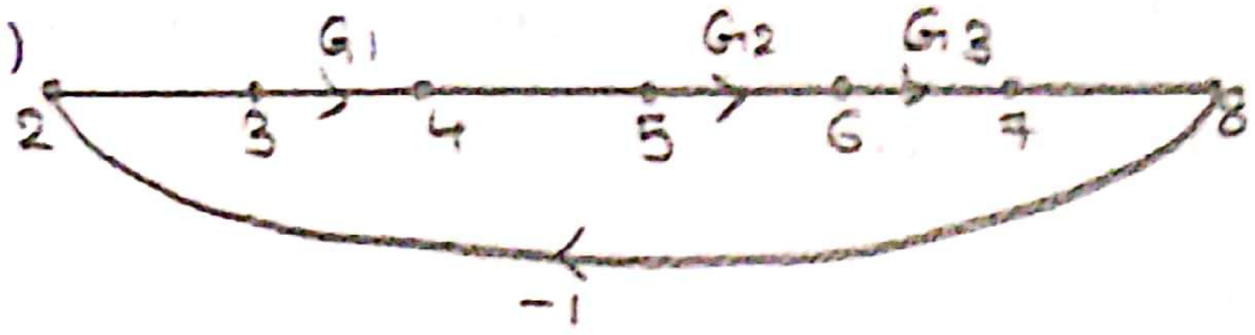
$P_{11} = -G_1G_2H_1$

(ii)



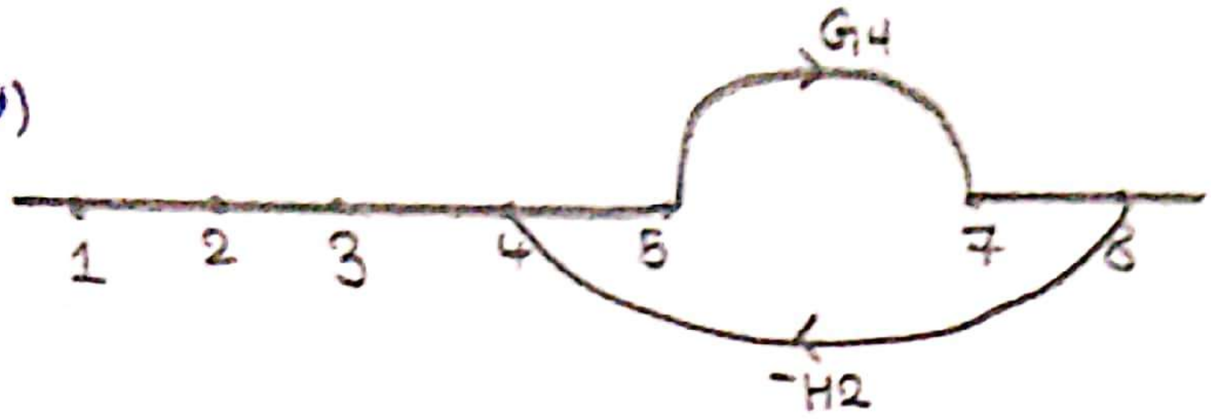
$P_{21} = -G_2G_3H_2$

(iii)



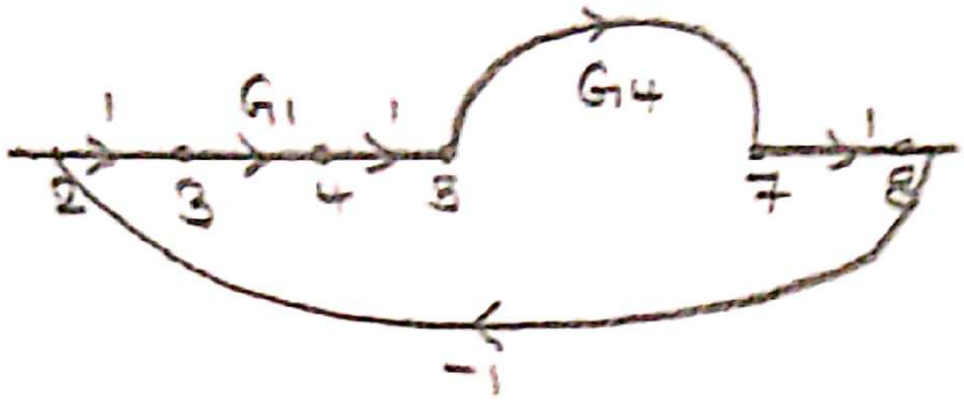
$P_{31} = -G_1G_2G_3$

(iv)



$P_{41} = -G_4H_2$

(v)



$P_{51} = -G_1G_4$

Two non-touching loops:-

No combination of two non touching loops

calculation of Δ and Δ_K :-

$$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41} + P_{51}) + 0$$

$$= 1 - (-G_1G_2H_1 - G_2G_3H_2 - G_1G_2G_3 - G_4H_2 - G_1G_4)$$

$$= 1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 + G_4H_2 + G_1G_4.$$

$\Rightarrow \Delta_1 = 1$;

$\Delta_2 = 1$;

Then the transfer function,

$$T = T(s) = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

$$= \frac{1}{\Delta} [P_1\Delta_1 + P_2\Delta_2]$$

$$= \frac{G_1G_2G_3(1) + G_1G_4(1)}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 + G_4H_2 + G_1G_4}$$

$$\therefore T(s) = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 + G_4H_2 + G_1G_4}$$

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7. Construct the signal flowgraph for a set of equations and find overall gain using mason's gain formula.

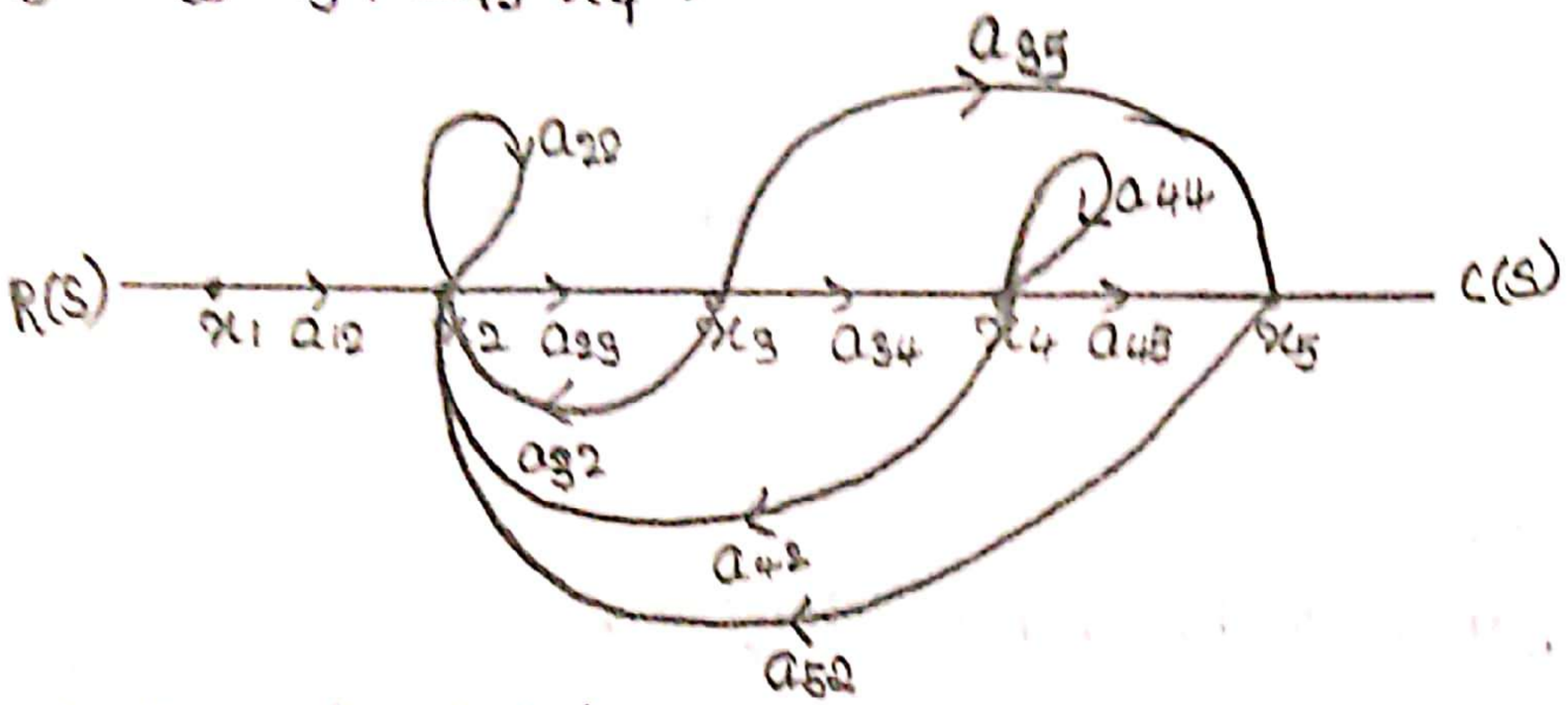
$x_2 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5$

$x_3 = a_{23}x_2$

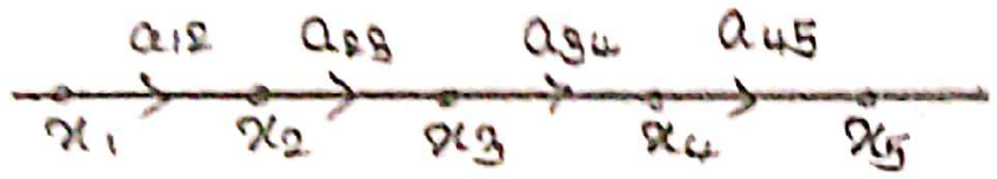
$x_4 = a_{34}x_3 + a_{44}x_4$

$x_5 = a_{35}x_3 + a_{45}x_4$.

Sol:-



No of forward path(K) = 2

(i) 

$P_{11} = a_{12} \cdot a_{23} a_{34} \cdot a_{45}$

(ii) 

$P_{21} = a_{12} \cdot a_{23} \cdot a_{35}$

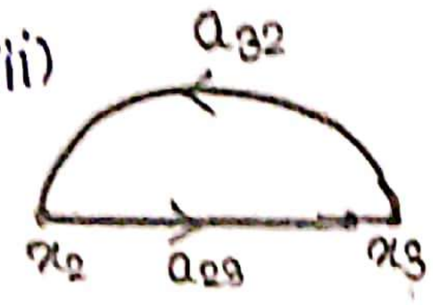
Individual loops:-

(i) 

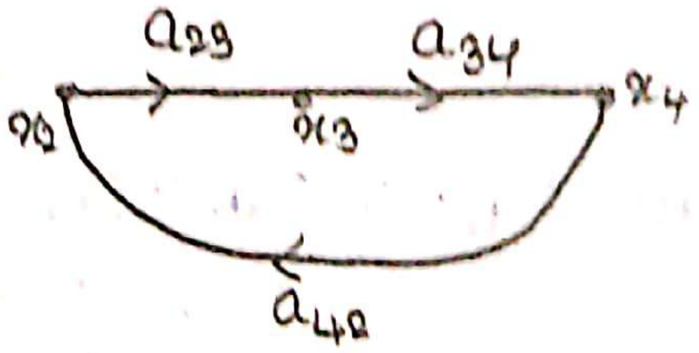
$P_{11} = a_{22}$

(ii) 

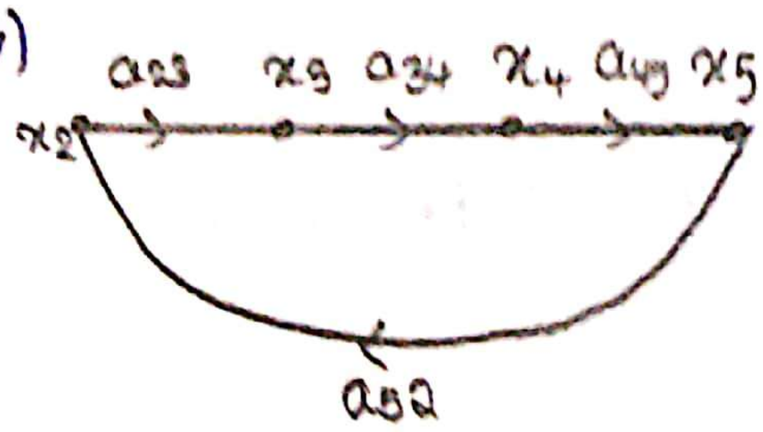
$P_{21} = a_{44}$

(iii) 

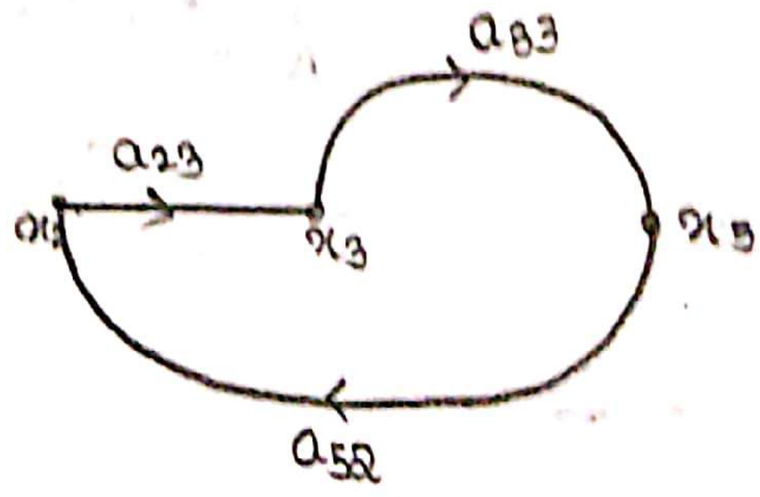
$P_{31} = a_{23} \cdot a_{32}$

(iv) 

$P_{41} = a_{23} \cdot a_{34} \cdot a_{42}$.

(v) 

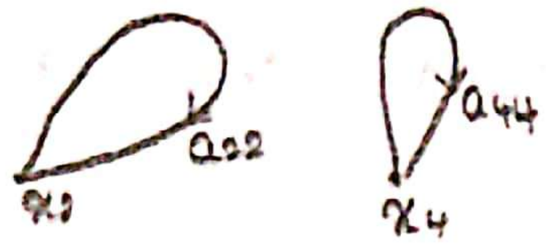
$P_{51} = a_{23} \cdot a_{34} \cdot a_{45} \cdot a_{52}$

(vi) 

$P_{61} = a_{23} \cdot a_{35} \cdot a_{52}$.

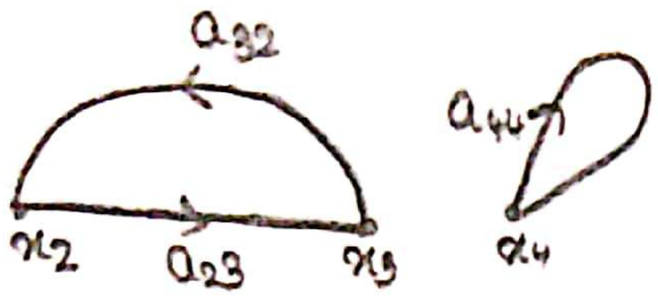
Non Touching loops:-

(i)



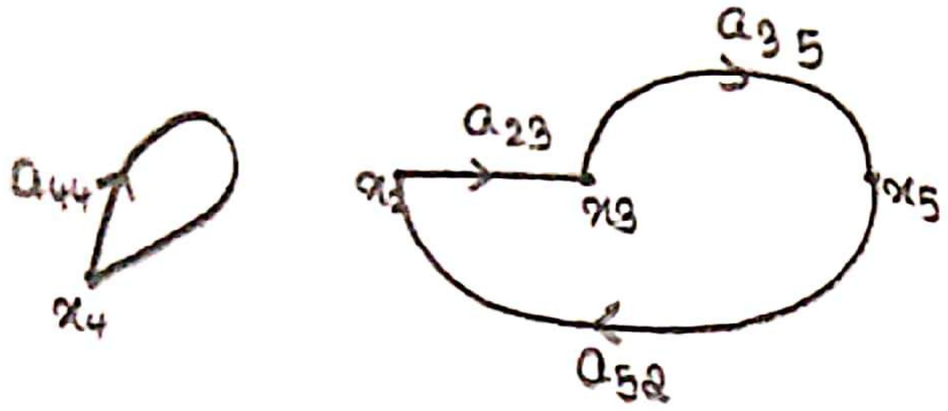
$P_2 = a_{22} \cdot a_{44}$

(ii)



$P_{22} = a_{23} \cdot a_{32} \cdot a_{44}$

(iii)



$P_{32} = a_{23} \cdot a_{35} \cdot a_{52} \cdot a_{44}$

calculation of Δ and Δ_K :-

$$\Delta = 1 - \left[\begin{array}{c}\text{sum of individual}\\ \text{loop}\end{array}\right] + \left[\begin{array}{c}\text{sum of products of two}\\ \text{non touching loops}\end{array}\right] - \cdots$$

$$\Delta = 1 - [P_{11} + P_{21} + P_{31} + P_{41} + P_{51} + P_{61}] + [P_{12} + P_{22} + P_{32}]$$

$$\Delta = 1 - [a_{22} + a_{44} + a_{32} \cdot a_{23} + a_{23}a_{34}a_{42} + a_{23}a_{34}a_{45}a_{52} + a_{23}a_{35}a_{52}] + [a_{22}a_{44} + a_{23}a_{32}a_{44} + a_{23}a_{35}a_{52}a_{44}]$$

$\Rightarrow \Delta_1 = 1$;

$\Delta_2 = 1 - a_{44}$;

Then the transfer function,

$$T = T(s) = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

$$= \frac{1}{\Delta} [P_1 \Delta_1 + P_2 \Delta_2]$$

$$= \frac{a_{12}a_{23}a_{34} + a_{45} + a_{12}a_{23}a_{35}(1 - a_{44})}{1 - [a_{22} + a_{44} + a_{23}a_{32} + a_{23}a_{34}a_{42} + a_{23}a_{34}a_{45}a_{52} + a_{23}a_{35}a_{52}) + [a_{22}a_{44} + a_{23}a_{32}a_{44} + a_{23}a_{35}a_{52}a_{44}]}$$

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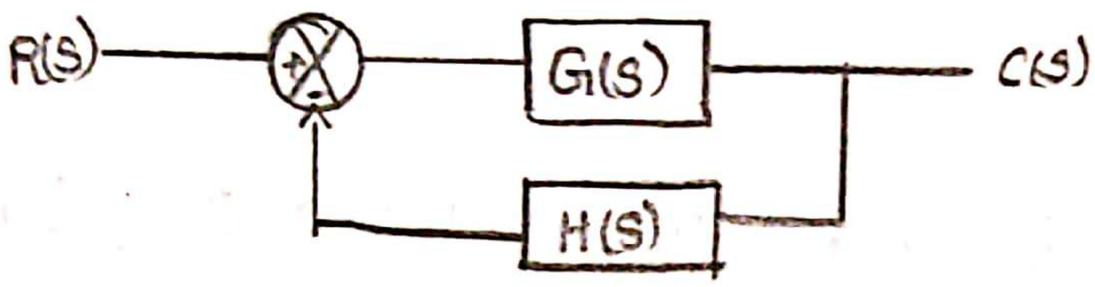
$$\therefore \Pi(\Theta) = \frac{a_{12}a_{23}a_{34}a_{45} + a_{12}a_{31}a_{33} - a_{12}a_{31}a_{35}a_{44}}{1 - [a_{22}+a_{44}+a_{23}a_{32}+a_{13}a_{34}a_{42}+a_{23}a_{34}a_{45}a_{32}+a_{24}a_{35}a_{42}]+}$$

$$[a_{22}a_{44}+a_{23}a_{32}a_{44}+a_{23}a_{35}a_{52}a_{44}].$$

UNIT-II
TIME RESPONSE ANALYSIS

Introduction:-

- The time response of a system is the output of a closed loop system as a function of time. It is denoted by $c(t)$. The time response can be obtained by solving the differential equations governing the system.
- consider a closed loop system as given below



From the above transfer function, the output in s-domain is given by the product of transfer function and input $R(s)$.

- Then on taking inverse laplace transform of this product, the domain response $c(t)$ is obtained as

From figure $\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$

$\therefore$ Response $C(S) = R(S) \cdot M(S)$

$= R(S) \cdot \frac{G(S)}{1+G(S)H(S)}$

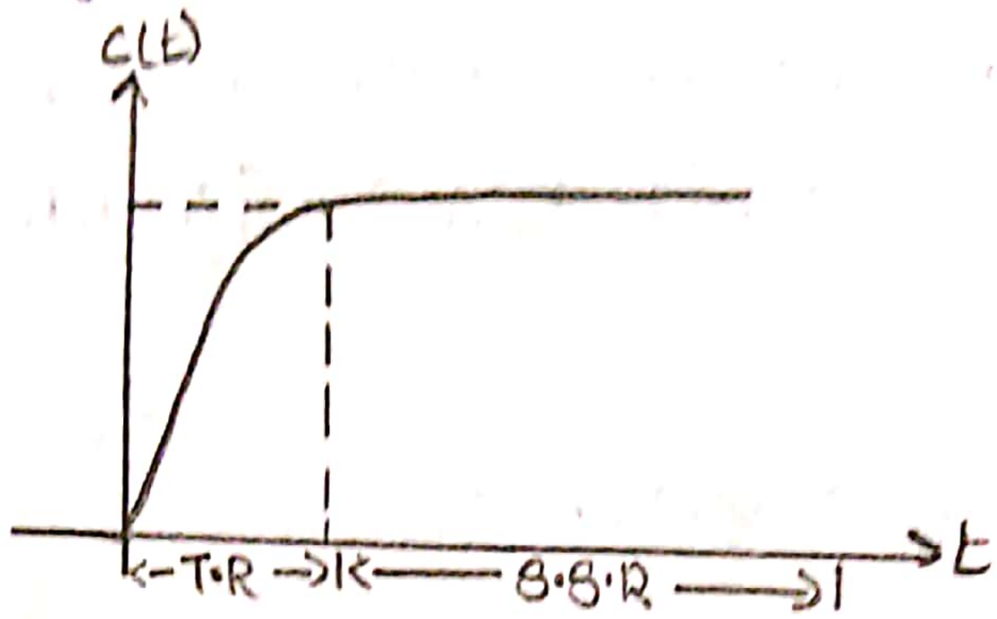
Applying inverse laplace transform on both sides

$c(t) = L^{-1}[C(S)] = L^{-1}[R(S) \cdot M(S)]$

$= L^{-1}\left[R(S) \cdot \frac{G(S)}{1+G(S)H(S)}\right]$

- The time response of control system consist of two parts
  1. The transient Response
  2. The steady state Response

- the transient response shows the output of the system when the input changes from one state to another state.
- the steady state response shows the output as time (t) approaches to infinity (constant).



Test signals (Transient) :-

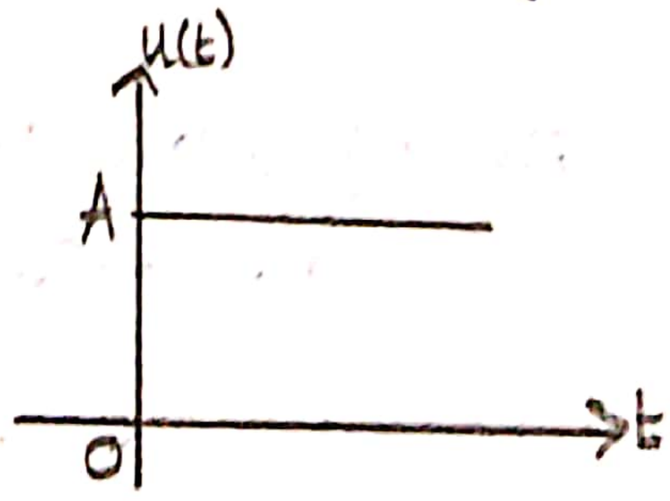
- Test signals is input required to predict the response of a system.
- The characteristics of actual input signals are sudden shock, a constant velocity and constant acceleration. The commonly used test input signals are impulse, step, ramp and parabolic.

(i) Step signal :-

The step signal is a signal whose value changes from 0 to point A at $t=0$, then remains constant at A for $t>0$. The step signal resembles an actual steady input to a system. A special case of step signal in which A is unity.

Mathematically, it is represented as

$$u(t) = A \quad t \geq 0$$
$$= 0 \quad t < 0.$$

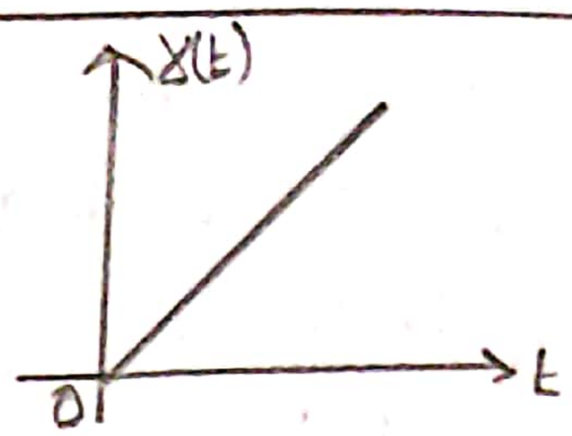


(ii) Ramp signal :-

It is a signal whose values increases with time from an initial value of zero at $t=0$, the ramp signal resembles a constant velocity input to the system. A special case of ramp signal in which A is unity.

Mathematically, it is represented as

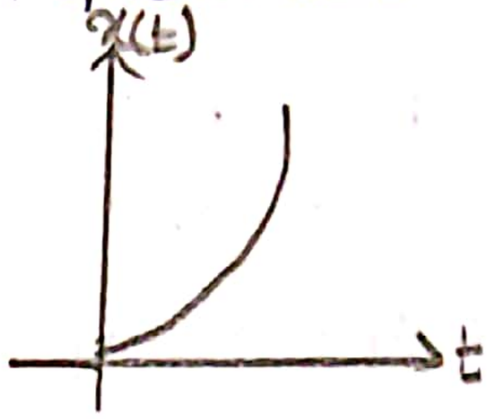
$$r(t) = At \quad t \geq 0$$
$$= 0 \quad t < 0.$$



(iii) parabolic signal :-

The instantaneous value varies a square of the time from an initial value of zero at $t=0$. Now take a look on sketch of the signal w.r.t the time resembles a parabola. the parabolic signal indicates a constant acceleration input to the system. A special case of parabolic signal in which A is unity.

$$r(t) = At^2/2 \quad t \geq 0$$
$$= 0 \quad t < 0$$

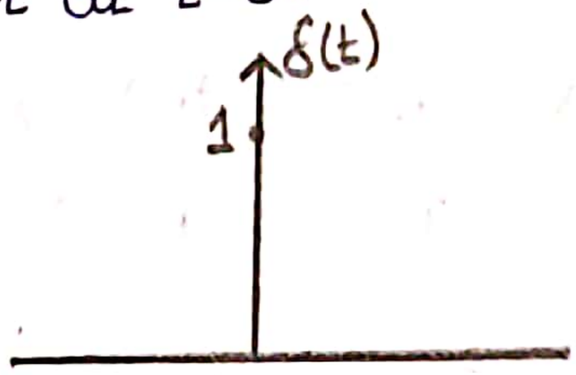


(iv) Impulse signal :-

A signal which is available for very short duration is called impulse response. Ideal impulse signal is a unit impulse signal which is defined as a signal having zero values at all time s except at $t=0$.

Mathematically it is represented as

$$\delta(t) = 1 \quad t = 0$$
$$= 0 \quad t \neq 0.$$



Tabulation of test signal in both time and s-domain :-

Test signals	Time Domain input $r(t)$	s-Domain $R(s)$
1. Step signal	A	A/s
2. unit step signal	1	$1/s$
3. Ramp signal	At	A/s^2
4. unit ramp signal	t	$1/s^2$
5. parabolic signal	$At^2/2$	A/s^3
6. unit parabolic signal	$t^2/2$	$1/s^3$

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Order of a system:-

- The input output relationship of a control system can be expressed by a differential equation. The order of a system is given by the order of differential equation governing the system.
- If the system is governed by n^{th} order differential equation, the system is called n^{th} order of the system.
- Now the transfer function of a system is obtained by taking laplace transform on both sides governing the system.

$$T(S) = \frac{P(S)}{Q(S)} = \frac{b_0 S^m + b_1 S^{m-1} + b_2 S^{m-2} + \dots + b_{m-1} S + b_m}{a_0 S^n + a_1 S^{n-1} + a_2 S^{n-2} + \dots + a_{n-1} S + a_n}$$

- The order of the system is given by the maximum power of S in the denominator polynomial $Q(S)$

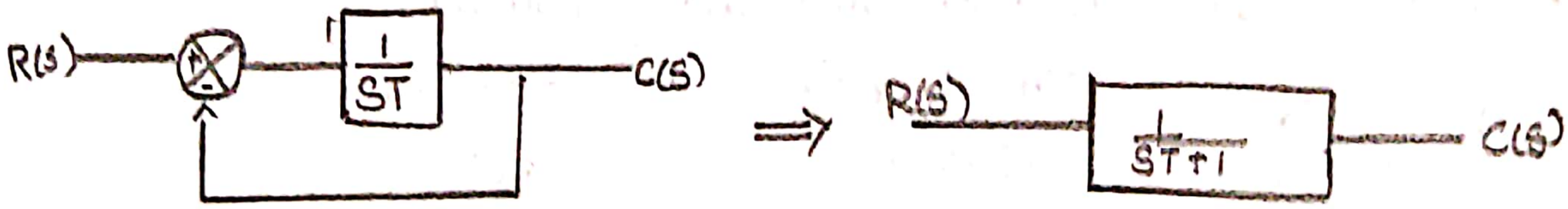
$$Q(S) = a_0 S^n + a_1 S^{n-1} + a_2 S^{n-2} + \dots + a_{n-1} S + a_n$$

Now, n is the order of the system, when $n=0$, the system is zero order system.

When, $n=1$, the system is 1st order system

$n=2$, the system is 2nd order system.

Response of 1st order unit step input:-



$$T(S) = \frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)} = \frac{\frac{1}{ST}}{1 + \frac{1}{ST}} = \frac{1}{ST+1}.$$

$\Rightarrow$ The response of Transfer Function $= \frac{C(S)}{R(S)} = \frac{1}{1+ST}$.

The step input signal $x(t) = 1$, $R(S) = \frac{1}{S}$.

the response in S-domain $C(s) = R(s) \cdot \frac{1}{sT+1}$

$$= \frac{1}{S} \cdot \frac{1}{ST+1}$$

$$= \frac{1}{S(ST+1)} = \frac{1}{ST\left(S+\frac{1}{T}\right)} = \frac{1/T}{S\left(S+\frac{1}{T}\right)}.$$

By applying partial Fractions

$$C(S) = \frac{\frac{1}{T}}{S\left(S+\frac{1}{T}\right)} = \frac{A}{S} + \frac{B}{S+\frac{1}{T}}$$

$$C(S)\Big|_{S=0} = \frac{\frac{1}{T}}{S\left(S+\frac{1}{T}\right)} = \frac{A\left(S+\frac{1}{T}\right)+B\cdot S}{S\left(S+\frac{1}{T}\right)}$$

$$C(S) = \frac{1}{T} = A\left(0+\frac{1}{T}\right)+B\cdot 0$$

$$\frac{1}{T} = A\cdot\frac{1}{T} \Rightarrow \boxed{A=1.}$$

$$C(S)\Big|_{S=-\frac{1}{T}} = \frac{1}{T} = A\left[\frac{-1}{T}+\frac{1}{T}\right]+B\left[\frac{-1}{T}\right]$$

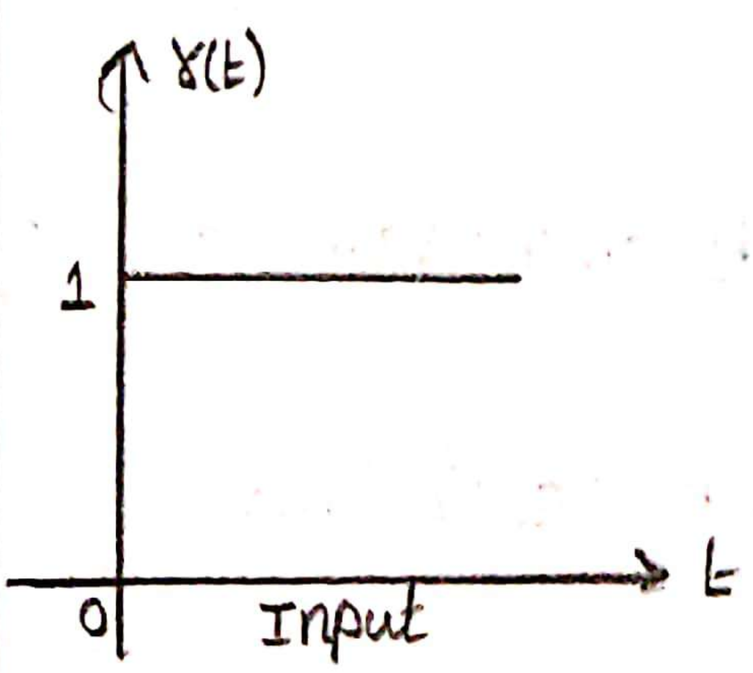
$$\frac{1}{T} = B\left[\frac{-1}{T}\right] \Rightarrow \boxed{B=-1}$$

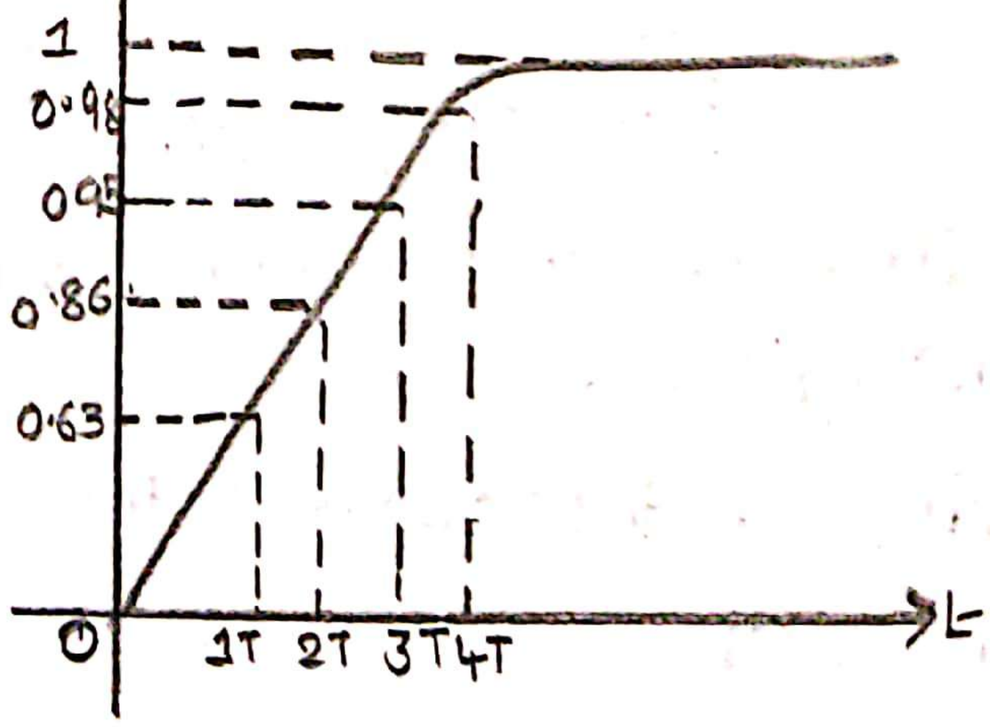
$$\therefore C(S) = \frac{1}{S} - \frac{1}{S+\frac{1}{T}}.$$

Apply inverse laplace Transform

$$L^{-1}[C(S)] = L^{-1}\left[\frac{1}{S} - \frac{1}{S+\frac{1}{T}}\right]$$

$$\therefore C(t) = 1 - e^{-t/T}.$$





When,

$t = 0\tau \Rightarrow C(t) = 1 - e^{-0\tau/\tau} = 1 - e^{0} = 1 - 1 = 0$

$t = 1\tau \Rightarrow C(t) = 1 - e^{-1\tau/\tau} = 1 - 0.367 = 0.63$

$t = 2\tau \Rightarrow C(t) = 1 - e^{-2\tau/\tau} = 1 - 0.135 = 0.865$

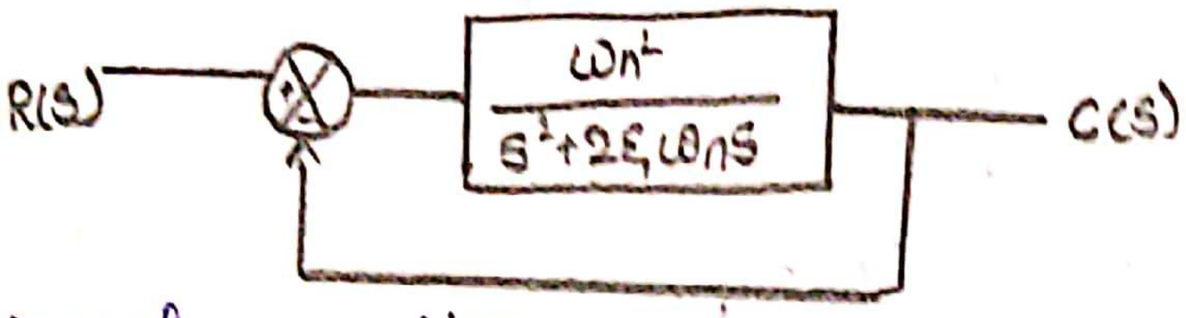
$t = 3\tau \Rightarrow C(t) = 1 - e^{-3\tau/\tau} = 1 - 0.045 = 0.950$

$t = 4\tau \Rightarrow C(t) = 1 - e^{-4\tau/\tau} = 1 - 0.018 = 0.981$

$t = 5\tau \Rightarrow C(t) = 1 - e^{-5\tau/\tau} = 1 - 0.006 = 0.99$

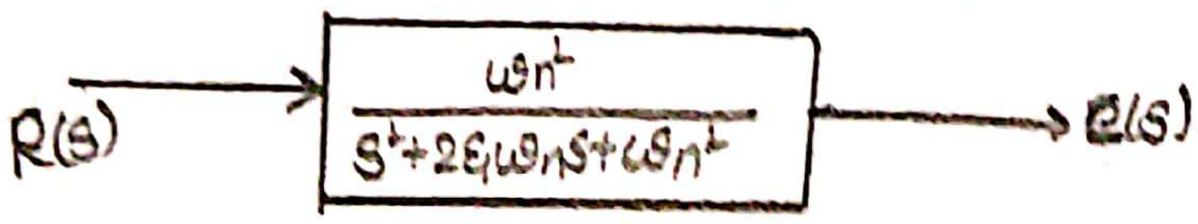
$t = \infty\tau \Rightarrow C(t) = 1 - e^{-\infty\tau/\tau} = 1 - 0 = 1.$

Response of 2nd order system :-



then the transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{\omega_n^2}{s^2+2\xi\omega_n s}}{1+\frac{\omega_n^2}{s^2+2\xi\omega_n s}} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}.$$



Where, ω_n = undamped natural Frequency (rad/sec)

ξ = damping ratio = $\frac{\text{Actual Damping}}{\text{critical Damping}}$

- the standard form of 2nd order system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

- the response $C(t)$ of 2nd order system depends on the value of damping ratio (ξ).
- Depending on the value of ξ, the system can be classified into following 4 cases :-

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(i) $\xi = 0$ => undamped system

(ii) $0 < \xi < 1$ => underdamped system

(iii) $\xi = 1$ => critically damped system

(iv) $\xi > 1$ => overdamped system

- The characteristic equation of the 2nd order system is $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$. It is a quadratic equation and roots of this equation by $s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4(1)(\omega_n^2)}}{2(1)}$$

$$= \frac{-2\xi\omega_n \pm \sqrt{4\omega_n^2(\xi^2 - 1)}}{2}$$

$$= \frac{-2\xi\omega_n \pm 2\omega_n\sqrt{\xi^2 - 1}}{2}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}.$$

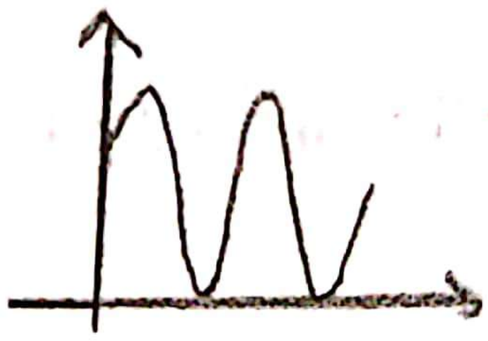
(i) When $\xi = 0$

$$\Rightarrow = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$= -0.\omega_n \pm \omega_n\sqrt{0 - 1}$$

$$= \pm\omega_n(\sqrt{-1})$$

$$= \pm j\omega_n$$ (i.e., Roots are purely imaginary and then the system is undamped).



(ii) When $\xi = 1$

$$\Rightarrow = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$= -1.\omega_n \pm \omega_n\sqrt{1 - 1}$$

$$= -\omega_n$$ (i.e., Roots are real and equal then the system is critically damped).

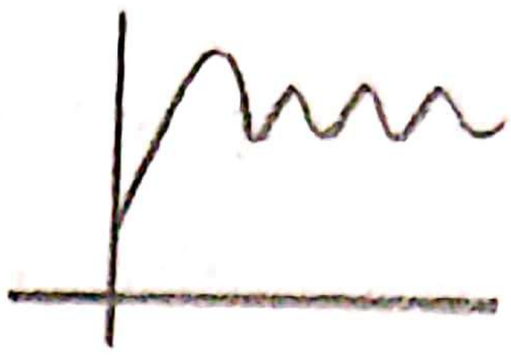
(iii) When $0<\xi<1$ =>

$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$

$= -\xi\omega_n \pm \omega_n\sqrt{-1}\sqrt{1-\xi^2}$

$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

$= -\xi\omega_n \pm j\omega_d$ (i.e., the roots are complex conjugate)



(iv) When $\xi>1$

$\Rightarrow -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$ (i.e., the roots are real and unequal).

Response of 2nd order undamped system for unit step input :-

We know that

$$\text{Transfer Function} = \frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}$$

The unit step input $x(t)=1$; $R(S)=1/S$

The response in S-domain $C(S) = R(S)\cdot\frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}$

$$= \frac{1}{S}\cdot\frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}$$

Here, the undamped system, $\xi=0$

$$= \frac{\omega_n^2}{S(S^2+\omega_n^2)}$$

By applying partial fractions

$$C(S) = \frac{\omega_n^2}{S(S^2+\omega_n^2)} = \frac{A}{S} + \frac{B}{S^2+\omega_n^2}$$

$$\frac{\omega_n^2}{S(S^2+\omega_n^2)} = \frac{A(S^2+\omega_n^2)+B\cdot S}{S(S^2+\omega_n^2)}$$

$C(S)|_{S=0} = \omega_n^2 = A(0+\omega_n^2)+B\cdot 0$

$\omega_n^2 = A\,\omega_n^2$

$\boxed{A=1}$

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$$C(s)\Big|_{s=j\omega n} = \omega n^2 = A\left[(j\omega n)^2 + \omega n^2\right] + B\left[j\omega n\right]$$

$$\omega n^2 = A\left[-\omega n^2 + \omega n^2\right] + B\, j\omega n$$

$$\omega n^2 = B\left[j\omega n\right]$$

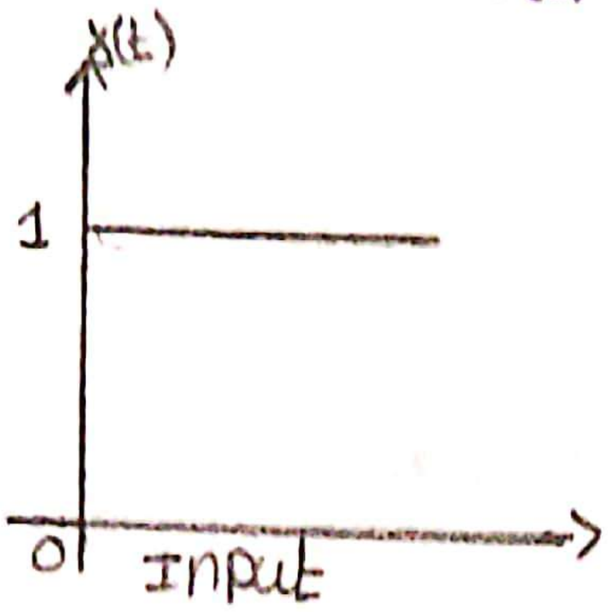
$$B = \frac{\omega n}{j} = -j\omega n$$

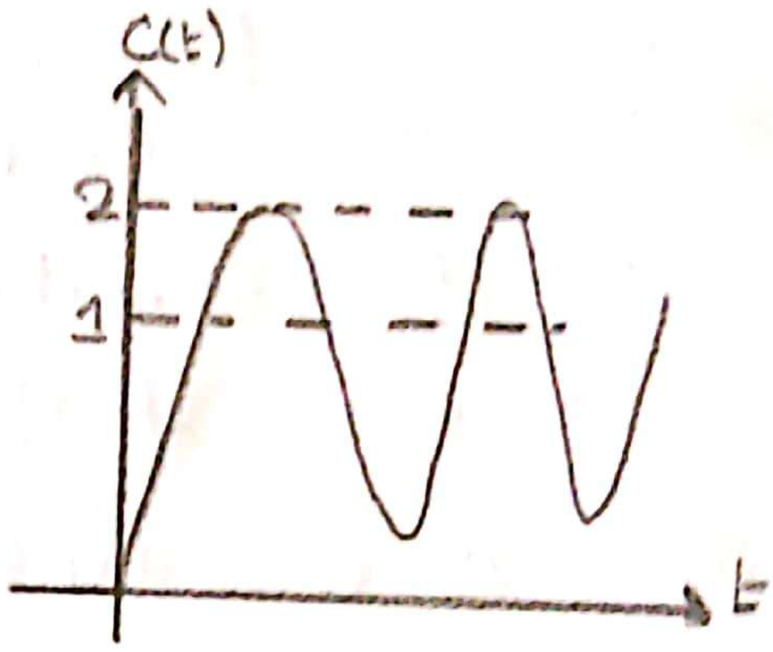
$$\therefore \boxed{B = -s}$$

$$\therefore C(s) = \frac{1}{s} - \frac{s}{s^2+\omega n^2}$$

Apply inverse Laplace transform on both sides

$$c(t) = 1 - \cos\omega n t.$$





Response of 2nd order underdamped system for unit step input :-

- The standard form of closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega n^2}{s^2 + 2\xi\omega n s + \omega n^2}$$

- For underdamped system, $0 < \xi < 1$ and roots of the denominator or complex conjugate are

$$= -\xi\omega n \pm \omega n\sqrt{\xi^2 - 1}$$

$$= -\xi\omega n \pm \omega n\sqrt{-1}\sqrt{1-\xi^2}$$

$$= -\xi\omega n \pm j\omega d$$

since $\xi < 1$, ξ^2 is also < 1 and so $1 - \xi^2$ is always positive.

- The response in s-domain $C(s) = R(s) \cdot \frac{\omega n^2}{s^2 + 2\xi\omega n s + \omega n^2}$

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then, the unit step input $r(t)=1$, $R(s)=\frac{1}{s}$

$$C(s)=\frac{1}{s}\cdot\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$=\frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

- The condition is underdamped system $0<\xi<1$

By applying partial fractions

$$C(s)=\frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}=\frac{A}{s}+\frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

(i) $$C(s)\Big|_{s=0}=\frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}=\frac{A(s^2+2\xi\omega_n s+\omega_n^2)+(Bs+C)s}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$\omega_n^2=A(0+0+\omega_n^2)+(B\cdot 0+C)\cdot 0$$

$$\omega_n^2=A\omega_n^2 \Rightarrow \boxed{A=1}$$

(ii) comparing coefficients of s^2 and s terms equal to zero

$$\omega_n^2=As^2+A2\xi\omega_n s+A\omega_n^2+Bs^2+Cs$$

s^2 :- $0=A+B$, $0=1+B$, $B=-1$

s :- $0=A2\xi\omega_n+C$, $0=2\xi\omega_n+C$, $C=-2\xi\omega_n$

$$\therefore C(s)=\frac{1}{s}+\frac{(-1)\cdot s-2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$=\frac{1}{s}-\frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2} \quad \text{——— (1)}$$

Let us add and subtract $\xi^2\omega_n^2$ in denominator of 2nd term

$$C(s)=\frac{1}{s}-\frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2+\xi^2\omega_n^2-\xi^2\omega_n^2}$$

$$=\frac{1}{s}-\frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2(1-\xi^2)}$$

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$$C(s) = \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}$$

$$= \frac{1}{s} - \left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} + \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right] \quad \text{——(2)}$$

Let us multiply and divide the ω_d in denominator of 3rd term

$$= \frac{1}{s} - \left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} + \frac{\xi\omega_n \cdot \omega_d}{[(s+\xi\omega_n)^2+\omega_d^2]\omega_d}\right]$$

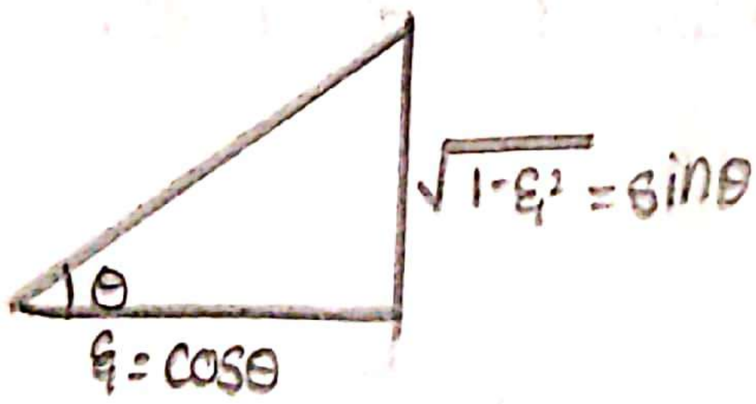
$$= \frac{1}{s} - \left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} + \frac{\xi\omega_n}{\omega_d} \cdot \frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right]$$

·Apply inverse laplace transform

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t + \frac{\xi\omega_n}{\omega_n\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin\omega_d t$$

$$= 1 - e^{-\xi\omega_n t}\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin\omega_d t$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right]$$



$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t\right]$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\theta+\omega_d t). \quad \text{(For unit step)}$$

$$c(t) = A\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\theta+\omega_d t)\right] \quad \text{(For step)}$$

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Response of 2nd order critically damped system for unit step input :

- the standard form of closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For critically damped $\xi = 1$

Then the unit step input $r(t) = 1$; $R(s) = \frac{1}{s}$

Response $C(s) = R(s) \cdot \frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s} \cdot \frac{\omega_n^2}{(s+\omega_n)^2}$

$$= \frac{\omega_n^2}{s(s+\omega_n)^2} \cdot$$

By applying partial fraction

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$\omega_n^2 = A(s+\omega_n)^2 + B \cdot s + C \cdot s\,(s+\omega_n)$$

$$\omega_n^2 = As^2 + A2s\omega_n + A\omega_n^2 + B \cdot s + C \cdot s^2 + C \cdot s\omega_n$$

$$C(s)|_{s=0} = \omega_n^2 = 0 + 0 + A\omega_n^2 + 0$$

$$\boxed{A = 1}$$

comparing coefficient of s^2 and s

s^2 :-

$0 = A + C$

$0 = 1 + C$

$\boxed{C = -1}$

s :-

$0 = A2\omega_n + B + C\omega_n$

$= 2\omega_n + B - \omega_n$

$= \omega_n + B$

$B = -\omega_n$

$$\therefore C(s) = \frac{1}{s} + \frac{-\omega_n}{(s+\omega_n)^2} + \frac{-1}{s+\omega_n}$$

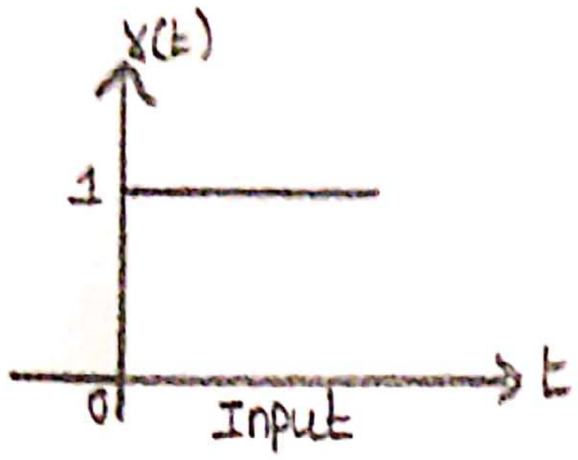
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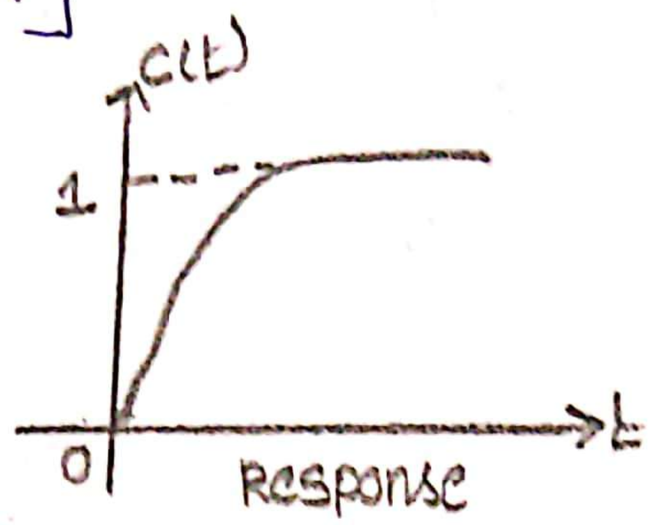
$$C(s) = \frac{1}{s} - \left[\frac{\omega_n}{(s+\omega_n)^2} + \frac{1}{s+\omega_n}\right]$$

Apply inverse laplace transform on both sides

$$c(t) = 1 - \omega_n t e^{-\omega_n t} - e^{-\omega_n t}$$

$$\therefore c(t) = 1 - e^{-\omega_n t}[t\omega_n + 1]$$





Response of 2nd order overdamped system for unit step input:-

The standard form of closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

- For overdamped system ($\xi > 1$) the roots of the denominator transfer function or distinct, let the roots of denominator be s_a, s_b
- Now assume $s_1 = -s_a$, $s_2 = -s_b$

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$s_1 = -\xi\omega_n + \sqrt{\xi^2 - 1}$$

$$s_2 = -\xi\omega_n - \sqrt{\xi^2 - 1}$$

$$\therefore s_1 = -s_a = \xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$$

$$s_2 = -s_b = \xi\omega_n + \omega_n\sqrt{\xi^2 - 1}.$$

Then the unit step signal $x(t) = 1$; $R(s) = \frac{1}{s}$

Response $C(s) = R(s) \cdot \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

$$= \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

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$$= \frac{\omega n^2}{S(S+S_1)(S+S_2)}$$

By applying partial Fractions

$$C(S) = \frac{A}{S} + \frac{B}{S+S_1} + \frac{C}{S+S_2}$$

$$A = C(S) \times S \Big|_{S=0} = \frac{\omega n^2}{S(S+S_1)(S+S_2)} \times S$$

$$= \frac{\omega n^2}{S_1 S_2}$$

$$= \frac{\omega n^2}{(\xi\omega n - \omega n\sqrt{\xi^2-1})(\xi\omega n + \omega n\sqrt{\xi^2-1})}$$

$$= \frac{\omega n^2}{\xi^2\omega n^2 + \xi\omega n^2\sqrt{\xi^2-1} - \xi\omega n^2\sqrt{\xi^2-1} - \omega n^2(\xi^2-1)}$$

$$= \frac{\omega n^2}{\omega n^2(\xi^2-\xi^2+1)} = 1$$

$$\therefore \boxed{A = 1}$$

$$B = C(S) \times (S+S_1) \Big|_{S=-S_1} = \frac{\omega n^2}{S(S+S_1)(S+S_2)} (S+S_1)$$

$$= \frac{\omega n^2}{-S(-S_1+S_2)} = \frac{-\omega n^2}{S_1(-S_1+S_2)}$$

$$= \frac{-\omega n^2}{(-\xi\omega n + \sqrt{\xi^2-1}\,\omega n + \xi\omega n + \omega n\sqrt{\xi^2-1})} \cdot \frac{1}{S_1}$$

$$= \frac{-\omega n^2}{2\omega n\sqrt{\xi^2-1}} = \frac{-\omega n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{S_1}$$

$$\therefore B = \frac{-\omega n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{S_1}$$

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$$C = C(S) \times (S+S_2)\Big|_{S=-S_2} = \frac{\omega_n^2}{S(S+S_1)(S+S_2)} \times (S+S_2)$$

$$= \frac{\omega_n^2}{-S_2(-S_2+S_1)} = \frac{-\omega_n^2}{S_2(-S_2+S_1)}$$

$$= \frac{-\omega_n^2}{(-\xi\omega_n - \omega_n\sqrt{\xi^2-1} + \xi\omega_n - \omega_n\sqrt{\xi^2-1})} \cdot \frac{1}{S_2}$$

$$= \frac{-\omega_n^2}{-2\omega_n\sqrt{\xi^2-1}} \cdot \frac{1}{S_2}$$

$$\therefore C = \frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{S_2}$$

$$\therefore C(S) = \frac{A}{S} + \frac{B}{S+S_1} + \frac{C}{S+S_2}$$

$$= \frac{1}{S} + \frac{\left(\frac{-\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{S_1}\right)}{S+S_1} + \frac{\left(\frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{S_2}\right)}{S+S_2}$$

$$= \frac{1}{S} - \frac{\omega_n}{(2\sqrt{\xi^2-1} \cdot S_1)(S+S_1)} + \frac{\omega_n}{(2\sqrt{\xi^2-1} \cdot S_2)(S+S_2)}$$

$$= \frac{1}{S} - \frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{S_1} \cdot \frac{1}{S+S_1} + \frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{S_2} \cdot \frac{1}{S+S_2}$$

Apply inverse laplace transform

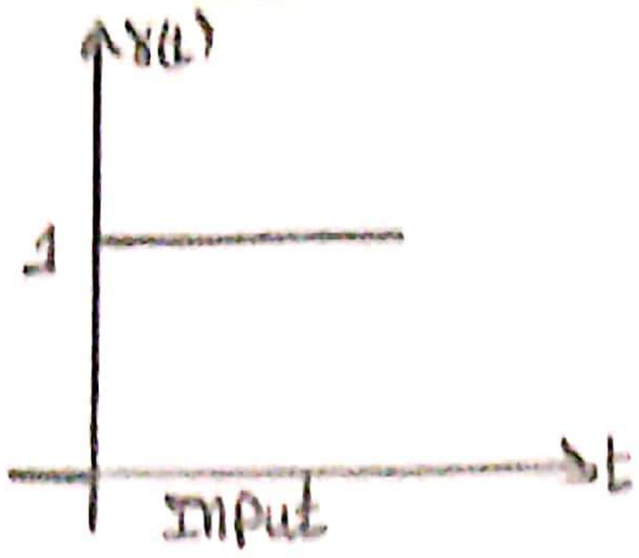
$$C(t) = 1 - \frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{S_1} e^{-S_1 t} + \frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{S_2} e^{-S_2 t}$$

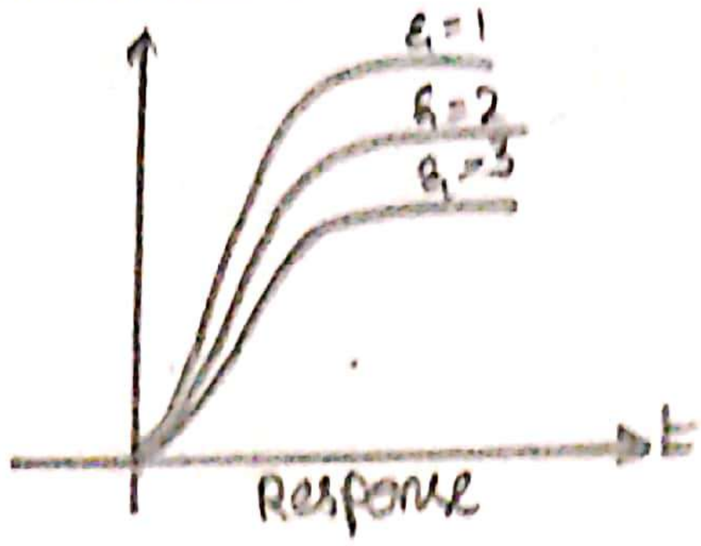
$$= 1 - \frac{\omega_n}{2\sqrt{\xi^2-1}} \left[\frac{1}{S_1} e^{-S_1 t} + \frac{1}{S_2} e^{-S_2 t}\right]$$

where $S_1 = \xi\omega_n - \omega_n\sqrt{\xi^2-1}$

$S_2 = \xi\omega_n + \omega_n\sqrt{\xi^2-1}$

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- the response has no oscillations but it takes longer time for the response to reach the final steady state value.

Time domain specifications:-

- The desired performance characteristics of control system are specified in terms of domain specifications.
- The transient response of a system to a unit step input depends on the initial conditions.

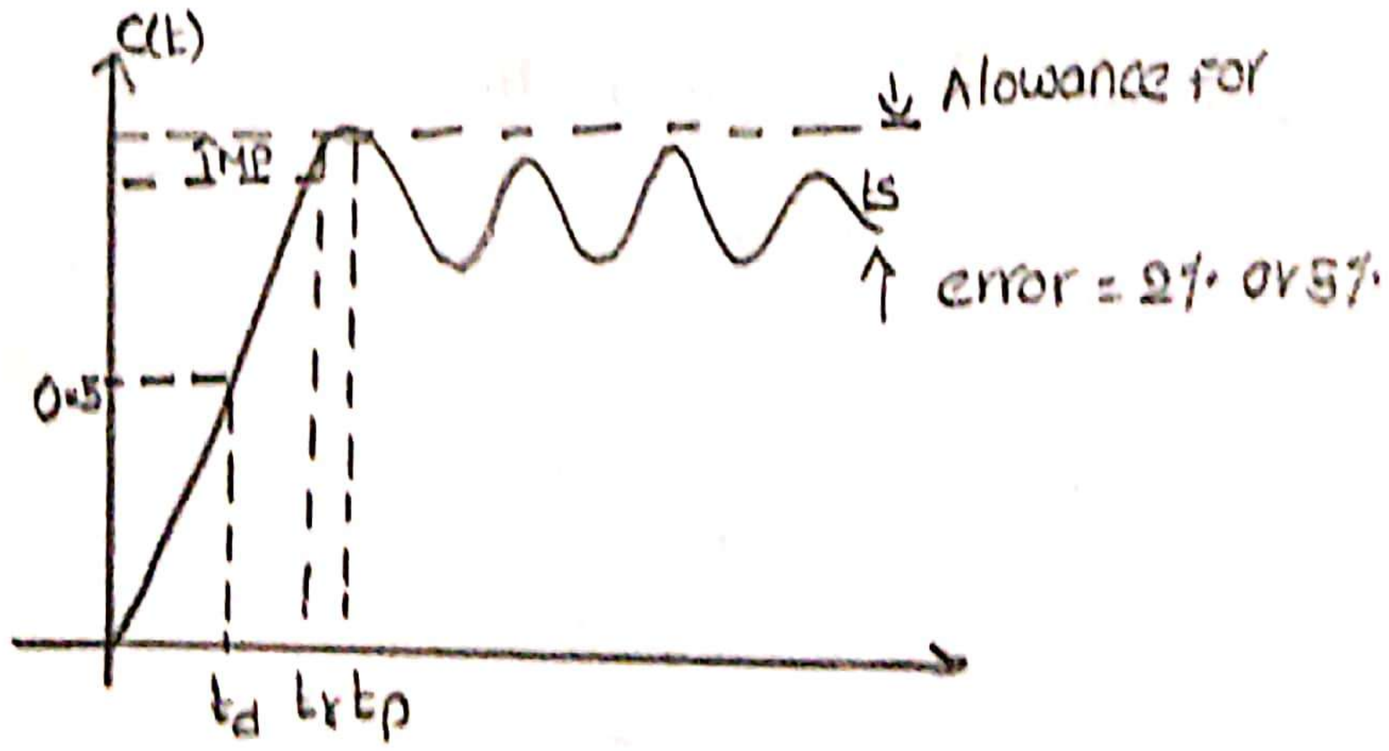
(i) Delay time (t_d)

(ii) Rise time (t_r)

(iii) peak time (t_p)

(iv) peak/Maximum overshoot (Mp)

(v) settling time (t_s)



- The time domain specifications are defined as follows:

Delay time:- It is the time taken for response to reach 50% of the final value for the very first time.

Rise time:- It is the time taken for response to reach rise 0 to 100% for the very first time.

- For underdamped system, the rise is 0 to 100%
- For overdamped system, the rise is 10% to 90%.
- For 5% to 95% the system is critically damped.

Peak time:- It is the time taken for the response to reach peak value, the very first time.

Peak/Maximum overshoot:- It is defined as the ratio of maximum peak value to final peak value, where the maximum peak value is measured from final value.

$$M_p = \frac{C(t_p) - C(\infty)}{C(\infty)} \quad ; \quad \%M_p = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100$$

where, $C(t_p)$ = Max. value of peak ;

$C(\infty)$ = Final peak value.

Settling value:- It is defined as the time taken by a response to reach and stay within specified exxor.

- It is usually expressed as % of Final value. The usual tolerable error is 2% or 5% of the Final value.

⇒Rise time:- The response of underdamped system of 2nd order system is

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

$$\therefore C(t) = C(t_\alpha) = 1 \Rightarrow t = t_\alpha = '$$

$$1 = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t_\alpha + \theta)$$

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$$\theta = -\frac{e^{-\xi\omega_n t_\delta}}{\sqrt{1-\xi^2}} \sin(\omega_d t_\delta + \theta)$$

$$-e^{-\xi\omega_n t_\delta} \sin(\omega_d t_\delta + \theta) = 0$$

$$e^{-\xi\omega_n t_\delta} \neq 0$$

But $\sin(\omega_d t_\delta + \theta) = 0$

$$\omega_d t_\delta + \theta = \sin^{-1}(0)$$

$$\omega_d t_\delta + \theta = \pi$$

$$\omega_d t_\delta = \pi - \theta$$

$$t_\delta = \frac{\pi - \theta}{\omega_d} \text{ in secs} \qquad \because \theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

peak time (t_p):-

- The unit step response of underdamped 2nd order system is

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

Differentiating $C(t)$ w.r.t to 't'

$$\frac{d}{dt} C(t) = 0 \Big|_{t=t_p} \Rightarrow$$

$$\frac{d}{dt} C(t) = 0 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} (-\xi\omega_n) \sin(\omega_d t + \theta) - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cos(\omega_d t + \theta)\omega_d$$

$$= \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[\xi\omega_n \sin(\omega_d t + \theta) - \cos(\omega_d t + \theta)\omega_d\right]$$

$$= \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \left[\xi\omega_n \sin(\omega_d t + \theta) - \cos(\omega_d t + \theta)\omega_n\sqrt{1-\xi^2}\right]$$

$$= \frac{\omega_n e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \left[\xi \sin(\omega_d t + \theta) - \cos(\omega_d t + \theta)\sqrt{1-\xi^2}\right].$$

$$= \frac{\omega_n e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \left[\sin(\omega_d t + \theta)\cos\theta - \cos(\omega_d t + \theta)\sin\theta\right]$$

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$$0 = \frac{\omega_n e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin\left[(\omega_d t_p + \theta) - \theta\right]$$

$$\sin(\omega_d t_p) = 0$$

$$\omega_d t_p = \sin^{-1}(0) = \pi$$

$$\therefore \quad t_p = \frac{\pi}{\omega_d}$$

peak overshoot:-

- The unit step response of underdamped 2nd order system is

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

$$\%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

At $t = \infty$,

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

$$c(\infty) = 1 - \frac{e^{-\xi\omega_n(\infty)}}{\sqrt{1-\xi^2}} \sin(\omega_d \infty + \theta)$$

$$c(\infty) = 1 - 0 = 1$$

$\Rightarrow c(t) = c(t_p)$

$$c(t_p) = 1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \theta)$$

$$= 1 - \frac{e^{-\xi\omega_n\left(\frac{\pi}{\omega_d}\right)}}{\sqrt{1-\xi^2}} \sin\left(\omega_d \frac{\pi}{\omega_d} + \theta\right)$$

$$= 1 - \frac{e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin(\pi + \theta)$$

$$= 1 - \frac{e^{-\xi \frac{\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} (-\sin\theta)$$

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$$= 1 + \frac{e^{-\zeta\pi/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}} \quad (\sqrt{1-\zeta^2})$$

$$= 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\%M_p = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}} - 1 \times 100$$

$$\therefore \%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

Settling time (t_s):-

The response of 2nd order underdamped system is

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

If error is 2%.

$$\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$$

$$e^{-\zeta\omega_n t_s} = 0.02$$

$$-\zeta\omega_n t_s = \ln(0.02) = -4$$

$$\zeta\omega_n t_s = 4$$

$$t_s = \frac{4}{\zeta\omega_n}$$

$$\therefore t_s = 4T$$

If error is 5%.

$$e^{-\zeta\omega_n t_s} = 0.05$$

$$-\zeta\omega_n t_s = \ln(0.05)$$

$$-\zeta\omega_n t_s = -3$$

$$t_s = \frac{3}{\zeta\omega_n}$$

$$\therefore t_s = 3T$$

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problems:-

1. The unity feedback of system of open loop transfer function $G_1(s)=\frac{k}{s(s+10)}$. Determine Gain(k), so that the system will have a damping of 0.5, this value of k determine the peak overshoot for end time T, a unit step input.

Sol:- Given $G_1(s)=\frac{k}{s(s+10)}$

W.k.T closed loop transfer function is $\frac{G_1(s)}{1+G_1(s)H(s)}$

$$= \frac{\frac{k}{s(s+10)}}{1+\frac{k}{s(s+10)}\cdot 1} = \frac{k}{s(s+10)+k} = \frac{k}{s^2+10+k} \quad \text{---(1)}$$

The 2nd order transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \quad \text{---(2)}$$

By comparing eq-(1) and eq-(2)

$2\xi\omega_n = 10$ $\qquad$ $\omega_n^2 = k$

$2\times 0.5\times\omega_n = 10$ $\qquad$ $k = 10^2$

$\therefore \omega_n = 10$ $\qquad$ $\therefore k = 100$

peak overshoot $(M_p) = e^{-\xi\pi/\sqrt{1-\xi^2}}$

$$= e^{-0.5\pi/\sqrt{1-(0.5)^2}} = 0.163 \text{ sec} = 16.3\%$$

peak time $(t_p) = \frac{\pi}{\omega_d}$

$$= \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{10\sqrt{1-(0.5)^2}} = 0.362 \text{ sec} = 36.2\%.$$

2. A unity feedback has an open loop transfer function $G_1(s)=\frac{10}{s(s+2)}$. Find the rise time, %Mp, tp and settling time for a step input of 12 units.

30|:- Given $G_1(S) = \frac{10}{S(S+2)}$

W.K.T closed loop transfer function is $\frac{G(S)}{1+G_1(S)H(S)}$

$$= \frac{\frac{10}{S(S+2)}}{1+\frac{10}{S(S+2)}} = \frac{10}{S(S+2)+10} = \frac{10}{S^2+2S+10} \quad \text{——(1)}$$

The 2nd order Transfer Function is

$$\frac{C(S)}{R(S)} = \frac{\omega n^2}{S^2+2\xi\omega n S+\omega n^2} \quad \text{——(2)}$$

By comparing eq-(1) & eq-(2)

$2\xi\omega n = 2$ $\qquad$ $\omega n^2 = 10$

$\xi\omega n = 1$ $\qquad$ $\omega n = \sqrt{10} = 3.16$

$\xi(3.16) = 1$

$\therefore \xi = 0.316$

Rise time $(t_r) = \frac{\pi-\theta}{\omega_d}$; $\omega_d = \omega n\sqrt{1-\xi^2}$

$\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$ $\qquad$ $= 3.16\sqrt{1-(0.316)^2}$

$= \tan^{-1}\left(\frac{\sqrt{1-(0.316)^2}}{0.316}\right)$ $\qquad$ $= 2.99$

$= 71.57$

(i) $t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-71.57}{2.99} = \frac{-68.42}{2.99} = -22.88$ sec

(ii) peak time $(t_p) = \frac{\pi}{\omega_d} = \frac{\pi}{\omega n\sqrt{1-\xi^2}} = \frac{\pi}{2.99} = 1.05$ sec

(iii) $M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = e^{-0.316\times\pi/\sqrt{1-(0.316)^2}} = 0.351$

$\%M_p = \frac{0.351}{100}\times 12 = 4.21\%$

(iv) Settling time :-

If 2% $t_s = 4T = 4$

If 5% $t_s = 3T = 3(1) = 3$

$\because T = \frac{1}{\xi\omega n} = \frac{1}{0.316\times 3.16} = 1$

3. Obtain the response of unity feedback system whose open loop transfer function $G(S) = \frac{4}{S(S+5)}$ and when the input is unit step signal.

Sol:- Given $G(S) = \frac{4}{S(S+5)}$

w.k.t closed loop transfer function is

$$= \frac{G(S)}{1+G(S)H(S)} = \frac{\frac{4}{S(S+5)}}{1+\frac{4}{S(S+5)}} = \frac{4}{S(S+5)+4} = \frac{4}{S^2+5S+4}$$

The unit step input $x(t) = 1$; $R(S) = \frac{1}{S}$.

The Response in s-domain $C(S) = R(S) \cdot \frac{4}{S^2+5S+4}$

$$= \frac{1}{S} \cdot \frac{4}{S^2+5S+4} = \frac{4}{S(S^2+5S+4)}.$$

By applying partial Fraction

$$C(S) = \frac{4}{S(S+1)(S+4)} = \frac{A}{S} + \frac{B}{S+1} + \frac{C}{S+4}$$

$$= \frac{A(S+1)(S+4) + B \cdot S(S+4) + C \cdot S(S+1)}{S(S+1)(S+4)}$$

$$4 = A(S+1)(S+4) + BS(S+4) + C \cdot S(S+1)$$

$C(S)|_{S=0}$ => $4 = A(0+1)(0+4) + 0 + 0$

$4 = 4A \Rightarrow A = 1$

$C(S)|_{S=-1}$ => $4 = A(-1+1)(-1+4) + B(-1)(-1+4) + C(-1)(-1+1)$

$4 = -3B \Rightarrow B = \frac{-4}{3}$.

$C(S)|_{S=-4}$ => $4 = A(-4+1)(-4+4) + B(-4)(-4+4) + C(-4)(-4+1)$

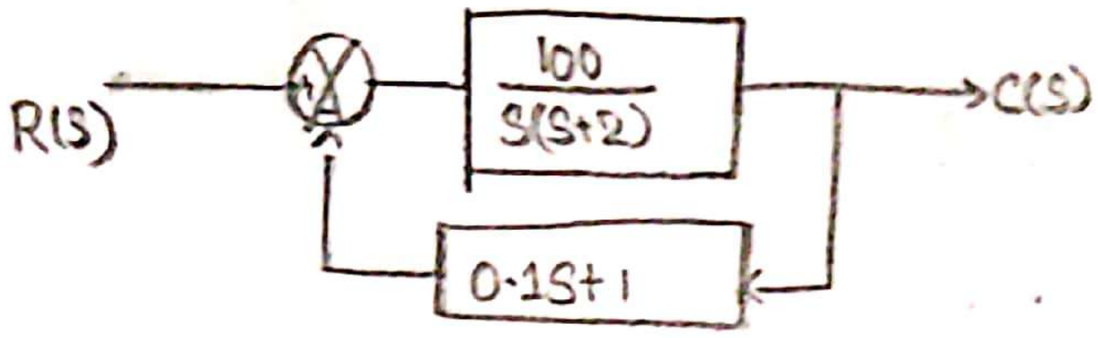
$4 = 12C \Rightarrow C = \frac{1}{3}$

$$\therefore C(S) = \frac{1}{S} - \frac{4}{3(S+1)} + \frac{1}{3(S+4)}$$

Apply inverse laplace Transform

$$\therefore C(t) = 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t}.$$

4. A positional control system with velocity feedback is shown in Fig. What is the response of the system for unit step input signal



Sol:- Given $G(S) = \frac{100}{S(S+2)}$; $H(S) = 0{\cdot}1S + 1$

W.K.T Closed loop transfer function $\frac{G(S)}{1+G(S)H(S)}$

$$= \frac{\frac{100}{S(S+2)}}{1+\frac{100}{S(S+2)}[0{\cdot}1S+1]} = \frac{100}{S(S+2)+100(0{\cdot}1S+1)} = \frac{100}{S^2+12S+100}.$$

The unit step response $x(t) = 1$; $x(S) = \frac{1}{S}$

$$C(S) = R(S) \cdot \frac{100}{S^2+12S+100}$$

$$= \frac{1}{S} \cdot \frac{100}{S^2+12S+100} = \frac{100}{S(S^2+12S+100)}$$

characteristic equation is $S(S^2+12S+100) = 0$

The roots are $S = 0, -6+8i, -6-8i$.

Applying partial Fractions

$$\frac{100}{S(S^2+12S+100)} = \frac{A}{S} + \frac{BS+C}{S^2+12S+100}$$

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$$100 = A(S^2+12S+100) + S(BS+C)$$

$$100 = AS^2+12AS+A100+BS^2+CS$$

comparing S^2, S and constants of both sides.

S^2 :-

$0 = A+B$

$0 = 1+B$

$B = -1$

C :-

$100 = 100A$

$A = 1$

S :-

$0 = 12A+C$

$0 = 12+C$

$C = -10$

$$\therefore C(S) = \frac{1}{S} + \frac{(-S-10)}{S^2+12S+100} = \frac{1}{S} - \frac{S+12}{S^2+12S+100}$$

$$= \frac{1}{S} - \frac{(S+6+6)}{S^2+12S+100}$$

$$= \frac{1}{S} - \frac{(S+6)}{S^2+12S+36+64} - \frac{6}{S^2+12S+36+64}$$

$$= \frac{1}{S} - \frac{(S+6)}{(S+6)^2+8^2} - \frac{6}{(S+6)^2+8^2}$$

$$= \frac{1}{S} - \frac{(S+6)}{(S+6)^2+8^2} - \frac{6}{8}\cdot\frac{8}{(S+6)^2+8^2}$$

By applying inverse laplace transform

$$c(t) = 1 - e^{-6t}\cos 8t - \frac{6}{8}e^{-6t}\sin 8t$$

$$\therefore c(t) = 1 - e^{-6t}\left[\cos 8t + \frac{6}{8}\sin 8t\right].$$

5. A unity feedback control system is characterized by the following open loop transfer function $G(S) = \frac{0.4S+1}{S(S+0.6)}$. Determine the transient response for unit step signal and also determine maximum over shoot and peak time.

Sol:- Given $G(S) = \frac{0.4S+1}{S(S+0.6)}$

W.K.T closed loop Transfer Function $= \frac{G(S)}{1+G(S)H(S)}$

$$= \frac{\frac{0.4S+1}{S(S+0.6)}}{1+\frac{0.4S+1}{S(S+0.6)}} = \frac{0.4S+1}{S^2+0.6S+0.4S+1} = \frac{0.4S+1}{S^2+S+1}$$

The unit step response $x(t)=1 \; ; x(S)=\frac{1}{S}$

$$C(S) = R(S) \cdot \frac{0.4S+1}{S^2+S+1}$$

$$= \frac{1}{S} \cdot \frac{0.4S+1}{S^2+S+1} = \frac{0.4S+1}{S(S^2+S+1)}.$$

By applying partial Fraction

$$C(S) = \frac{0.4S+1}{S(S^2+S+1)} = \frac{A}{S} + \frac{BS+C}{S^2+S+1}$$

$$0.4S+1 = A(S^2+S+1)+S(BS+C)$$

$C(S)\big|_{S=0}$ $\Rightarrow$ $0.4(0)+1 = A(0+0+1)+0$

$1 = A \Rightarrow \boxed{A=1}$

By comparing coefficients of S^2 and S

S^2:- $0 = A+B$, $0 = 1+B$, $B = -1$

S:- $0.4 = A+C$, $0.4 = 1+C$, $C = 0.4-1 = -0.6$.

$$\therefore C(S) = \frac{1}{S} + \frac{-S-0.6}{S^2+S+1}$$

$$= \frac{1}{S} - \frac{S+0.6}{S^2+S+1}$$

$$= \frac{1}{S} - \frac{S+0.6}{S^2+S+0.25+0.75}$$

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$$= \frac{1}{S} - \frac{S+0.6}{(S^2+2\times0.5S+0.5)^2+0.75}$$

$$= \frac{1}{S} - \frac{S+0.5+0.1}{(S+0.5)^2+0.75}$$

$$C(S) = \frac{1}{S} - \frac{S+0.5}{(S+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}} \cdot \frac{\sqrt{0.75}}{(S+0.5)^2+0.75}$$

By applying inverse laplace transform

$$C(t) = 1 - e^{-0.5t}\cos\sqrt{0.75}\,t - \frac{0.1}{\sqrt{0.75}}\, e^{-0.5t}\sin\sqrt{0.75}\,t$$

$$= 1 - e^{-0.5t}\left[0.1155\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right].$$

Now, By comparing characteristic equation with denominator

$$S^2+2\xi\omega_n S+\omega_n^2 = S^2+S+1$$

$$\omega_n^2 = 1 \qquad 2\xi\omega_n = 1$$

$$\omega_n = 1 \qquad 2\xi = 1$$

$$\xi = \frac{1}{2} = 0.5$$

$$M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = e^{-0.5\pi/\sqrt{1-0.5^2}} = 0.163$$

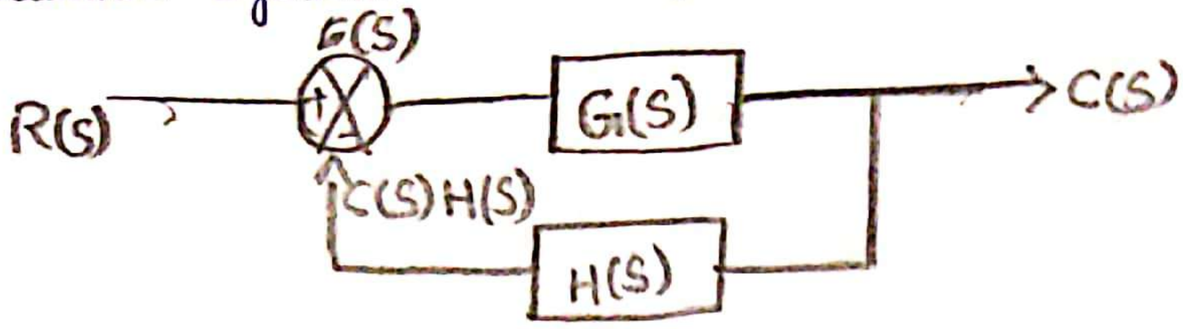
$$\therefore \%M_p = 0.613\times100 = 16.3\%$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{1\times\sqrt{1-0.5^2}} = 3.62 \text{ sec.}$$

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Steady state Error :-

- It is defined as error signal when $t \to \infty$.
- It measures system accuracy.



$$E(S) = R(S) - C(S)\,H(S) \quad \text{——(1)}$$

$$C(S) = E(S)\;G(S) \quad \text{——(2)}$$

Sub Eq-(2) in eq-(1)

$$E(S) = R(S) - E(S)\,G(S)\,H(S)$$

$$E(S) + E(S)G(S)H(S) = R(S)$$

$$E(S)\,[1+G(S)H(S)] = R(S)$$

$$E(S) = \frac{R(S)}{1+G(S)H(S)}$$

We know that $e(t) = L^{-1}[E(S)] = L^{-1}\left[\frac{R(S)}{1+G(S)H(S)}\right]$

- The steady state error is defined as the value of $e(t)$ when $t \to \infty$

$$\therefore e_{ss} = \lim_{t\to\infty} e(t)$$

- The Final value theorem of laplace transform states that if

$$F(S) = L.T\,[f(t)]$$

then $f(t) = \lim_{S\to 0} f(S)$

using Final value theorem of steady state error e_{ss},

$$e_{ss} = \lim_{S\to 0} e(t)$$

$$\therefore e_{ss} = \lim_{S\to 0} S\cdot E(S)$$

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Type of system:-

- This type number is specified open loop transfer function $G(s)H(s)$, the no.of poles of loop transfer function lying at origin decides the type number of the system.

$$G(s)H(s) = K\frac{P(s)}{Q(s)} = K\cdot\frac{(s+z_1)(s+z_2)}{s^N(s+p_1)(s+p_2)}\cdots$$

$z_1, z_2 \cdots$ are zeroes of system

$p_1, p_2 \cdots$ are poles of system

N = No.of poles at origin

- Simplifying the value of N decides the type number of the control system if $N=0$, then Type '0' system.

$N=1$ then Type '1' system.

$N=2$ then Type '2' system.

Static Error constants:-

- It depends on type error and static constants, signal is corresponding

(i) positional Error constant $(K_p) = \lim_{s\to 0} G(s)H(s)$

(ii) velocity error constant $(K_v) = \lim_{s\to 0} s\,G(s)H(s)$

(iii) Acceleration Error constant $(K_a) = \lim_{s\to 0} s^2 G(s)H(s)$.

Here K_p, K_a, K_v are static error constants.

(i) steady state error signal is unit step signal:-

$$e_{ss} = \lim_{s\to 0} s\,E(s)$$

$$e_{ss} = \lim_{s\to 0} s\frac{R(s)}{1+G(s)H(s)}$$

the unit step i/p signal $x(t)=1$; $R(s)=\frac{1}{s}$

$$e_{ss} = \lim_{s\to 0} s\,\frac{1/s}{1+G(s)H(s)}$$

$$= \lim_{s\to 0} \frac{1}{1+G(s)H(s)}$$

$$e_{ss} = \frac{1}{1 + \lim_{s\to 0} G_1(s)H(s)} = \frac{1}{1+K_p}$$

For Type '0' system:

$$K_p = \lim_{s\to 0} G_1(s)H(s)$$

$$= \lim_{s\to 0} K\frac{(s+z_1)(s+z_2)}{s^1(s+p_1)(s+p_2)}$$

$$= K\cdot\frac{z_1 z_2}{p_1 p_2}$$

K_p = constant

$$\therefore e_{ss} = \frac{1}{1+\text{constant}} = \frac{1}{\text{constant}} = \text{constant}$$

For Type '1' system :-

$$K_p = \lim_{s\to 0} G_1(s)H(s)$$

$$= \lim_{s\to 0} K\frac{(s+z_1)(s+z_2)}{s^1(s+p_1)(s+p_2)}$$

$$K_p = \frac{1}{0} = \infty$$

$$\therefore e_{ss} = \frac{1}{1+\infty} = \frac{1}{\infty} = 0.$$

Steady state error input is unit ramp signal:-

$$e_{ss} = \lim_{s\to 0} s\cdot E(s)$$

$$e_{ss} = \lim_{s\to 0} s\frac{R(s)}{1+G_1(s)H(s)}$$

the unit ramp signal $x(t) = 1$; $R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s\to 0} s\frac{1/s^2}{1+G_1(s)H(s)}$$

$$= \lim_{s\to 0} \frac{1}{s(1+G_1(s)H(s))}$$

$$= \frac{1}{\lim_{s\to 0} sG_1(s)H(s)}$$

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$$= \frac{1}{1+K_v}$$

For Type '0' system

$$K_v = \lim_{s\to 0} s\, G(s)H(s)$$

$$= \lim_{s\to 0} s\, \frac{K(s+z_1)(s+z_2)}{s^0(s+p_1)(s+p_2)}$$

$$K_v = 0$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$

For Type '1' system

$$K_v = \lim_{s\to 0} s\, G(s)H(s)$$

$$= \lim_{s\to 0} s\, k\, \frac{(s+z_1)(s+z_2)}{s^1(s+p_1)(s+p_2)}$$

$$= k\, \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots}$$

$$K_v = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{\text{constant}} = \frac{1}{\text{constant}} = \text{constant}$$

For Type-'2' system

$$K_v = \lim_{s\to 0} s^1 k\, \frac{(s+z_1)(s+z_2)}{s^1(s+p_1)(s+p_2)}$$

$$= 0$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty.$$

Steady state error when input is unit parabolic signal:-

$$e_{ss} = \lim_{s\to 0} s\, E(s)$$

$$= \lim_{s\to 0} s\, \frac{R(s)}{1+G(s)H(s)}$$

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The unit parabolic signal $x(t) = 1$; $R(s) = \frac{1}{s^3}$

$$e_{ss} = \lim_{s \to 0} s \frac{1/s^3}{1 + G(s)H(s)}$$

$$= \frac{1}{1 + \lim_{s \to 0} s^2 G(s)H(s)}$$

$$= \frac{1}{1 + K_a}.$$

For TYPE '0' system

$$K_a = \lim_{s \to 0} s^2 G(s)H(s)$$

$$= \lim_{s \to 0} s^2 K \frac{(s+z_1)(s+z_2)}{s^0(s+p_1)(s+p_2)}$$

$$= 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

For TYPE '1' system

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$= \lim_{s \to 0} s^2 K \frac{(s+z_1)(s+z_2)}{s(s+p_1)(s+p_2)}$$

$$= 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

For TYPE '2' system

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$= \lim_{s \to 0} s^2 K \frac{(s+z_1)(s+z_2)}{s^2(s+p_1)(s+p_2)}$$

$$= K \frac{z_1 z_2}{p_1 p_2}$$

$$= \text{constant}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\text{constant}} = \text{constant}.$$

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Steady state Error for various Types of Inputs:-

Type Number	Steady state Inputs		
	unit step	unit Ramp	unit parabolic
Type-0	constant $\left(\frac{1}{1+k_p}\right)$	∞	∞
Type-1	0	constant $\left(\frac{1}{k_v}\right)$	∞
Type-2	0	∞	$\left(\frac{1}{k_a}\right)$ constant
Type-3	0	∞	∞

problems:-

1. For the open loop Transfer Function given below. Explain what type of input signal give rise to constant steady state and calculate their values.

(i) $G(S) = \frac{20(S+2)}{S(S+1)(S+3)}$ (ii) $G(S) = \frac{10}{(S+2)(S+3)}$ (iii) $G(S) = \frac{10}{S^2(S+1)(S+2)}$

Sol:- (i) Given $G(S) = \frac{20(S+2)}{S(S+1)(S+3)}$

Assume unity feedback $H(S) = 1$

∵ since, it is a type-1 system. Therefore the velocity error constant is to be calculated (since it is having one pole at origin, so it is type-1 system).

Now, Steady state error $e_{ss} = \frac{1}{K_v}$

$$\therefore K_v = \lim_{S \to 0} S\, G(S)\, H(S)$$

$$= \lim_{S \to 0} S \frac{20(S+2)}{S(S+1)(S+3)}$$

$$= \frac{20 \times 2}{1 \times 3} = \frac{40}{3}$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{\frac{40}{3}} = \frac{3}{4.0} = 0.075.$$

(ii) Given $G_1(s) = \frac{10}{(s+2)(s+3)}$

since there is no poles at origin, it is type-0 system

$$K_p = \lim_{s \to 0} G_1(s)H(s)$$

$$= \lim_{s \to 0} \frac{10}{(s+2)(s+3)}$$

$$= \frac{10}{6}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\frac{10}{6}} = \frac{6}{16} = 0.375.$$

(iii) Given $G(s) = \frac{10}{s^2(s+1)(s+2)}$

Since there are two poles at origin, it is type-2 system.

$$K_a = \lim_{s \to 0} s^2 G(s)H(s)$$

$$= \lim_{s \to 0} s^2 \frac{10}{s^2(s+1)(s+2)}$$

$$= 5$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{5} = 0.2.$$

2. For a unity feedback control system, the open loop transfer function $G(s) = \frac{10(s+2)}{s^2(s+1)}$. Find

(i) the position, velocity, Acceleration

(ii) the steady state error when the input is R(s) when $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$.

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Sol:- (i) Given $G(S) = \frac{10(S+2)}{S^2(S+1)}$

kp :-

$$K_p = \lim_{S\to 0} G(S)H(S)$$

$$= \lim_{S\to 0} \frac{10(S+2)}{S^2(S+1)}$$

$$= \infty$$

$$\therefore e_{ss} = \frac{1}{1+k_p} = \frac{1}{1+\infty} = \frac{1}{\infty} = 0$$

kv :-

$$K_v = \lim_{S\to 0} S\, G(S)H(S)$$

$$= \lim_{S\to 0} S \frac{10(S+2)}{S^2(S+1)}$$

$$= \infty$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

ka :-

$$K_a = \lim_{S\to 0} S^2 G(S)H(S)$$

$$= \lim_{S\to 0} S^2 \frac{10(S+2)}{S^2(S+1)}$$

$$= 20$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{20} = 0.05$$

(i) Given $R(S) = \frac{3}{S} - \frac{2}{S^2} + \frac{1}{3S^3}$

$$\therefore e_{ss} = \lim_{S\to 0} S \frac{R(S)}{1+G(S)H(S)}$$

$$e_{ss} = \lim_{S\to 0} S \frac{\frac{3}{S} - \frac{2}{S^2} + \frac{1}{3S^3}}{1+\frac{10(S+2)}{S^2(S+1)}}$$

$$= \lim_{S\to 0} \frac{\frac{9S^2 - 2S + 1}{3S^3}}{\frac{S^2(S+1)+10(S+2)}{S^2(S+1)}}$$

$$e_{ss} = \lim_{s\to 0} s \times \frac{9s^2-6s+1}{3s} \times \frac{(s+1)}{s^3+s+10s+20}$$

$$= \lim_{s\to 0} \frac{(9s^2-6s+1)(s+1)}{3(s^3+11s+60)}$$

$$= \lim_{s\to 0} \frac{9s^3+3s^2-7s+1}{3s^3+33s+60}$$

$$\therefore e_{ss} = \frac{1}{60} = 0.016.$$

3. The open loop Transfer Function of a control system with unity Feedback is $G(s) = \frac{10}{s(0.1s+1)}$. Evaluate the static error constants of the system and obtain the steady state error of the system when subjected to an input given by the polynomial $x(t) = a_0 + a_1 t + a_2 t^2/2$.

Sol:- Given $G(s) = \frac{10}{s(0.1s+1)}$

kp :-

$$k_p = \lim_{s\to 0} G(s)H(s)$$

$$= \lim_{s\to 0} \frac{10}{s(0.1s+1)} = \frac{10}{0} = \infty$$

kv :-

$$k_v = \lim_{s\to 0} s\, G(s)H(s)$$

$$= \lim_{s\to 0} \cancel{s} \frac{10}{\cancel{s}(0.1s+1)} = \frac{10}{1} = 10$$

ka :-

$$k_a = \lim_{s\to 0} s^2 G(s)H(s)$$

$$= \lim_{s\to 0} \cancel{s^2} \frac{10}{\cancel{s}(0.1s+1)} = 0$$

(ii) Given $x(t) = a_0 + a_1 t + a_2 t^2/2$

In s-domain $x(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}$

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$$\therefore e_{ss} = \lim_{s\to 0} s \frac{R(s)}{1+G(s)H(s)}$$

$$= \lim_{s\to 0} s \frac{\frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}}{1 + \frac{10}{s(0.1s+1)}}$$

$$= \lim_{s\to 0} s \frac{\frac{a_0 s^2 + a_1 s + a_2}{s^3}}{\frac{s(0.1s+1)+10}{s(0.1s+1)}}$$

$$= \lim_{s\to 0} \frac{a_0 s^2 + a_1 s + a_2}{s} \times \frac{0.1s+1}{0.1s^2+s+10}$$

$$= \lim_{s\to 0} \frac{(a_0 s^2 + a_1 s + a_2)(0.1s+1)}{s(0.1s^2+s+10)}$$

$$= \lim_{s\to 0} \frac{0.1a_0 s^3 + 0.1a_1 s^2 + 0.1a_2 s + a_0 s^2 + a_1 s + a_2}{0.1s^3 + s^2 + 10s}$$

$$= \lim_{s\to 0} \frac{(0.1a_0)s^3 + (0.1a_1 + a_0)s^2 + (0.1a_2 + a_1)s + a_2}{0.1s^3 + s^2 + 10s}$$

$$\therefore e_{ss} = \frac{a_2}{0} = \infty .$$

4. Consider a unity feedback system with a closed loop transfer function $\frac{C(s)}{R(s)} = \frac{ks+b}{s^2+as+b}$. Determine open loop transfer function $G(s)$. Show that steady error with unit ramp input is given by $\frac{(a-k)}{b}$.

Sol:- Given closed loop transfer function

$$M(s) = \frac{C(s)}{R(s)} = \frac{ks+b}{s^2+as+b}$$

$$\Rightarrow M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$M(s)\left[1+G(s)H(s)\right] = G(s)$$

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$$G(S) = M(S) + M(S)G(S)H(S)$$

$$G(S) - M(S)G(S)H(S) = M(S)$$

$$G(S)\left[1 - M(S)H(S)\right] = M(S)$$

$$G(S) = \frac{M(S)}{1 - M(S)H(S)}$$

$$= \frac{\frac{KS+b}{S^2+aS+b}}{1 - \frac{KS+b}{S^2+aS+b}} = \frac{KS+b}{S^2+aS-KS}$$

$$\therefore G(S) = \frac{KS+b}{S^2+S[a-K]}$$

(ii) unit ramp signal = $\frac{a-K}{b}$

$$K_V = \lim_{S \to 0} S\ G(S)H(S)$$

$$= \lim_{S \to 0} \cancel{S}\ \frac{K(S+b)}{\cancel{S}[S+(a-K)]}$$

$$= \lim_{S \to 0} \frac{KS+b}{S+a-K} = \frac{b}{a-K}$$

$$\therefore e_{ss} = \frac{1}{K_V} = \frac{1}{\frac{b}{a-K}} = \frac{a-K}{b}.$$

UNIT-3 ; CONCEPTS OF STABILITY AND ROOT LOCUS

Impulse response and stability

The term stability refers to the stable working condition of a control system. Every working system is designed to be stable. In a stable system, the response or output is predictable, finite and stable for a given input (or for any changes in input or for any changes in system parameters.

The different definitions of the stability are the following

1. For a bounded input signal, if the output has constant amplitude oscillations then the system may be stable or unstable under some limited constraints. Such a system is called limitedly stable.
2. If a system output is stable for all variations of its parameters, then the system is called absolutely stable systems.
3. If a system output is stable for a limited range of variations of its parameters, then the system is called conditionally stable system.

BIBO stability

A linear relaxed system is said to have BIBO stability if every bounded (finite) input result in a bounded (finite) output.

* A condition for BIBO stability can be obtain from convolution operation define by eq"

$$r(t) = \int_{-\infty}^{\infty} m(\gamma)\, s(t-\gamma)\, d\gamma$$

Here t = dummy variable used for integration

Location of poles on s-plane for stability

* The closed transfer function m(s) can be expressed as ratio of two polynomials in "s" The denominator polynomial of closed loop Transfer function is called characteristic equation.

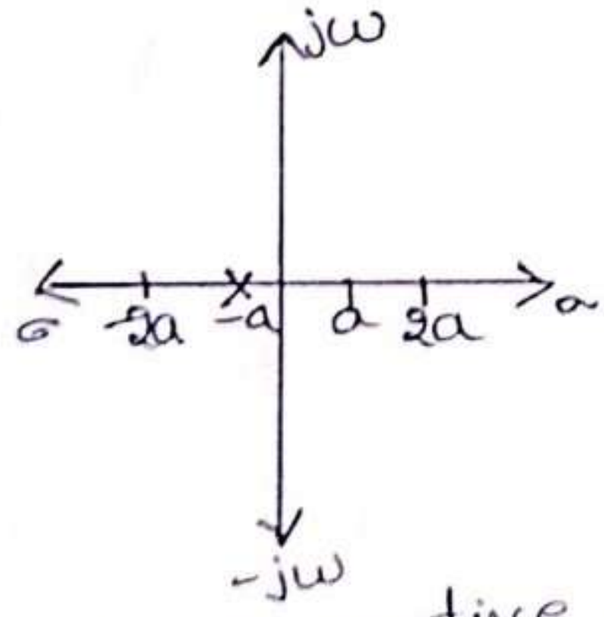
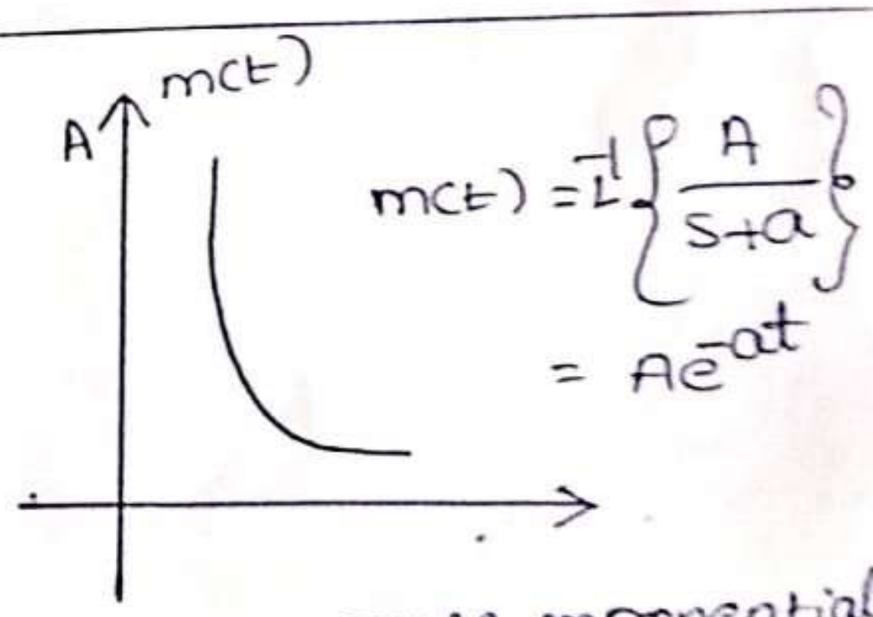
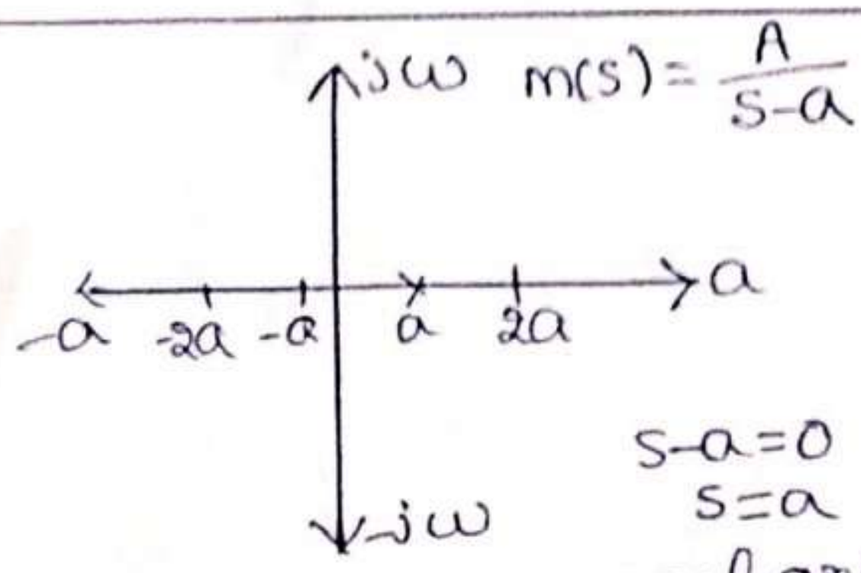
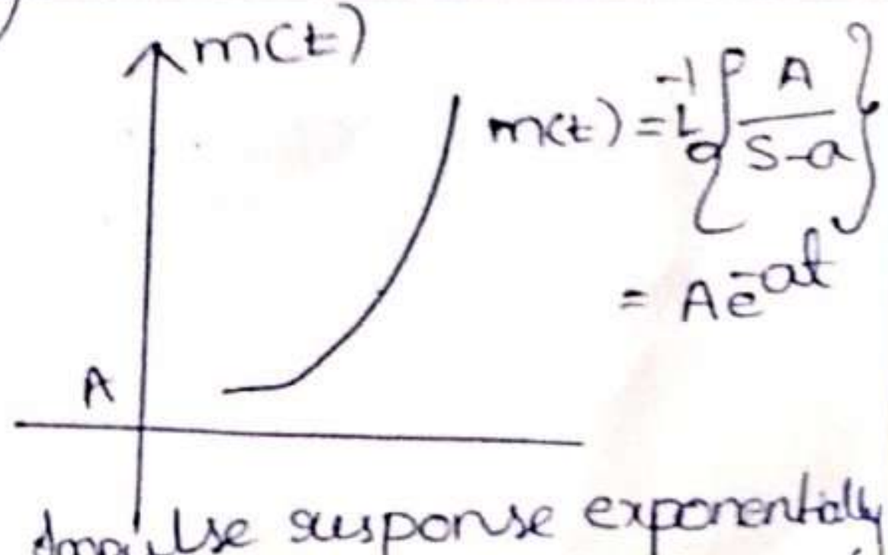
* The root of characteristic equation are poles of closed loop transfer function m(s) is given by

$$M(S) = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

$$= \frac{(s+z_1)(s+z_2) + \cdots + (s+z_m)}{(s+p_1)(s+p_2) + \cdots + (s+p_n)}$$

polynomials $z_1, z_2, z_3 \cdots z_m$ are zeroes.

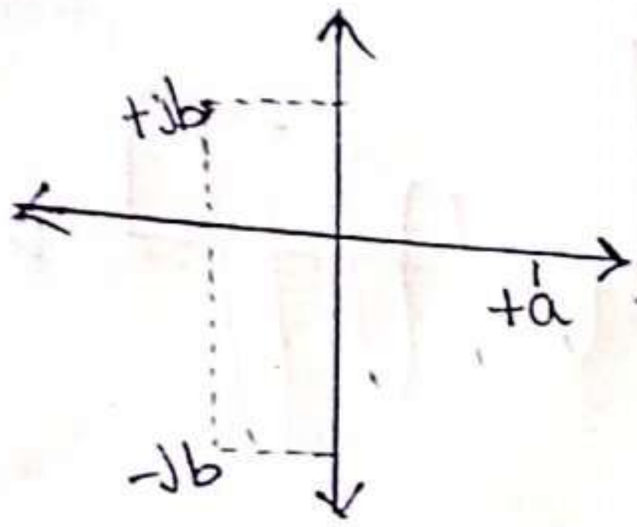
pdynomials $p_1, p_2, p_3 \cdots p_n$ are poles.

Transfer function m(s) & location of roots on s-plane	Impulse response m(t)
①  $m(s) = \frac{A}{s+a}$ $s+a=0$ $s=-a$ Roots are negative real axis	 $m(t) = L^{-1}\left\{\frac{A}{s+a}\right\}$ $= Ae^{-at}$ Impulse response exponentially decaying then it is stable system.
②  $m(s) = \frac{A}{s-a}$ $s-a=0$ $s=a$ roots are positive real axis	②  $m(t) = L^{-1}\left\{\frac{A}{s-a}\right\}$ $= Ae^{at}$ Impulse response exponentially growing then it is unstable system.

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$$M(s) = \frac{A}{s+a+jb} + \frac{A^*}{s+a-jb}$$



$s+a+jb=0$

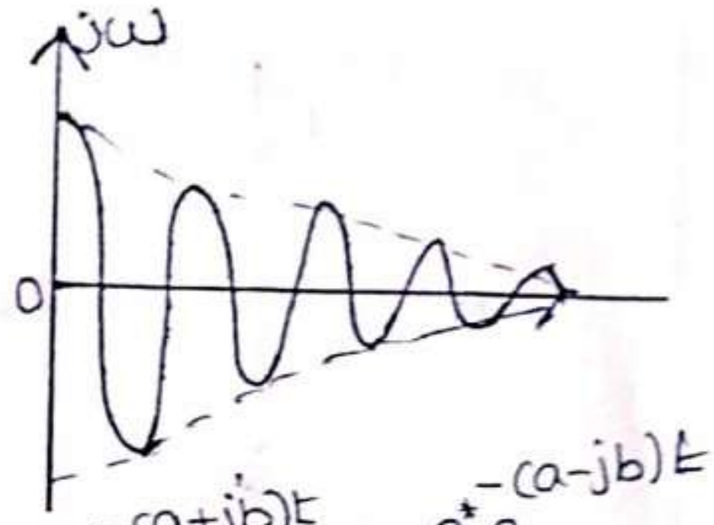
$s=-(a+jb)$

$s+a-jb=0$

$s=-(-a-jb)$

complex conjugate roots on left half of s-plane

③ $m(t) = L^{-1}\left\{\frac{A}{s+a+j\omega} + \frac{A^*}{s+a-j\omega}\right\}$

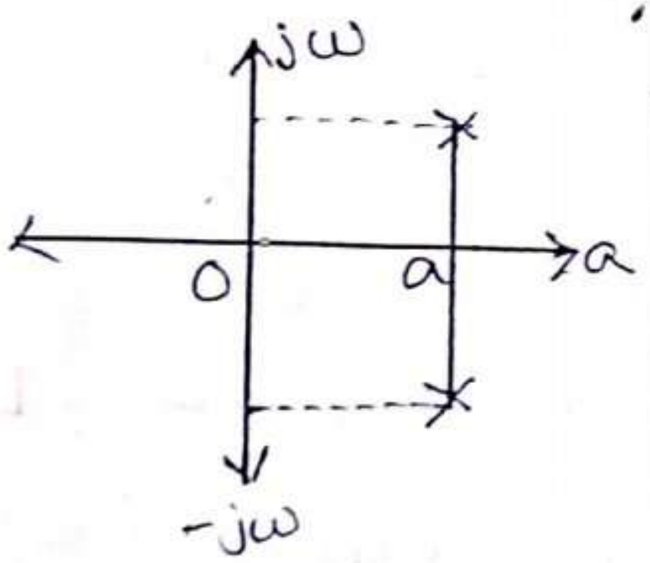


$\rightarrow Ae^{-(a+jb)t} + A^*e^{-(a-jb)t}$

$= 2Ae^{-at}\cos bt$

Impulse response is damped sinusoidal stable system.

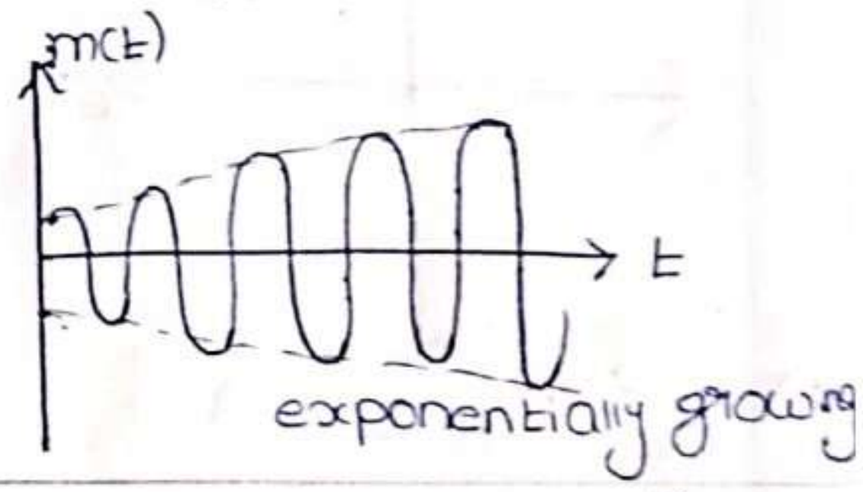
④ $M(s) = \frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}$



④ $m(t) = L^{-1}\left\{\frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}\right\}$

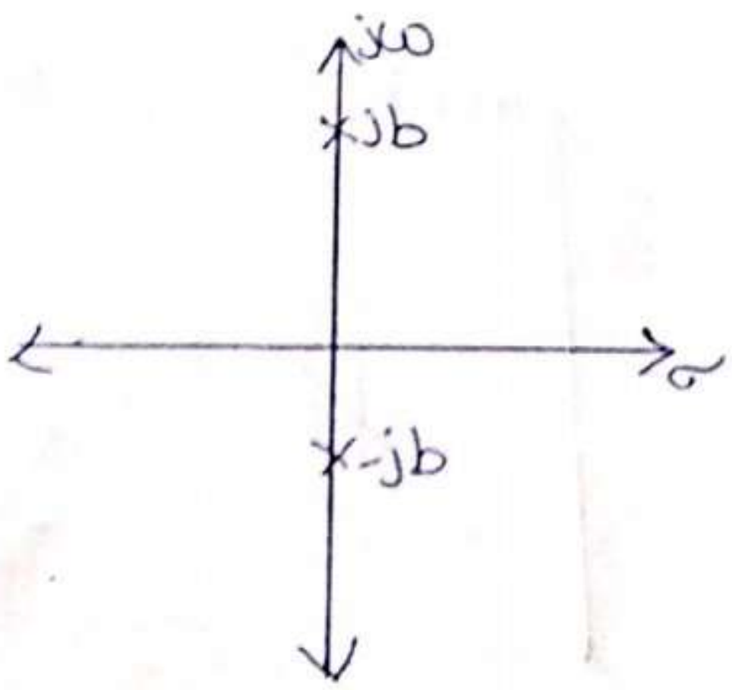
$= Ae^{-(a+jb)t} + A^*e^{-(-a+jb)t}$

$= 2Ae^{at}\cos bt$



exponentially growing

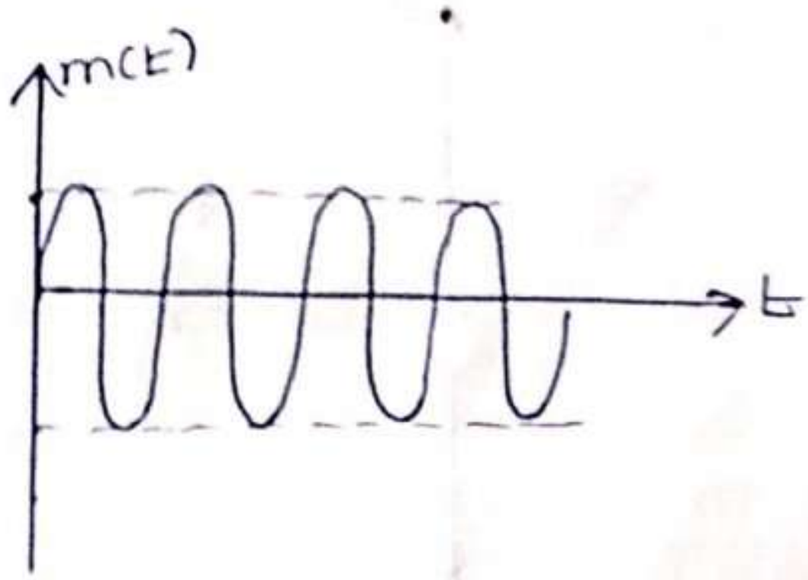
⑤ $m(s) = \frac{A}{s+jb} + \frac{A^*}{s-jb}$



⑤ $m(t) = L^{-1}\left\{\frac{A}{s+jb} + \frac{A^*}{s-jb}\right\}$

$= Ae^{jbt} + A^*e^{jbt}$

$= 2A\cos bt$

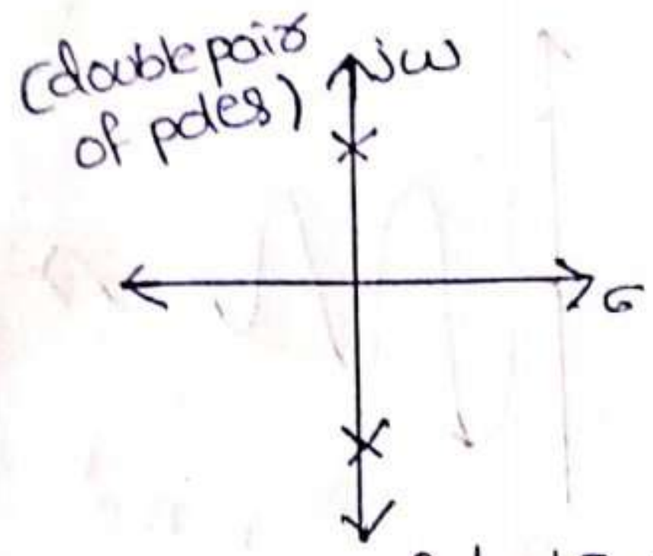


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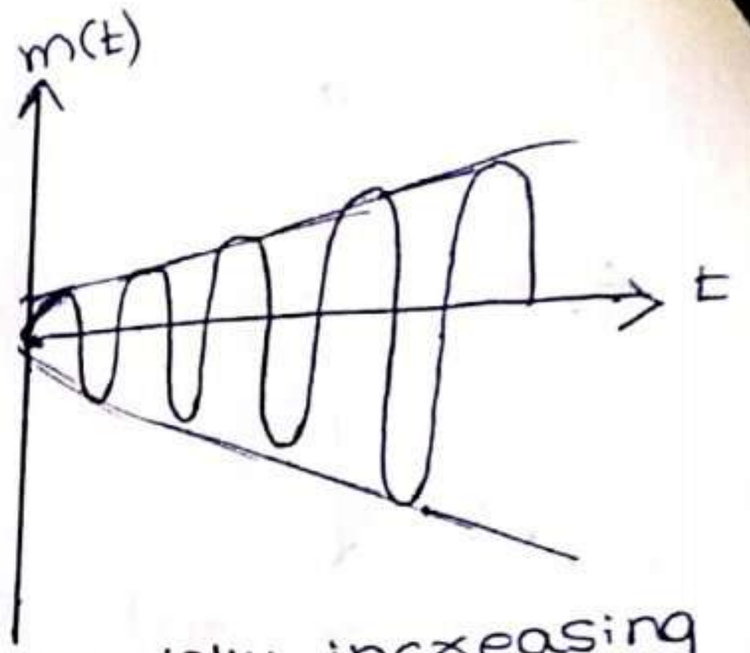
no sign changes in the

⑥ $m(s) = \frac{A}{(s+jb)^2} + \frac{A^*}{(s+jb)^2}$



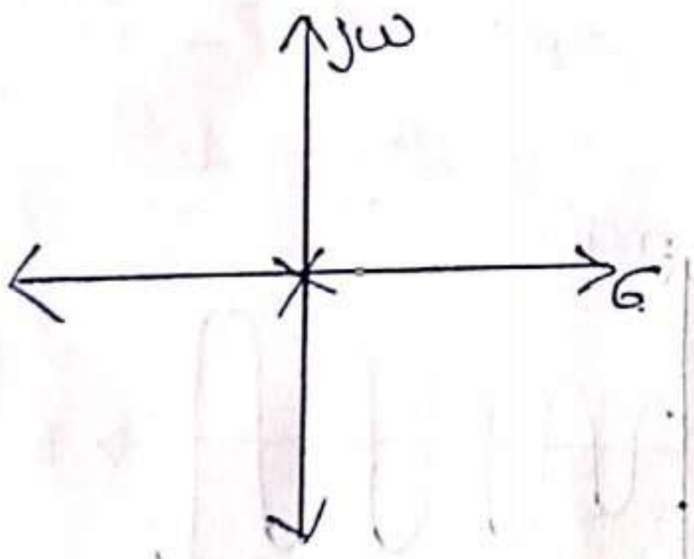
Double pair of loots on imaginary axis.

$m(t) = L^{-1}\left\{ \frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2} \right\}$



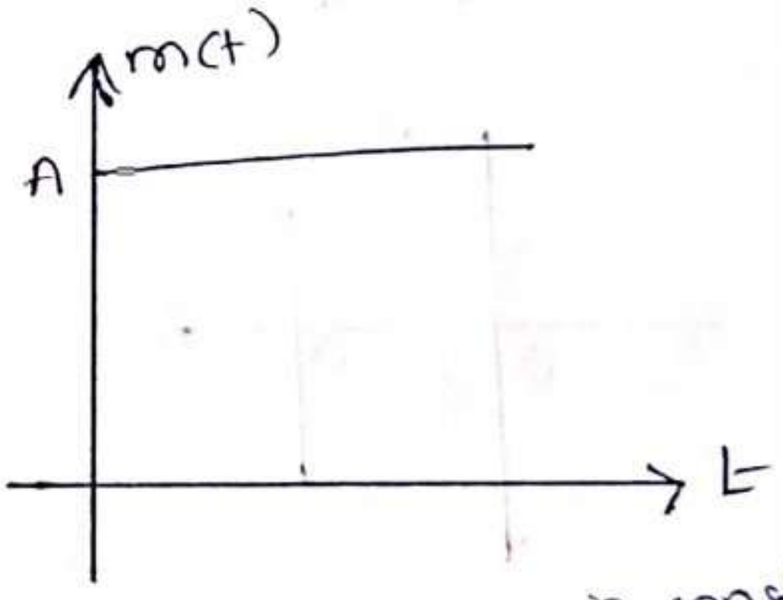
Exponentially increasing system is unstable.

① $m(s) = \frac{A}{s}$



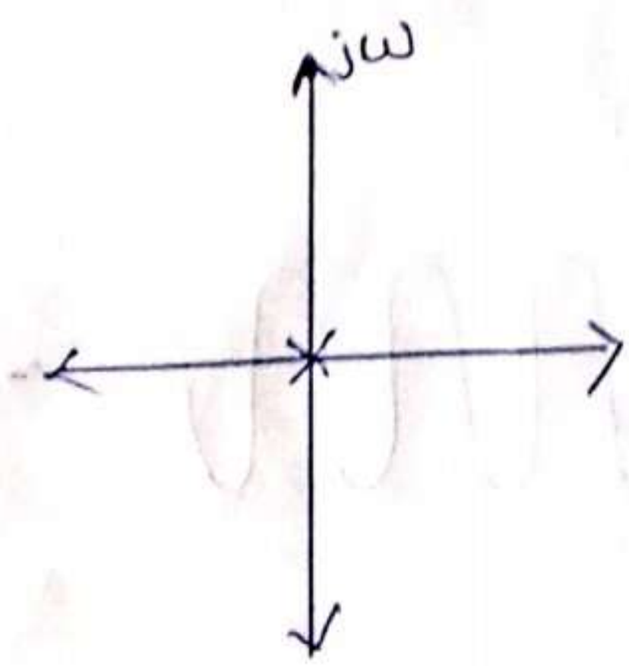
roots at origin

$m(t) = L^{-1}\left\{ \frac{A}{s} \right\}$

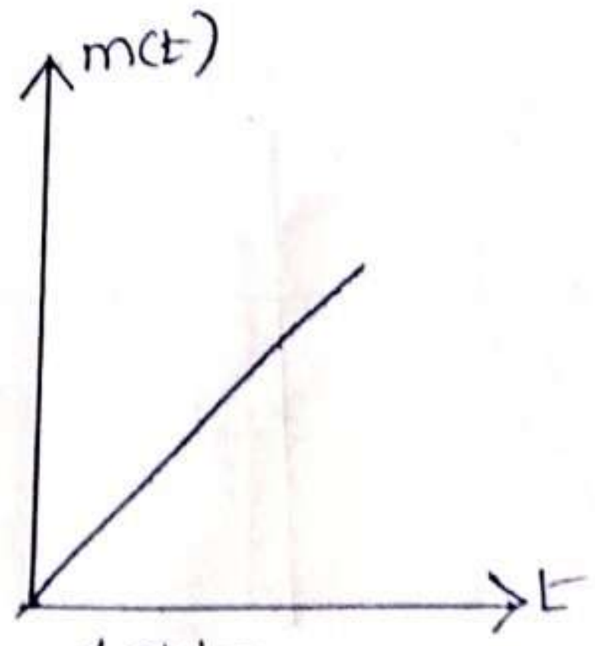


Impulse response is constant
The system is marginally stable

⑧ $m(s) = \frac{A}{s^2}$



⑧ $m(t) = L^{-1}\left\{ \frac{A}{s^2} \right\}$



unstable
impulse response is linearly increasing.

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As long as F < Fcritical, object Regains its original position but if F > Fcritical, object will come out stability depends on Certain Conditions of the system hence system is called

Conditionally stable system.

* Routh - Hurwitz Criterion :-

This Represents a Method of Determining the location of Poles of a Characteristic equation with Respect to the left half and Right half of the s-plane without actually solving the equation.

The T.F of any linear closed loop system can be Represented as,

$$\frac{C(S)}{R(S)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots\ldots + b_n}{a_0 s^n + a_1 s^{n-1} + \ldots\ldots + a_n} = \frac{B(S)}{F(S)}$$

where a and b are Constants.

To find the closed loop poles we equate F(S) = 0. This equation is called the characteristic equation of the System.

i.e, $\Rightarrow F(S) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots + a_n = 0.$

Thus, the Roots of the characteristic equation are the closed loop poles of the System which decide the stability of the System.

* Determine the Stability of the given characteristic equation by Hurwitz method.

$$F(S) : s^3 + s^2 + s^1 + 1 = 0.$$

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4. $F(s) = s^3+s^2+s+1 = 0$.

$a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 4.$

Highest order $n = 3$.

$$H = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix}$$

(D_1, D_2, D_3)

3X3

$|D_1| = |1| = 1$

$|D_2| = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = -3.$

$|D_3| = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix} = -12$

[∵ The number of roots will be determined by the highest degree of C.E.]

As D_2 and D_3 are negative, given System is unstable.

As there is one sign change so, one root lies in the right half of s-plane and the Remaining two lies in left half of the s-plane.

Disadvantages:

1. For Higher order Systems, to Solve the determinants of higher order is very Complicated and time Consuming.

2. Difficult to Predict marginal Stability of the System.

3. For the System to be Stable, all the above determinants must be positive

* Routh's Stability Criterion :

It is also called as Routh's array Method or Routh's Hurwitz Method.

Routh suggested a method of tabulating the coefficients of Characteristic Equation in a Particular way Tabulation of the coefficients gives an array called Routh's Array

Consider the general characteristic equation

$$F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0.$$

Method of Forming an array :-

s^n	a_0	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	
s^{n-3}	c_1	c_2	c_3	
$\vdots$	$\vdots$			
s^0	a_n			

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - b_3 a_1}{b_1}.$$

$$c_3 = \frac{b_1 a_7 - a_1 \times b_1}{b_1}$$

* Routh's Criterion :-

The Routh's stability criterion can be stated as,

"The necessary and sufficient condition for stability is that all of the elements in the first column of the Routh Array be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of the Routh array corresponds to the number of roots of the characteristic equation in the right half of s-plane".

In the process of constructing the Routh Array the missing terms are considered as zeros. Also, all the elements of any row can be multiplied (or) divided by a positive constant to simplify the computational works.

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In the construction of Routh Array, one may come across the following 3 cases

Case-I :- Normal Routh Array (Non-zero elements in its first Column of Routh Array).

Case-II :- A row of all zeros.

Case-III :- First element of a row is zero but some (or) other elements are not zero.

Case-I :- Normal Routh Array :-

In this case there is no difficulty in forming Routh Array. The Routh Array can be constructed as explained above. The sign changes are noted to find the number of roots lying on the right half of S-plane and the stability of the system can be estimated.

In this case if there is no sign changes in the first Column of Routh array then all the roots are lying on the left half of S-plane and the system is stable

If there is sign change in the first Column of Routh array, then the system is unstable and the number of roots lying on the Right half of S-plane is equal to the number of sign changes. The Remaining roots are on left half of S-plane

Case-II :- A Row of All Zeros :-

An all zero row indicates the existance of an even polynomial as a factor of the given characteristic equation. This even polynomial factor is called the "Auxillary polynomial". The Co-efficients of Auxillary polynomial will always be elements

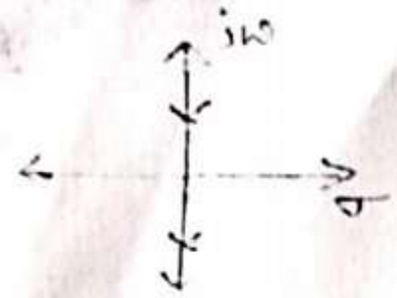
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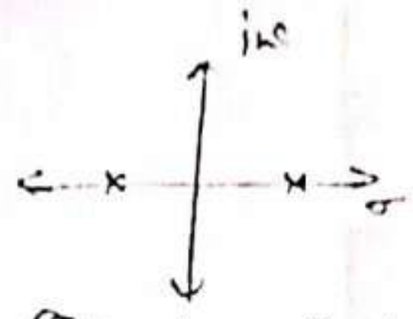
If any array above the zero of zero is in the array

The roots of even polynomial occurs in pairs that are equal in magnitude and opposite in sign. Hence these roots can be:

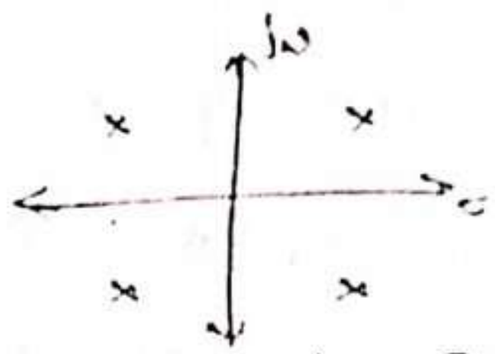
Purely imaginary, Purely Real (or) Complex.



(a) Purely imaginary.



(b) Purely Real



(c) Complex Roots.

Method - 1 :

1. Determine the Auxillary Polynomial A(s).
2. Differentiate the Auxillary Polynomial with respect to s to get

$$\frac{d\,A(s)}{ds}$$

3. A Row of zeros is Replaced with coefficients of $\frac{dA(s)}{ds}$.
4. Continue the Construction of the array in the usual manner.

(a) If there is a ~~sing~~ sign changes in the first column of the Routh Array then the System is unstable. The number of roots lying on the right half of s-plane is equal to the number of sign changes. The number of Roots lying on the imaginary axis can be estimated from the roots of auxillary pdynomial. The Remaining roots are lying on the left half of s-plane.

(b) If there is no sign changes in the first column of the Routh array then all zero row indicate the existance of Purely imaginary roots and so the System is "limitedly" (or) "marginally stable"

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The roots of auxillary equation lies on the imaginary axis and the remaining roots lies on the left-half of S-plane.

Method-2 :-

1. Determine the auxillary polynomial A(s).
2. Divide the characteristic equation by Auxillary polynomial.
3. Construct Routh array using the coefficients of quotient polynomial.
4. The array is interpreted as follows.

(a) If there is a sign change in the first column of Routh array of quotient polynomial then the system is unstable. The number of roots of quotient polynomial lying on right half of S-plane is given by number of sign changes in first column.

The roots of auxillary polynomial are directly calculated to find whether they are purely imaginary (or) purely Real (or) Complex.

The total number of roots on the Right half of S-plane is given by the sum of number of sign changes and the number of roots of auxillary polynomial with positive Real part. The number of roots on imaginary axis can be estimated from the roots of auxillary polynomial. The remaining roots are lying on the left half of S-plane.

(b) If there is no sign change in the first column of the Routh array of quotient polynomial then the signal is limitedly bvs marginally stable. since there is no sign change all the roots of quotient polynomial are lying on the left half of Splane

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The roots of auxillary polynomial are directly calculated to find whether they are purely imaginary (or) Real (or) Complex. The number of roots lying on imaginary axis and on the right half of S-plane can be estimated from the roots of auxillary polynomial. The Remaining roots are lying on the left half of S-plane.

Case III : First Element of a Row is zero :

While Constructing a Routh Array, if a zero is encountered as first element of a row then all the elements of a next row will be infinite. To overcome this problem let $0 \to \varepsilon$ and Complete the Construction of array in the usual way.

Finally let $\varepsilon \to 0$ and determine the values of the elements of the array which are fractions of ε. The Resultant array is interpreted as follows.

(a) If there is no sign change in the first Column of the Routh array and if there is no row with all zeros, then all the roots are lying on the left half of s-plane and the System is stable.

(b) If there is a sign changes in the first Column of Routh array and there is no row with all zeros, then Some of the roots are lying on the right half of S-plane and the System is unstable. The number of roots lying on the right half of s-plane is equal to number of sign changes and the Remaining roots are lying on the left half of S-plane.

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(c) If there is a row of all zeros after letting $\varepsilon \to 0$, then there is a possibility of roots on imaginary axis. Determine the auxillary polynomial and divide the characteristic equation by auxillary polynomial to eliminate the imaginary roots. The Routh array is constructed using the coefficients of auxillary polynomial and the characteristic equation is interpreted as explained in Method-2 of Case-II Polynomial.

* Problems:

1. Using Routh criterion, determine the stability of system Represented by the characteristic equation, $s^4+8s^3+18s^2+16s+5=0$. Comment the location of roots of C.E.

Sol: given C.E $\Rightarrow$ $s^4+8s^3+18s^2+16s+5=0$.

The given characteristic equation is of 4th order and so, it has 4 roots. Since the highest power is even number, first the row of routh array starts with the coefficients of even powers and from the second row using the coefficients of odd powers of s.

$\Rightarrow$

s^4	:	1	18	5
s^3	:	8	16	
s^2	:	16	5	
s^1	:	1.7		
s^0	:	5		

$s^2 = \frac{1\times18-2\times1}{1}, \frac{1\times5-0\times1}{1}$

$= 16. \quad 5$

$\therefore s^2 : 16 \quad 5$

$s^1 = \frac{16\times2-5\times1}{16}$

$= 1.6875 \cong 1.7$

$s^0 = \frac{1.7\times5-0\times16}{1.7}$

$= 5.$

$\therefore s^0 : 5$

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It is observed that all the elements of the Routh Array are positive and there is no sign change. Hence all the roots are lying on the left half of s-plane and the System is stable.

Result:

1. The System is stable
2. Since, no sign change, all the roots are lying on left half of the s-plane.

(2) $s^3 + 6s^2 + 11s + 6 = 0$.

Sol: order = 3, Therefore, 3 roots would be obtained.

$\Rightarrow$

s^3	:	1	11
s^2	:	6	6
s^1	=	10	
s^0	:	6.	

s^1 : $\frac{6 \times 11 - 1 \times 6}{6} = \frac{60}{6}$
$= 10$,

s^0 : $\frac{10 \times 6 - 6 \times 0}{10} = 6$

$\therefore$ The System is stable since there is no sign change in the first column of Routh Array table. Therefore, all the roots are lying on the half (left) of s-plane.

3. $s^5 + 2s^4 + 6s^3 + 3s^2 + 2s + 1 = 0$.

Sol: The highest order of given characteristic equation is 5.

So, it should contains 5 roots.

$\Rightarrow$

s^5	:	1	6	2
s^4	:	2	3	1
s^3	:	1.5	1.5	
s^2	:	2.33	1	
s^1	:	-0.429		
s^0				

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∴ There is two sign changes in the first Column of Routh Array table. So, the System is unstable.

→ As there are two sign changes, the two roots are lying on the right half of s-plane and the Remaining roots are lying on the left half of s-plane.

4. $8s^5 + 4s^4 + 30s^3 + 25s^2 + 62s - 44 = 0$.

Sol: The given Characteristic Equation = $8s^5 + 4s^4 + 30s^3 + 25s^2 + 62s - 44 = 0$

As the highest order of C.E is 5 so, it contains 5 roots.

The highest power of C.E is of odd number so, the Routh Array starts with odd power coefficients in decreasing order.

s^5	:	8	30	62
s^4	:	4	25	-44
s^3	:	-20	150	
s^2	:	55	-44	
s^1	:	134		
s^0	:	-44.		

s^3 : $\frac{4\times30 - 25\times8}{4}$, $\frac{4\times62 + 8\times44}{4}$

-20 , 150.

s^2 : $\frac{-20\times25 - 150\times4}{-20}$, $\frac{-20\times-44}{-20}$

$= \frac{-1100}{-20}$, -44.

$= 55$

$s^1 = \frac{55\times150 - (-20\times-44)}{55}$

$= 134$

$s^0 = \frac{134\times-44}{134} = -44$.

As there are 3 sign changes in the first Column of the Routh Array table. So, the System is unstable.

As there are 3 sign changes in first Column thus, the 3 roots are lying on the right half of s-plane and Remaining 2 roots lies on the left half of s-plane.

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Alternative Method of C.E $s^6+2s^5+s^4+2s^3+4s+3s^2+5=0$ ①

Replace s in the characteristic equation by $1/z$ and transform the Poles in the S-plane to the Z-plane. The characteristic equation in the z-plane is,

$$\Rightarrow \left(\frac{1}{z}\right)^6+2\left(\frac{1}{z}\right)^5+\left(\frac{1}{z}\right)^4+2\left(\frac{1}{z}\right)^3+4\left(\frac{1}{z}\right)+3\left(\frac{1}{z}\right)^2+5=0$$

$$\Rightarrow 5z^6+4z^5+3z^4+2z^3+z^2+2z+1=0.$$

Now, Construct the Routh Array table for C.E in Z-plane.

s^6	5	3	1
s^5	4	2	2
s^4	0.5	-1.5	1
s^3	14	-6	
s^2	-1.2857	1	
s^1	4.88	0	
s^0	1		

from this method ~~also~~, we found two sign changes in the first Columns of Routh Array table So, two roots are lies on the right half of s-plane and the remaining roots are lies on the left half of s-plane.

2. Hain Method :

given Characteristic equation, $s^6+2s^5+s^4+2s^3+3s^2+4s+15=0$

As, the order of the given Characteristic equation is 6. So, it Contains totally 6 Roots. The Routh table is formed as follows

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s^6	:	1	1	3	5
s^5	:	2	2	4	
s^4	:	0	1	5	
s^3					

The first element of s^4 row is zero. So, the system is unstable. Now, Replacing the value zero with the positive integer. ε.

$\therefore$ s^6	:	1	1	3	5
s^5	:	2	2	4	
s^4	:	ε	1	5	
s^3	:	$\frac{\varepsilon \times 2 - 2 \times 1}{\varepsilon}$	$\frac{4\varepsilon - 10}{\varepsilon}$		
s^2	:	$\frac{-4\varepsilon^2 + 12\varepsilon - 2}{\varepsilon}$	5		
s^1	:	$\frac{\left(\frac{-4\varepsilon^2+12\varepsilon-2}{\varepsilon}\right)\left(\frac{4\varepsilon-10}{\varepsilon}\right) - 5\left(\frac{2\varepsilon-2}{\varepsilon}\right)}{\frac{-4\varepsilon^2+12\varepsilon-2}{\varepsilon}}$		0.	
s^0	:	5.			

Now, apply $\varepsilon \to 0$. then the array will be.

s^6	:	1	1	3	5
s^5	:	2	2	4	
s^4	:	0	1	5	
s^3	:	$-\infty$	$-\infty$		
s^2	:	$-\infty$	5		
s^1	:	$-\infty$	0		
s^0	:				

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Thus the above array we conclude that there are two sign changes. So the system is unstable, two roots lies in the right half of s-plane and remaining roots lies on the left half of s-plane.

3. $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$.

Sol.

s^5	:	1	2	3
s^4	:	1	2	5
s^3	:	0	-2	

Replace $0 \to \varepsilon$ we get,

s^5	:	1	2	3
s^4	:	1	2	5
s^3	:	ε	-2	
s^2	:	$\frac{2\varepsilon+2}{\varepsilon}$	5	
s^1	:	$\frac{-(5\varepsilon^2+4\varepsilon+4)}{2\varepsilon+2}$		
s^0	:	5.		

Now, Replace $\varepsilon \to 0$ we get,

s^5	:	1	2	3
s^4	:	1	2	5
s^3	:	0	-2	
s^2	:	1	-5	
s^1	:	-2		
s^0	:	5		

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There are two sign changes in Routh Array after Replacing ε by 0. So, the System is unstable. Therefore, two Roots are lying on the right half of s-plane. and remaining will lies on the left half of s-plane.

Alternative Method :

Transform the characteristic equation in the s-plane to z-plane by placing s by $1/z$ and applying Routh's test. The C.E in z-plane is.

$$\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^4 + 2\left(\frac{1}{z}\right)^3 + 3\left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^2 + 5 = 0.$$

$$\Rightarrow 5z^5 + 3z^4 + 2z^3 + 2z^2 + z + 1 = 0.$$

i.e

z^5	:	5	2	1
z^4	:	3	2	1
z^3	:	-1.33	-0.66	
z^2	:	0.5	1	
z^1	:	2		
z^0	:	1		

from this method also we observed two sign changes. Therefore, the System is unstable. and the two roots are lying on the Right half of s-plane and the other roots are lying on the left half of s-plane.

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Write Case 1 : Problems

Write characteristic equation of a system is $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$

Sol: The characteristic equation is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$.

The given characteristic polynomial is 7th order equation and so it has 7 roots since the highest power of 's' is odd number, form the first row of array using the coefficients of odd powers of 's' and the of second row using the coefficients of even powers of 's' as shown below,

s^7	:	1	24	24	23
s^6	:	9	24	24	15

Divide the s^6 row with 3.

s^7	:	1	24	24	23
s^6	:	3	8	8	5
s^5	:	1	1	1	
s^4	:	1	1	1	
s^3	:	0	0		
s^2	:	0.5	1		
s^1	:	-3			
s^0	:	1			

$s^5 : \frac{8\times24-8\times1}{3}, \frac{3\times24-8\times1}{3}, \frac{3\times23-5\times1}{3}$

s^5 : 21.33 21.33 21.33

s^5 : 1 1 1

$s^4 : \frac{1\times8-1\times3}{1}, \frac{1\times8-1\times3}{1}, \frac{1\times5-0\times3}{1}$

$s^4 = 5 \quad 5 \quad 5$

s^4 : 1 1 1

$s^3 : \frac{1\times1-1\times1}{1}, \frac{1\times1-1\times1}{1}$

: 0 0.

Auxillary polynomial $A(s) = \frac{s^4+s^2+1}{4s^3+2s}$

Differentiate w.r.t. s.

$\frac{dA}{ds} = 4s^3+2s$

$= 4s^3+1s$

s^2 : 2 1

s^1 : $\frac{2\times1-1\times1}{2}$, $\frac{2\times1-1\times0}{2}$

: 0.5 1

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In examining the first column elements of Routh array, it is found that there are two sign changes. Hence, the two roots are lying on the Right half of s-plane and so, the System is unstable

Now, The Row of all zeros indicate the possibility of roots on imaginary axis. This can be tested by Evaluating the roots of Auxillary Polynomial.

The auxillary polynomial $= s^4 + s^2 + 1$

Now, Put $s^2 = x$.

$$\Rightarrow x^2 + x + 1 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \pm j\frac{\sqrt{3}}{2} = 1\angle 120^\circ \text{ (or) } 1\angle -120^\circ$$

$$s = \sqrt{x} = \pm(0.5 \pm j\,0.866) \text{ (or) } \pm(0.5 - j\,0.866)$$

$\therefore$ The System is unstable. As there are two sign changes therefore, two roots are lying on the right half of s-plane and Remaining are lies on the left half of Splane

[i.e. $s = \pm\sqrt{x} = \pm\sqrt{1\angle 120^\circ}$
$= \pm\sqrt{1}\angle\frac{120^\circ}{2}$
$= \pm 1\angle 60^\circ$]

Alternate Method :-

s^7	:	1	24	23
s^6	:	3	8	5
s^5	:	1	1	
s^4	:	1	1	
s^3	:	0	0	

The Auxillary polynomial $A(s) = s^4 + s^2 + 1 = 0$.

Now, Divide the characteristic Equation with Auxillary polynomial to get Polynomial and Continue the Routh Array

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with the quotient polynomial.

∴ The Auxillary polynomial $A(s) \Rightarrow s^4 + s^2 + 1 = 0$.

Quotient polynomial is,

$$s^3 + 9s^2 + 23s + 15 = 0.$$

$$\begin{array}{r|l} s^4 + s^2 + 1 & s^3 + 9s^2 + 23s + 15 \\ \hline s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 & \\ \underline{s^7} \quad \underline{s^5} \quad \underline{s^3} & \\ \hline 9s^6 + 23s^5 + 24s^4 + 23s^3 + 24s^2 + 23s + 15 & \\ \underline{9s^6} \quad \underline{9s^4} \quad \underline{9s^2} & \\ \hline 23s^5 + 15s^4 + 23s^3 + 15s^2 + 23s + 15 & \\ \underline{23s^5} \quad \underline{23s^3} \quad \underline{23s} & \\ \hline 15s^4 + 15s^2 + 15 & \\ \underline{15s^4 + 15s^2 + 15} & \\ \hline 0 & \end{array}$$

$$\Rightarrow \begin{array}{lcc} s^3 & : & 1 \quad 23 \\ s^2 & : & 9 \quad 15 \\ s^2 & : & 3 \quad 5 \\ s^1 & : & 21.33 \\ s^0 & : & 5 \end{array}$$

$$s^1 = \frac{3 \times 23 - 5 \times 1}{3} = 21.33$$

The elements of column-1 of quotient polynomial are all positive and there is no sign change. Hence all the roots of quotient polynomial are lying on the left half of s-plane. To determine the stability, the roots of auxillary polynomial should be calculated.

The Auxillary Equation $A(s) = s^4 + s^2 + 1 = 0$.

Put $s^2 = x$.

$$\Rightarrow x^2 + x + 1 = 0. \quad \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm j\sqrt{3}}{2} = \frac{-1 + j\sqrt{3}}{2} \text{ (or) } \frac{-1 - j\sqrt{3}}{2}$$

$$= 1 \angle 120^\circ \text{ (or) } 1 \angle -120$$

$$\Rightarrow s = \pm\sqrt{x} = \pm\sqrt{1 \angle 120^\circ}, \pm\sqrt{1 \angle -120}$$

$$= \pm\sqrt{1}\,\sqrt{\angle 120}, \pm\sqrt{1}\,\sqrt{\angle -120}$$

$$= \pm 1 \angle 120/2, \pm 1 \angle -120/2$$

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$= \frac{1}{2}(0.5 + j0.866)$, $\frac{1}{2}(0.5 - j0.866)$.

The roots of auxillary equation are complex and has quadrantal symmetry. Two roots of auxillary equation are lying on the right half of s-plane and the other two on the left half of s-plane.

∴ The roots of C.E are given by the roots of auxillary equation and the quotient equation thus, the two roots are lying on the right half of s-plane. Therefore, the system is unstable. The Remaining roots are lying on the left half of s-plane.

2. Comment on stability of the system with the following C.E.

$$D(s) = s^6 + s^5 + 7s^4 + 6s^3 + 31s^2 + 25s + 25.$$

Sol: given, characteristic equation

$$D(s) = s^6 + s^5 + 7s^4 + 6s^3 + 31s^2 + 25s + 25 = 0$$

The given characteristic equation is of the order 6. So, it has 6 roots. Since, the highest power of 's' is even number & the Routh Array start with the even power coefficients in the first row, then and the second row using the coefficients of odd powers of 's' as shown below.

s^6	:	1	7	31	25
s^5	:	1	6	25	
s^4	:	1	6	25	→ This coefficients of powers will forms a Auxillary polynomial.
s^3	:	0	0.		

∴ The auxillary polynomial $A(s) = s^4 + 6s^2 + 25$.

$\frac{d \cdot A(s)}{ds} = 4s^3 + 12s.$

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Now Substitute the Coefficients of the differential Auxiliary Polynomial, we get.

s^4	:	1	7	31	25
s^5	:	1	6	25	
s^{21}	:	1	6	25	
s^3	:	4	12		
s^2	:	3.	25		
s^1	:	$-64/3$			
s^0	:	25			

$$\epsilon' = \frac{3\times12 - 4\times25}{3}, \quad \frac{36-100}{3} = \frac{-64}{3}$$

Here, the two sign changes are observed in first Column So, the two roots are lying on the right half of s-plane. and remaining roots are lying on left-half of s-plane. Thus, the system is unstable.

As the Row of Zeros indicates, there may exists a purely imaginary, purely real (or) Purely Complex. roots. These will get with the help of Auxillary polynomial.

$$\therefore \quad s^4 + 16s^2 + 25 = 0.$$

$$\Rightarrow \text{Put } s^2 = x. \quad \Rightarrow \quad x^2 + 6x + 25 = 0.$$

$$x = \frac{-6 \pm \sqrt{36 - 25\times4}}{2} = \frac{-6 \pm \sqrt{-64}}{2} = \frac{-6 \pm j8}{2}$$

$$= -3 \pm j4. = s$$

$$= 5\angle 126.86, \; 5\angle -126.86$$

$$\Rightarrow s^2 = x \Rightarrow s = \sqrt{x} = \pm\sqrt{x} = \sqrt{5\angle 126.86}, \; \sqrt{5\angle -126.86}$$

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$$= \pm\left(\sqrt{5} \cdot \angle\frac{126.86}{2}\right) \quad , \pm\left(\sqrt{5} \ \angle\frac{126.86}{2}\right)$$

$$= \pm(2.23 \ \angle 63.43) \quad . \pm(2.23 \ \angle -63.43)$$

$$= \pm (1 \pm j2) \ , \ \pm(1 \pm j2) .$$

$$= \pm (1+j2), \ \pm (1-j2) .$$

Ans. Alternative Method:

s^6	:	1	7 31 25
s^5	:	1	6 25
s^4	:	1	6 25
s^3	:	0	0.

Auxillary polynomial $A(s) = s^4+6s^2+25=0$.

Now, Divide the characteristic polynomial with Auxillary polynomial, we will get one quotient polynomial. Then construct the Routh array for the quotient polynomial.

$$s^4+6s^2+25 \ \Big| \ s^2+s+1$$

$$s^6+s^5+7s^4+6s^3+31s^2+25s+25$$

$$s^6 \quad 6s^4 \quad 25s^2 \quad 25s$$

$$s^5+s^4+6s^3+6s^2+25s+25$$

$$s^5 \quad 6s^3 \quad 6s^2 \quad 25s$$

$$s^4+6s^2+25$$

$$s^4+6s^2+25$$

$$0$$

∴ The quotient polynomial $q(s) = s^2+s+1 = 0$.

s^2	:	1	1
s^1	:	1	
s^0	:	1	

$$s^0 = \frac{1\times1-1\times0}{1} = 1$$

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As there are no sign changes in the quotient polynomial, Due to the presence of a row of zeros, there may exists an imaginary roots. Thus will get from the Auxillary polynomial

$$A(s) = s^4 + 6s^2 + 25 = 0.$$

$\Rightarrow$ Put $s^2 = x$.

$$\Rightarrow x^2 + 6x + 25 = 0.$$

$$\Rightarrow x = \frac{-6 \pm \sqrt{36-100}}{2} = \frac{-6 \pm j8}{2}$$

$$= -3 \pm j4., \; -3 - j4$$

$$= 5\angle 126.86, \; 5\angle -126.86$$

Now, $s^2 = x \Rightarrow s = \sqrt{x} = \pm\sqrt{x} = \sqrt{5\angle 126.86}, \sqrt{5\angle -126.86}$

$$= \pm\sqrt{5}\angle\frac{126.86}{2}, \; \pm\sqrt{5}\angle\frac{-126.86}{2}$$

$$= \pm(2.23\angle 63.43), \pm(2.23\angle -63.43)$$

$$= \pm(1+j2), \pm(1-j2).$$

Now the number of roots on right half of s-plane will be decided that the number of positive Real part roots is obtained from Auxillary polynomial.

So, As there are two positive Real part root so, the two roots are lies on right half of s-plane and the remaining roots lies on left half of s-plane.

Therefore, the System is unstable.

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3 $+s^6+8s+12s+20s^2+16s+16.$

7. given characteristic equation.

$\Rightarrow s^6+2s^5+8s^4+12s^3+20s^2+16s+16.$

s^6	:	1	8	20	16
s^5	:	2	12	16	
s^4	:	2	12	16	→ Auxillary polynomial A(s).
s^3	:	0	0	0	

$\therefore A(s) = 2s^4+12s^2+16.$

$= s^4+6s^2+8.$

$\Rightarrow A(s) = s^4+6s^2+8$

$\frac{dA(s)}{ds} = 4s^3+12s.$

s^6	:	1	8	20	16
s^5	:	2	12	16	
s^4	:	2	12	16	
s^3	:	4	12		
s^2	:	3	8		
s^1	:	0.33			
s^0	:	8			

Here, all the elements in the first columns are positive. And, we got a Row of zeros. So, the system is marginally (or) limitedly stable. The Roots of Auxillary polynomial are,

$A(s) \Rightarrow s^4+6s^2+8=0.$

Put, $s^2=x \Rightarrow x^2+6x+8=0.$

$\Rightarrow x = \frac{-6 \pm \sqrt{36-32}}{2} = \frac{-6 \pm \sqrt{4}}{2}$

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$$= \frac{-6 \pm 2}{2} \Rightarrow \frac{-6+2}{2}, \frac{-6-2}{2}$$

$$\Rightarrow -2, -4$$

$$\therefore S = \pm\sqrt{x} = \pm\sqrt{-2}, \pm\sqrt{-4}$$

$$= \pm j\sqrt{2}, \pm j\sqrt{4}$$

$$= \pm j\sqrt{2}, \pm j2$$

$$= \pm j1.414, \pm j2$$

As we got the two distinct purely imaginary roots. So, the system is marginally (or) limitedly stable.

Alternate Method :-

given characteristic equation $\Rightarrow s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$

Now,	s^6	1	8	20	16
	s^5	1	6	8	
	s^4	1	6	8	
	s^3	0	0	0.	

$\therefore$ The Auxillary polynomial $A(s) \Rightarrow s^4+6s^2+8=0$.

Divide the main characteristic equation with Auxillary equation to obtain the quotient polynomial.

$$s^4+6s^2+8 \;\Big|\; s^2+2s+2$$

$$s^6+2s^5+8s^4+12s^3+20s^2+16s+16$$

$$-s^6 \quad -6s^4 \quad -8s^2$$

$$2s^5+2s^4+12s^3+12s^2+16s+16$$

$$-2s^5 \quad -12s^3 \quad -16s$$

$$2s^4+12s^2+16$$

$$2s^4+12s^2+16 \;/\; (1)$$

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∴ The quotient polynomial $Q(s) = 3-1 / 5-1$

$$\Rightarrow \begin{array}{ll} s^2 & 1 \quad 1 \quad 2 \\ s^1 & 2 \quad 1 \\ s^0 & 2 \end{array}$$

In the first column of quotient polynomial also there are no sign changes. So, to know (or) determine the system is stable (or) unstable we have to find the roots for the Auxillary polynomial.

$\therefore A(s) \Rightarrow s^4 + 6s^2 + 8 = 0$

Put $s^2 = x$

$\Rightarrow x^2 + 6x + 8 = 0.$

$$\Rightarrow x = \frac{-6 \pm \sqrt{36-32}}{2}$$

$$= \frac{-6 \pm \sqrt{4}}{2}$$

$$= \frac{-6 \pm 2}{2} \Rightarrow \frac{-6+2}{2}, \quad \frac{-6-2}{2}$$

$\Rightarrow -2, -4.$

$\therefore s = \pm\sqrt{x}$

$= \pm\sqrt{-2}, \quad \pm\sqrt{-4}$

$= \pm j\sqrt{2}, \quad \pm j\sqrt{4}$

$= \pm j\sqrt{2}, \quad \pm j2$

$= \pm j1.414, \pm j2$

Here, there are two imaginary roots on both the sides of imaginary axis. And these are distinct roots. So, the system is Marginally stable (or) Limitedly stable

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4. Determine the range of K for stability of unity feedback system whose open loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}.$$

Sol: Given open loop T.F $G(s) = \frac{K}{s(s+1)(s+2)}$.

We know that, the closed loop transfer function is,

$$T.F = \frac{G(s)}{1+G(s)H(s)}. \qquad [\because H(s) = 1]$$

$$\Rightarrow T.F = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{K}{s(s+1)(s+2)+1+K}.$$

Now, the characteristic equation is $1+G(s)H(s) = 0$.

$$\Rightarrow \text{i.e.} \quad s(s+1)(s+2)+K = 0.$$

$$\Rightarrow s(s^2+3s+2)+1K=0$$

$$\Rightarrow s^3+3s^2+2s+K = 0.$$

Now, the Routh Array is constructed as follows,

s^3	:	1	2
s^2	:	3	K
s^1	:	$\frac{6-K}{3}$	
s^0	:	K	

for, the system to be stable there should not be any sign change in the elements of first Column. Hence, choose the value of K so that the first Column elements are positive.

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from 5th Row, for the System to be stable, $K>0$.

from 5th row, for the System to be stable, $\frac{6-K}{1}>0$

$\Rightarrow 6-K>0$

$\Rightarrow 6>K \Rightarrow K<6$.

∴ The value of 'K' should be in the range of $0<K<6$

for the System to be stable

6. Open loop transfer function with unity feed back of a System is given by $G_1(S) = \frac{K}{(S+2)(S+4)(S^2+6S+25)}$. Determine the value of K which will cause sustained oscillation in the closed loop system.

Sol The closed loop T.F $= \frac{G(S)}{1+G(S)} = \frac{\frac{K}{(S+2)(S+4)(S^2+6S+25)}}{1+\frac{K}{(S+2)(S+4)(S^2+6S+25)}}$

$= \frac{K}{(S+2)(S+4)(S^2+6S+25)+1+K}$.

Now, the characteristic equation is,

$(S+2)(S+4)(S^2+6S+25)+1+K=0$.

$\Rightarrow (S^2+6S+8)(S^2+6S+25)+1K=0$.

$\Rightarrow S^4+12S^3+69S^2+198S+200+K=0$.

∴

S^4 :	1	69	$200+K$
S^3 :	12	198	
S^2 :	52.5	$200+K$.	
S^1 :	$\frac{666.25-K}{52.5}$		
S^0 :	$200+K$.		

S^1 : $\frac{52.5\times16.5-200+K}{52.5}$

$= \frac{666.25-K}{52.5}$

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For the System to be stable there should be no sign changes in first Column and Chang the Value of K so, that the first elements of first Column are positive.

∴ for from S¹ row ⇒ $666.25 - K > 0$

⇒ $K < 666.25$

from S° row, ⇒ $200 + K > 0$ ⇒ $200 + K > 0$

⇒ $-K < +200$

⇒ $K > -200$

∴ The Range of K for the System to be stable is,

$0 < K < 666.25$.

Now, the System will oscillate when $K = 666.25$ then the S¹ row becomes zero, which indicates the possibility of Imaginary roots on S-plane. A System will oscillates when the roots are on imaginary axis and no root is on Right half of s-plane.

when $K = 666.25$, then the auxiliary equation is,

$A(S) = 52.5s^2 + (200 + K) = 0$.

⇒ $52.5s^2 + 200 + 666.25 = 0$

⇒ $52.5s^2 + 866.25 = 0$

⇒ $s^2 = \frac{-866.25}{52.5} = -16.5$

⇒ $s = \pm\sqrt{-16.5}$

$= \pm j\frac{4.06}{\Omega}$ Similar to $\pm j\Omega$.

∴ when $K = 666.25$ the System will oscillate and the

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frequency of oscillation is = 1.06 rad/sec

(ii) Unity feedback control system is characterized by open loop transfer function. $G(s) = \frac{K(s+13)}{s(s+3)(s+7)}$.

i. Calculate the range of values of K for the system to be stable

ii. What is the marginal value of K for stability? Determine the frequency of oscillations.

The characteristic equation is $G_T(s) = 1 + G(s)H(s) = 0$.

$$\Rightarrow 1 + \frac{K(s+13)}{s(s+3)(s+7)} = 0$$

$$\Rightarrow s(s+3)(s+7) + K(s+13) = 0.$$

$$\Rightarrow s^3 + 10s^2 + (21+K)s + 13K = 0.$$

∴ Routh Array is tabulated as,

s^3	1	$21+K$
s^2	10	$13K$
s^1	$\frac{10(21+K) - 13K}{10} = \frac{210 - 3K}{10}$	
s^0	$13K$	

i, For the system to be stable, the first column elements of Routh table is should be positive.

from the s^0 row, $13K > 0$.

$\Rightarrow K > 0$.

from the s^1 row, $210 - 3K > 0$.

$\Rightarrow 3K < 210$

$\Rightarrow K < 70$

∴ Hence, the range of values of K for stability is $0 < K < 70$.

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(14). the system will oscillate for $K = 70$, then the s' row will becomes zero, which indicates the possibility of imaginary roots.

when $K = 70$ the Auxillary equation is,

$$A(S) \Rightarrow 10s^2 + 13K = 0$$

$$\Rightarrow 10s^2 + 13 \times 70 = 0 \quad (\because K = 70)$$

$$\Rightarrow 10s^2 = -13 \times 70$$

$$\Rightarrow s^2 = \frac{-13 \times 70}{10}$$

$$\Rightarrow s^2 = \sqrt{\frac{-13 \times 70}{10}} = \pm j\sqrt{\frac{13 \times 70}{10}}$$

$$= \pm j\, 9.53.$$

$\therefore$ The frequency of oscillation is $\omega = 9.53$ rad/sec

(x) $2s^5 + s^4 + s^3 + 5s^2 + 3s + 5 = 0$

Sol:-

s^5	2	5	3
s^4	2	5	5
s^3	0	-2	

Now Replace an Multiple integral ε in place of zero.

s^3	ε	-2	
s^2	$\frac{5\varepsilon+4}{\varepsilon}$	ε	
s^1	$\frac{-2\left(\frac{5\varepsilon+4}{\varepsilon}\right) - \varepsilon^2}{\frac{5\varepsilon+4}{\varepsilon}}$		
s^0	ε		

Now Replace 0 in place of multiple integral ε

s^5	2	5	3
s^4	2	5	5
s^3	0	-2	
s^2	∞	0	
s^1	$-\infty$		
s^0	0		

∴ TWO sign changes. So, the TWO Roots lies in the Right half of s-plane and Remaining roots in R.H. splane.

∴ System is unstable.

B.N.Prasad

s^3 $\frac{2\times5 - 2\times5}{2} = 0$

$\frac{2\times3 - 2\times5}{2} = \frac{6-10}{2} = \frac{-4}{2}$

s^4	2	5	5
s^3	ε	-2	
s^2	$\frac{5\varepsilon+4}{\varepsilon}$	ε	
s^1	-2		

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1) $s^6+4s^5+s^4-3s^3+s^2+33+15=0$

s^6	: 2	1	15
s^5	: 4	-32	3
s^4	: 17	49.5	15
s^3	: -43.64	-0.52	
s^2	: +9.29	15	
s^1	: 12.76		
s^0	: 15		

s^4 : 17 ; 49.5 ; 15

s^3 : $\frac{17\times-32-4\times49.5}{17}$; $\frac{17\times49.5-4\times15}{17}$; $\frac{17\times0-4\times0}{17}$

$\frac{-544-198}{17}=-43.64$

s^2 : $\frac{-43.64\times49.5+17\times0}{-43.64}$; $\frac{-43.64\times15+17\times0}{-43.64}$

$\frac{-2160.18+8.84}{-43.64}$; $\frac{+43.64\times15+49.29\times0}{49.29}$

$\frac{651.6-25.6}{49.29}$

s^1 : 12.76

s^0 : $\frac{12.76\times15-0}{12.76}$

$=15$

As there is two sign changes. So two roots lies in right half of s-plane & Remaining roots lies in left half of s-plane.

The system is unstable

$s^6+3s^5+5s^4+9s^3+8s^2+6s+4=0$

s^6	: 1	5	8	4
s^5	: 3	9	6	
s^4	: 2	6	4	
s^3	: 0	0		

s^4 : $\frac{3\times5-9}{3}$; $\frac{3\times8-1\times6}{3}$; $\frac{3\times4-}{3}$

2 ; 6 ; 4

s^3 : $\frac{2\times9-3\times6}{2}$; $\frac{2\times6-3\times4}{2}$

0 ; 0

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A Row of all zeros.

Now, the Auxillary polynomial $A(s) = 2s^4 + 6s^2 + 4$

$\therefore \frac{dA(s)}{ds} = 8s^3 + 12s$

$\therefore$

s^4 :	2	6	4
s^3 :	8	12	
s^2 :	3	4	
s^1 :	1.33		
s^0 :	4		

$s^2 : \frac{8\times6 - 2\times12}{8} ; \frac{8\times4-0}{8}$

$3 \quad : \quad 4$

$s^1 : \frac{3\times12 - 8\times4}{3}$

$\frac{36-32}{3} = \frac{4}{3}$

$s^0 : \frac{1.33 \times 4}{1.33}$

As there is no sign changes, even though we can't judge stability of a System

As the Row becomes zero, there is a possibility of purely imaginary, Purely Real (or) Complex roots. This can be obtained Auxillary polynomial.

$2s^4 + 6s^2 + 4 = 0$

let $s^2 = x \quad \therefore 2x^2 + 6x + 4 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - 4\times2\times4}}{2\times2}$

$= \frac{-6 \pm \sqrt{36-32}}{4} = \frac{-6 \pm \sqrt{4}}{4}$

$= \frac{-6 \pm 2}{4} = \frac{-6+2}{4} ; \frac{-6-2}{4}$

$= \frac{-4}{4} ; \frac{-8}{4}$

$\therefore s^2 = -1, -2 \Rightarrow s = \sqrt{-1}, \sqrt{2} = \pm j1, \pm j2 = -1, -2$

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1) Alternate Method :-

Sol: Substitute $s = \frac{1}{z}$ in given characteristic equation

we get, $2\left(\frac{1}{z}\right)^5 + 2\left(\frac{1}{z}\right)^4 + 5\left(\frac{1}{z}\right)^3 + 5\left(\frac{1}{z}\right)^2 + 3\times\frac{1}{z} + 5 = 0$

$\Rightarrow 5z^5 + 3z^4 + 5z^3 + 5z^2 + 2z + 2 = 0$

Now Routh - Array.

$\Rightarrow$

z^5	5	5	2
z^4	3	5	2
z^3	-3.33	1.33	
z^2	0.801	2	
z^1	6.98		
z^0	2		

$\frac{3\times5 - 5\times5}{3}$; $\frac{3\times2 - 5\times2}{3}$

-3.33 $\frac{6-10}{3}$

z^2 : $\frac{-3.33\times2 + 1.33\times3}{-3.33}$; $\frac{-3.33\times2 - 0}{-3.33}$

z^1 : $\frac{0.801\times-1.33 + 3.33\times2}{0.801}$

$= \frac{-1.065 + 6.66}{0.801}$

$= 6.98$

z^0 : $\frac{6.98\times2 - 0}{6.98}$

$= 2$

There are TWO sign changes, so, two roots are lying in right half s-plane and remaining roots are lying in left half of s-plane

The System is unstable

(c) Alternate Method for $s^6+3s^5+5s^4+9s^3+8s^2+6s+4=0$

Sol:

s^6 : 1 5 8 4

s^5 : 3 9 6

s^4 : 2 6 4 $\Rightarrow$ 1 3 2

s^3 : 0 0

Now, the Auxillary polynomial $A(s) = 2s^4+6s^2+4=0$.

Now, Divide the given characteristic equation with Auxillary equation

$$2s^4+6s^2+4 \;\Big|\; s^2+3s+2$$

$s^6+3s^5+5s^4+9s^3+8s^2+6s+4$

$2s^6$ (−) $3s^4$ (−) $2s^2$ (−)

$3s^5+2s^4+9s^3+6s^2+6s+4$

$3s^5$ (−) $9s^3$ (−) $6s$ (−)

$2s^4+6s^2+4$

(−) $2s^4+6s^2+4$

0.

Now, quotient polynomial $Q(t) = s^2+3s+2$

s^2 : 1 2

s^1 : 3

s^0 :

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As there is no sign changes in first column of Routh Array of Quotient polynomial and we have got a Row of Zeros. So, the System will be limited by stable. From the Auxillary polynomial the purely imaginary Roots will be find out.

$$A(s) = s^4 + 3s^2 + 2 = 0$$

$\Rightarrow$ let, $s^2 = x \Rightarrow x^2 + 3x + 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 2\times4\times1}}{2}$$

$$= \frac{-3 \pm \sqrt{9-8}}{2}$$

$$= \frac{-3 \pm \sqrt{1}}{2}$$

$$= \frac{-3 \pm 1}{2} = \frac{-3+1}{2}, \frac{-3-1}{2}$$

$$= \frac{-2}{2}, \frac{-4}{2}$$

$$= -1, -2$$

$\therefore s = \sqrt{x} = \sqrt{-1}, \sqrt{-2}$

$= \pm j1, \pm j2$

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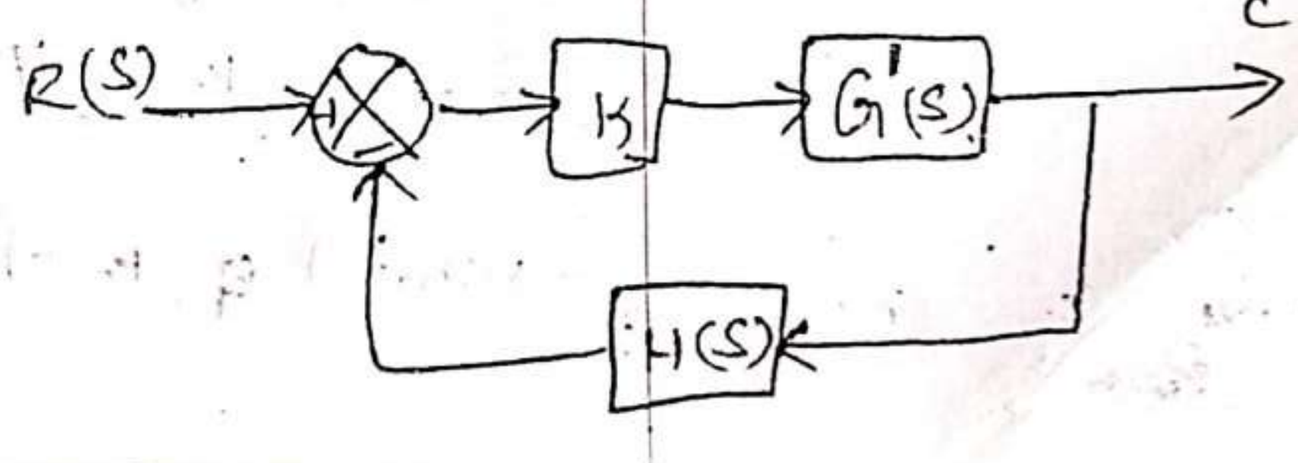
Root locus

1. The Root locus Technique was introduced by N.R. Evans. an American scientist in 1948. This is a graphical method in which movement of Poles in the S-plane is sketched when a particular parameter of a system is varied from "zero to infinity". Note that the parameter is usually the gain but any other parameter may be varied. But for Root locus method, "gain" is assumed to be a parameter which is to be varied from zero to infinity.

* Basic Concept of Root locus:-

In general, the characteristic equation of a closed loop system is given by,

$$1 + G_1(S)\,H(S) = 0$$



For the Root locus, the gain K is assumed to be a variable parameter and is a part of forward path of the closed loop System.

$\therefore \quad G(s) = K \cdot G'(s)$.

Where K = Gain of the amplifier in the forward path or also called System Gain.

Now, the characteristic equation becomes.

$1 + G(s)H(s) = 0 \quad$ i.e $\quad 1 + K \cdot G'(s)H(s) = 0$.

which Contains 'K' as a variable parameter.

Now, if gain K is varied from $-\infty$ to ∞ then for each seperate value of 'K' we will get seperate set of locations of the roots for the characteristic equation. If all such locations are joined the resulting locus is called Root locus.

Definition:-

The Root locus can be defined as, the locus of closed loop poles obtained when a System gain 'K' is varied from 0 to $-\infty$ is called Root locus.

Angle and Magnitude Conditions:-

For a general closed loop C.E

$1 + G(s)H(s) = 0$.

i.e $\quad G(s)H(s) = -1$

As S-plane is Complex we can write above eq.

$$\boxed{G(s)H(s) = -1 + j0.}$$

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... 's' values can be expressed as 'a+jw' i.e, G(s)H(s) term. is also a complex one. So for any value of 's' if it has to be on the root locus, it must satisfy the above equation

As both sides of above equations are in Rectangular form, Convert both sides into Polar form and then equate angle and magnitudes of both sides. This gives two condition called as

1. Angle Condition 2. Magnitude Condition.

* Angle Condition:

$$G(s)\,H(s) = -1+j0.$$

Equating angles of both sides,

$$\Rightarrow \angle G(s)\,H(s) = \pm(2q+1)\,180^\circ \qquad q = 0,1,2,\ldots$$

This means the angle of G(s)H(s) should be an odd multiple of 180°. If this condition is met, then for any root, lies on the Root locus path.

For any root of 's' this condition should satisfy other than the root does not belongs to Root locus.

∴ Angle Condition can be stated as,

$|\angle G(s)\,H(s)|$ = for any value of 's' which is the root of equation $1+G(s)H(s)=0$ is,

$= \pm(2q+1)\,180^\circ$ where, $q = 0,1,2,3,\ldots$

= odd Multiple of 180°.

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decided that among the 3 roots $s = -0.75$ is one of the root of characteristic equation using range and irrgularities conditions.

* Rules for the Construction of Root locus:-

Rule-1:- The Root locus is symmetrical about Real axis.

Rule-2:- Each branch of root locus starts (or originates from a Pole and terminates at either a finite zero (or) a zero at infinity. The number of branches determines based on the difference of finite open loop poles to the finite number of zeros.

Rule-3:- Segments of the real axis having an odd number of poles and zeros to their right are the parts (or branches of Root locus.

Rule-4:- To find the angle of Asymptotes which is given by, $\phi = \frac{\pm 180(2q+1)}{n-m}$; $q = 0, 1, 2, \ldots (n-m)$.

where, n = number of poles and

m = number of zeros.

Rule-5:- To find the Centroid which is a point on Real axis from which the asymptotic angles have to be drawn is given by,

$$\sigma_A = \frac{\text{Sum of no of Poles} - \text{Sum of no of zeros}}{n-m}.$$

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Rule 6: The breakaway and break in points of root locus are determined from the roots of the equation $\frac{dK}{ds} = 0$.

Rule 7: The angle of departure from a complex pole is given by $180 - \phi_1 + \phi_2$

where ϕ_1 is the net angle contribution to the complex pole from other poles and ϕ_2 is net angle contribution to the complex pole from the zeros.

Similarly, the angle of arrival is given by,

$$180^\circ - \phi_2 + \phi_1.$$

ϕ_1 = net angle contribution of poles and ϕ_2 is net angle contribution to the complex zero from other zeros.

Rule-8: The intersection point of root locus on imaginary axis can be determined by the use of R-H criterion (or) by letting $s = j\omega$ in characteristic Equation and equating Real part and imaginary part to zero, to solve ω and K.

And the value of ω is the intersection point on imaginary axis and K is the gain at the intersection point.

* Procedure to Construct the Root locus:

Step-1: Locate the poles and zeros of $G(s), H(s)$ on the s-plane. The root locus branch start from open loop poles and terminates at zeros.

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Step - 2 :- Determine the Root locus on the Real axis.

Step - 3 :- Determine the asymptotes of root locus branches and meeting point of asymptotes with real axis.

Step - 4 :- Find the breakaway and break in point

Step 5 :- If there is a Complex pole then determine the angle of departure from the Complex pole. If there is a Complex zero then determine the angle of arrival at Complex zero.

Step - 6 :- Find the points where the root loci may cross the imaginary axis.

Step - 7 :- Take a series of Test points in the broad neighbourhood of the origin of the s-plane, and adjust the test point to satisfy angle criterion. Sketch the root locus by joining the test point with the help of smooth curve

Problems :-

1. Sketch the root locus of the system whose open loop transfer function is, $G(s) = \frac{K}{s(s+2)(s+4)}$.

Sol :- 1. Locate poles and zeros.

The poles of open loop transfer function of equation, $s(s+2)(s+4)=0$ is $-2, -4, 0$.

$\therefore$ number of poles $n = 3$

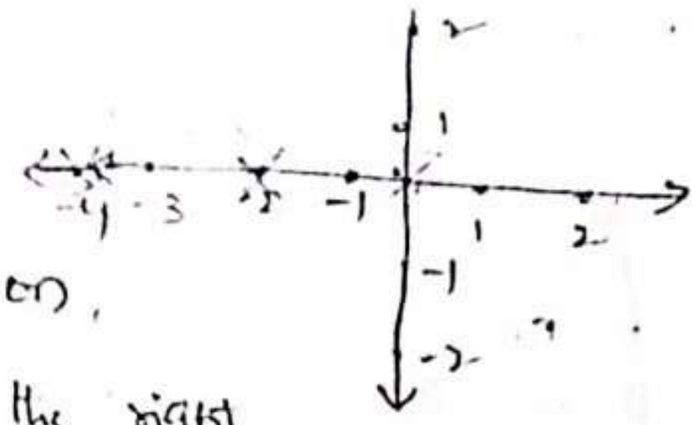
and zeros $= 0$. ($\because$ There is no 's' term in Numerator)

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Step 2:

There are 3 Poles on Real axis. Choose a test point on real axis between $S=0, -2$. To the right of this point the total number of real poles and zeros is one which is odd number. Hence, the real axis between $S=0$ & $S=-2$ will be a Part of Root locus.



Similarly. $S=-4$, $S=-\infty$ there exists a 3 number of Poles which is odd number. Therefore, from $S=-4$ to $S=-\infty$ will be a Part of Root locus.

3. Angle of Asymptotes $= \frac{\pm 180\,(2q+1)}{n-m}$ $(q=0,1,2,\cdots n-m)$

$\Rightarrow q=0 \Rightarrow \frac{\pm 180\,(1)}{3-0} = \pm 60°$

$q=1 \Rightarrow \frac{\pm 180\,(2+1)}{3} = \pm 180°$

$q=2 \Rightarrow \frac{\pm 180\times 5}{3} = \pm 300 = \mp 60°$.

4. Centroid $= \frac{\text{Sum of Poles} - \text{Sum of zeros}}{\text{Poles} - \text{zeros}}$

$= \frac{0-2-4-0}{3} = \frac{-6}{3} = -2$.

5. To find breakaway Point or Breakin point

Closed loop T.F $= \frac{G(S)}{1+G(S)} = \frac{\frac{K}{S(S+2)(S+4)}}{1+\frac{K}{S(S+2)(S+4)}}$

$= S(S+2)(S+4)+K=0$

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$$\Rightarrow \quad s(s+2)(s+4)+K=0$$

$$\Rightarrow \quad K = -(s^3+6s^2+8s).$$

Now, differentiate above eqn w.r.t. s. and equate to zero.

$$\Rightarrow \frac{dK}{ds} = -(3s^2+12s+8) = 0$$

$$\Rightarrow \quad -(3s^2+12s+8) = 0$$

$$\Rightarrow \quad s = \frac{-12 \pm \sqrt{12^2-4\times3\times8}}{2\times3}$$

$$= -0.845 \text{ (or) } -3.154.$$

✓ ✗

Now checks for K.

when $s = -0.845$, the value of K is,

$$K = -(s^3+6s^2+8s)$$

$$= -[(-0.845)^3 + 6(-0.845)^2 + (8\times-0.845)]$$

$$= \underline{3.08}$$

Since, K is positive for $s = \underline{-0.845}$ so, $s = -0.845$ is a actual Breakaway point.

when $s = -3.154$

$$K = -[(-3.154)^3 + 6(-3.154)^2 + 8(-3.154)]$$

$$= \underline{-3.08}$$

Since K is negative therefore, $s = -3.154$ is not a Breakaway point.

6. Since, there is no Complex pole (or) Complex zero.

So, there is no angle of departure (or) angle of Arrival.

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:) Crossing point on Imaginary axis.

Characteristic equation is given by,

$$s^3 + 6s^2 + 8s + K = 0.$$

Put $s = j\omega$

$$\Rightarrow (j\omega)^3 + 6(j\omega)^2 + 8j\omega + K = 0$$

$$\Rightarrow -j\omega^3 - 6\omega^2 + 8j\omega + K = 0.$$

Now, Equate Real Part & imaginary Parts to zero.

$$\Rightarrow -j\omega^3 + 8j\omega = 0$$

$$+8j\omega = +j\omega^3$$

$$\Rightarrow \omega^2 = 8$$

$$\omega = \pm\sqrt{8}$$

$$= \pm 2.8$$

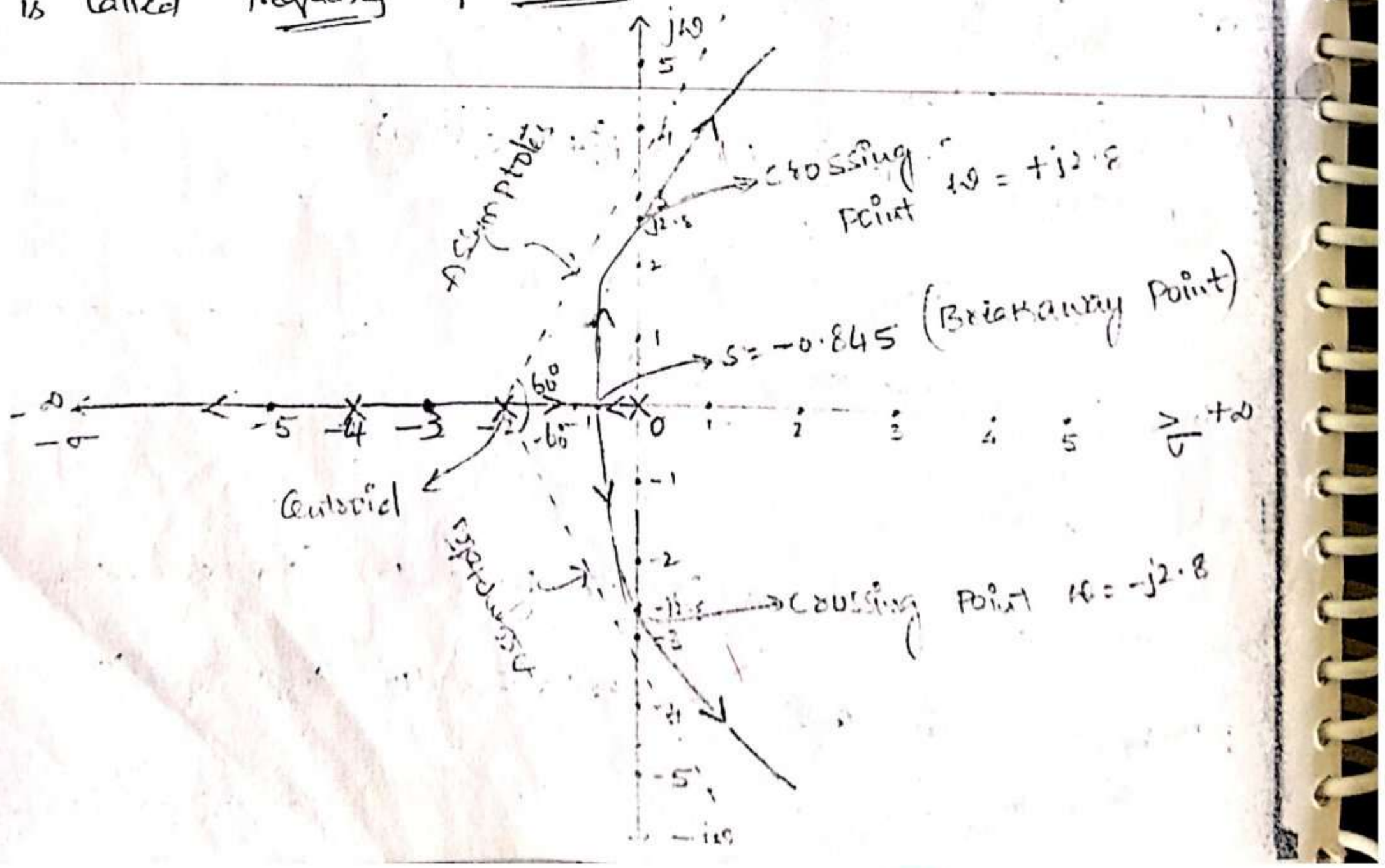
$$+6\omega^2 = +K$$

$$K = 6\omega^2$$

$$= 6 \times 8$$

$$= 48$$

The value of $\omega = \pm j2.8$ is the point that crosses the locus on the imaginary axis for $K = 48$. and value $\omega = \pm 2.8$ is called frequency of oscillation.



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Sketch the root locus of the system with loop transfer function $G(s)H(s) = \frac{K}{s(s+2)(s^2+s+1)}$ $\quad$ T.F $= \frac{K(s+1)}{s(s^2+4s+11)}$

Sol: Given T.F $\Rightarrow G(s)H(s) = \frac{K}{s(s+2)(s^2+s+1)}$ $\quad \left[\because \frac{-1 \pm \sqrt{1-4}}{2} \Rightarrow -0.5 \pm j\frac{\sqrt{3}}{2}\right]$

1. The number of poles are $0, -2, -0.5 \pm j\frac{\sqrt{3}}{2}$

The number of zeros $= 0$.

2. As we got 4 poles so, there must exists 4 Root locus branches.

3. The Angle of Asymptotes $= \frac{180°(2q+1)}{n-m}$ $\quad [\because q = 0, \dots, n-m = 0,1,2,3,4]$

for $q=0 \Rightarrow \frac{\pm 180°}{4} = \pm 45°$

$q=1 \Rightarrow \frac{180° \times 3}{4-0} = \pm 135°$

$q=2 \Rightarrow \frac{5 \times 180°}{4} = \pm 225°$.

$q=3 \Rightarrow \frac{7 \times 180°}{4} = \pm 315°$.

4. Centroid $= \frac{\text{Sum of the poles} - \text{Sum of zeros}}{\text{no of poles} - \text{no of zeros}}$

$= \frac{0 - 2 - 0.5 + j\frac{\sqrt{3}}{2} - 0.5 - j\frac{\sqrt{3}}{2} - 0}{4}$

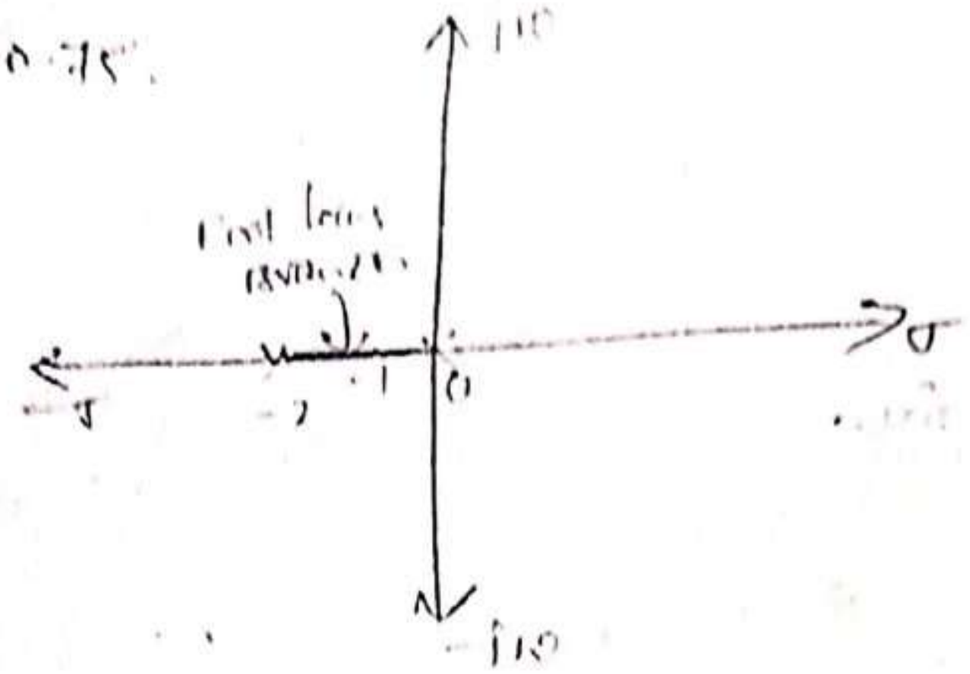
Now, taking the only real parts of poles & zeros.

$\Rightarrow \sigma = \frac{0 - 2 - 0.5 - 0.5 - 0}{4}$

$= \frac{-3}{4} = -0.75$

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$$\Rightarrow K = -\left(s^4 + 3s^3 + 3s^2 + 2s\right) \quad \text{---(4)}$$

$$\Rightarrow \frac{dK}{ds} = 0$$

$$\Rightarrow -4s^3 - 9s^2 - 6s - 2 = 0 \quad \Rightarrow -\left(4s^3 + 9s^2 + 6s + 2\right) = 0$$

$$\Rightarrow \therefore s = -1.455,\ -0.397 + j0.43,\ -0.397 - j0.430$$

Two complex root with real part

Substitute $s = -1.455$ in (4)

$$\Rightarrow K = -\left[(-1.455)^4 + 3(-1.455)^3 + 3(-1.455)^2 + 2(-1.455)\right]$$

$$= -\left[4.481 - 9.211 + 6.35 - 2.91\right]$$

$$= -(-1.319) = 1.319.$$

$\therefore$ As the 'K' value is positive, the root $s = -1.455$ can be a breakaway point.

Now, Check for $s = -0.397 + j0.430$

$$\Rightarrow K = -\left[(-0.397)^4 + 3(-0.397)^3 + 3(-0.397)^2 + 2(-0.397)\right]$$

$$= -\left[0.024 + (-0.187) + 0.492 - 0.794\right]$$

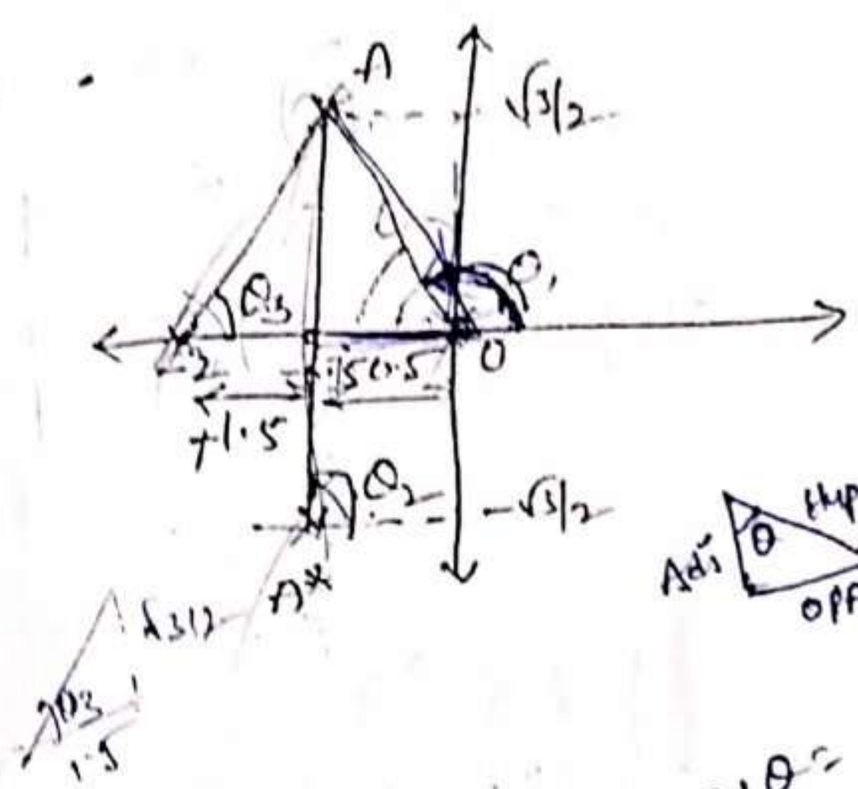
As the Root $s = -0.397 + j0.430$ will not be a breakaway point because this root does not lies in below.. the k

$\therefore$ $S = -1.455$ will be actual Breakaway point

6. The angle of Departure from the complex pole.

$\Rightarrow$ $S = -0.5 \pm j\frac{\sqrt{3}}{2}$

Now, Draw the vectors from all the other poles and zeros to the Complex pole $S = -0.5 \pm j\sqrt{3}/2$.



$\therefore$ $\theta_1 = 180^\circ - \tan^{-1}\frac{\sqrt{3}}{2\times 0.5}$

$= 120^\circ$.

$\theta_2 = 90^\circ$.

$\theta_3 = \tan^{-1}\frac{\sqrt{3}}{2\times 1.5}$

$= 30^\circ$.

$\therefore$ Angle of Departure $\phi_d = 180^\circ - (\theta_1 + \theta_2 + \theta_3)$

$= 180^\circ - (120 + 90 + 30)$

$= 180^\circ - 240^\circ$

$= -60^\circ$

$\cos\theta = \frac{Adj}{Hyp}$

$\tan\theta = \frac{opposite}{Adjecat}$

Now, the angle of departure from Complex pole $S = -0.5 + j\frac{\sqrt{3}}{2}$ is -60°. and from $S = -0.5 - j\frac{\sqrt{3}}{2}$ is $-(-60^\circ) = 60^\circ$.

7. Calculating the crossing point.

Characteristic equation is, $1 + G(S)H(S) = 0$

$\Rightarrow 1 + \frac{K}{S(S+2)(S^2+S+1)} = 0$

$\Rightarrow S^4 + 3S^3 + 3S^2 + 2S + K = 0$.

Now, Substitute the $S = j\omega$ in above equation we get,

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$$\Rightarrow (j\omega)^4 + 5(j\omega)^3 + 5(j\omega)^2 + 2j\omega + K = 0$$

$$\Rightarrow \omega^4 - 5j\omega^3 - 5\omega^2 + 2j\omega + K = 0.$$

Equating the Real & Imaginary parts to zero. we get,

$$-5j\omega^3 + 2j\omega = 0$$

$$\Rightarrow +5j\omega^3 = +2j\omega$$

$$\Rightarrow \omega^2 = \frac{2}{5}$$

$$\Rightarrow \omega = \sqrt{\frac{2}{5}}$$

$$= \pm j\,0.8165.$$

$j\omega$.

$$\omega^4 - 5\omega^2 + K = 0$$

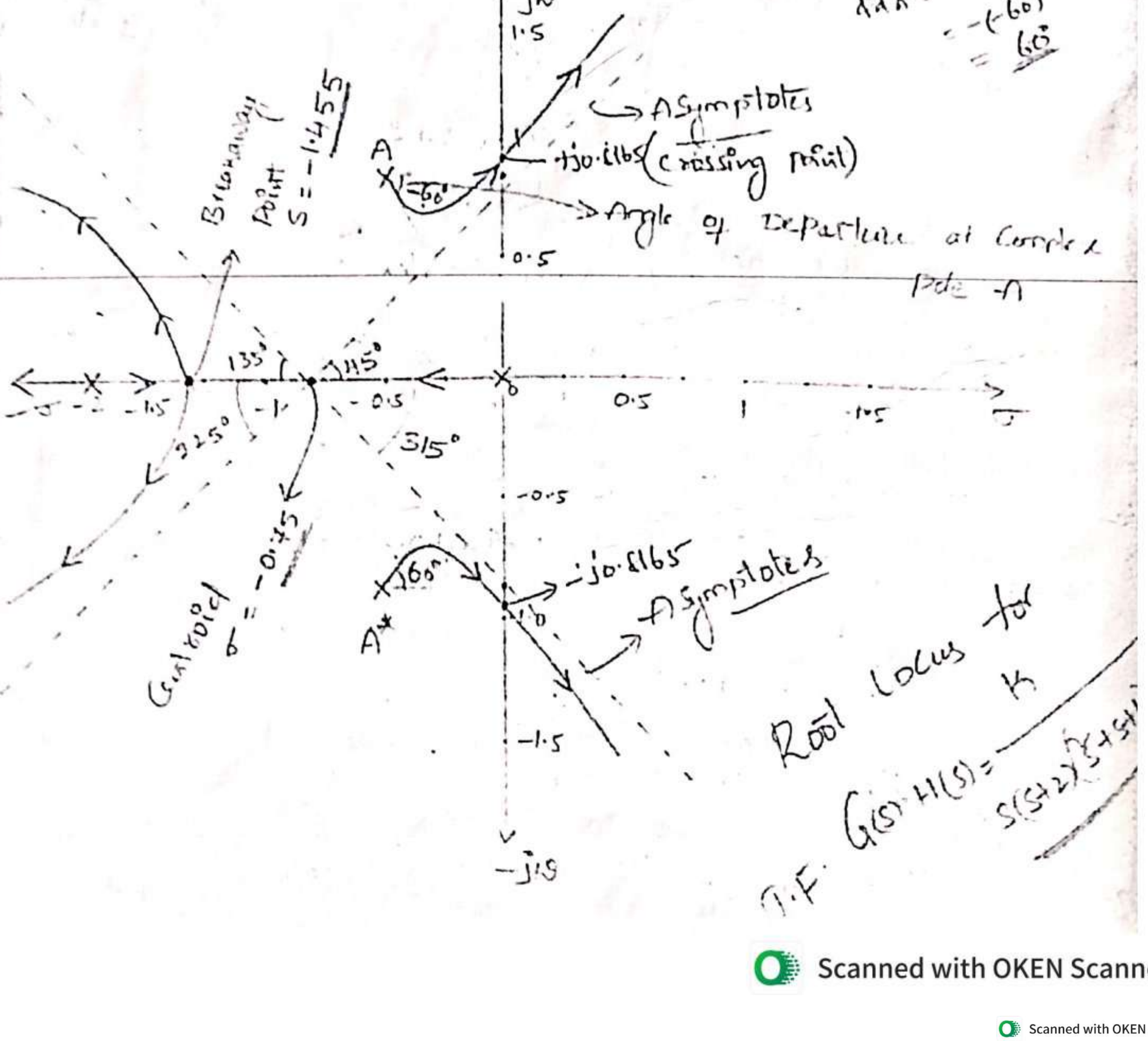
$$\left(\frac{2}{5}\right)^2 - 5\times\frac{2}{5} + K = 0$$

$$\Rightarrow \frac{4 - 9\times 2}{9} + K = 0$$

$$\Rightarrow K = \frac{-14}{9} = +\frac{14}{9}$$

$$\therefore K = 14/9.$$

Now, Plotting the Root locus.



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3 Sketch the Root locus for $G(s)H(s) = \frac{K}{s(s+1)(s+2)(s+3)}$

Sol: 1. given, $G(s)H(s) = \frac{K}{s(s+1)(s+2)(s+3)}$

The number of poles $= 0, -1, -2, -3 = 4$

number of zeros $= 0$

2 As we get 4 poles So, there exists 4 Root locus branches.

3 Angle of Asymptotes.

$$\Rightarrow \frac{\pm 180(2q+1)}{n-m} \qquad q = 0, 1, 2, 3$$

for $q=0 \Rightarrow \frac{\pm 180^\circ \times 1}{4} = \pm 45^\circ$.

$q=1 \Rightarrow \frac{\pm 180^\circ \times 3}{4} = \pm 135^\circ$.

$q=2 \Rightarrow \frac{\pm 180 \times 5}{4} = \pm 225^\circ$

$q=3 \Rightarrow \frac{\pm 180^\circ \times 7}{4} = \pm 315^\circ$.

$q=4 \Rightarrow \frac{\pm 180 \times 9}{4} = \mp 45^\circ$.

4. Centroid

$$\sigma = \frac{\text{Sum of poles} - \text{Sum of zeros}}{\text{number of Poles} - \text{number of zeros}}$$

$$= \frac{0-2-1-3-0}{4} = \frac{-6}{4} = \frac{-3}{2}$$

$$= -1.5.$$

5. finding the Breakaway point.

The C.E $= 1 + G(s) = 0 \Rightarrow 1 + \frac{K}{s(s+1)(s+2)(s+3)} = 0$

$\Rightarrow s(s+1)(s+2)(s+3) + K = 0.$

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$$\Rightarrow K = -s^4 - 6s^3 - 11s^2 - 6s \quad \text{---- (A)}$$

Now, $\frac{dK}{ds} = -\left[4s^3 + 18s^2 + 22s + 6\right]$

And equate $\frac{dK}{ds} = 0$

$$\Rightarrow 4s^3 + 18s^2 + 22s + 6 = 0$$

$$\Rightarrow s = -0.381,\ -2.61,\ -1.5$$

Now Substitute the Value $s = -0.381$ in (A)

$$\Rightarrow K = -\left[(-0.381)^4 + 6(-0.381)^3 + 11(-0.381)^2 + 6(-0.381)\right]$$

$$= -\left[(+0.021) - 0.331 + 1.596 = 2.286\right]$$

$$= -(-1) = 1. \quad \text{(Positive)}$$

Now, Checks for, $s = -2.61$

$$\Rightarrow K = -\left[(-2.61)^4 + 6(-2.61)^3 + 11(-2.61)^2 + 6(-2.61)\right]$$

$$= -\left[46.4 - 106.67 + 74.933 - 15.66\right]$$

$$= -(-0.997) \cong 1. \quad \text{(Positive)}$$

for $s = -1.5$

$$\Rightarrow K = -\left[(-1.5)^4 + 6(-1.5)^3 + 11(-1.5)^2 + 6(-1.5)\right]$$

$$= -\left[5.06 - 20.25 + 24.75 - 9\right]$$

$$= -[0.56] = -0.56. \quad \text{(Negative)}$$

Now, for the Roots $s = -0.381$ and $s = -2.61$ the 'K' Value is positive and for $s = -1.5$ it is Negative.

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Therefore, $s = -0.381$ and $s = -2.61$ are the actual Breakaway points.

6. Crossing point.

The open loop transfer function of a unity f/b system is given by.

$$G(s) = \frac{K(s+9)}{s(s^2+4s+11)}$$

- Sketch the root locus of the system

$$\Rightarrow s^4 + 6s^3 + 11s^2 + 6s + 18 = 0.$$

Now, Substitute $s = j\omega$ in above equation,

$$\Rightarrow (j\omega)^4 + 6(j\omega)^3 + 11(j\omega)^2 + 6j\omega + 18 = 0.$$

$$\Rightarrow \omega^4 - 6j\omega^3 - 11\omega^2 + 6j\omega + K = 0.$$

$$\Rightarrow \omega^4 - 11\omega^2 + K = 0$$

$$1 - 11 + K = 0$$

$$K = 10.$$

$$-6j\omega^3 + 6j\omega = 0.$$

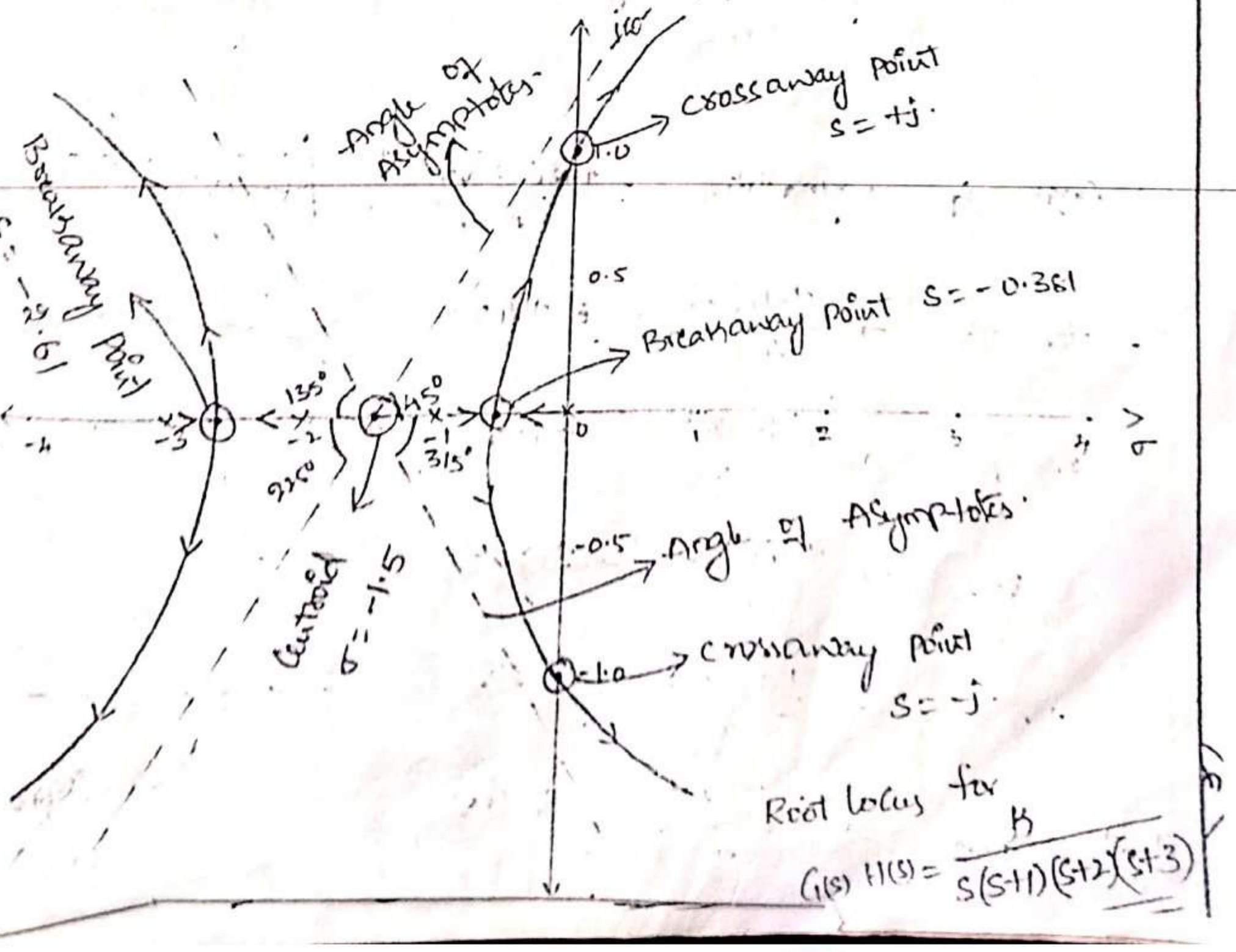
$$-6j\omega^3 = -6j\omega$$

$$\omega^2 = 1$$

$$\omega = \pm\sqrt{1}$$

$$= \pm 1.$$

Now, the Complete Root locus is,



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Sol: Given, $G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$

$\left[\frac{-4 \pm \sqrt{16-80}}{2} \Rightarrow -2 \pm j4\right]$

1. Number of poles $= 0, -4, -2 \pm j4$.

$= 4$.

Number of zeros $= 0$.

2. As we got the 4 poles so, there should be 4 root locus branches, will exist

3. Centroid $= \sigma = \frac{\text{Sum of poles} - \text{Sum of zeros}}{\text{No of poles} - \text{no of zeros}}$

$= \frac{-4 - 2+j4 - 2 - j4 - 0}{4}$

$= \frac{-8}{4} = \underline{-2}$

$\therefore \sigma = -2$

4. Angle of asymptotes $= \frac{\pm 180^\circ (2q+1)}{n-m}$ $\left[q = 0, 1, \ldots, n-m\right]$

$\therefore$ for $q=0 \Rightarrow \frac{\pm 180^\circ \times 1}{4} = \pm 45^\circ$

for $q=1 \Rightarrow \frac{\pm 180^\circ \times 3}{4} = \pm 135^\circ$

$q=2 \Rightarrow \frac{\pm 180^\circ \times 5}{4} = \pm 225^\circ$

$q=3 \Rightarrow \frac{\pm 180^\circ \times 7}{4} = \pm 315^\circ$

and, $q=4 \Rightarrow \frac{\pm 180^\circ \times 9}{4} = \mp 45^\circ$

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Breakaway Point.

The characteristic equation. $1+G(s)H(s)=0$.

$$\Rightarrow 1+\frac{K}{s(s+4)(s^2+4s+20)}=0$$

$$\Rightarrow s(s+4)(s^2+4s+20)+K=0.$$

$$\Rightarrow K=-\left[(s^2+4s)(s^2+4s+20)\right] \quad \text{---} (1)$$

$$\Rightarrow K=-(s^4+8s^3+36s^2+80s). \quad \text{---} (A)$$

Differentiate above equation w.r.t. s. we get,

$$\frac{dK}{ds}=-(4s^3+24s^2+72s+80)=0$$

$$\Rightarrow 4s^3+24s^2+72s+80=0$$

on dividing by 4 we get

$$\Rightarrow s^3+6s^2+18s+20=0.$$

The Roots for above equation can also be find by trail and Error Method (or) from Calculator.

$s=-2, \quad -2\pm j2.449.$

Here, we got the roots as Complex except $s=-2$,

$\therefore$ ~~neglecting~~ the ~~Complex roots~~ ~~and~~ now checking for.

in

the Root $s=-2$

Cheq. for $-2\pm j2.449$

K = positive

$$\Rightarrow K=-\left[s^4+8s^3+36s^2+80s\right]$$

$$=-\left[(-2)^4+8(-2)^3+36\times4+80\times-2\right]$$

$$=-\left[16-64+144-160\right]$$

$$=-[-64]=\underline{64.} \quad \text{(positive)}$$

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When K = 2, we get the value of K as positive. So, the critical boundary point is $S = -2$, $\rightarrow \pm j2.115$.

As we get a Complex pole. So, there is a need to find the (A)th Calculate angle of Departure.

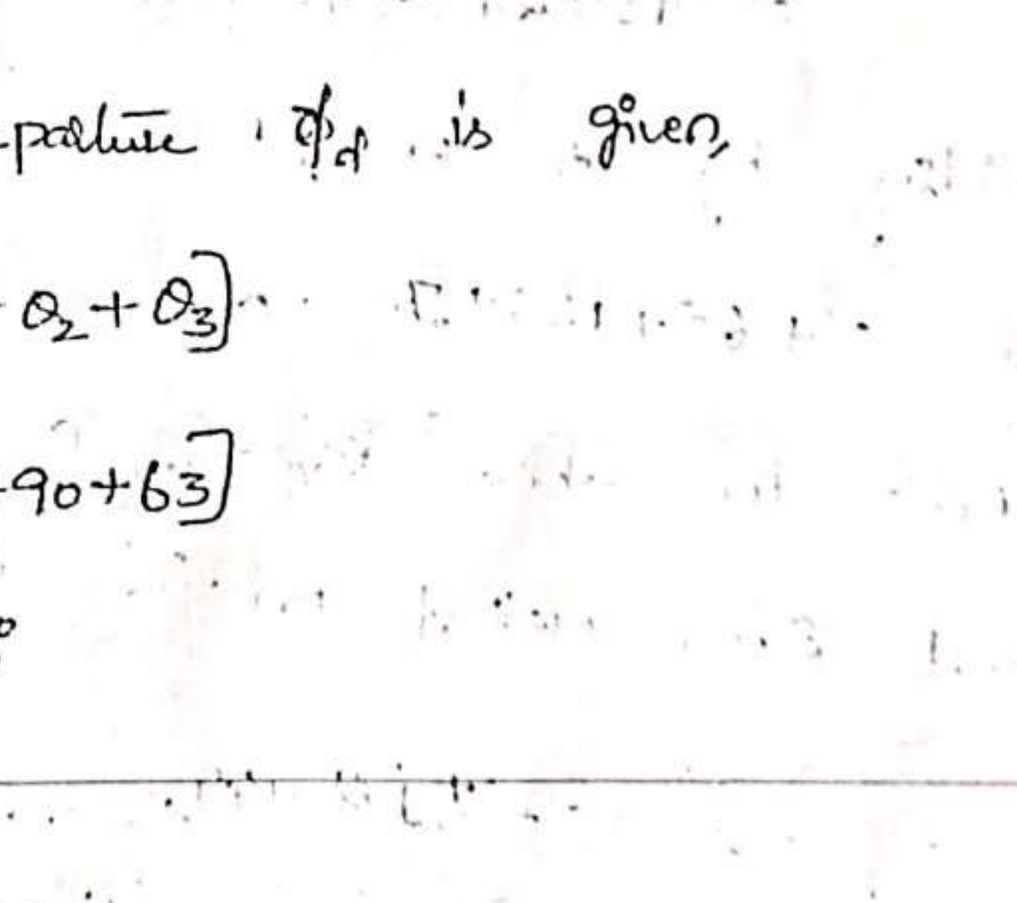
$$\theta_1 = 180^\circ - \tan^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \theta_1 = \underline{117^\circ}$$

$$\theta_2 = \underline{90^\circ}$$

$$\theta_3 = 180^\circ - \tan^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \theta_3 = \underline{63^\circ}$$



Now, the angle of departure ϕ_d is given,

$$\phi_d = 180^\circ - [\theta_1 + \theta_2 + \theta_3]$$

$$= 180^\circ - [117 + 90 + 63]$$

$$= 180^\circ - 270^\circ$$

$$\Rightarrow \phi_d = \underline{-90^\circ}$$

$\therefore$ The angle of departure from the Complex pole is $\underline{-90^\circ}$.

for the Complex pole A^* $= -(-90^\circ)$

$$= \underline{90^\circ}$$

7. Crossing point on imaginary axis.

from the characteristic Equation.

$$\Rightarrow s^4 + 8s^3 + 36s^2 + 80s + K = 0.$$

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Now, Substitute $S=j\omega$ in above equation,

$\Rightarrow (j\omega)^4 + 8(j\omega)^3 + 36(j\omega)^2 + 80j\omega + K = 0.$

$\Rightarrow \omega^4 - 8j\omega^3 - 36\omega^2 + 80j\omega + K = 0.$

Now, Equate the real part and imaginary part to zero.

$\Rightarrow \omega^4 - 36\omega^2 + K = 0$

$(10)^2 - 36\times10 + K = 0$

$100 - 360 + K = 0$

$\Rightarrow K = \underline{260}.$

$-8j\omega^3 + 80j\omega = 0$

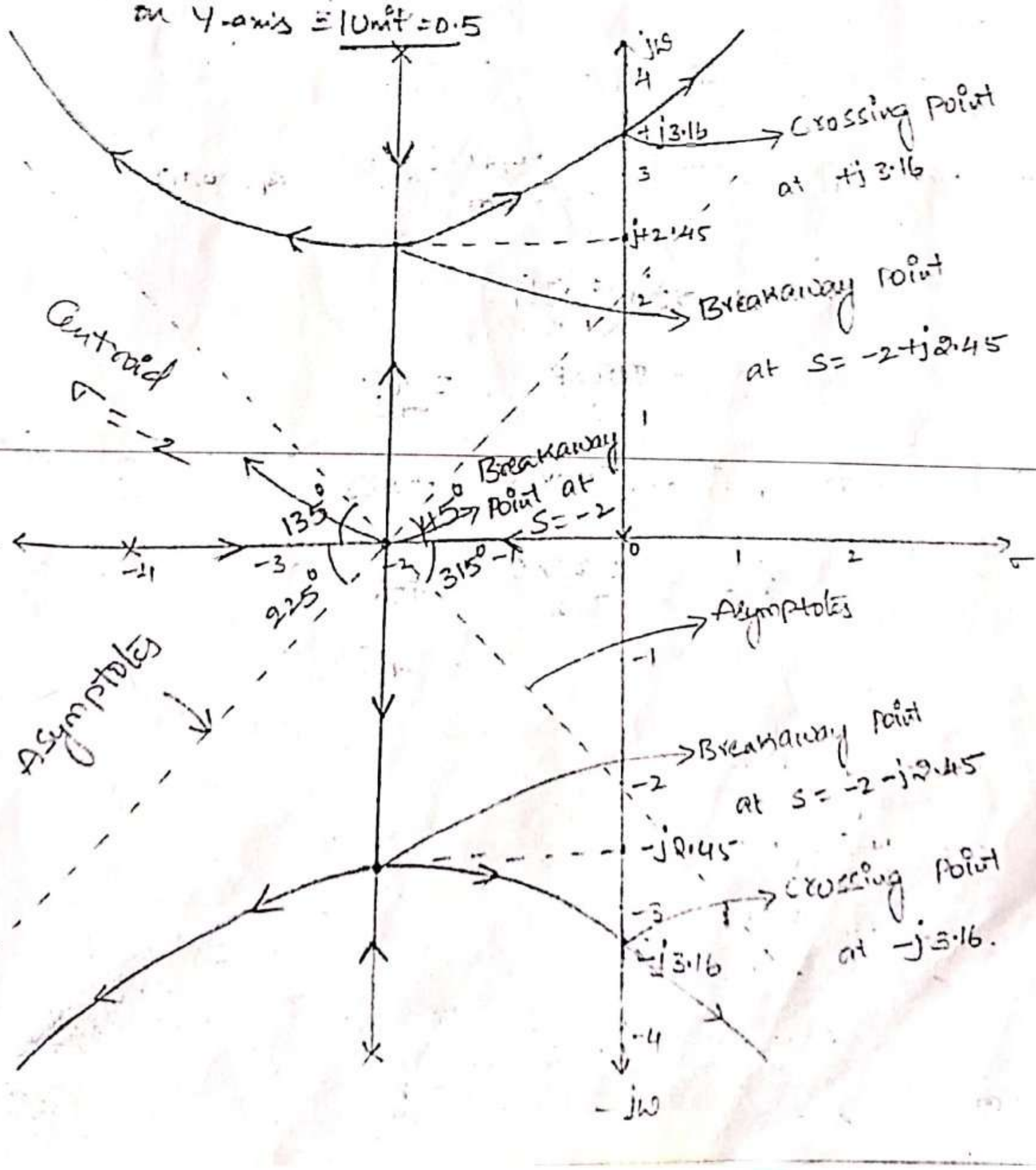
$\Rightarrow -8j\omega^{\cancel{3}\,2} = -\frac{80j\omega}{10}$

$\Rightarrow \omega^2 = 10.$

$\Rightarrow \omega = \pm\sqrt{10}$

$\therefore \boxed{\omega = \pm 3.16}$

Scale: on x-axis 1
on y-axis = 1 Unit = 0.5



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$s^2(s+9)$.

Ex. given, $G(s) = \frac{K(s+1)}{s^2(s+9)}$

1. The number of Poles = 0, 0, −9.

= 3.

Number of Zeros = −1.

2. As we got 3 Poles So, the 3 Root locus branches will exist.

3. Centroid = $\sigma = \frac{\text{Sum of poles} - \text{Sum of zeros}}{\text{No of poles} - \text{No of zeros}}$

$= \frac{0-0-9+1}{3-1} = \frac{-8}{2}$

$= -4.$

$\therefore \sigma = -4.$

4. Angle of asymptotes $= \frac{2q+1}{n-m}(\pm 180°)$ $q = 0, 1, 2$

for $q=0 \Rightarrow \pm 180\left(\frac{2q+1}{n-m}\right)$

$= \frac{\pm 180 \times 1}{2} = \pm 90°$

for $q=1 \Rightarrow \frac{\pm 180 \times 3}{2} = \pm 270°$

$q=2 \Rightarrow \frac{\pm 180° \times 5}{2} = 450°$

$= \mp 90°$

5. Breakaway Point

$C.E = 1 + G(s) = 0$

$= 1 + \frac{K(s+1)}{s^2(s+9)} = 0$

$= s^2(s+9) + Ks + 5K = 0$

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$$\Rightarrow K = -\left[\frac{s^2(s+9)}{s+1}\right] \quad \text{— (A)}$$

$\therefore$ Now, $\frac{dK}{ds} = 0$

$$\Rightarrow \frac{dK}{ds} = -\left[\frac{s^2(s+9) - (s+1)(3s^2+18s)}{(s+1)^2}\right] = 0$$

$$= -\left[\frac{s^3+9s^2-(3s^3+18s^2+3s^2+18s)}{(s+1)^2}\right] = 0$$

$$= -\left[\frac{-2s^3-12s^2-18s}{(s+1)^2}\right] = 0$$

$$\Rightarrow 2s^3+12s^2+18s = 0.$$

$$s = \underline{0, 0, -3}.$$

Now, Substitute $s = -3$ in (A)

$$\Rightarrow K = -\left[\frac{s^2(s+9)}{s+1}\right]$$

$$= -\left[\frac{9(-3+9)}{-3+1}\right]$$

$$= +\frac{9(6)^3}{42}$$

$$K = \underline{27} \quad \text{(positive)}.$$

$\therefore$ when $\underline{s = -3}$, the K value is positive. So, the root $s = -3$ is the actual breakaway point

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6. As there is no complex pole, Hence, there is no need to find the angle of departure (or) angle of arrival.

7. Crossing point-

-from the C.E $\Rightarrow$ $1+G(s)H(s)=0$

$$\Rightarrow 1+\frac{K(s+1)}{s^2(s+9)}=0$$

$$\Rightarrow s^2(s+9)+Ks+K=0$$

$$\Rightarrow s^3+9s^2+Ks+K=0$$

Now, Substitute $s=j\omega$ in above equation,

$$\Rightarrow (+j\omega)^3+9(j\omega)^2+Kj\omega+K=0$$

$$\Rightarrow -j\omega^3-9\omega^2+Kj\omega+K=0.$$

$$\Rightarrow -j\omega^3+Kj\omega=0 \quad \Big| \quad -9\omega^2+K=0$$

$$\Rightarrow -j\omega^{3-2}=-Kj\omega \quad \Big| \quad \Rightarrow -9(K)+K=0$$

$$= \omega^2=K. \quad \Big| \quad \Rightarrow K=9K$$

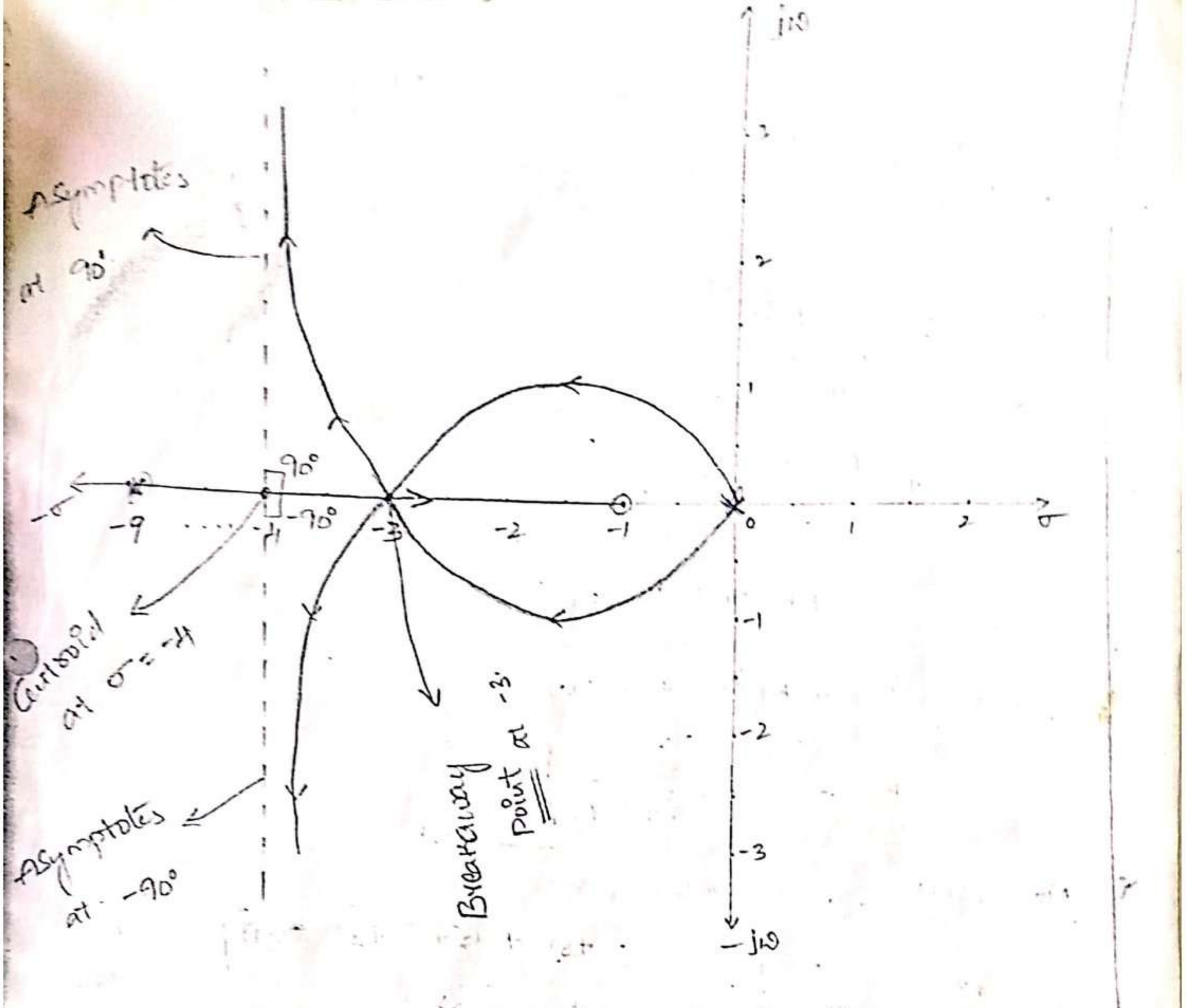
$\Rightarrow$

∴ Where 'K' is the gain of the System and this gain of the System Can't be zero.

Hence, there is no intersection point on imaginary axis.

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(5) Sketch the Root locus for a Transfer function.

$$G(s) = \frac{K(s+2)(s+1)}{s(s-1)}.$$

Sol^n given, $G(s) = \frac{K(s+2)(s+1)}{s(s-1)}$.

1. Number of poles = 0, 1.

   Number of zeros = -2, -1.

2. As there are two poles so, there exist two Root locus branches will exist on the Real axis.

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3. Asymptotes $= \frac{1\&2\,(2q+1)}{n-m}$ [where $n-m=0$]

$\therefore$ There is no Asymptotes

4. Breakaway Point

$\Rightarrow 1+G(s)H(s)=0$

$\Rightarrow 1+\frac{k(s+2)(s+1)}{s(s-1)}=0$

$\Rightarrow s(s-1)+k(s+2)(s+1)=0$

$\Rightarrow s^2-s+k(s^2+3s+2)=0$

$\Rightarrow k=-\left(\frac{s^2-s}{s^2+3s+2}\right)$ ——— (A)

Now, $\frac{dk}{ds}=0$

$\Rightarrow \frac{dk}{ds}=-\left[\frac{-(s^2-s)(2s+3)+(s^2+3s+2)(2s-1)}{(s^2+3s+2)^2}\right]$

$=-\left[\frac{-2s^3-s^2+3s+2s^3+5s^2+s-2}{(s^2+3s+2)^2}\right]=0$

$\left[\frac{d}{ds}\left(\frac{u}{v}\right)=\frac{u'v-uv'}{v^2}\right]$

$\Rightarrow -\left[\frac{2(2s^2+2s-1)}{(s^2+3s+2)^2}\right]=0$

$\Rightarrow 2(2s^2+2s-1)=0$

$\Rightarrow 2s^2+2s-1=0$

$\Rightarrow s=0.36,\ -1.36$

Substitute $s=-1.36$ in (A)

$\Rightarrow k=\left[\frac{-(s^2-s)}{s^2+3s+2}\right]$

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$$K = -\left[\frac{(-1.36)^2 + 1.56}{(-1.36)^2 + 3(-1.36) + 2}\right]$$

$$\Rightarrow K = -\left[\frac{3.206}{-0.2304}\right]$$

$$= -(-13.93)$$

$$\therefore K = 13.93. \quad \text{(positive)}$$

Now, for $s = 0.36$

$$K = -\left[\frac{(0.36)^2 - 0.36}{(0.36)^2 + 3(0.36) + 2}\right]$$

$$= -\left[\frac{0.1296 - 0.36}{0.1296 + 1.08 + 2}\right]$$

$$= -\left[\frac{-0.2304}{3.2096}\right]$$

$$\therefore K = 0.071 \quad \text{(positive)}$$

$\therefore s = 0.36$ is lies between the two poles 0 and 1 so it is called as Breaks away point, where as the $s = -1.36$ lies between two zeros. so, it is called as Breaks in point

5. As we don't have a complex pole so, no need to find the angle of departure

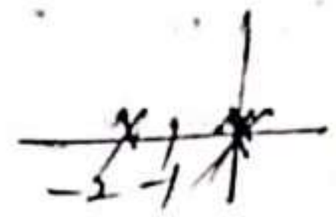
6. Crossing Point.

$\Rightarrow s^3 - s + Ks^2 + 3Ks + 2K = 0$

$s^2(s+2)$

$s=0, 0, -2$

Now, Substitute $s = j\omega$ in above equation.



$\Rightarrow (j\omega)^3 - j\omega + K(j\omega)^2 + 3Kj\omega + 2K = 0.$

$\Rightarrow -\omega^2 - K\omega^2 + 2K = 0$ | $-\omega + 3K\omega = 0$

$-\omega^2(1+K) + 2K = 0$ | $\Rightarrow \omega = 3K\omega$

$\Rightarrow \omega^2(1+K) = 2K$ | $\Rightarrow K = 1/3.$

$\Rightarrow \omega^2 = \frac{2K}{1+K}$

$= \frac{2/3}{1+1/3} = 0.5$

$\Rightarrow \omega = \pm\sqrt{0.5} = \pm 0.7$

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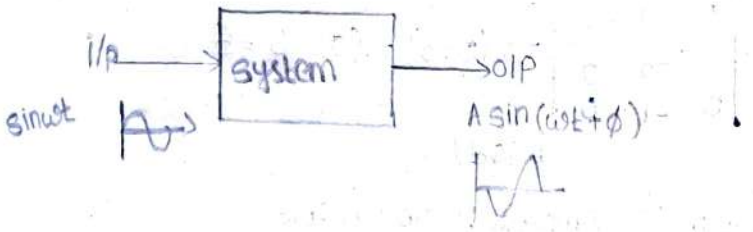
UNIT-4

Frequency response Analysis

- Frequency response is the steady state response of a system to the system is a sinusoidal input
- Consider a linear time invariant LTI system if Let $x(t)$ is input sinusoidal signal
- The response or output $y(t)$ is also a sinusoidal signal of same Frequency but with different magnitude & phase angle.
- The Frequency response of a system is normally obtained by varying the Frequency of the input signal by keeping the magnitude of the input signal at a constant value.
- In the system transfer Function $T(s)$, if s is replaced by "$j\omega$" is called sinusoidal transfer Function.

Advantages:-

1. Frequency response test easily to perform
2. Transfer Function can be obtained by the frequency response of the system
3. It is used to find the absolute and relative stability of the system
4. The operation required for to obtain Frequency response are simple and easy to use.



Disadvantages:-

1. obtain Frequency response particularly it time consuming.
2. The high time constant, this method is not convenient.
3. This method is considered as "old and outdated". when compared

with methods develop for digital computer simulation and modelling.

Frequency domain specifications:-

The performance and characteristics of a control system in Frequency domain or measured in terms of Frequency domain specifications

(i) Resonant peak (M_r) It is identified as the maximum value of the magnitude of the closed loop transfer function is called resonant peak.

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

(ii) Resonance Frequency:- The frequency at the

The Frequency at which resonance or peak occurs is called resonance Frequency.

$$\omega_r = \frac{1}{\omega_n\sqrt{1-\xi^2}}$$

(iii) Bandwidth:- It is defined as the range of frequencies for which the gain is more than -3dB. The Frequency at which the gain is -3dB is called cutoff Frequency

(iv) cut-off rate:- The slope of the log magnitude curve near the cut-off Frequency is called cut-off rate. It indicates the ability of the system to differentiate the signal from noise.

(v) Gain Margin (kg):- It is defined as the reciprocal of the openloop transfer function at phase crossover Frequency.

- The phase crossover Frequency can be represented by ω_{pc}.
- The Frequency at which the phase open loop transfer Function is 180° is called phase crossover Frequency ω_{pc}

$$kg = \frac{1}{|G(j\omega)|}$$

$$K = 20\log kg = 20\log\frac{1}{|G(j\omega)|}$$

$$\therefore K = -20\log|G(j\omega)|$$

Phase Margin (γ):-

- It is defined as amount of additional phase lag at gain crossover frequency required to bring the system to the verge in the stability.
- Gaincrossover frequency is the frequency at which the magnitude of the open loop transfer function is unity (or) zero dB.
- The phase margin is obtained by adding to the phase angle "ϕ" of the open loop transfer function at the gain of the cross over frequency

$$\gamma = 180^\circ + \phi_{gc}$$

Imp

Bode plot:-

- It is a frequency response plot of the transfer function. A bode plot consists of two subplots
  - (i) Magnitude plot
  - (ii) phase plot
- The bode plot can be drawn both openloop and closed loop system. It is usually drawn to the open loop system
- The standard representation of the logarithmic of the open loop transfer function $G(j\omega)$ having magnitude

$$20 \log |G(j\omega)|$$

where the base of logarithm is "10".

- The unit used in the representation of magnitude is decibels(dB).

Advantages:-

The main advantage of bode plot is the multiplication of magnitude can be converted to addition.

- consider the open loop transfer function

$$G(s) = \frac{K(1+sT_1)}{s(s1+sT_2)(1+sT_3)}$$

put $s = j\omega$

$$G(j\omega) = \frac{K \cdot 1 + j\omega T_1}{j\omega(1+j\omega T_2)(1+j\omega T_3)}$$

$a+ib$
Mag $\sqrt{a^2+b^2}$
Phase $\tan^{-1}\left(\frac{b}{a}\right)$
$= \frac{\text{imaginary}}{\text{real}}$

$$= \frac{k\sqrt{1+\omega^2 T_1^2}\ \tan^{-1}(\omega T_1)}{\angle 90^\circ\, \omega\sqrt{1+\omega^2 T_2^2}\cdot \tan^{-1}(\omega T_2)\cdot\sqrt{1+\omega^2 T_3^2}\tan^{-1}(\omega T_3)}$$

$$\angle G_1(j\omega) = k\tan^{-1}(\omega T_1) - 90^\circ - \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_3)$$

$\Rightarrow$ Magnitude $K = 20\log|G(j\omega)|$

$$|G(j\omega)| = 20\log\left[\frac{k\sqrt{1+\omega^2 T_1^2}}{\omega\sqrt{1+\omega^2 T_2^2}\cdot\sqrt{1+\omega^2 T_3^2}}\right]$$

$$= 20\log\sqrt{1+\omega^2 T_1^2} + 20\log k/\omega + 20\log\left(\frac{1}{\sqrt{1+\omega^2 T_2^2}}\right) + 20\log\left(\frac{1}{\sqrt{1+\omega^2 T_3^2}}\right)$$

$$= 20\log\frac{k}{\omega} + 20\log\sqrt{1+\omega^2 T_1^2} - 20\log\sqrt{1+\omega^2 T_2^2} - 20\log\sqrt{1+\omega^2 T_3^2}.$$

Basic Factor of $G(j\omega)$:-

The basic factors that vary frequently occurs in a typical values of Transfer Function $G(j\omega)$ or

(i) constant gain (k)

(ii) Integral Factor $\frac{k}{j\omega}$ (or) $\frac{k}{(j\omega)^n}$.

(iii) Derivative Factor $k(j\omega)$ or $k(j\omega)^n$.

(iv) First order in denominator $\frac{1}{1+j\omega T}$ (or) $\frac{1}{(1+j\omega T)^m}$

(v) First order in numerator Factor $1+j\omega T$ (or) $(1+j\omega n)^m$

Procedure:-

Step-1:- convert the transfer function into bode plot form (or) time constant form. The bode form of the transfer function is

$$G(s) = \frac{K(1+sT_1)}{(1+sT_2)(1+sT_3)}.$$

List the corner frequency in the increasing order and prepare a table shown in below:-

Term	corner Frequency	Slope(dB)	change in slope (dB)

Step-2:-

- For the Frequency factor "$1+j\omega T$" the Frequency ($\omega = \frac{1}{T}$) is called "corner Frequency (ω_c)"
- In the above table enter 'K' (or) $\frac{K}{j\omega}$ (or) $K(j\omega)$ as the first term and the other term in the increase order of corner Frequency.
- Then enter the corner Frequency, slope contributed by each term and change in slope at every corner Frequency.

Step-3:- change an arbitary Frequency (ω_1) which is lesser than the lowest corner Frequency. Now calculate the dB, magnitude $K(j\omega)^m$ (or) $\frac{K}{(j\omega)^m}$ at ω_1 and at the corner Frequency.

Step-4:- Then calculate the gain at every corner Frequency one by one using the formula,

Gain at ω_y = change in gain from ω_x to ω_y + gain at ω_x

$\therefore A = \{\text{slope from } \omega_x \text{ to } \omega_y \times \log\left(\frac{\omega_y}{\omega_x}\right)\} + \text{gain at } \omega_x$

Step-5:- choose an arbitrary Frequency (ω_n) which is greater than the highest corner Frequency. calculate the gain at "ω_n" by using the Formula at in above step

Step-6:- Now in a semilog graph take the required range of Frequencies on x-axis and the range of dB magnitude on y-axis. Then make all points obtain from step 3,4,5 on the graph and joining the points by straight line.

Step-7:-

Take another axis in the graph on y-axis, where the magnitude plot is drawn in this y-axis marked the desired

range of phase angles. After choosing proper scale from the calculated values of ω and phase angle. Mark the all points on the graph joining the points by a smooth curve.

1. plot the bode diagram for the following transfer function and obtain the gain and phase cross over frequency.

$$G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$$

Sol: Given $G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$

put $s = j\omega$

$$G(j\omega) = \frac{10}{j\omega(1+0.4j\omega)(1+0.1j\omega)}$$

corner frequencies

$\frac{1}{1+0.4j\omega} \Rightarrow \omega_{c_1} = \frac{1}{T_1} = \frac{1}{0.4} = 2.5$ rad/sec

$\frac{1}{1+0.1j\omega} \Rightarrow \omega_{c_2} = \frac{1}{T_2} = \frac{1}{0.1} = 10$ rad/sec.

Here Following the below table

Term	corner Frequency ω_c	Slope	change in slope dB/dec
$\frac{10}{j\omega}$	-	-20	
$\frac{1}{1+0.4j\omega}$	$\omega_{c_1} = \frac{1}{0.4} = 2.5$ rad/s	-20	-20-20=-40
$\frac{1}{1+0.1j\omega}$	$\omega_{c_2} = \frac{1}{0.1} = 10$ r/s	-20	-40-20=-60

$\omega_L = 0.1$ rad/s ; $\omega_{c_1} = 2.5$ r/s ; $\omega_{c_2} = 10$ r/s ; $\omega_n = 50$ r/s

$\omega_L = 0.1$ r/s

$\Rightarrow \omega = \omega_L = 0.1$ r/s

$A = 20 \log |G(j\omega)|$

$= 20 \log \left|\frac{10}{j\omega}\right| = 20 \log \left|\frac{10}{0.1}\right| = 40$ dB

$\Rightarrow \omega = \omega_{c_1} = 2.5$ r/s

$A = 20 \log \left|\frac{10}{j\omega}\right| = 20 \log \left|\frac{10}{2.5}\right| = 12$ dB

$\omega = \omega_{c2} = 10$ r/s

A = slope changes from ω_{c1} to $\omega_{c2} \times \log\left(\frac{\omega_{c2}}{\omega_{c1}}\right) + A$ at ω_{c1}

$= -40 \times \log\left(\frac{10}{2.5}\right) + 12 = -12$ dB

$\Rightarrow \omega = \omega_n = 50$ r/s

A = slope changes from ω_{c2} to $\omega_n \times \log\left(\frac{\omega_n}{\omega_{c2}}\right) + A$ at ω_{c2}

$= -60 \times \log\left(\frac{50}{10}\right) + (-12) = -53$ dB

Magnitude plot:-

$\omega_{\downarrow} = 0.1$ r/s $\Rightarrow A = 40$ dB

$\omega_{c1} = 2.5$ r/s $\Rightarrow A = 12$ dB

$\omega_{c2} = 10$ rad/sec $\Rightarrow A = -12$ dB

$\omega_n = 50$ rad/sec $\Rightarrow A = -53$ dB

phase plot:-

$\angle G(j\omega) = -90° - \tan^{-1}(0.4\omega) - \tan^{-1}(0.1\omega)$

ω	$\phi \angle G(j\omega)$
0.1	-92.80°
1	-117.5
2.5	-149.03°
5	-180°
10	-210.90°
30	-240.80°
50	-258.82°

2. sketch the bode plot for the following functions:-

$G(s)H(s) = \frac{Ks^2}{(1+0.25s)(1+0.025s)}$. Find the K at $\omega_{gc} = 10$ rad/sec

Given $G(s)H(s) = \frac{Ks^2}{(1+0.25s)(1+0.025s)}$

put $s = j\omega$

$= \frac{K(j\omega)^2}{(1+0.25j\omega)(1+0.025j\omega)}$

corner Frequencies

$\frac{1}{1+0.25j\omega} \Rightarrow \omega_{c1} = \frac{1}{T_1} = \frac{1}{0.25} = 4$ rad/sec

$\frac{1}{1+0.025j\omega} \Rightarrow \omega_{c2} = \frac{1}{T_2} = \frac{1}{0.025} = 40$ rad/sec

Here the Following table:-

Term	corner Frequency	slope (dB)	change in slope (dB/dec)
$K(j\omega)^2$	$s=20$	40	
$\frac{1}{1+0.25j\omega}$	$\omega_{c1} = \frac{1}{T_1} = \frac{1}{0.25} = 4$ rad/sec	-20	40-20 = 20
$\frac{1}{1+0.025j\omega}$	$\omega_{c2} = \frac{1}{T_2} = 40$	-20	20-20 = 0

$\omega_1 = 1$ rad/sec ; $\omega_{c1} = 4$ r/s ; $\omega_{c2} = 40$ rad/sec ; $\omega_n = 100$ rad/sec.

$\Rightarrow \omega = \omega_1 = 1$ r/s

$A = 20 \log|G(j\omega)|$ $K=1$

$= 20 \log|K(j\omega)^2| = 20 \log|K(1)^2| = 0$

$\Rightarrow \omega = \omega_{c1} = 4$ r/s

$A = 20 \log|G(j\omega)| = 20 \log|K(4)^2| = 24K$ $K=1$

$\Rightarrow \omega = \omega_{c2} = 40$ r/s

A = slope changes from ω_{c1} to $\omega_{c2} \times \log\left(\frac{\omega_{c2}}{\omega_{c1}}\right) + A$ at ω_{c1}

$= 20 \times \log\left(\frac{40}{4}\right) + 24K = 20 + 24K = 44$ dB

$\Rightarrow \omega = \omega_n = 100$ r/s

A = slope changes from ω_{c_2} to $\omega_n \times \log\left(\frac{\omega_n}{\omega_{c_2}}\right) + A$ at ω_{c_2}

$= 0 \times \log\left(\frac{100}{40}\right) + 24 = 44$ dB.

Magnitude plot:-

$\omega_d = 1$ rad/sec $\Rightarrow$ A = 0 dB

$\omega_{c_1} = 4$ rad/sec $\Rightarrow$ A = 24 dB

$\omega_{c_2} = 40$ rad/sec $\Rightarrow$ A = 44 dB

$\omega_n = 100$ rad/sec $\Rightarrow$ A = 44 dB.

phase plot :-

$\angle G(j\omega) = 180 - \tan^{-1}(0.25\omega) - \tan^{-1}(0.025\omega)$

ω	ϕ $\angle G(j\omega)$
1	164°
2	150°
4	129°
20	74°
40	50°
70	33°
100	24°

Now, $20 \log K = -32$

$\log K = \frac{-32}{20}$

$K = 10^{\left[\frac{-32}{20}\right]} = 0.025$ dB

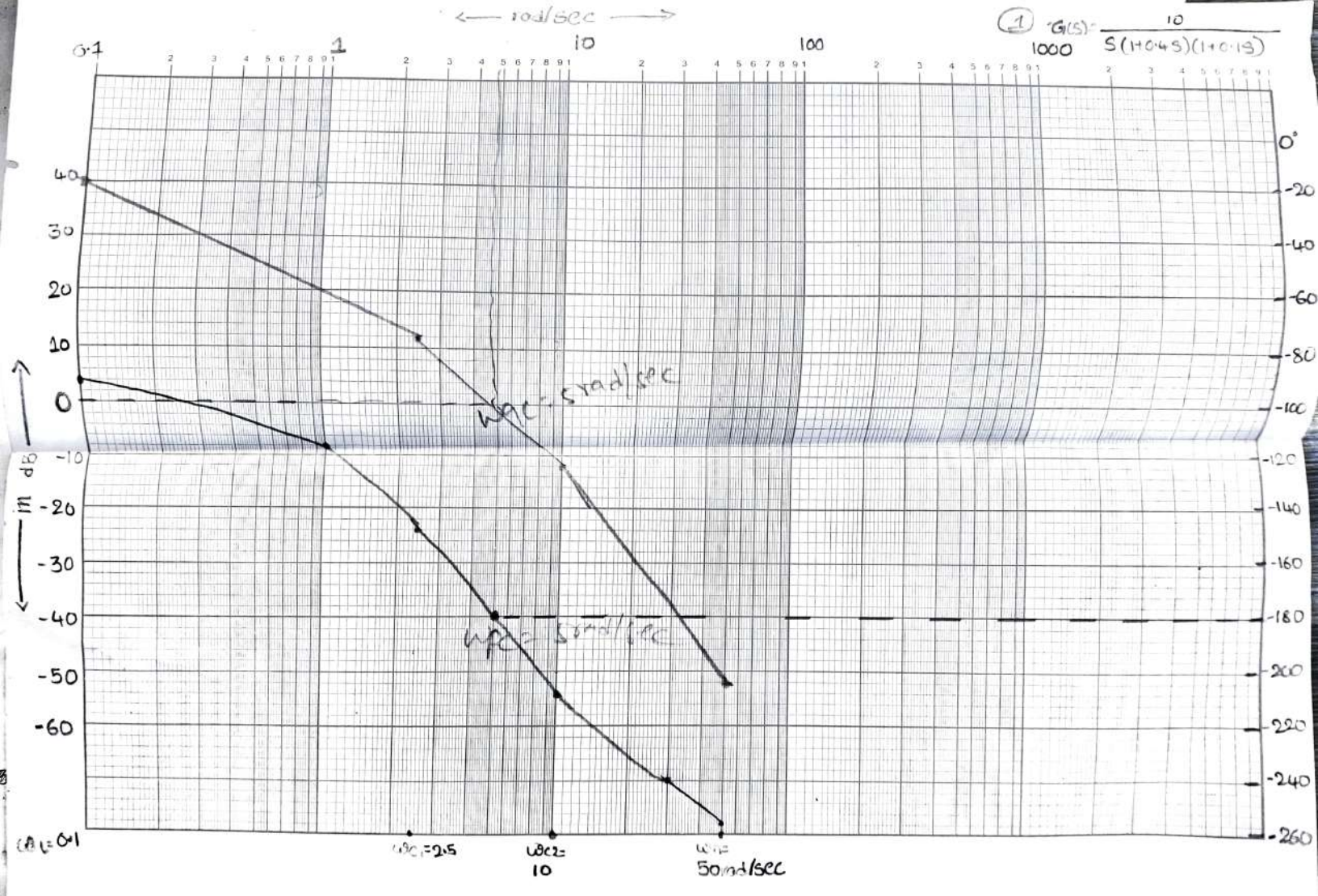
Now, magnitudes are,

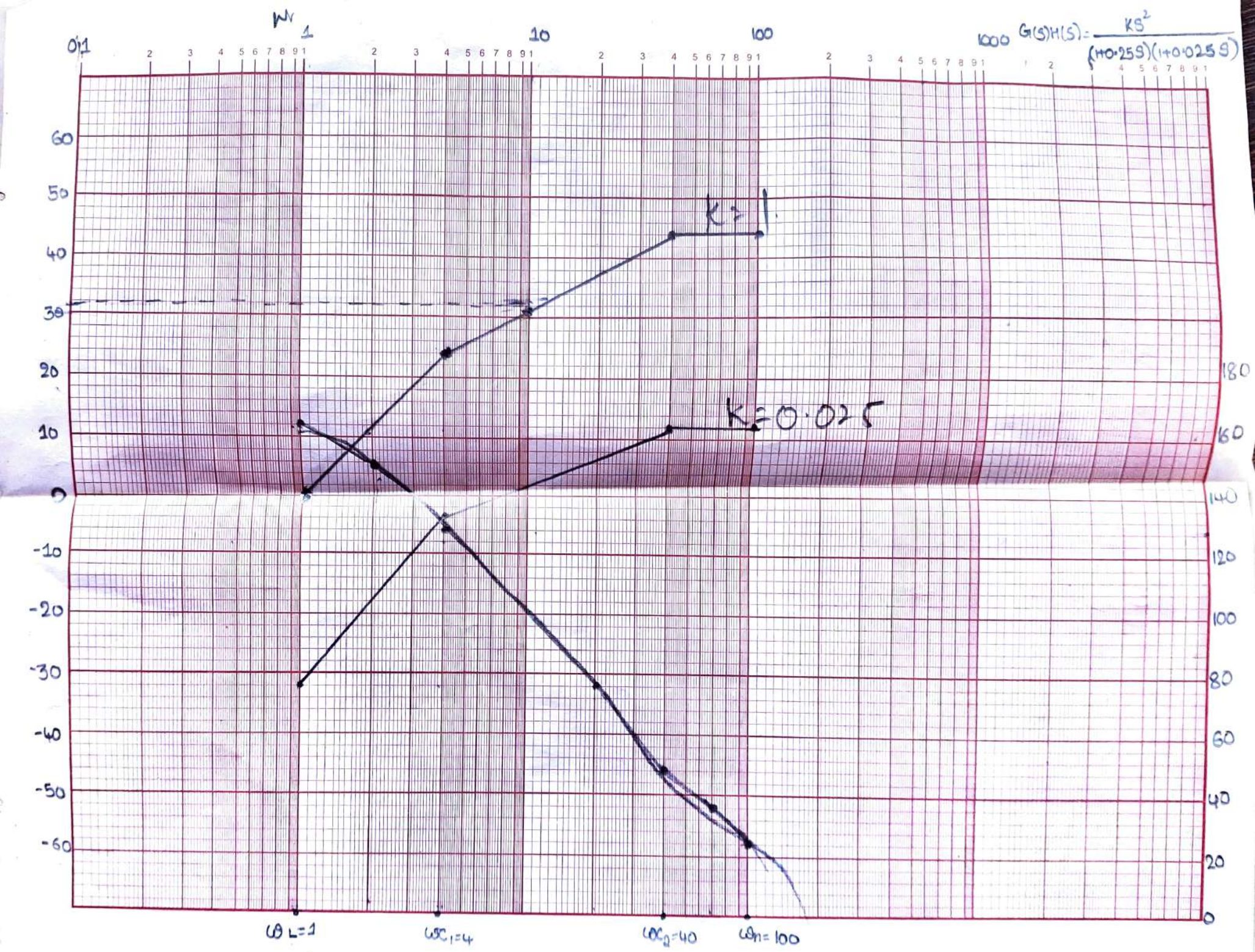
$\omega_d = 0 - 32 = -32$ dB

$\omega_{c_1} = 24 - 32 = -8$ dB

$\omega_{c_2} = 44 - 32 = 12$ dB

$\omega_n = 44 - 32 = 12$ dB.

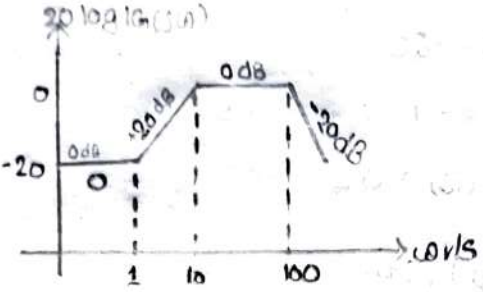

← rad/sec →
0.1
1
10
100
1000
(1) G(s) = 10 / S(1+0.4S)(1+0.1S)
40
30
20
10
0
-10
-20
-30
-40
-50
-60
m dB
0°
-20
-40
-60
-80
-100
-120
-140
-160
-180
-200
-220
-240
-260
ω = 0.1
ωc1 = 2.5
ωc2 = 10
ω = 50 rad/sec
ωc1 = 5 rad/sec
ωc2 = 5 rad/sec


ω
0.1
1
10
100
1000
G(S)H(S) = KS² / (1+0.25S)(1+0.025S)
K=1
K=0.025
60
50
40
30
20
10
0
-10
-20
-30
-40
-50
-60
180
160
140
120
100
80
60
40
20
0
ω L=1
ωc1=4
ωc2=40
ωn=100

very imp Gate

Transfer Function From Bode plot:-

1. consider bode plot (magnitude plot) in figure Find the transfer function



corner Frequencies, ω = 0, 1, 10 and 100

$$H(s) = \frac{k(s+z_1)(s+z_2)\cdots(s+z_n)}{s^n(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= \frac{k\left(1+\frac{s}{z_1}\right)\left(1+\frac{s}{z_2}\right)\cdots\left(1+\frac{s}{z_n}\right)}{s^n\left(1+\frac{s}{p_1}\right)\left(1+\frac{s}{p_2}\right)\cdots\left(1+\frac{s}{p_n}\right)}$$

Here when slope is increasing by +20 dB/dec then zero's is present.

When slope is decreased by -20 dB/dec then poles are present

- ω = 1 rad/sec ⇒ slope is added i.e., +20 dB/dec ⇒ zero is exist

  at $\omega = 1$

  $\therefore \left(1+\frac{s}{z_1}\right) = (1+s/1) = (1+s)$ in numerator

- ω = 10 rad/sec ⇒ slope is 0 dB/dec is added at ω = 10 r/s i.e., pole exist

  $\therefore \left(1+\frac{s}{p_1}\right) = \left(1+\frac{s}{10}\right)$

- ω = 100 rad/sec ⇒ slope is added -20 dB/dec ; poles is exist at ω = 100 r/s

  $\therefore \left(1+\frac{s}{p_2}\right) = \left(1+\frac{s}{100}\right)$

$$\therefore H(s) = \frac{k(1+s)}{(1+s/10)(1+s/100)}$$

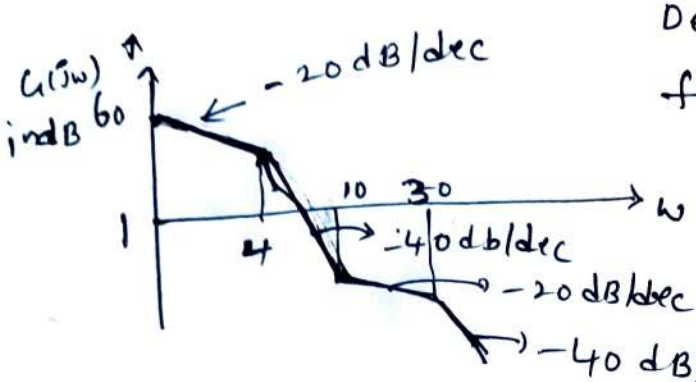
$$= \frac{k(1+s)}{\frac{1}{10}(10+s)\frac{1}{100}(s+100)}$$

$$= \frac{k(1+s)}{\frac{1}{1000}(10+s)(100+s)} = \frac{k1000(s+1)}{(s+10)(s+100)} \quad \text{---(1)}$$

Determination of transfer function from Bode Plot.

Ex: determine the transfer function of the system whose asymptotic magnitude plot is as shown below.

(Pb)



(1) Pole at origin.

(2) Pole with $w = 4$ rad/sec

(3) zero with $w = 10$ rad/sec

(4) Pole with $w = 30$ rad/sec.

starting point $= 20 \log k = 60$ dB

$\therefore k = 1000$

$$G(j\omega) = \frac{1000\left(1+\frac{s}{10}\right)}{s\left(1+\frac{s}{4}\right)\left(1+\frac{s}{30}\right)}$$

$$= \frac{1000\left(\frac{10+s}{10}\right)}{s\,\frac{1}{4}\left(\frac{4+s}{4}\right)\frac{1}{30}\left(\frac{30+s}{30}\right)}$$

$$= \frac{1000\left(\frac{10+s}{10}\right)}{\frac{1}{120}\,s\,(4+s)\,(30+s)}$$

$$G(j\omega) = \frac{12000\,(s+10)}{s(s+4)(s+30)}$$

polar plot :-

- The polar plot is also called Nquist it is currently plotted on a polar graph sheet and frequency response is considered from 0 to ∞.
- The polar plot of a simusoidal transfer function $G(j\omega)$ is a plot of $|G(j\omega)|$ vs the phase angle of $G(j\omega)$ to polar co-ordinate has ω is varied from 0 to ∞ Here polar plot

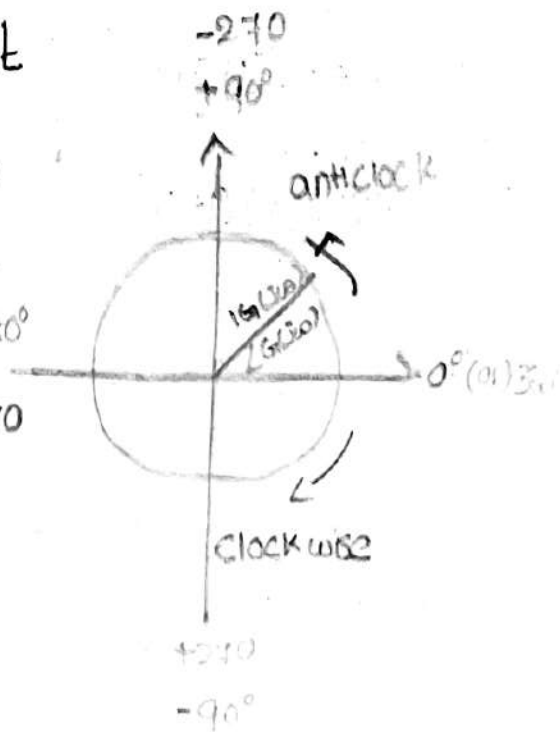
polar plot

Nquist plot

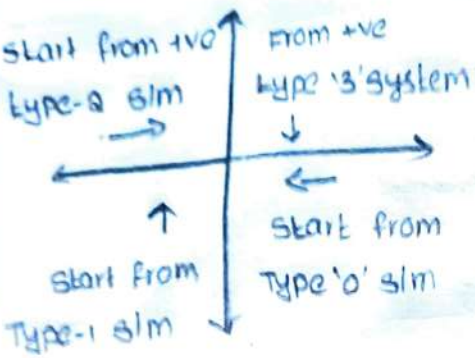
phase plot

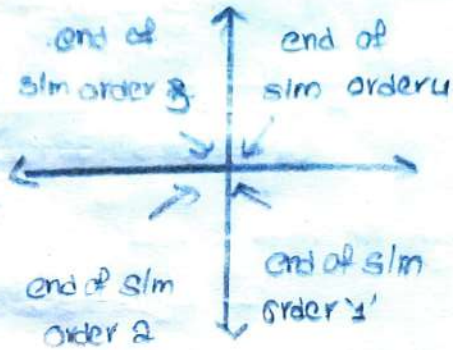
- In polar graph sheet +ve phase angle is measured in anticlock wise direction the return axis is called -ve phase angle is measured in dockwise

x-axis



- the alternate of $G(j\omega)$ can be expressed in rectangular coordinate as "$G(j\omega) = G_2(j\omega) + jG_1(j\omega)$"
- To convert the polar coordinates to rectangular coordinates



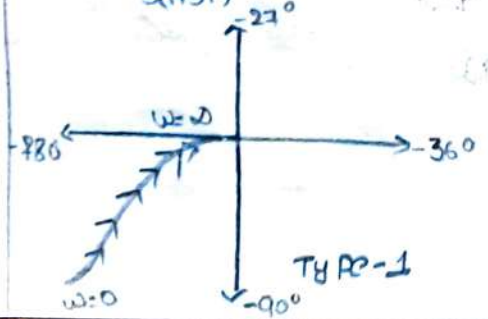


polar plot type and order of a system:-

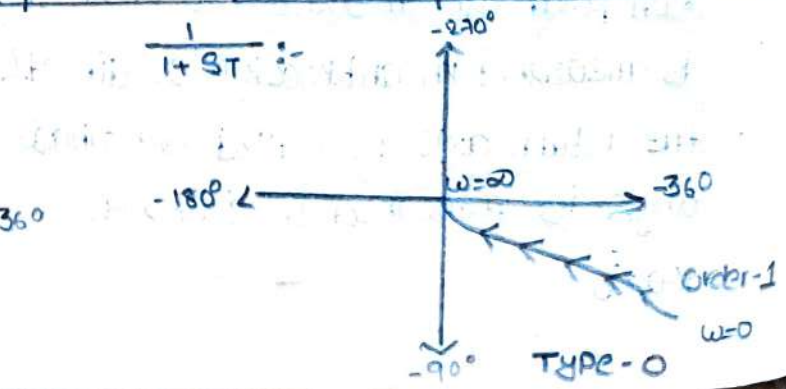
	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$	Type	order
$\frac{1}{s(1+sT)}$	$\frac{1}{\omega\sqrt{1+(\omega T)^2}}$	$-90 - \tan^{-1}(\omega T)$	1	2
$\frac{1}{1+sT}$	$\frac{1}{\sqrt{1+(\omega T)^2}}$	$-\tan^{-1}(\omega T)$	0	1
$\frac{1}{s(1+sT)(1+sT)}$	$\frac{1}{\omega\sqrt{1+(\omega T)^2}\sqrt{1+(\omega T)^2}}$	$-90^\circ - \tan^{-1}(\omega T) - \tan^{-1}(\omega T)$	1	3
$\frac{1}{s^2(1+sT)(1+sT)}$	$\frac{1}{\omega^2\sqrt{1+(\omega T)^2}\sqrt{1+(\omega T)^2}}$	$-180^\circ - \tan^{-1}(\omega T) - \tan^{-1}(\omega T)$	2	4

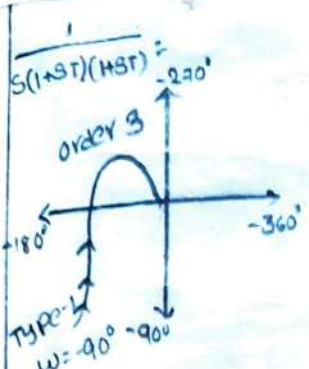
	Magnitude		phase	
At ω =	0	∞	$\omega=0$	$\omega=\infty$
$\frac{1}{s(1+sT)}$	∞	0	-90°	-180°
$\frac{1}{1+sT}$	1	0	0	-90°
$\frac{1}{s(1+sT)(1+sT)}$	∞	0	-90°	-270°
$\frac{1}{s^2(1+sT)(1+sT)}$	∞	0	-180°	-360°

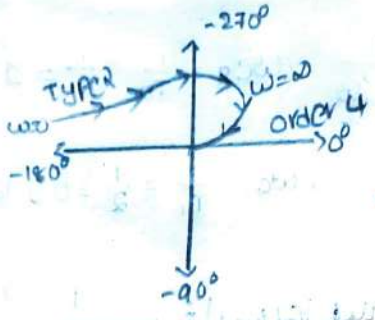
For s/m $\frac{1}{s(1+sT)}$:-



$\frac{1}{1+sT}$:-







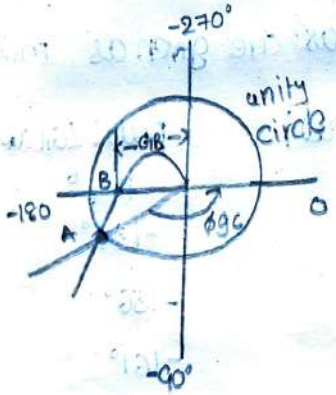
Gain Margin and phase Margin:-

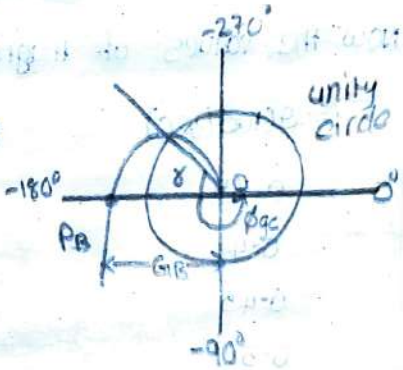
The gain margin is defined as the inverse of the magnitude of G(jω) at phase cross over Frequency. The phase cross over Frequency is Frequency at which the phase of G(jω) is 180

$$\text{Gain Margin } (k_g) = \frac{1}{|G(j\omega)_{pc}|}$$

The phase margin is defined as the phase angle of G(jω) at gain cross over Frequency. the gain cross over Frequency at which the magnitude of G(jω) is unity.

phase margin $\gamma = 180° + \phi_{gc}$; ϕ_{gc} = phase angle of G(jω)





1. The open loop transfer function for a unity feedback system is given as $G(s) = \frac{1}{s(1+s)(1+2s)}$. sketch the polar plot and determine the phase and gain Margin

A. Given $G(s) = \frac{1}{s(1+s)(1+2s)}$

$s = j\omega$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)}$$

Corner Frequencies

$\frac{1}{1+j\omega} \Rightarrow \omega_{c_2} = \frac{1}{T_2} = \frac{1}{1} = 1$ rad/sec

$G(s) = \frac{1}{s^2(1+s)(1+2s)}$

$\frac{1}{1+2j\omega} \Rightarrow \omega_{c_1} = \frac{1}{T_1} = \frac{1}{2} = 0.5$ rad/sec

$\Rightarrow$ Magnitude $|G(j\omega)| = \frac{1}{\omega^2\sqrt{(1+\omega^2)}\sqrt{1+(2\omega)^2}}$

$= \frac{1}{\omega^2\sqrt{(1+\omega^2)(1+4\omega^2)}}$

$= \frac{1}{\omega^2\sqrt{1+4\omega^2+\omega^2+4\omega^4}} = \frac{1}{\omega^2\sqrt{1+5\omega^2+4\omega^4}}$

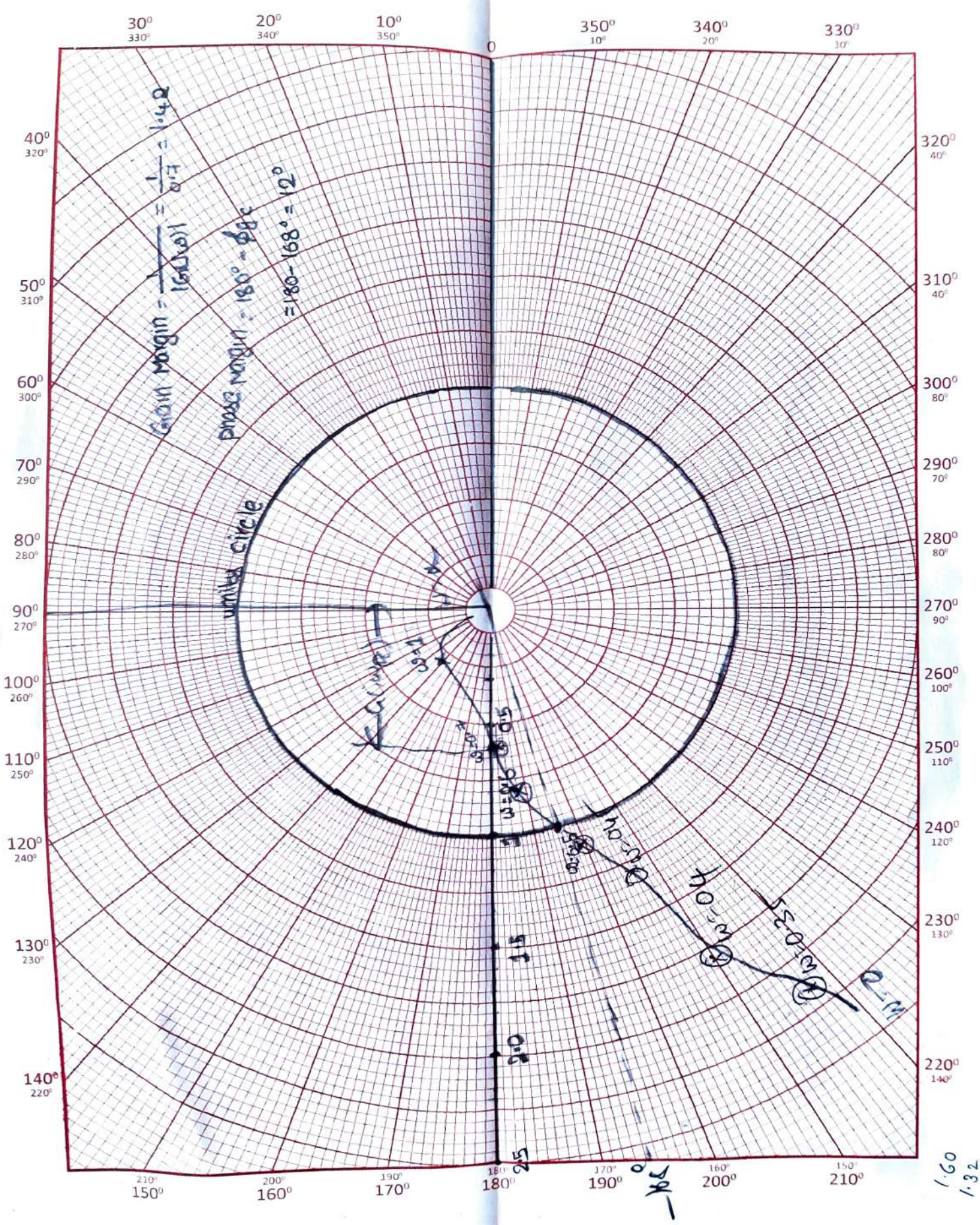
Now, phase is

$\angle G(j\omega) = \frac{1}{\angle 90 + \tan^{-1}(\omega) + \tan^{-1}(2\omega)}$

$= -90^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$

Now the values of magnitude and phase are given as follows

ω (rad/sec)	Magnitude $\lvert G(j\omega)\rvert$	phase $\angle G(j\omega)$
0.35	2.2	-144°
0.4	1.8	-150°
0.45	1.5	-156°
0.5	1.2	-161°
0.6	0.9	-171°
0.7	0.6	-179°
1	0.3	-198°


Gain Margin = 1/0.42 = 1.42
Phase Margin = 180° - 168° = 12°
unit circle
0.5
0.25
0.2
0.15
0.1
0.05
0.03
0.02
0.01
0.005
0.003
0.002
0.001
0.0005
0.0003
0.0002
0.0001
1.60
1.32

		0.31

2) The open loop transfer function, of a unity fb s/m is given by $G(s) = \frac{1}{s^2(1+s)(1+2s)}$. Sketch the polarplot and determine, Gain & phase margin

A) Given,

$$G(s) = \frac{1}{s^2(1+s)(1+2s)}$$

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega)(1+2j\omega)}$$

$$= \frac{1}{(-\omega^2)(1+j\omega)(1+2j\omega)}$$

$\omega_{c_1} = \frac{1}{2} = 0.5$ rad/sec ; $\omega_{c_2} = \frac{1}{1} = 1$ rad/sec

$$|G(j\omega)| = \frac{1}{\omega^2\sqrt{(1+\omega^2)}\sqrt{1+4\omega^2}}$$

$$= \frac{1}{\omega^2\sqrt{1+4\omega^4+\omega^2+4\omega^2}}$$

$$|G(j\omega)| = \frac{1}{\omega^2\sqrt{1+5\omega^2+4\omega^4}}$$

Now phase,

$$\angle G(j\omega) = -180^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega).$$

ω	magnitude	phase
0.35	6.3	-234°
0.4	4.5	-240°
0.45	3.3	-246°
0.5	2.5	-251°
0.6	1.5	-261°
0.7	0.9	$-269^\circ = -270^\circ$
1.	0.3	-288°

3) The Openloop T.F of a unity fb sys is given by, $G(s) = \frac{1}{s(1+s)^2}$. Sketch the polar plot.

A) Given, $G(s) = \frac{1}{s(1+s)^2}$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)^2}$$

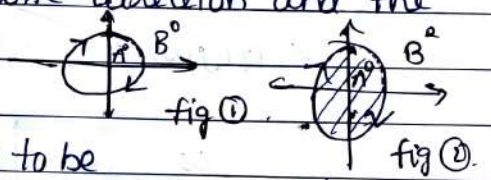
Type = 1
Order = 3.

$$= \frac{1}{j\omega(1+\omega^2+2j\omega)}$$

$\omega_{c_1} = \frac{1}{1} = 1$ rad/sec

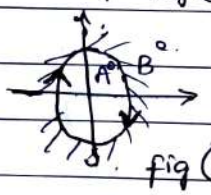
Nyquist Stability Criterian :- Nyquist Stability Criterian to draw the complete frequency response of open loop transfer function. It is frequency used to apply nyquist criterian.

Encirclement :- A point or a region encircled by a closed path if it is found inside the path with figure (1). The point "A" is encircled in the clockwise direction and the point "B" is not encircled.



Enclosed :-

Any point or a region is said to be Enclosed by a closed path. If it is found to lie to the right of the path, when the path is transverse in the prescribed direction,



The shaded region in figure ② and figure ③ are the regions enclosed by the closed ~~by the~~ path.

In figure ② , the point 'A' is enclosed by a closed path and the point 'B' is not enclosed by closed path.

In figure ③ the point 'A' is not enclosed by a closed path but the point 'B' is enclosed by closed path.

(1) Draw the nyquist plot for the sys $G(s)H(s) = \frac{1}{1+s}$.

Ans Given, $G(s)H(s) = \frac{1}{1+s}$

put $s = j\omega$.

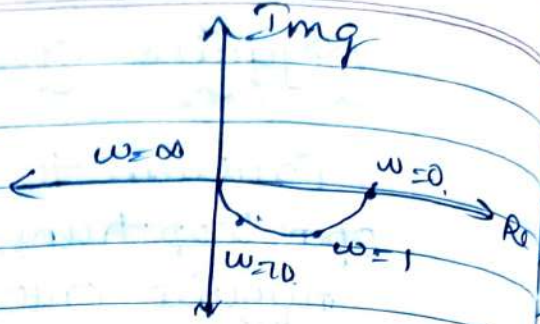
$$G(j\omega)H(j\omega) = \frac{1}{1+j\omega} \times \frac{1-j\omega}{1-j\omega}$$

$$= \frac{1-j\omega}{1+\omega^2}$$

$$G(j\omega)H(j\omega) = \frac{1}{1+\omega^2} - j\frac{\omega}{1+\omega^2}.$$

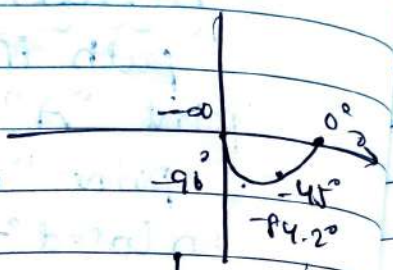
$$\text{Real} = \frac{1}{1+\omega^2} \; ; \quad \text{img} = \frac{-\omega}{1+\omega^2}$$

ω	Re	Im
0	1	0
1	0.5	-0.5
10	0.01	-0.1
∞	0	0



$$|G(j\omega)| = \frac{1}{1+j\omega} = \frac{1}{\sqrt{1+\omega^2}}$$

$$\angle G(j\omega) = -\tan^{-1}(\omega)$$



ω	0	1	10	∞
$\|G(j\omega)\|$	1	0.707	0.1	0
$\angle G(j\omega)$	0	-45°	-84.2°	-90°

Compensation/compensator:-

A compensator in a control system is a device that improves system performance by adjusting its response to achieve desired stability and accuracy

Need of compensation:-

- To obtain a desired performance of the system we use compensating n/w's. These are applied to the network system in the form of feedback or forward path gain adjustment.

2. compensate a unstable s/m to make it stable

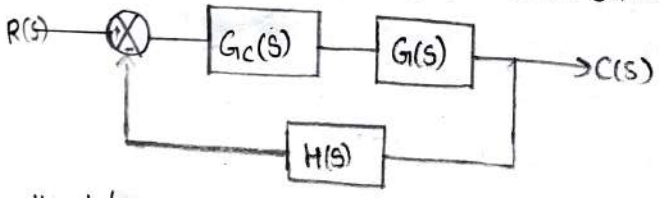
3. used to minimize overshoot

4. These networks increases the steady state accuracy of the system

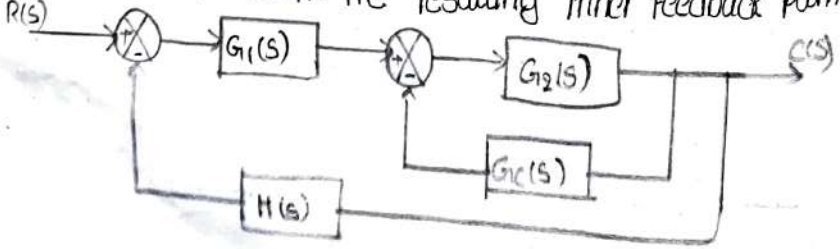
5. compensation n/w's also introduce poles & zero's in the s/m, there by causes changes in the transfer function of the system. Due to this performance specifications of the system change.

Methods of compensation:-

1. series compensation:- In series compensation, the compensator with transfer function, $G_c(s)$ is placed in series with plant.



2. Feedback/parallel compensation :- In parallel compensation, some element is feedback to the input and compensator with transfer function $G_c(s)$ placed in the resulting inner feedback path.



Note:- The compensator may be electrical, mechanical, hydraulic, pneumatic can other type of device or network. some electrical/ electronic compensator used are lead, lag and lead lag compensator.

Lag compensator:- If a sinusoidal signal is applied to a lag network, then in steady state the o/p will have a phase lag w.r.t input.

- It introduces a pole enhancing steady state accuracy, but potentially slowing down the system response.
- It is generally a LPF with high Frequency signals are attenuated.
- In lag compensator a pole at $s = -1/\beta T$ and zero at $s = -1/T$.

Here, $\beta > 1$ so the zero is located to the left of the pole the negative axis.

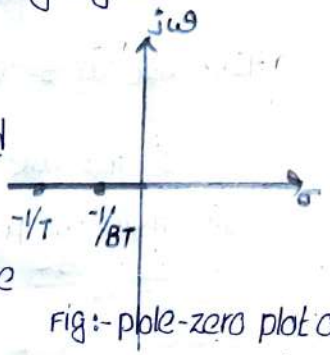
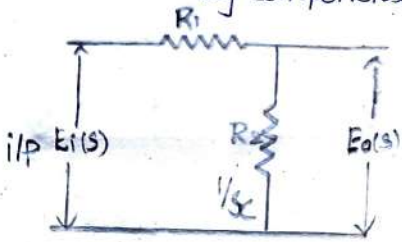


Fig:- pole-zero plot of lag compensators.

Transfer is

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{\beta} \frac{\left(s+\frac{1}{T}\right)}{\left(s+\frac{1}{\beta T}\right)}$$



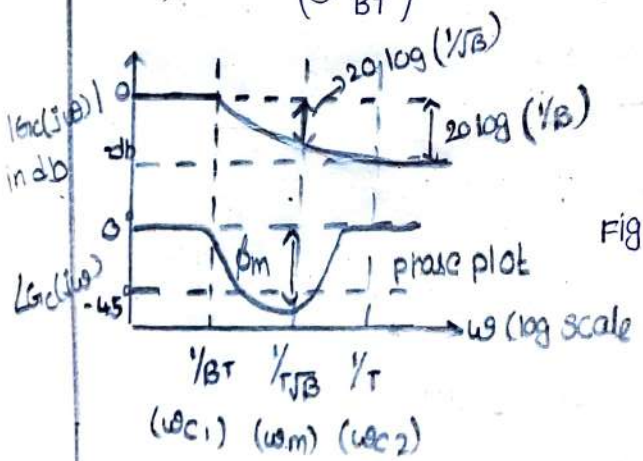


Fig:- Bode plot

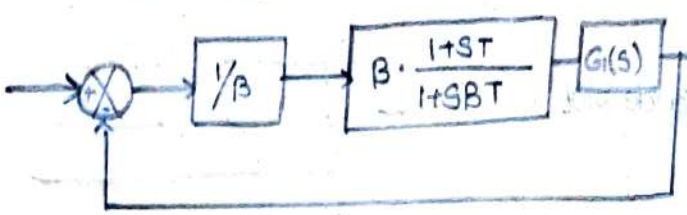


Fig:- Block diagram of lag compensator.

Frequency phase by

$\omega_m = \sqrt{\omega_{c1}\omega_{c2}}$

$\sqrt{\frac{1}{\beta T}\frac{1}{T}} = \frac{1}{T\sqrt{\beta}}$

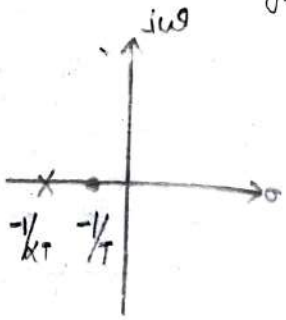
Lead compensator:

If a sinusoidal signal is applied to the lead network, then in steady state o/p will have a phase lead with respect to the i/p

The lead compensation increases the bandwidth which improves the speed of response and reduces the amount of overshoot.

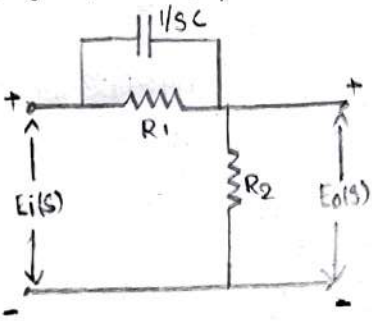
It provide a unstable system to stable system

- Lead compensator is basically a HPF and it amplifies high frequency noise signals.



Here zero at $s=-1/T$ & pole at $s=-1/\alpha T$.

- $\alpha<1$, so the ~~pole~~ zero is closer to the origin than the pole.



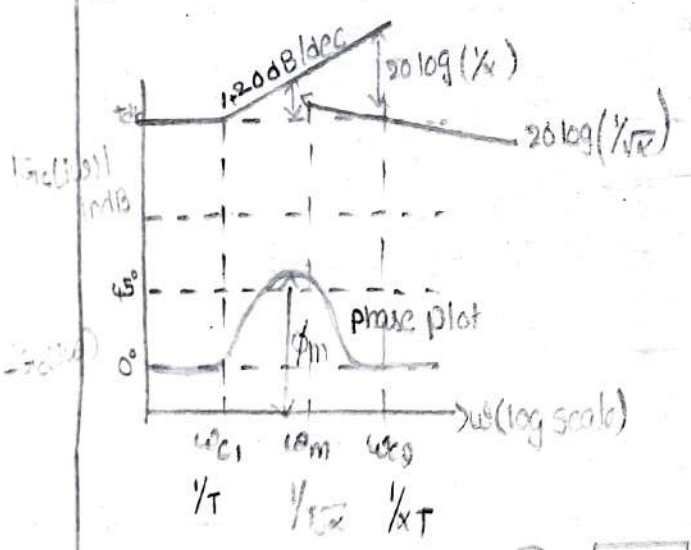


Fig:- Bode plot.



Fig:- bbck diagram

Lag-Lead compensator:-

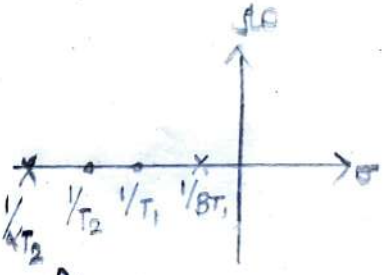
If a sinusoidal signal is applied, both phase lag and phase lead occurs in the o/p, but in different frequency regions.

phase lag occurs in LF region and phase lead occurs in HF region i.e., the phase angle varies from lag to lead as the frequency is increased from zero to infinity

- Lead compensator basically increases between & speeds up the response decreases max. overshoot in the step response
- Lag compensator increases the bw frequency gain and thus i improves the steady state accuracy of the system but reduces the speed of responses due to reduced bandwidth.
- Lag-Lead compensation passes two poles and two zero's and so such compensation increases the order of the system by two, unless cancellation of poles and zero's occurs in the compensate system

Transfer Function $= G_c(s) = \frac{(s+\frac{1}{T_1})}{(s+\frac{1}{\beta T_1})} \cdot \frac{(s+\frac{1}{T_2})}{(s+\frac{1}{\alpha T_2})}$

(Lag) (Lead)



When $\beta > 1$ and $\alpha << 1$

$E_i(s)$ = i/p vol

$E_o(s)$ = o/p vol

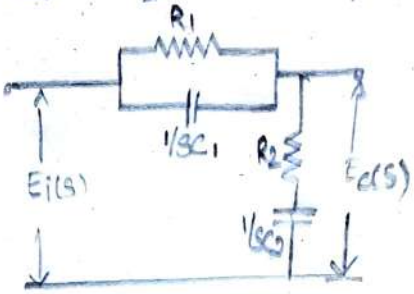


Fig:- electrical lag-lead compensator

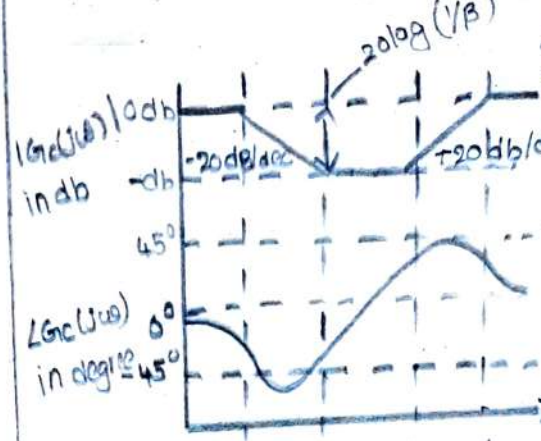


Fig:- Bode plot of Lag-lead comp.

Unit-5

State Space Analysis and Design

State space Analysis Approach is a powerfull technique for the analysis and design of Control Systems. The state space Analysis is a Modern approach and also easier for analysis using Computers. The Conventional Methods of analysis employs the transfer function of the System.

The Drawbacks of the Transfer function Model are

* TF is defined under zero initial conditions.

1. Transfer function is applicable to linear time invariant Systems.
2. Transfer function analysis is restricted to single input and single output Systems.
3. Does not provide the information Regarding the internal State of the System

→ The State space Analysis can be applied for any type of the Systems. The analysis can be carried with initial conditions and can be carried on multiple input and multiple output Systems.

Definitions :

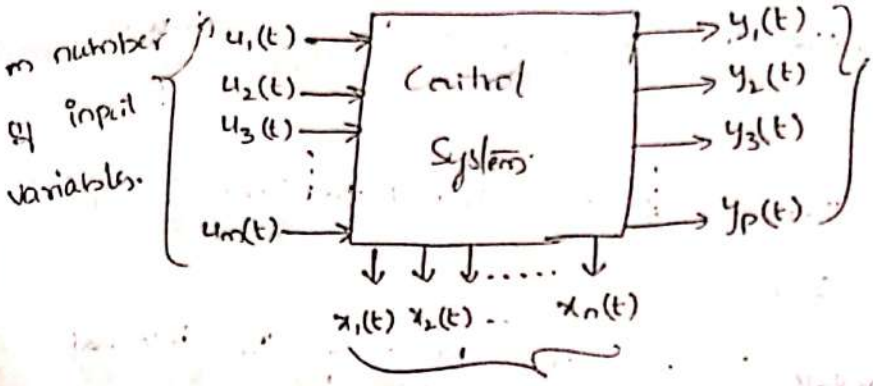
State : The state is the condition of the System at any instant of time 't'.

State variable : A set of variable which Describes the state of the System at any time instant is called state variable

* State Space Formulation:

The state of a dynamic system is a minimal set of variables such that the knowledge of these variables at $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$, completely determines the behaviour of the system for $t > t_0$.

In the state variable formulation of a system, in general a system consists of m-inputs, p-outputs and n-state variables.



Input vector $U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$, output vector $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$

State variable vector $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

* State Model of a linear System :-

The state model of a system consists of the state equation and the output equation. The state equation of a system is a function of state variables and inputs as defined below.

The State Equation can be arranged in the form of 'n' number of first order differential equations as below.

$$\frac{dx_1}{dt} = \dot{x}_1 = f_1(x_1, x_2, \ldots x_n : u_1, u_2, \ldots, u_m)$$

$$\frac{dx_2}{dt} = \dot{x}_2 = f_2(x_1, x_2, \ldots x_n : u_1, u_2, \ldots, u_m)$$

$$\vdots$$

$$\frac{dx_n}{dt} = \dot{x}_n = f_n(x_1, x_2, \ldots x_n : u_1, u_2, \ldots, u_m)$$

The above equations can be written as,

$$\boxed{\dot{x}(t) = f[x(t), u(t)]}$$

For, linear time invariant systems the first derivatives of the state variables can be expressed as a linear combination of state variables and inputs.

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \ldots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \ldots + b_{2m}u_m$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \ldots + b_{nm}u_m$$

When above equations are written in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & b_{2m} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \quad \text{(A)}$$

In general the above equation can be expressed as,

$$\dot{x} = Ax + Bu \quad \text{(B)} \qquad \boxed{\dot{x}(t) = Ax(t) + Bu(t)}$$

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Where $x(t)$ = State vector matrix

$U(t)$ = Input vector matrix

A = System Matrix

B = Input matrix.

The output at any time are functions of state variables and inp

$\therefore$ output vector $Y(t) = f[x(t), U(t)]$

i.e $y_1 = c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \ldots + d_{1m}u_m$

$y_2 = c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \ldots + d_{2m}u_m$

$\vdots$

$y_p = c_{p1}x_1 + c_{p2}x_2 + \ldots + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \ldots + d_{pm}u_m$

Now, writing in Matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \ldots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1m} \\ d_{21} & d_{22} & \ldots & d_{2m} \\ \vdots & & & \\ d_{p1} & d_{p2} & \ldots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

Hence, $\Rightarrow Y(t) = C\,X(t) + D\,U(t)$ ——— (B)

$Y(t)$ = Output Vector Matrix

C = output matrix.

from the equations (A) and (B)

(A) $\Rightarrow \dot{X} = A\,X(t) + B\,U(t)$ ——— (C)

(B) $\Rightarrow Y = C\,X(t) + D\,U(t)$.

$\dot{X} = A\,X(t) + B\,U(t)$ → State Equation

$Y = C\,X(t) + D\,U(t)$ → Output Equation

} State Model

* State Diagram:

The Pictorial Representation of a state Model of the System is called State Diagram. The state Diagram of the System can be either in Blocks Diagram form (or) signal flow graph form.

The State diagram of a state model is constructed using three basic elements Scalar, Adder, and Integrator.

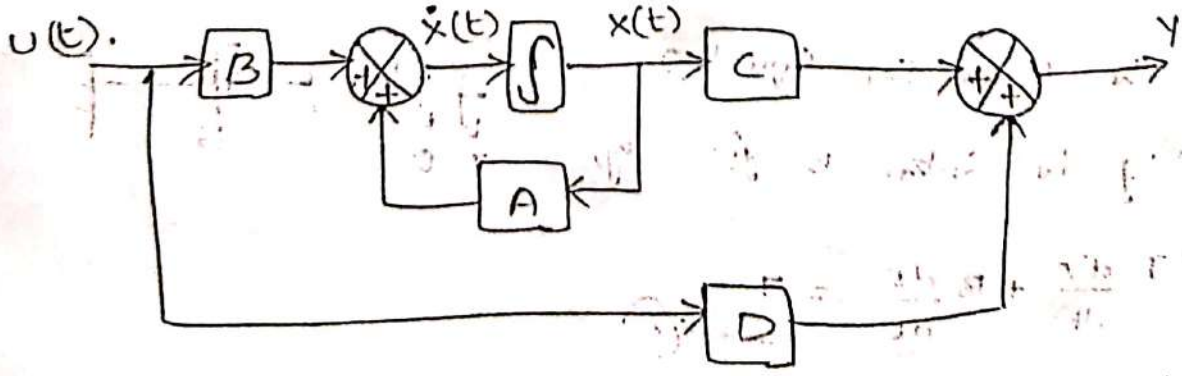
Scalar: The Scalar is used to multiply a signal by a constant. The input signal $x(t)$ is multiplied by a scaler a to give the output, $a\,x(t)$.

Adder: The adder is used to add two or more signals. The output of the adder is sum of incoming signals.

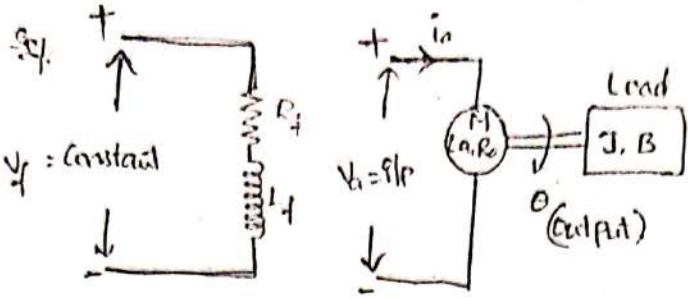
Integrator: The integrator is used to integrate the signal. They are used to integrate the derivatives of state variables to get the state variables.

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

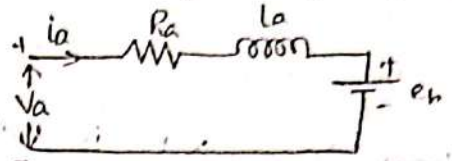
$$y(t) = C\,x(t) + D\,u(t).$$



⇒ Determine the State model of Armature Controlled Dc motor.



Now, the equivalent circuit of Armature is given by,



By Kirchoff's voltage law,

$$\Rightarrow i_a R_a + L_a \frac{di_a}{dt} + e_b = V_a \quad \text{---(1)}$$

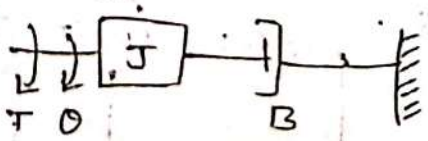
The back emf of DC Machine is Proportional to speed of shaft

$$\therefore e_b \propto \frac{d\theta}{dt} \Rightarrow e_b = K_b \frac{d\theta}{dt} \quad \text{---(2)}$$

And Torque of DC Machine is Proportional to flux and Current. Since flux is Constant in this System, the torque is Proportional to i_a alone

$$\Rightarrow T \propto i_a \Rightarrow T = K_t i_a \quad \text{---(3)}$$

The Mechanical System of the motor is shown in figure below,



Now, the Differential Equation governing the System is given by,

$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = T \quad \text{---(4)}$$

Now, Substitute (2) in (1)

$$\Rightarrow i_a R_a + L_a \frac{di_a}{dt} + e_b = V_a \Rightarrow i_a R_a + L_a \frac{di_a}{dt} + K_b \frac{d\theta}{dt} = V_a \quad \text{---(5)}$$

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Similarly Substitute (3) in (1)

$$\Rightarrow J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = k_t \cdot i_a \quad \text{---- (6)}$$

Now the equations (5) and (6) are the Differential equations governing the Armature controlled DC Motor.

Now, let choose, i_a, ω, and θ as state variables to model the armature controlled DC Motor.

$$\dot{x}_1 = i_a, \quad x_2 = \omega = \frac{d\theta}{dt}, \quad x_3 = \theta.$$

The input to the motor is armature voltage V_a and let $V_a = u$, where u is the general notation for input variable

Substituting the state variables in (5)

$$\Rightarrow (5) = x_1 R_a + L_a \frac{dx_1}{dt} + k_b x_2 = u.$$

$$\text{let, } \frac{dx_1}{dt} = \dot{x}_1 \Rightarrow x_1 R_a + L_a \cdot \dot{x}_1 + k_b x_2 = u$$

$$\Rightarrow \dot{x}_1 = \frac{-R_a}{L_a} x_1 - \frac{k_b}{L_a} x_2 + \frac{1}{L_a} u \quad \text{---- (7)}$$

Similarly,

$$(6) \Rightarrow J\frac{d^2x_3}{dt^2} + B\frac{dx_3}{dt} = k_t \cdot x_1$$

$$\text{let } \frac{d^2x_3}{dt^2} = \dot{x}_2 \text{ and } \frac{dx_3}{dt} = x_2$$

$$\therefore J\dot{x}_2 + Bx_2 = k_t \cdot x_1.$$

$$\Rightarrow \dot{x}_2 = \frac{k_t}{J} x_1 - \frac{B}{J} x_2 \quad \text{---- (8)}$$

we have assumed $x_3 = \theta$

$$\Rightarrow \frac{dx_3}{dt} = \dot{x}_3 \text{ and } \frac{d\theta}{dt} = x_2$$

$$\Rightarrow \dot{x}_3 = x_2$$

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from the (7) (8) (9) equations

$$\dot{x}_1 = -\frac{R_a}{L_a} x_1 - \frac{k_b}{L_a} x_2 + \frac{1}{L_a} u.$$

$$\dot{x}_2 = \frac{K_t}{J} x_1 - \frac{B}{J} x_2$$

$$\dot{x}_3 = x_2$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R_a}{L_a} & -\frac{k_b}{L_a} & 0 \\ \frac{K_t}{J} & -\frac{B}{J} & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} \\ 0 \\ 0 \end{bmatrix} u.$$

$$\dot{X} = A X + B U.$$

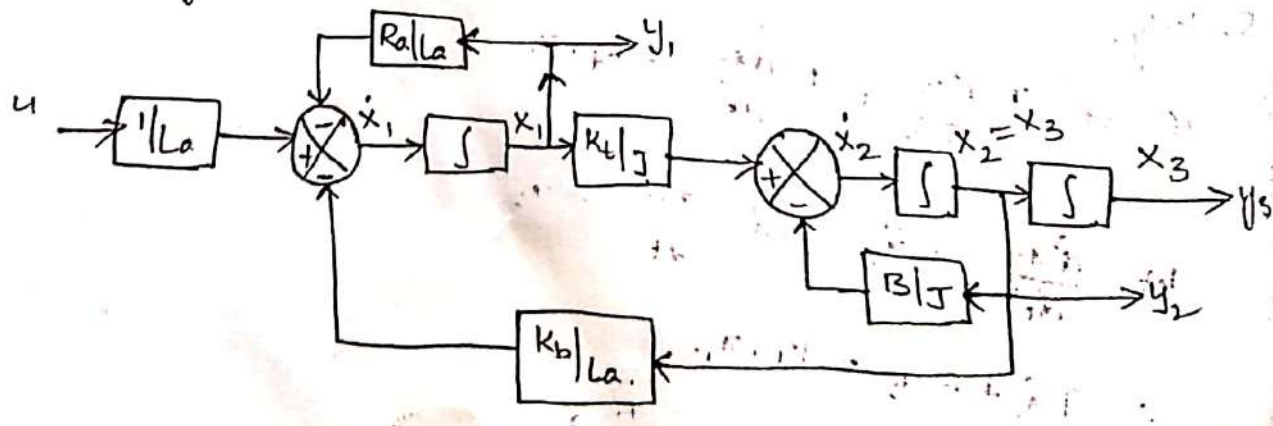
The desired outputs will be i_a, ω and θ.

$$\therefore y_1 = i_a, \quad y_2 = \omega = \frac{d\theta}{dt}, \quad y_3 = \theta$$

$$\Rightarrow y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3.$$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow Y = CX.$$

Blocks Diagram Representation

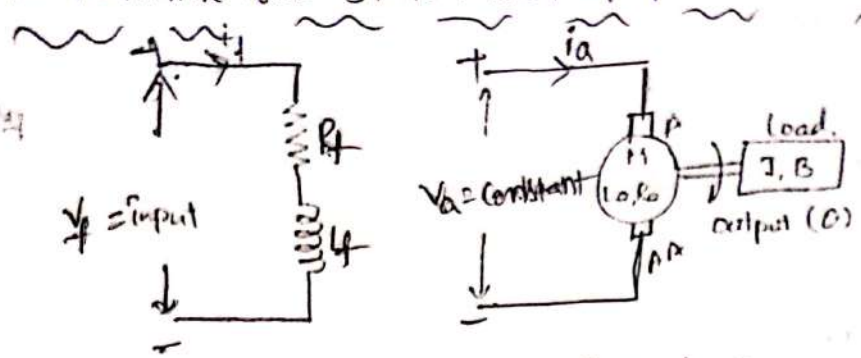


State equation $\dot{X} = AX + BU$

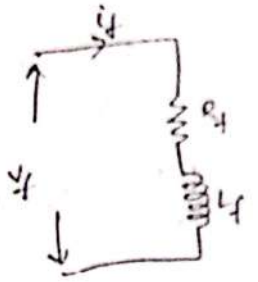
Output equation $Y = CX + DU$

} Combinely called as STATE MODEL

Determine the state model of field controlled DC motor:



Now the equivalent circuit of field is,



By Kirchoff's Voltage law,

$$R_f i_f + L_f \frac{di_f}{dt} = V_f \quad \text{— (1)}$$

The Torque of DC Motor is directly proportional to flux and armature current. Since armature current is constant, but flux is proportional to field current. So, the torque is directly proportional to field current

$$\therefore T \propto i_f \Rightarrow T = K_f i_f \quad \text{— (2)}$$

From the Mechanical System:

$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = T \quad \text{— (3)}$$

Substitute (2) in (3)

$$\Rightarrow J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = K_f i_f \quad \text{— (4)}$$

Now, (1) and (4) are the Differential equations governing the System.

Let us choose, i_f, ω, θ are the state variables and are related to the general notation of state variables x_1, x_2 and x_3.

$$\therefore x_1 = i_f, \quad x_2 = \omega = \frac{d\theta}{dt}, \quad x_3 = \theta \text{ and } V_f = u(\text{input})$$

On Substituting these values in (1) and (4) Equations.

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① $\Rightarrow R_f x_1 + L_f \frac{dx_1}{dt} = u$

$\Rightarrow R_f x_1 + L_f \dot{x}_1 = u$

$\Rightarrow \dot{x}_1 = \frac{-R_f}{L_f} x_1 + \frac{1}{L_f} u$ — ⑤

④ $\Rightarrow J \frac{d^2x_3}{dt^2} + B \frac{dx_3}{dt} = K_{tf} x_1$

Let, $\frac{d^2x_3}{dt^2} = \dot{x}_2$ and $\frac{dx_3}{dt} = x_2$

$\Rightarrow J\dot{x}_2 + B x_2 = K_{tf} x_1$

$\Rightarrow \dot{x}_2 = -\frac{K_{tf}}{J} x_1 - \frac{B}{J} x_2$ — ⑥

$x_3 = \theta$

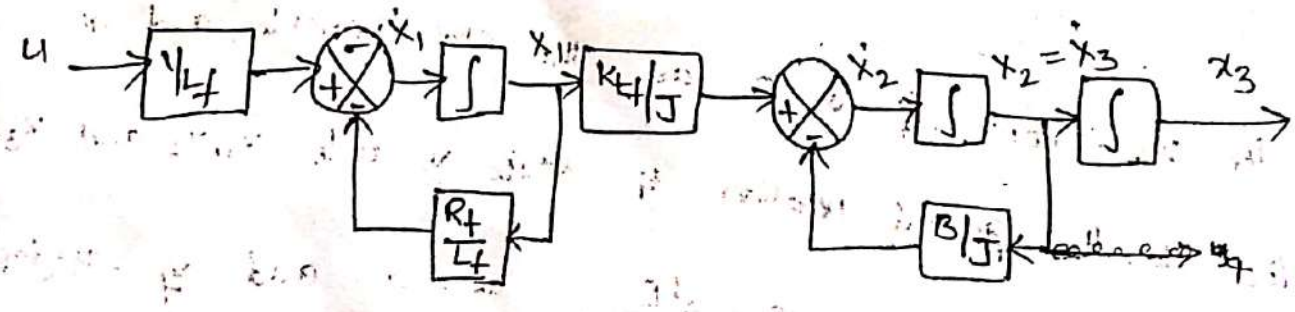
$\frac{dx_3}{dt} = \frac{d\theta}{dt}$

$\Rightarrow \dot{x}_3 = x_2$ — ⑦

Using the ⑤, ⑥ and ⑦ equations, the state model is written as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{-R_f}{L_f} & 0 & 0 \\ \frac{K_{tf}}{J} & -\frac{B}{J} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_f} \\ 0 \\ 0 \end{bmatrix} u$$

$\dot{X} = A X + B U$



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* Transfer function of State Model :-

∴ State model Consists of state equation and output Equation.

State equation ⇒ $\dot{x} = Ax + Bu$

⇒ $\dot{x}(t) = Ax(t) + Bu(t)$ ——— (1)

Output Equation ⇒ $y(t) = Cx(t) + Du(t)$ ——— (2)

On taking the laplace Transform for equation (1)

⇒ $sX(s) = AX(s) + BU(s)$.

⇒ $sX(s) - AX(s) = BU(s)$.

⇒ $X(s)[sI - A] = BU(s)$ ——— (A)

∴ [s is not matrix to make it matrix we multiply with Identity Matrix I]

from (2) ⇒ $Y(s) = CX(s) + DU(s)$ ——— (3)

Now, Substitute $X(s)$ from (A) in (3)

⇒ $Y(s) = C[sI - A]^{-1}BU(s) + DU(s)$.

$= U(s)[C[sI - A]^{-1}B + D]$

∴ The $\boxed{T.F = \frac{Y(s)}{U(s)} = C[sI - A]^{-1}B + D.}$

* State Space Representation Using phase Variables :-

The State Model using phase variables can be easily determined if the system model is already known in the differential Equation (or) transfer function form. There are 3 methods of modelling a System using phase variables and they are explained in the following Cases.

1.

Problem-1

Consider the following n^{th} order differential equation relating the output to the input of a system.

$$y^{n} + a_1 y^{(n-1)} + \cdots + a_{n-2}\ddot{y} + a_{n-1}\dot{y} + a_n y = bu.$$

By choosing the output y and their derivatives as state variables, we get,

$$x_1 = y \Rightarrow y = x_1$$

$$x_2 = \dot{y} \Rightarrow \dot{y} = \dot{x}_1 = x_2$$

$$x_3 = \ddot{y} \Rightarrow \ddot{y} = \dot{x}_1 = \dot{x}_2 = x_3$$

$$\vdots$$

$$x_n = y^{(n-1)} \Rightarrow \dot{x}_n = y^{n}$$

$$\dddot{y} = \dot{x}_1 = \dot{x}_2 = \dot{x}_3 = x_4$$

Now, on substituting the state variables in the differential equation,

$$\dot{x}_n + a_1 x_n + a_2 x_{n-1} + \cdots + a_{n-2} x_3 + a_{n-1} x_2 + a_n x_1 = bu.$$

$$\therefore \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - a_{n-2} x_3 - \cdots - a_2 x_{n-1} - a_1 x_n + bu.$$

The state equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - a_{n-2} x_3 - \cdots - a_2 x_{n-1} - a_1 x_n + bu.$$

On arranging the above equations in the Matrix form we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u$$

Here, the Matrix A (System matrix) has a very special form. It has all 1's in the upper off-diagonal. its last row is comprised of the negative of the coefficients of the original differential equation and all other elements are zero. This form of Matrix A is known as

"BUSH FORM" (or) "COMPANION FORM."

we have assumed $y = x_1$

$$\therefore y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Method-2

Consider the following nth order differential equation governing the output $y(t)$ to the input $u(t)$ of a system

$$\overset{n}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{m}{u} + b_1 \overset{(m-1)}{u} + \cdots b_{m-1}\dot{u} + b_m u$$

let $n = m = 3$.

$$\Rightarrow \dddot{y} + a_1 \ddot{y} + a_2 \dot{y} + a_3 y = b_0 \dddot{u} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u$$

Now, taking the laplace transform we get,

$$\Rightarrow s^3 Y(s) + a_1 s^2 Y(s) + a_2 s Y(s) + a_3 Y(s) = b_0 s^3 U(s) + b_1 s^2 U(s) + b_2 s U(s) + b_3 U(s)$$

$$\Rightarrow Y(s)\left(s^3 + a_1 s^2 + a_2 s + a_3\right) = U(s)\left(b_0 s^3 + b_1 s^2 + b_2 s + b_3\right)$$

Now, $$\frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$= \frac{s^3\left[b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}\right]}{s^3\left(1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}\right)} = \frac{b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}}{1 - \left(\frac{-a_1}{s} + \frac{a_2}{s^2} - \frac{a_3}{s^3}\right)}$$

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The transfer function of a system with four forward paths and three feedback loops is given by,

$$T(s) = \frac{P_1 + P_2 + P_3 + P_4}{1 - (P_{11} + P_{12} + P_{13})}$$

which is similar to the Mason's gain formula.

Here, P_1, P_2, P_3, P_4 are the forward paths and is given by,

$$P_1 = b_0, \quad P_2 = b_1/s, \quad P_3 = b_2/s^2, \quad P_4 = b_3/s^3.$$

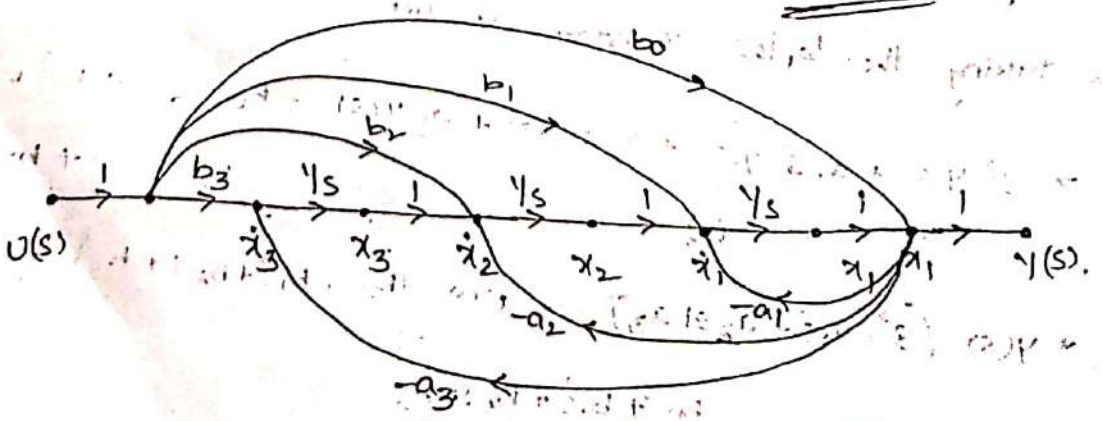
and P_{11}, P_{12}, P_{13} denotes the touching loops each other is given by,

$$P_{11} = \frac{-a_1}{s}, \quad P_{12} = \frac{-a_2}{s^2}, \quad P_{13} = \frac{-a_3}{s^3}$$

Hence for the transfer function above, a signal flow graph can be constructed as shown below.

Now, let assign state variables at the output of each integrator in the in the signal flow graph. Hence at the input of each integrator, the first derivative of the state variable will be available. The state equations are formed by summing all the incoming signals to the nodes.,

(Direct decomposition method)



At node $\dot{x}_1 \Rightarrow \dot{x}_1 = -a_1(x_1 + b_0 u) + x_2 + b_1 u$

$$= -a_1 x_1 + x_2 + (b_1 - a_1 b_0) u \quad \text{---} (1)$$

At node = 2

$\Rightarrow \dot{x}_2 = -a_2(x_1 + b_0 u) + x_3 + b_2 u$

$\therefore \dot{x}_2 = -a_2 x_1 + x_3 + (b_2 - a_2 b_0) u$ ——— (2)

At node $\dot{x}_3$

$\Rightarrow \dot{x}_3 = -a_3(x_1 + b_0 u) + b_3 u$

$\therefore \dot{x}_3 = -a_3 x_1 + (b_3 - a_3 b_0) u$. ——— (3)

and the output equation is given by the sum of the incoming signals to output node.

$\therefore y = x_1 + b_0 u$. ——— (4)

∴ The state Model can be Represented as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \end{bmatrix} u$$

and $y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 u$.

Method-3.

Consider the differential equation governing the output $y(t)$ to the input $u(t)$ of a system.

$$\dddot{y} + a_1 \ddot{y} + a_2 \dot{y} + a_3 y = b_0 \dddot{u} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u$$

Taking the laplace transform for above equation,

$$Y(s)s^3 + a_1 s^2 Y(s) + a_2 s\, Y(s) + a_3 Y(s) = \left[b_0 s^3 + b_1 s^2 + b_2 s + b_3\right] U(s).$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

Let $\frac{Y(s)}{U(s)} = \frac{Y(s)}{X_1(s)} \times \frac{X_1(s)}{U(s)}$.

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Where $\frac{X_1(s)}{U(s)} = \frac{1}{s^3+a_1s^2+a_2s+a_3}$ and $\frac{Y(s)}{X_1(s)} = b_0s^3+b_1s^2+b_2s+b_3$

Now $\frac{X_1(s)}{U(s)} = \frac{1}{s^3+a_1s^2+a_2s+a_3}$

$X_1(s)\left[s^3+a_1s^2+a_2s+a_3\right] = U(s)$.

$s^3x_1(s) + a_1s^2x_1(s) + a_2s\,x(s) + a_3x(s) = U(s)$.

On taking inverse laplace transform,

$\dddot{x}_1 + a_1\ddot{x}_1 + a_2\dot{x}_1 + a_3x_1 = u$ ——— (A)

Let the state variables be, x_1, x_2, x_3.

where, $x_2 = \dot{x}_1$, $x_3 = \ddot{x}_1 = \dot{x}_2$, $\dot{x}_3 = \dddot{x}_1$

On substituting state variables in equation (A)

(A) $\Rightarrow$ $\dot{x}_3 + a_1x_3 + a_2x_2 + a_3x_1 = u$

$\Rightarrow$ $\dot{x}_3 = -a_3x_1 + (-a_2x_2) - a_1x_3 + u$. ——— (C)

∴ The state equations are

$$\left.\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= -(a_3x_1+a_2x_2+a_1x_3)+u\end{aligned}\right\} \text{——— (B)}$$

Now $\frac{Y(s)}{X_1(s)} = b_0s^3+b_1s^2+b_2s+b_3$

$\Rightarrow Y(s) = b_0s^3x_1(s) + b_1s^2x_1(s) + b_2s\,x_1(s) + b_3x_1(s)$.

On taking inverse laplace transform,

$\Rightarrow y = b_0\dddot{x}_1 + b_1\ddot{x}_1 + b_2\dot{x}_1 + b_3x_1$

$= b_0\dot{x}_3 + b_1x_3 + b_2x_2 + b_3x_1$

Now substitute eqn (C) in above equation we get,

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$$y = b_0(-a_3x_1 - a_2x_2 - a_1x_3 + u) + b_1x_3 + b_2x_2 + b_3x_1$$

$$= (b_3 - a_3b_0)x_1 + (b_2 - a_2b_0)x_2 + (b_1 - a_1b_0)x_3 + b_0u. \quad \text{(7)}$$

Now, the state model from the equations (5) and (6) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} (b_3 - a_3b_0) & (b_2 - a_2b_0) & (b_1 - a_1b_0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [b_0]u.$$

*** Problems:**

1. Construct the state model for a system characterized by a differential equation. $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0.$

Sol: given DE $= \frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0.$

let us choose 'y' and their derivaties as state variables.

let $y = x_1.$

$$\dot{y} = \dot{x}_1 = x_2$$

$$\ddot{y} = \dot{x}_1 = \dot{x}_2 = x_3$$

$$\dddot{y} = \ddot{x}_1 = \dot{x}_2 = \dot{x}_3.$$

Now Substitute the above variables in D.E

$$\Rightarrow \dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + u = 0$$

$$\Rightarrow \dot{x}_3 = -6x_3 - 11x_2 - 6x_1 - u$$

Therefore, the state equations are,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

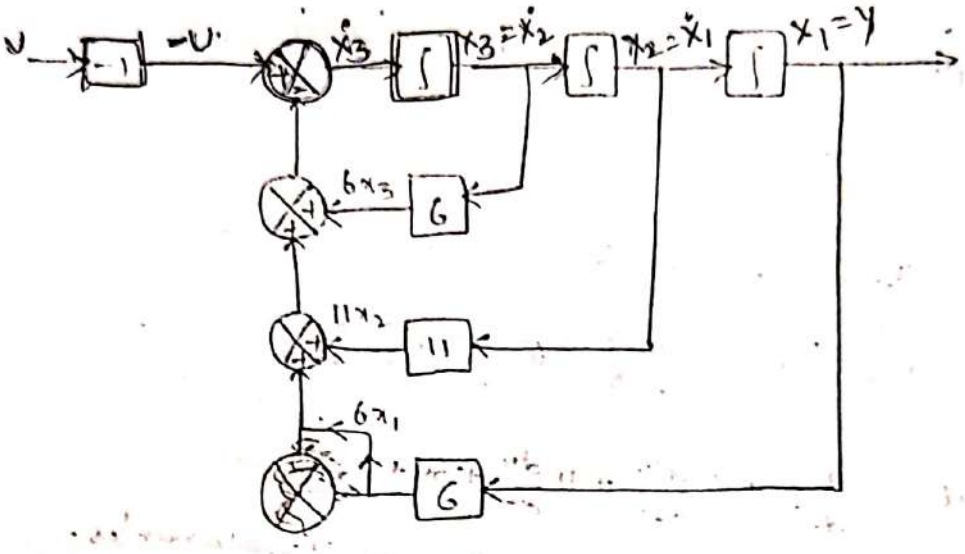
$$\dot{x}_3 = -6x_3 - 11x_2 - 6x_1 - u$$

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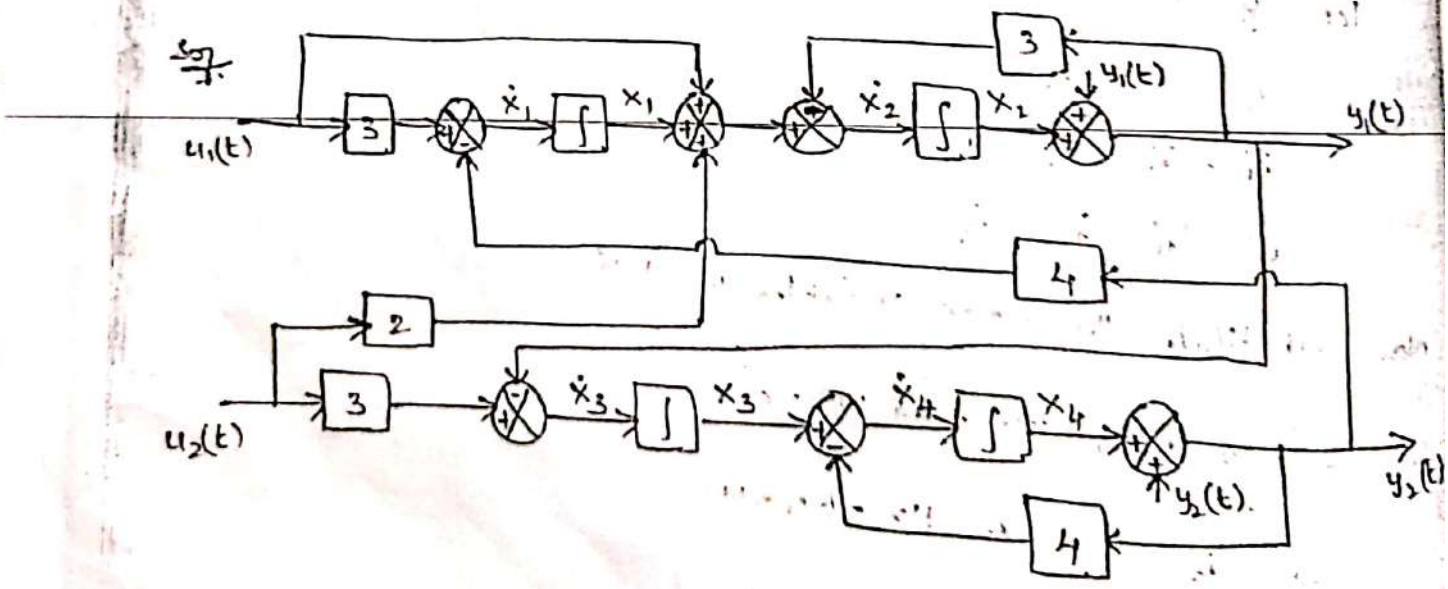
$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u$$

Here output $y = x_1$

$$\therefore y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



(2) The state Diagram of a system is shown below. Assign state variables and obtain the state Model of the system.



Sol: Since there are 4 integrators in the state diagram, we can assign 4 state variables. The state variables can be assigned to the output of the integrators. Hence, at the input of integrator, the

Equations are formed by summing all the incoming signals to the input of the integrator and equating to the corresponding first derivative of the system

$$\dot{x}_1 = -4x_1 + 3u_1$$

$$\dot{x}_2 = x_1 - 3x_2 + u_1 + 2u_2$$

$$\dot{x}_3 = -x_2 + 3u_2$$

$$\dot{x}_4 = x_3 - 4x_4.$$

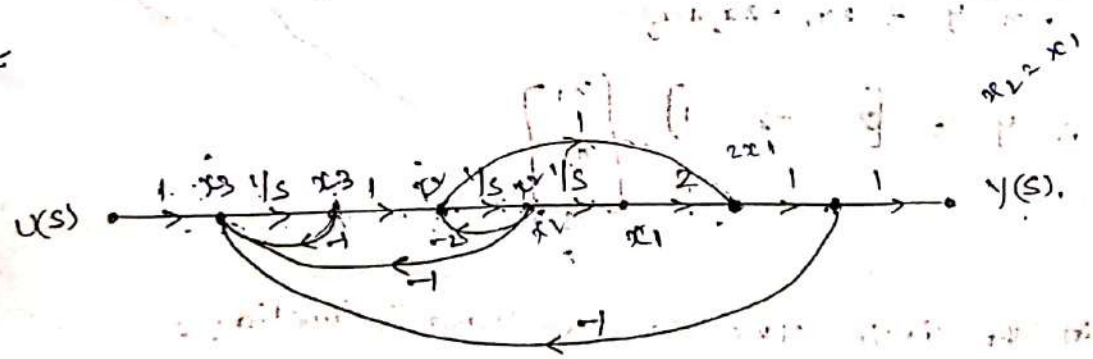
The output equations are $y_1 = x_2$, $y_2 = x_4$.

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & -3 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

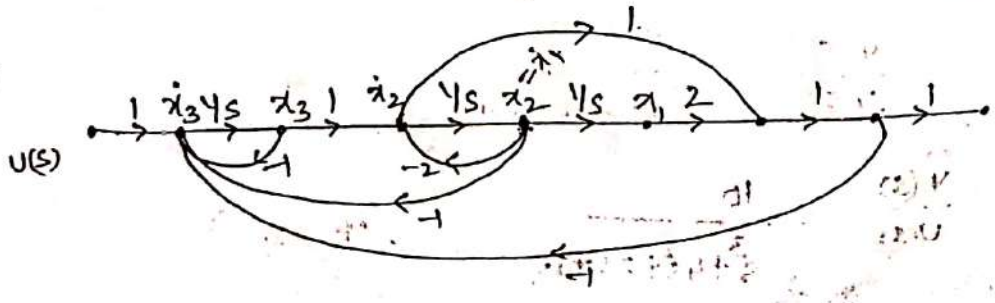
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

3. Obtain the state Model for the signal flow graph below.?

Sol:



Sol:



Taking the signals coming to node-5

$$\dot{x}_1 = x_2 \quad \text{---} (1)$$

Similarly,

$$\dot{x}_2 = -2x_2 + x_3 \quad \text{---} (2)$$

$$\dot{x}_3 = -(\dot{x}_2 + 2x_1) - x_2 - x_3 + 4$$

$$= -\dot{x}_2 - 2x_1 - x_2 - x_3 + 4$$

Now, Substitute $\dot{x}_2 = -2x_2 + x_3$ in above eq.

$$\Rightarrow \dot{x}_3 = -(-2x_2 + x_3) - 2x_1 - x_2 - x_3 + 4$$

$$= 2x_2 - x_3 - 2x_1 - x_2 - x_3 + 4$$

$$\therefore \dot{x}_3 = -2x_1 + x_2 - 2x_3 + 4 \quad \text{---} (3)$$

from the (1). (2). (3) equations the state Model is obtained as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

The output Equation is, $y = 2x_1 + x_2$

Substitute $x_2 = -2x_2 + x_3$

$$\Rightarrow y = 2x_1 - 2x_2 + x_3.$$

$$\therefore y = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

4. Obtain the State Model for the Transfer function

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3 + 4s^2 + 2s + 1}$$

Sol: given $\frac{Y(s)}{U(s)} = \frac{10}{s^3 + 4s^2 + 2s + 1}$ (Method-1)

$\Rightarrow Y(s)\left[s^3+4s^2+2s+1\right] = 10\,U(s)$.

$\Rightarrow s^3 Y(s) + 4s^2 Y(s) + 2s\,Y(s) + Y(s) = 10\,U(s)$.

$\Rightarrow$ on taking inverse laplace transform we get,

$\Rightarrow \dddot{y} + 4\ddot{y} + 2\dot{y} + y = 10u$ — (1)

Now, $y = x_1$

$\dot{y} = \dot{x}_1 = x_2$

$\ddot{y} = \ddot{x}_1 = \dot{x}_2 = x_3$

$\dddot{y} = \dddot{x}_1 = \ddot{x}_2 = \dot{x}_3$

(1) $\Rightarrow \dot{x}_3 + 4x_3 + 2x_2 + x_1 = 10u$

$\Rightarrow \dot{x}_3 = 10u - 4x_3 - 2x_2 - x_1$

∴ Now, the state equations are,

$\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, $\dot{x}_3 = 10u - 4x_3 - 2x_2 - x_1$.

output Equation $y = x_1$.

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

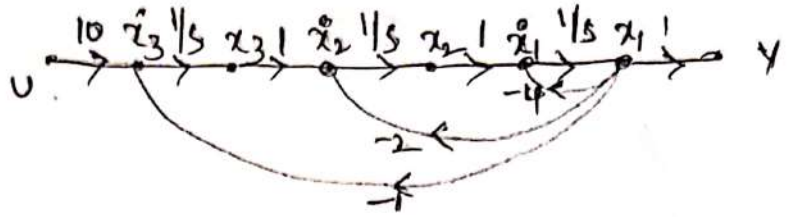
Method-2

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$$

$$= \frac{10/s^3}{1-\left(\frac{-4}{s} - \frac{2}{s^2} - \frac{1}{s^3}\right)}$$

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The signal flow graph from a transfer function consists of one forward path and three individual loop gains $-4/s$, $-2/s^2$, $-1/s^3$.



The state equations from the above signal flow graph.

$$\dot{x}_1 = -4x_1 + x_2$$

$$\dot{x}_2 = -2x_1 + x_3$$

$$\dot{x}_3 = -x_1 + 10u.$$

output equation $y = x_1$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u.$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Method-3:

given $\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$

$$\Rightarrow \frac{Y(s)}{X_1(s)} \times \frac{X_1(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$$

$$\Rightarrow \frac{Y(s)}{X_1(s)} = 10, \quad \frac{X_1(s)}{U(s)} = \frac{1}{s^3+4s^2+2s+1}$$

$$\Rightarrow Y(s) = 10\,X_1(s), \quad X_1(s)\left[s^3+4s^2+2s+1\right] = U(s)$$

$\Rightarrow$ on taking inverse laplace transform we get,

$$\Rightarrow y = 10x_1, \quad \dddot{x}_1 + 4\ddot{x}_1 + 2\dot{x}_1 + x_1 = 4$$

$$\dot{x}_3 + 4x_3 + 2x_2 + x_1 = 4$$

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Now, the state equations are,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -4x_3 - 2x_2 - x_1 + u$$

and output equation $y = 10x_1$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

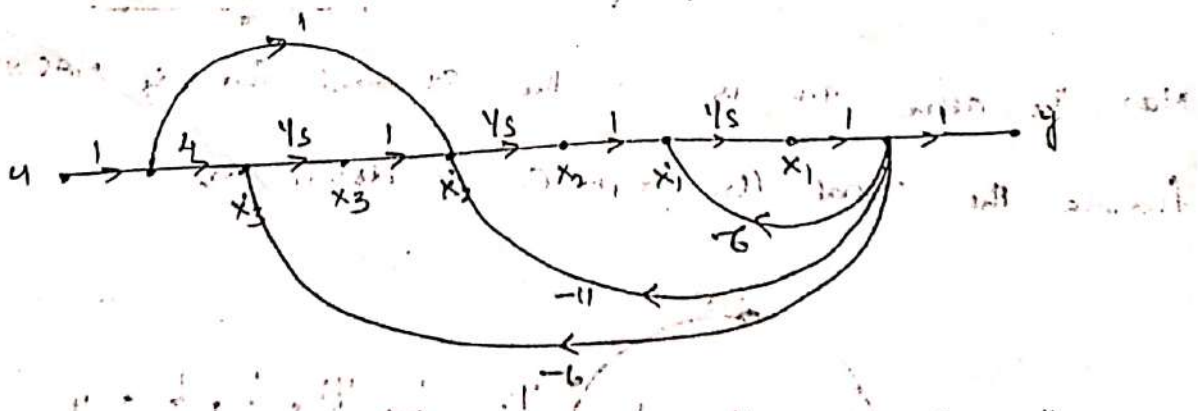
5. Find the state Model for the system with. $T.F = \dfrac{s+4}{s^3+6s^2+11s+6}$

Sol: given $T.F = \dfrac{Y(s)}{U(s)} = \dfrac{s+4}{s^3+6s^2+11s+6}$.

Now, Divide the denominator and Numerator with s^3

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\frac{1}{s^2} + \frac{4}{s^3}}{1 + \frac{6}{s} + \frac{11}{s^2} + \frac{6}{s^3}} = \frac{\frac{1}{s^2} + \frac{4}{s^3}}{1 - \left[\frac{-6}{s} - \frac{11}{s^2} - \frac{6}{s^3}\right]}$$

∴ From the above equation we have two forward paths and the terms in denominator represents the three loops as shown in the figure



Now, the differential equations should be attained from the above signal flow graph.

The state equations are represented from the signal flow graph, i.e,

$$\dot{x}_1 = -6x_1 + x_2$$

$$\dot{x}_2 = -11x_1 + x_3 + u$$

$$\dot{x}_3 = -6x_1 + 4u.$$

From the above state equations the model will be written as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \cdot u.$$

and $y = x_1 \Rightarrow y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

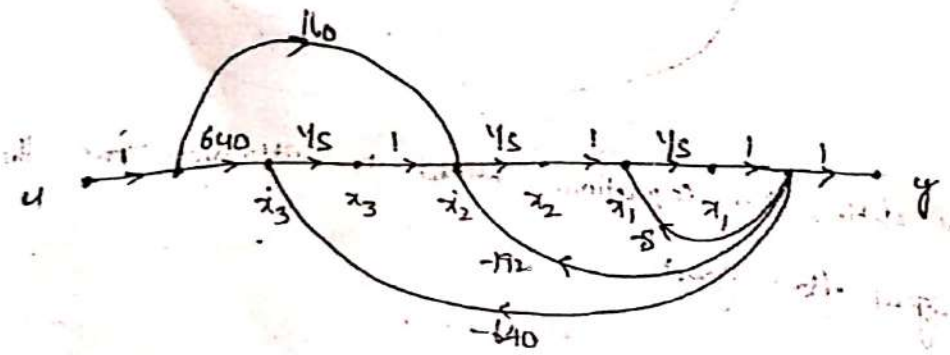
6. The closed loop T.F is given by, $\frac{Y(s)}{U(s)} = \frac{160(s+4)}{s^3+8s^2+192s+640}$. Obtain the state model using signal flow graph.

Sol: given, $T.F = \frac{Y(s)}{U(s)} = \frac{160(s+4)}{s^3+8s^2+192s+640}$

Now, divide the numerator and denominator with s^3.

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\frac{160}{s^2} + \frac{4\times 160}{s^3}}{1 + \frac{8}{s} + \frac{192}{s^2} + \frac{640}{s^3}} = \frac{\frac{160}{s^2} + \frac{640}{s^3}}{1 - \left(\frac{-8}{s} - \frac{192}{s^2} - \frac{640}{s^3}\right)}.$$

Now the above T.F is in the standard form of MASON's gain formula, the signal flow graph can be drawn as :



Let x_1, x_2, x_3 be the state variables.

The state equation and the output equations are written from the signal flow graph is given by,

$$\dot{x}_1 = -8x_1 + x_2$$

$$\dot{x}_2 = -192x_1 + x_3 + 160u$$

$$\dot{x}_3 = -640x_1 + 640u$$

and $y = x_1$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -8 & 1 & 0 \\ -192 & 0 & 1 \\ -640 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 160 \\ 640 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

7. $\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = 2\ddot{u} + 6\dot{u} + 5u$

Sol: given that $\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = 2\ddot{u} + 6\dot{u} + 5u$.

$$\Rightarrow \frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = 2\frac{d^2u}{dt^2} + 6\frac{du}{dt} + 5u$$

Now, apply inverse laplace transform,

$$\Rightarrow s^3Y(s) + 4s^2Y(s) + 5sY(s) + 2Y(s) = 2s^2U(s) + 6sU(s) + 5U(s)$$

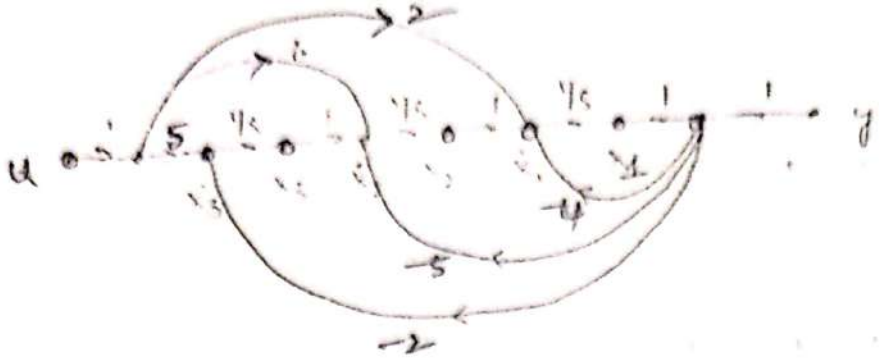
$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{2s^2 + 6s + 5}{s^3 + 4s^2 + 5s + 2}$$

Dividing the denominator and Numerator with s^3.

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\frac{2}{s} + \frac{6}{s^2} + \frac{5}{s^3}}{1 - \left(\frac{-4}{s} - \frac{5}{s^2} - \frac{2}{s^3}\right)}$$

From the above transfer function, obtain the signal flow graph.

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From the above flow graph.

The state equations are $\dot{x}_1 = -4x_1 + x_2 + 2u$

$\dot{x}_2 = -5x_1 + x_3 + 6u$

$\dot{x}_3 = -2x_1 + 5u$

and $y = x_1$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ -5 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Q. A State Model is characterised by.

$$\dot{x} = \begin{bmatrix} -1 & -4 & -1 \\ -1 & -6 & -2 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Soln From the above State Model find. T.F

$$A = \begin{bmatrix} -1 & -4 & -1 \\ -1 & -6 & -2 \\ -1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Now, the transfer function $= C\left[SI - A\right]^{-1} B$.

$$\left[SI - A\right] = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -4 & -1 \\ -1 & -6 & -2 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} S+1 & 4 & 1 \\ 1 & S+6 & 2 \\ 1 & 2 & S+3 \end{bmatrix}$$

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$|SI - A| = s^3 + 10s^2 + 18s + 16.$

$[SI - A]^{-1} = \frac{Adj\,[SI - A]}{|SI - A|}.$

Co-factors, $(SI - A) = \begin{bmatrix} S+1 & 4 & 1 \\ 1 & S+6 & 2 \\ 1 & 2 & S+3 \end{bmatrix}$

$C_{11} = S+6(S+3) - 4$
$= s^2 + 9s + 14$

$C_{12} = 1(S+3) - 2 \times 1$
$= S+1$

$C_{13} = 2 - (S+6)$
$= -S - 4$

Similarly, $C_{21} = 4S + 10$ and $C_{31} = -s + 2$

$C_{22} = s^2 + 4s + 2$

$C_{23} = 2s - 2$

$C_{32} = 2s + 1$

$C_{33} = s^2 + 7s + 2$

Now, ~~Cofactor Matrix~~ $[SI - A]^T$

Now, $Adj\,[SI - A] = \begin{bmatrix} +(s^2+9s+14) & -(4s+10) & +(-s+2) \\ -(s+1) & +(s^2+4s+2) & -(2s+1) \\ +(-s-4) & -(2s-2) & +(s^2+7s+2) \end{bmatrix}$

Now, $T.F = C \cdot [SI - A]^{-1} B$

$= [1 \;\; 1 \;\; 1] \frac{1}{s^3 + 10s^2 + 18s + 6} \begin{bmatrix} s^2+9s+14 & -(4s+10) & (-s+2) \\ -(s+1) & (s^2+4s+2) & -(2s+1) \\ (-s-4) & -(2s-2) & s^2+7s+2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$= \frac{1}{s^3 + 10s^2 + 18s + 6} \underset{1\times3}{[1 \;\; 1 \;\; 1]} \begin{bmatrix} -5s - 8 \\ s^2 + 2s + 1 \\ s^2 + 5s + 4 \end{bmatrix}_{3\times1} \rightarrow (1 \times 1)$

$\therefore T.F = \frac{2s^2 + 2s + (-3)}{\ldots\ldots}$

(9) Given $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. find the characteristic equation of the system and then find the roots.

Sol. As we know that the characteristic equation will be taken or considered as the denominator term of the obtained transfer function. From the T.F of the state model

$$\Rightarrow \text{T.F} = C.[SI-A]^{-1}B + D \quad (\text{or}) \quad C[SI-A]^{-1}B$$

$$= C \frac{\text{Adj}(SI-A)}{|SI-A|} B.$$

Now from the above Transfer function the characteristic equation is $|SI-A| = 0$

$$|SI-A| = \left| S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} s-1 & -2 & -1 \\ 0 & s-1 & -3 \\ -1 & -1 & s-1 \end{vmatrix}$$

$$= s-1\left[(s-1)^2-3\right] + 2(-3) - 1(s-1)$$

$$= s-1\left(s^2-2s+1-3\right) - 6 - s + 1$$

$$= s^3 - 2s^2 - 2s - s^2 + 2s + 2 - s - 5$$

$$\therefore |SI-A| = s^3 - 3s^2 - s + (-3).$$

So, the characteristic equation is, $s^3 - 3s^2 - s - 3 = 0$.

from the above equation the roots of 's' are

$3.525, \; -0.2625 \pm j\,0.884$

10. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix} \begin{bmatrix} \cdot \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cdot \\ x_2 \\ x_3 \end{bmatrix}$

find the T.F

Sol: given, $A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

$\therefore T.F = C\,[SI-A]^{-1} B$

$$|[SI-A]| = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix}$$

$$= \left| \begin{bmatrix} s+2 & -1 & 0 \\ 0 & s+3 & -1 \\ 3 & 4 & s+5 \end{bmatrix} \right|$$

$$= s^3 + 10s^2 + 35s + 41$$

$$[SI-A]^{-1} = \frac{\text{adj}\,(SI-A)}{|SI-A|}$$

$$\frac{\text{adj}\,(SI-A)}{|SI-A|} = \text{adj} \begin{bmatrix} s+2 & -1 & 0 \\ 0 & s+3 & -1 \\ 3 & 4 & s+5 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} s^2+8s+19 & s+5 & 1 \\ -3 & s^2+7s+10 & s+2 \\ -(3s+9) & -(4s+11) & s^2+5s+6 \end{bmatrix}}{s^3+10s^2+35s+41}$$

$\therefore$ The $T.F = C\,[SI-A]^{-1} B$

$$= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \frac{1}{s^3+10s^2+35s+41} \begin{bmatrix} s^2+8s+19 & s+5 & 1 \\ -3 & s^2+7s+10 & s+2 \\ -(3s+9) & -(4s+11) & s^2+5s+6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The $\boxed{T.F = \dfrac{s+2}{s^3+10s^2+35s+41}}$

$$\begin{vmatrix} 16 & +8 & 21 \\ 8 & -1 & 3 \end{vmatrix}.$$

If given, $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Now, $|\lambda I - A| = \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \right| = 0$

$$= \begin{vmatrix} \lambda-8 & 6 & -2 \\ 6 & \lambda-7 & 4 \\ -2 & 4 & \lambda-3 \end{vmatrix} = 0$$

$\Rightarrow \lambda - 8\left[(\lambda-7)(\lambda-3) - 16\right] - 6\left(6\lambda - 18 + 8\right) - 2\left[24 + 2(\lambda-7)\right] = 0$

$\Rightarrow (\lambda-8)\left(\lambda^2 - 10\lambda + 5\right) - 6(6\lambda - 10) - 2(2\lambda + 10) = 0$

$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda + 0 = 0$

$\lambda = 0, 3, 15$

$$[\lambda_1 I - A] = \begin{bmatrix} -8 & 6 & -2 \\ 6 & -7 & 4 \\ -2 & 4 & -3 \end{bmatrix} \qquad (\lambda_1 = 0)$$

$C_{11} = 21 - 16 = 5$

$C_{12} = -(-18+8) = 10$

$C_{13} = 24 - 14 = 16.$

$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$$[\lambda_2 I - A] = \begin{bmatrix} -5 & 6 & -2 \\ 6 & -4 & 4 \\ -2 & 4 & 0 \end{bmatrix} \qquad (\lambda_2 = 3)$$

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580, 89, A2, A8, B5, C7, D0, D4, 507, 89, 89

$C_{11} = +16$

$C_{12} = -(0+8)$

$= -8$

$C_{13} = 24 - 8$

$= 16$

$\therefore X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Similarly, $[\lambda_3 I - A] = \begin{bmatrix} 7 & 6 & -2 \\ 6 & 8 & +4 \\ -2 & 4 & 12 \end{bmatrix}$ $(\lambda_3 = 15)$

$C_{11} = 96 - 16$

$= 80$

$C_{12} = -(72+8)$

$= -80$

$C_{13} = 24 + 16$

$= 40$

$\therefore X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\therefore P = [X_1 \; X_2 \; X_3] = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$ [P = Modal Matrix]

Now, $P^{-1} = \frac{\text{adj}(P)}{|P|} = \begin{bmatrix} 0.111 & 0.222 & 0.222 \\ -0.222 & -0.111 & 0.222 \\ 0.222 & -0.222 & 0.111 \end{bmatrix}$

Now, Diagonalization of a Matrix A is given by:

$D = P^{-1} A P = \begin{bmatrix} 0.111 & 0.222 & 0.222 \\ -0.222 & -0.111 & 0.222 \\ 0.222 & -0.222 & 0.111 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.97 & 0 \\ 0 & 0 & 14.85 \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

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Diagonalize the matrix $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -4 & 1 \end{bmatrix}$

Now C.E $\Rightarrow |\lambda I - A| = 0$

$$\Rightarrow \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-8 & 8 & 2 \\ -4 & \lambda+3 & 2 \\ -3 & +4 & \lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow \lambda-8\left[(\lambda+3)(\lambda-1) - 8\right] - 8(-4\lambda+4-6) + 2\left[16 + 3(\lambda+3)\right] = 0$$

$$\Rightarrow \lambda-8\left(\lambda^2+2\lambda-3-8\right) - 8\left(-4\lambda+10\right) + 2\left(3\lambda-7\right) = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$\lambda = 3, 1, 2 \Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3.$

Now,

$$|\lambda_1 I - A| = \begin{bmatrix} -7 & 8 & 2 \\ -4 & 4 & 2 \\ -3 & 4 & 0 \end{bmatrix}$$

$C_{11} = -8, \quad C_{12} = -(0+6), \quad C_{13} = -16+12$

$= -6 \qquad\qquad = -4.$

$$\Rightarrow M_1 = \begin{bmatrix} -8 \\ -6 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}.$$

Similarly, $|\lambda_2 I - A| = \begin{bmatrix} -6 & 8 & 2 \\ -4 & 5 & 2 \\ -3 & 4 & 1 \end{bmatrix}$

$C_{11} = -3, \quad C_{12} = -(-4+6), \quad C_{13} = -16+15$

$= -2 \qquad\qquad = -1.$

$$\therefore M_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

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$|\lambda_3 I - A| =$

$$= \begin{bmatrix} -5 & 8 & 2 \\ -4 & 6 & 2 \\ -3 & 4 & 2 \end{bmatrix}$$

$C_{11} = 12 - 8 = 4$

$C_{12} = -(-8+16) = 2$

$C_{13} = -16+18 = 2$

$\therefore M_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$\therefore P = [M_1 \quad M_2 \quad M_3] = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

$\therefore$ Now, $P^{-1} = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}^{-1}$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$\therefore$ Now, Diagonalization of Matrix A

$= P^{-1} A P$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & -4 \\ 3 & -6 & 3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\Rightarrow$ Diagonalized Matrix should contain their Eigen values in as diagonal elements remaining elements will be zero.

13. Diagonalize the Matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Sol: given, $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$|\lambda I - A| = 0 \Rightarrow \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$\Rightarrow \begin{vmatrix} \lambda-1 & -1 & -3 \\ -1 & \lambda-5 & -1 \\ -3 & -1 & \lambda-1 \end{vmatrix} = 0$

$\Rightarrow (\lambda-1)\left[(\lambda-5)(\lambda-1)-1\right] + 1\left(-1(\lambda-1)-3\right) - (+3)\left[1+3(\lambda-5)\right] = 0$

$\Rightarrow (\lambda-1)(\lambda^2-5\lambda-\lambda+5-1) + (-\lambda+1-3) + 3(3\lambda-14) = 0$

$\Rightarrow (\lambda-1)(\lambda^2-6\lambda+4) + (-\lambda-2) - 9\lambda + 42 = 0$

$\Rightarrow \lambda^3-6\lambda^2+4\lambda-\lambda^2+6\lambda-4-\lambda-2-9\lambda+42 = 0$

$\Rightarrow \lambda^3-7\lambda^2+10\lambda+36 = 0$

$\therefore \lambda = -2, 3, 6.$

$[\lambda I - A] \Rightarrow \begin{bmatrix} -3 & -1 & -3 \\ -1 & -7 & -1 \\ -3 & -1 & -3 \end{bmatrix}$

$C_{11} = 21-1 = 20$

$C_{12} = -[3-3] = 0$

$C_{13} = 1-21 = -20$

$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

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$[\lambda_2 I - A] \Rightarrow \begin{bmatrix} 2 & -1 & -3 \\ -1 & -2 & -1 \\ -3 & -1 & 2 \end{bmatrix}$

$c_{11} = -4-1 = -5$

$c_{12} = -[-2-3] = 5$

$c_{13} = 1-6 = -5$

$\therefore X_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

Similarly, $[\lambda_3 I - A] \Rightarrow \begin{bmatrix} 5 & -1 & -3 \\ -1 & 1 & -1 \\ -3 & -1 & 5 \end{bmatrix}$

$c_{11} = 5-1 = 4$

$c_{12} = -(-5-3) = +8$

$c_{13} = 1+3 = 4$

$\therefore X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Now, $P = [X_1 \quad X_2 \quad X_3]$

$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 0.5 & 0 & -0.5 \\ -0.33 & 0.33 & -0.33 \\ 0.166 & 0.33 & 0.166 \end{bmatrix}$

Now, Diagonalization $D = P^{-1}AP$

$\Rightarrow D = \begin{bmatrix} 0.5 & 0 & -0.5 \\ -0.33 & 0.33 & -0.33 \\ 0.166 & 0.33 & 0.166 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2.97 & 0 \\ 0 & -0.006 & 5.95 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

Power of a Matrix:

As we know that $D = P^{-1}AP$.

Now, squaring on both sides we get,

$$D^2 = (P^{-1}AP)^2$$

$$= (P^{-1}AP)(P^{-1}AP)$$

$$= (P^{-1}A)(PP^{-1})(AP)$$

$$= P^{-1}A(I)(AP)$$

$$= (P^{-1}A)(AP)$$

$$\therefore \boxed{D^2 = P^{-1}A^2P}$$

Similarly, $D^3 = P^{-1}A^3P$.

in general $\Rightarrow \boxed{D^n = P^{-1}A^nP}$

$$\Rightarrow D^n = P^{-1}A^nP$$

$$\Rightarrow \boxed{A^n = PD^nP^{-1}}$$

4. Now, find A^4 for above problem for diagonalized real Matrix 'A'.

Sol: $D^4 = P^{-1}A^4P$.

$$\Rightarrow A^4 = PD^4P^{-1}$$

$$= \begin{bmatrix} 0.5 & 0 & -0.5 \\ -0.33 & 0.33 & -0.33 \\ 0.166 & 0.33 & 0.166 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow A^4 = \begin{bmatrix} 251 & 405 & 235 \\ 405 & 891 & 405 \\ 235 & 405 & 251 \end{bmatrix}$$

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(15) $A = \begin{bmatrix} -0.5 & -0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Show that

$P^{-1}AP$ is a diagonal matrix.

Sol: given, $A = \begin{bmatrix} -0.5 & -0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now, $P^{-1} = \frac{\text{adj } P}{|P|}$

$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now, $D = P^{-1} A P$.

$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.5 & -0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -3.8 \times 10^{-5} & 0 \\ -3.8 \times 10^{-5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$A = M^{-1} \Lambda M$

* Solution of State Equations without input or Excitation :

Consider a first order differential equation, with initial condition $x(0) = x_0$.

$$\frac{dx}{dt} = ax \quad ; \quad x(0) = x_0.$$

On Re-arranging the above equation.

$$\frac{dx}{x} = a \cdot dt.$$

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Now, on integrating the above equation

we get, $\int \frac{dx}{x} = \int a\,dt$

$\Rightarrow \log x = at + C$

$\Rightarrow x = e^{(at+C)}$

$\therefore x = e^{at} \cdot e^{C}$ ——— (1)

$\left(\because x(0) = e^{0 \cdot 0} \cdot e^{C} \right.$
$\Rightarrow x_0 = 1 \cdot e^{C}$
$\left. \Rightarrow e^{C} = x_0 . \right)$

When $t=0$ from the above equation, we get,

$x = x(0) = e^{C}$

given that, $x(0) = x_0 \therefore e^{C} = x_0$ ——— (2)

Now Substituting (2) in (1) we get,

$$\boxed{x = e^{at} \cdot x_0} \quad \text{——— (3)}$$

we know that, $e^{x} = \left[1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n \right]$ ——— (4)

from Eqs (3) and (4)

$$\Rightarrow X = e^{At} X_0 = \left[1 + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots + \frac{1}{n!}(At)^n + \cdots \right] X_0 .$$

——— (5)

Now, Consider the state equation with out input vector.

$$\dot{x}(t) = Ax(t) \quad ; \quad x(0) = X_0 \quad \text{——— (A)}$$

where, $x(0)$ is the initial condition vector.

Now, the solution of the Eq (5) can be written in the matrix equation as follows,

$$x(t) = A_0 + A_1 t + A_2 t^2 + \cdots + A_i t^i + \cdots \quad \text{——— (6)}$$

where, A_0, A_1, A_2, A_3 are matrices.

Now, differentiate the equation (6) we get

$$\dot{x}(t) = A_1 + 2A_2 t + 3A_3 t^2 + \cdots + iA_i t^{(i-1)} \cdots \quad \text{——— (7)}$$

on Multiplying the above equation (6) with A on both sides, we get

$\Rightarrow X(t) = A[A_0 + A_1 t + A_2 t^2 + \dots + A_n t^n + \dots]$ ——— (8)

From the equation (1)

$\dot{X}(t) = A X(t)$.

∴ equate the equations (7) and (8) we get,

$\boxed{A_1 = AA_0}$ ——— (9)

⇒ equating 't' coefficients

⇒ $2A_2 = AA_1$

⇒ $A_2 = \frac{1}{2} AA_1$ ——— (10)

Put $A_1 = AA_0$ in (10)

⇒ $\boxed{A_2 = \frac{1}{2} A^2 A_0}$ ——— (11)

⇒ equating t^2 coefficients we get

⇒ $3A_3 = AA_2$

$A_3 = \frac{1}{3} AA_2$ ——— (12)

Now, Substitute A_2 in (12) we get

⇒ $A_3 = \frac{1}{3} A \frac{1}{2} A^2 A_0$

$= \frac{1}{6} A^3 A_0$.

∴ $\boxed{A_3 = \frac{1}{3!} A^3 A_0}$.

Similarly, $A_4 = \frac{1}{4!} A^4 A_0$.

∴ $X(t) = \left[A_0 + AA_0 t + \frac{1}{2!} A^2 A_0 t^2 + \frac{1}{3!} t^3 A^3 A_0 + \dots + \frac{1}{n!} t^n A^n A_0 + \dots\right]$

$= \left[I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \dots + \frac{1}{n!} A^n t^n + \dots\right] A_0$

where, $A_0 = X_0$. [∵ from eqn (6) when $t = 0$]

⇒ $X(t) = \left[I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} (At)^3 + \dots\right] X_0$.

∴ $\boxed{X(t) = e^{At} X_0}$ ——— (13)

∴ The above equation is called Solution of State Equation.

and e^{At} is called state transition matrix and is denoted by $\phi(t)$.

* Properties of State transition Matrix:

1. $\phi(0) = e^{A \times 0} = I$ (unit matrix)
2. $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$

(or)

$\phi^{-1}(t) = \phi(-t)$. $\phi^{-1}(t) = \phi(-t)$.

3. $\phi(t_1 + t_2) = e^{At_1} \cdot e^{At_2} = \phi(t_1) \cdot \phi(t_2) = \phi(t_2) \cdot \phi(t_1)$.

* Compilation of State transition Matrix

Method-1 : Compilation of e^{At} using matrix Exponential

Method - 2 : Compilation of e^{At} using laplace transform

Method - 3 : " " " Canonical transformation.

Method - 4 : " " " Cayley Hamilton Theorem.

* Compilation of e^{At} by Exponential Method.

As we know,

$$e^{xt} = 1 + x \cdot t + \frac{1}{2!} x^2 t^2 + \frac{1}{3!} (xt)^3 + \frac{1}{4!} (xt)^4 + \dots$$

Similarly, $e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots$

where A = System matrix (or) state Matrix

e^{At} = State transition Matrix

I = Identity Unit Matrix.

* Compilation of e^{At} using laplace transform method.

Consider the state equation with out input vector

$$\dot{x}(t) = A x(t). \quad \text{---} (1)$$

on taking the laplace transform for the eqn (1)

$$\Rightarrow s X(s) - X(0) = A X(s).$$

$\Rightarrow sX(s) - AX(s) = X(0)$

$\Rightarrow (sI - A)X(s) = X(0)$

$\Rightarrow X(s) = [sI - A]^{-1} X(0)$.

Now, apply inverse laplace transform to above equation.

$\Rightarrow x(t) = L^{-1}[sI - A]^{-1} X(0)$. ——— (2)

$\Rightarrow$ Compare equation (2) with $x(t) = e^{At} x(0)$.

$$\boxed{e^{At} = L^{-1}[sI - A]^{-1}} \quad \text{——— (3)}$$

$\Rightarrow L(e^{At}) = [sI - A]^{-1}$

$\Rightarrow L[\phi(t)] = [sI - A]^{-1}$

$\Rightarrow \boxed{\phi(s) = [sI - A]^{-1}}$

where, $\phi(s)$ is called the Resolvent Matrix.

Problems.

1. Compute e^{At} by two methods. $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Sol: given, $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Method-1 :- $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$

$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$= \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}$.

$A^3 = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ -14 & -15 \end{bmatrix}$, $A^4 = \begin{bmatrix} -14 & -15 \\ 30 & 31 \end{bmatrix}$.

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}t + \frac{1}{2}t^2\begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 6 & 7 \\ -14 & -15 \end{bmatrix}t^3 + \frac{1}{24}\begin{bmatrix} -14 & -15 \\ 30 & 31 \end{bmatrix}t^4 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -2t & -3t \end{bmatrix} + \begin{bmatrix} -t^2 & -\frac{3}{2}t^2 \\ -3t^2 & \frac{7}{2}t^2 \end{bmatrix} + \begin{bmatrix} t^3 & \frac{7}{6}t^3 \\ -\frac{14}{6}t^3 & -\frac{15}{6}t^3 \end{bmatrix} + \begin{bmatrix} -\frac{14}{24}t^4 & -\frac{15}{24}t^4 \\ \frac{30}{24}t^4 & \frac{31}{24}t^4 \end{bmatrix} + \cdots$$

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$$= \begin{bmatrix} 1-t^2+t^3-\frac{7}{12}t^4+\cdots & t-\frac{3}{2}t^2+\frac{7}{6}t^3-\frac{5}{8}t^4+\cdots \\ -2t+3t^2-\frac{7}{3}t^3+\frac{5}{4}t^4-\cdots & 1-3t+\frac{7}{2}t^2-\frac{5}{2}t^3+\frac{31}{24}t^4+\cdots \end{bmatrix}$$

Here, the each term in the above matrix is the expansion of At.

The convergence of series is obtained by trail and error method.

$$e^{-t} = 1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4 = 1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{24}t^4-\cdots\cdots$$

$$e^{-2t} = 1-2t+\frac{1}{2!}(2t)^2-\frac{1}{3!}(2t)^3+\cdots\cdots = 1-2t+2t^2-\frac{4}{3}t^3+\frac{2}{3}t^4-\cdots\cdots$$

Now,

$$2e^{-t}-e^{-2t} = 2-2t+t^2-\frac{1}{3}t^3+\frac{1}{12}t^4-\cdots\cdots -1+2t-2t^2+\frac{4}{3}t^3-\frac{2}{3}t^4+\cdots\cdots$$

$$= 1-t^2+t^3-\frac{7}{12}t^4+\cdots\cdots$$

$$e^{-t}-e^{-2t} = 1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{24}t^4-\cdots\cdots -1+2t-2t^2+\frac{4}{3}t^3-\frac{2}{3}t^4+\cdots\cdots$$

$$= t-\frac{3}{2}t^2+\frac{7}{6}t^3-\frac{5}{8}t^4+\cdots\cdots$$

$$-2e^{-t}+2e^{-2t} = -2+2t-t^2+\frac{1}{3}t^3-\frac{1}{12}t^4+\cdots\cdots +2-4t+4t^2-\frac{8}{3}t^3+\frac{4}{3}t^4-\cdots\cdots$$

$$= -2t+3t^2-\frac{7}{3}t^3+\frac{5}{4}t^4-\cdots\cdots$$

$$-e^{-t}+2e^{-2t} = -1+t-\frac{1}{2}t^2+\frac{1}{6}t^3-\frac{1}{24}t^4+\cdots\cdots +2-4t+4t^2-\frac{8}{3}t^3+\frac{4}{3}t^4-\cdots\cdots$$

$$= 1-3t+\frac{7}{2}t^2-\frac{5}{2}t^3+\frac{31}{24}t^4-\cdots\cdots$$

$$\therefore e^{At} = \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}.$$

Method-2

Given: $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$e^{At} = \phi(t) = \mathcal{L}^{-1}[sI-A]^{-1} \Rightarrow \phi(s) = [sI-A]^{-1}$$

$$[sI-A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$|sI-A| \cdot \begin{vmatrix} s & 1 \\ -2 & s+3 \end{vmatrix} = s(s+3)+2$$

$$s^2+3s+2 = (s+2)(s+1)$$

Now, $[sI-A]^{-1} = \frac{\text{adj}[sI-A]}{|sI-A|}$ $\left(\because \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}_{\text{adj}}\right)$

$$= \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \Big/ (s+1)(s+2)$$

$$\therefore L^{-1}\{\phi(t)\} = \phi(s) = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+2)(s+1)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

By doing the partial fractions we get,

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$\Rightarrow s+3 = A(s+2) + B(s+1)$$

$$\Rightarrow A = 2 \quad (\because s = -1)$$

for $s = -2$, $B = -1$.

$$\frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$\Rightarrow 1 = A(s+2) + B(s+1)$$

for $s = -1$, $A = 1$

for $s = -2$, $B = -1$

$$\frac{-2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$-2 = A(s+2) + B(s+1)$$

for $s = -1$, $A = -2$

$s = -2$, $B = 2$

$$\frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$s = A(s+2) + B(s+1).$$

for $s = -1$, $A = -1$

$s = -2$, $B = 2$

$$\therefore \phi(s) = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{(s+2)} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

On taking the inverse laplace transform we get:

$$\therefore e^{At} = \phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

(2) $A_1 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$

Compute e^{At} ?

Sol: Given, $A_1 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$

Here, $A = A_1 + A_2$

$\therefore e^{At} = e^{(A_1+A_2)t} = e^{A_1 t} \cdot e^{A_2 t}$

we know that, $e^{At} = \phi(t) = L^{-1}[sI - A]^{-1}$.

$\therefore |sI - A_1| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \right| = \begin{bmatrix} s-\sigma & 0 \\ 0 & s-\sigma \end{bmatrix}$

$= (s-\sigma)^2$

Now, $[sI - A]^{-1} = \begin{bmatrix} (s-\sigma) & 0 \\ 0 & (s-\sigma) \end{bmatrix} \Big/ (s-\sigma)^2$

$= \begin{bmatrix} \frac{1}{s-\sigma} & 0 \\ 0 & \frac{1}{s-\sigma} \end{bmatrix}$

$\therefore e^{A_1 t} = L^{-1}[sI - A_1]^{-1} = \begin{bmatrix} e^{\sigma t} & 0 \\ 0 & e^{\sigma t} \end{bmatrix}$

$e^{A_2 t} = L^{-1}[sI - A_2]^{-1}$

$|sI - A_2| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \right| = \begin{vmatrix} s & -\omega \\ \omega & s \end{vmatrix}$

$= s^2 + \omega^2$

$\therefore [sI - A_2]^{-1} = \dfrac{\begin{bmatrix} s & \omega \\ -\omega & s \end{bmatrix}}{s^2 + \omega^2}$

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$$= \begin{bmatrix} \frac{s}{s^2+\omega^2} & \frac{\omega}{s^2+\omega^2} \\ \frac{-\omega}{s^2+\omega^2} & \frac{s}{s^2+\omega^2} \end{bmatrix}$$

Now, $e^{A_2 t} = L^{-1}\left[sI - A_2\right]^{-1}$

$$= L^{-1}\begin{bmatrix} \frac{s}{s^2+\omega^2} & \frac{\omega}{s^2+\omega^2} \\ \frac{-\omega}{s^2+\omega^2} & \frac{s}{s^2+\omega^2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

$$e^{At} = e^{A_1 t} \cdot e^{A_2 t}$$

$$= \begin{bmatrix} e^{-\sigma t} & 0 \\ 0 & e^{-\sigma t} \end{bmatrix} \cdot \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-\sigma t}\cos\omega t & e^{-\sigma t}\sin\omega t \\ -e^{-\sigma t}\sin\omega t & e^{-\sigma t}\cos\omega t \end{bmatrix}$$

3. A linear time invariant system is characterized by homogenius state equation. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Compute the solution of homogenius equation.

Sol^n given, $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution of homogenius equation $= x(t) = e^{At}x_0$.

where, $e^{At} = L^{-1}\left[sI - A\right]^{-1}$

$$\Rightarrow |sI - A| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix} \right|$$

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$= (s-1)^2$

$$[sI-A]^{-1} = \frac{\begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}}{(s-1)^2}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{+1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$\left\{ \because \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right\}$$

$$e^{At} = \phi(t) = L^{-1}[sI-A]^{-1}$$

$$= L^{-1} \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{+1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

$$\therefore e^{At} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

Now, Solution $= x(t) = e^{At} x_0$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^t \\ te^t \end{bmatrix}$$

$x(t) = e^{At} x_0$

V.V.Imp

* Concept of Controllability and Observability :-

Controllability :-

The Controllability verifies the usefullness of a state variable. In the Controllability Test, we can find, whether the state variable can be controlled to achieve the desired output & not. After determining the state model, the Controllability of a state variable is verified.

* Definition of Controllability :-

A System is said to be Completely Controllable if it is possible to transfer the system state from any initial state $x_0(t)$ to any other desired state $x(t_d)$ in specified finite time.

The Controllability of a state model can be tested by Kalmans Method and Gilberts Method.

i, Gilbert's Method :-

⇒ When the System Matrix has distinct eigen values, normal diagonalization is possible. In this the System Matrix can be diagonalized and the state model is converted to Canonical form → [When Repeated Eigen values obtained]

Consider the state model as below,

$$\dot{x} = AX + BU$$

$$y = CX + DU$$

The State Model can be converted to Canonical form by a transformation $X = MZ$.

Where, M = Modal matrix, Z = State variable Vector.

$\therefore \dot{X} = AX + BU$

$\Rightarrow M\dot{Z} = AMZ + BU$

$\Rightarrow \dot{Z} = M^{-1}AMZ + M^{-1}BU$

$\therefore \boxed{\dot{Z} = \Lambda Z + \bar{B}U}$ $\quad \left[\because \Lambda = M^{-1}AM,\ \bar{B} = M^{-1}B\right]$

Similarly $Y = CX$ $\quad [\because$ generally $D = 0]$

$Y = CMZ$

$\therefore \boxed{Y = \bar{C}Z}$ $\quad [\because \bar{C} = CM]$

Now, the Necessary and Sufficient Condition for Complete Controllability is that the Matrix $\bar{B}$ must have no Rows with all Zeros.

$\Rightarrow$ When the System Matrix has Repeated Eigen Values.

In this case, the System Matrix cannot be diagonalized but can converted to Jordan Canonical form.

Consider State Model $\dot{X} = AX + BU$

$Y = CX + DU$

The State Model can be transformed to Jordan Canonical form by a transformation $X = MZ$

∴ The transformed Model is given as similar to above case.

$$\dot{z} = \bar{J}z + \bar{B}u$$

$$y = \bar{C}z + Du$$

where, $\bar{J} = \bar{M}^{-1}AM$

$\bar{B} = \bar{M}^{-1}B$

$\bar{C} = CM$.

* Kalman's Method for Testing Controllability :-

Consider a state Equation $\dot{x} = Ax + Bu$, For this System, a Composite matrix Q_c can be formed such that,

$$\boxed{Q_c = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}}$$

where, n = order of the System (or) no. state variables.

In this case, the System is Completely Controllable, if the rank of the Composite matrix is "n".

The Rank of matrix Q_c is n, if the determinant of Composite Matrix Q_c is non zero, i.e, $|Q_c| \neq 0$.

Gilbert (selection)
Kalman (applicable to all systems)
Pontryagin
time consum

* Observability :-

Definition :-

A System is said to be Completely observable if every state $x(t)$ can be completely identified by measurements of output $y(t)$ over a finite time interval.

i, Gilbert's Method :-

Consider the state model, $\dot{x} = AX + BU$

$y = CX + DU.$

The state model can be transformed to canonical (or) Jordan canonical form by a transformation $X = MZ$.

Now the transformed state model is,

$\dot{z} = \Lambda Z + \bar{B}U$

$y = \bar{C}Z + DU$

(or)

$\dot{z} = JZ + \bar{B}U$

$y = \bar{C}Z + DU$

Where $\Lambda = M^{-1}AM$, if Eigen values are distinct

$J = M^{-1}AM$, if Eigen values are Repeated

$\bar{B} = M^{-1}B$ and $\bar{C} = CM.$

Now the necessary and sufficient condition for Complete observability is that none of the columns of the matrix $\bar{C}$ is zero.

ii) Kalman's Test for Observability :-

Consider a state Model $\dot{x} = AX + BU$

$Y = CX + DU$.

For this state model, a Composite Matrix Q_0 can be formed such that,

$$Q_0 = \begin{bmatrix} C^T & A^TC^T & (A^T)^2C^T & \ldots & (A^T)^{n-1}C^T \end{bmatrix}$$

where, n = Order of System (or) no. of state variables.

In this case, the System is Completely observable if the determinant of Q_0 is non-zero i.e, $|Q_0| \neq 0$.

* Problems :-

(1). A linear time invariant System is described by state model, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u$ & $Y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Determine whether Completely Controllable and Observable & no

Sol: i, Gilbert's Method :-

$$|\lambda I - A| = \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \right|$$

$$= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix}$$

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$$= \lambda\left[\lambda(\lambda+6)+11\right]+1\times 6 = 0$$

$$\lambda^3+6\lambda^2+11\lambda+6 = 0$$

$\therefore \lambda = -1, -2, -3$ (distinct eigen values)

$$[\lambda I - A] = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 6 & 11 & 5 \end{bmatrix}$$

Now, $C_{11} = 6, \quad C_{12} = -6, \quad C_{13} = 6$ $\quad \therefore X_1 = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ \\ \end{bmatrix}$

Similarly,

$$[\lambda_2 I - A] = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ 6 & 11 & 4 \end{bmatrix}$$

Now, $C_{11} = 3 \quad C_{12} = -6, \quad C_{13} = 12$ $\quad \therefore X_2 = \begin{bmatrix} 3 \\ -6 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

Now, $[\lambda_3 I - A] = \begin{bmatrix} -3 & -1 & 0 \\ 0 & -3 & -1 \\ 6 & 11 & -3 \end{bmatrix}$

$\therefore C_{11} = 2, \quad C_{12} = -6, \quad C_{13} = 18,$ $\quad \therefore X_3 = \begin{bmatrix} 2 \\ -6 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$

$\therefore$ Modal Matrix $M = [X_1 \; X_2 \; X_3]$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$1[-18+12]-1[-9+3]+1[-4+2] = -6+6-2 = -2$

$$\therefore M^{-1} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} -6 \end{bmatrix}$$

(1)

$$\Lambda = \bar{M}^{-1} A M$$

$$= \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

$$\bar{B} = \bar{M}^{-1} B$$

$$= \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\bar{C} = CM$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Now, It is observed that all the Elements of $\bar{B}$ Matrix are not all zeros. Hence the System is Completely Controllable

Similarly,

It is observed that the Elements of Columns of $\bar{C}$ Matrix are not all zeros. ∴ System is Completely Observable

Alternate Method :-

Kalmans test for Controllability :-

$$Q_C = [B \quad AB \quad A^2B]$$

given, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ 36 & 60 & 25 \end{bmatrix}$$

Now, $AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -12 \end{bmatrix}$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ 36 & 60 & 25 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \\ 50 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 25 \end{bmatrix}$$

$$\therefore Q_C = \left| \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -6 \\ 2 & -12 & 25 \end{bmatrix} \right|$$

$$= 1\,(0-4) = -4 \neq 0.$$

$\therefore$ System is Completely Controllable

Similarly, Kalman's test for observability :-

$$Q_o = \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T \end{bmatrix}$$

given, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$; $A^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$

$$(A^T)^2 = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -6 & 36 \\ 0 & -11 & 60 \\ 1 & -6 & 25 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$\because$ given $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$(A^T)^2 C^T = \begin{bmatrix} 0 & -6 & 36 \\ 0 & -11 & 60 \\ 1 & -6 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore Q_o = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1\,(1-0) - 0 + 0$$

$$= 1 \neq 0.$$

$\therefore |Q_o| \neq 0$ then the system is said to be Completely Observable

(2) Convert the given matrix into canonical form.

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

i) To find eigen values.

$$|[\lambda I - A]| = 0 \Rightarrow \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} \right| = 0$$

$$= \begin{vmatrix} \lambda - 4 & -1 & 2 \\ -1 & \lambda & -2 \\ -1 & 1 & \lambda - 3 \end{vmatrix}$$

$$= \lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$$

$\lambda = 1, 3, 3$ (Repeated values) $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 3$.

$$[\lambda_1 I - A] = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix}$$

~~$C_{11} = 0, \quad C_{12} = 0, \quad C_{13} = 0 \quad ; \quad \therefore X_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$~~

It gives null solution.

let, C_{21}, C_{22}, C_{23} be co-factors of $[\lambda_1 I - A]$ along 2nd row.

$C_{21} = -(0) = 0$ | $C_{22} = +(8) = 8$ | $C_{23} = -(-4) = 4$

$$\therefore X_1 = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$$

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$$[\lambda_2 I - A] = \begin{bmatrix} \lambda_2 - 4 & -1 & 2 \\ -1 & \lambda_2 & -2 \\ -1 & 1 & \lambda_2 - 3 \end{bmatrix}$$

$$C_{11} = + \begin{vmatrix} \lambda_2 & -2 \\ 1 & \lambda_2 - 3 \end{vmatrix} = \lambda_2(\lambda_2 - 3) + 2 = \lambda_2^2 - 3\lambda_2 + 2$$

$$C_{12} = - \begin{vmatrix} -1 & -2 \\ -1 & \lambda_2 - 3 \end{vmatrix} = (-1)\left[-1(\lambda_2 - 3) - 2\right] = \lambda_2 - 1$$

$$C_{13} = + \begin{vmatrix} -1 & \lambda_2 \\ -1 & 1 \end{vmatrix} = -1 + \lambda_2 = \lambda_2 - 1.$$

$$X_2 = \begin{bmatrix} \lambda_2^2 - 3\lambda_2 + 2 \\ \lambda_2 - 1 \\ \lambda_2 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \qquad (\because \lambda_2 = 3)$$

Similarly, eigen vector X_3 is given by,

$$X_3 = \begin{bmatrix} \frac{d}{d\lambda_2} C_{11} \\ \frac{d}{d\lambda_2} C_{12} \\ \frac{d}{d\lambda_2} C_{13} \end{bmatrix} = \begin{bmatrix} \frac{d}{d\lambda_2}(\lambda_2^2 - 3\lambda_2 + 2) \\ \frac{d}{d\lambda_2}(\lambda_2 - 1) \\ \frac{d}{d\lambda_2}(\lambda_2 - 1) \end{bmatrix} = \begin{bmatrix} 2\lambda_2 - 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore M = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \Rightarrow M^{-1} = \begin{bmatrix} 0 & 0.25 & -0.25 \\ -0.25 & -0.75 & 1.5 \\ 0.5 & 0.5 & -1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 0 & 1 & 1 \\ 0.5 & 1.5 & 3 \\ 0.5 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0.142 & -0.142 \\ 0.5 & -0.107 & 0.857 \\ 0.5 & 0.071 & -0.571 \end{bmatrix}$$

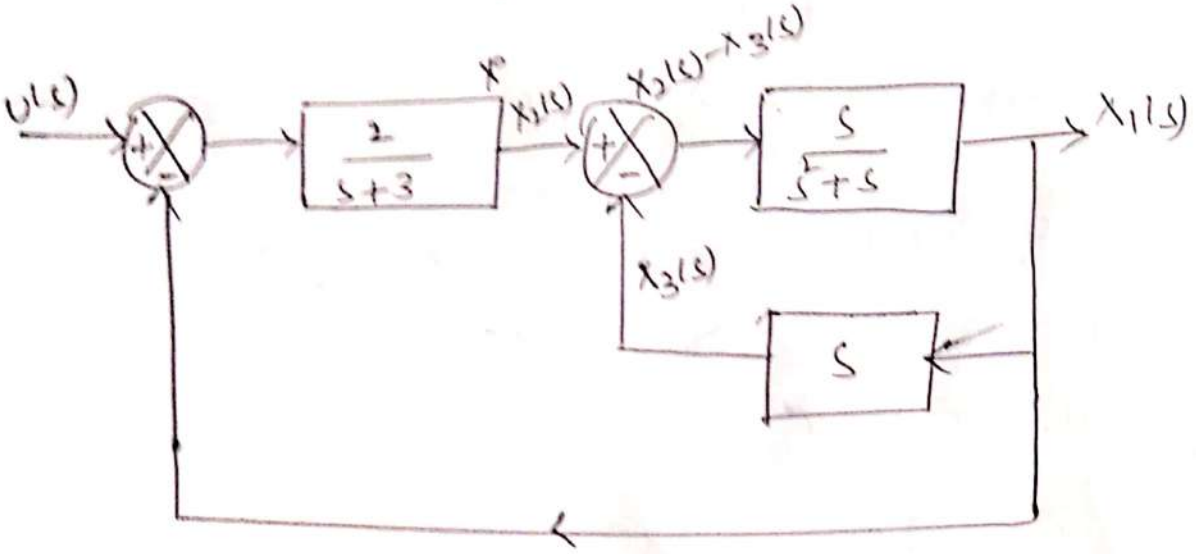
$$\therefore M^{-1}AM = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 \\ 8 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \cdots$$

M^{-1}

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$ → Jordan Blocks.

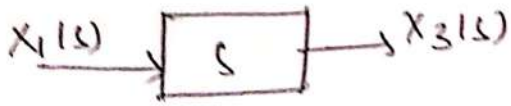
$$\therefore J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$ Canonical form of given Matrix 'A'.

(Pb) For the given system write the equation for, $\dot{x}_1, \dot{x}_2, \dot{x}_3$ also find whether the system is completly controllable (or) observable (or) not ?.



Sol:

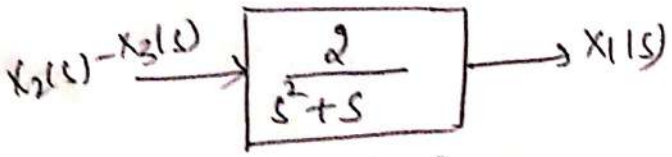
From the given figure,



$\Rightarrow X_3(s) = sX_1(s)$

Applying Inverse laplace transform

$$x_3 = \dot{x}_1 \quad \text{---} (1)$$



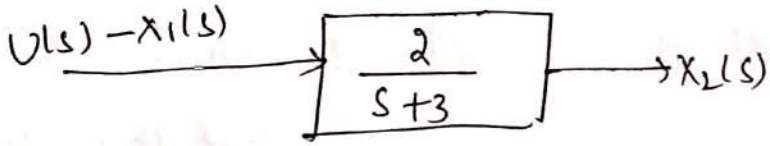
$$\left[X_2(s) - X_3(s)\right]\left[\frac{2}{s^2+5}\right] = X_1(s)$$

$$2X_2(s) - 2X_3(s) = s^2X_1(s) + 5X_1(s)$$

Apply Inverse laplace transform

$$2x_2 - 2x_3 = \ddot{x}_1 + \dot{x}_1 \quad \text{---} (2)$$

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$$U(s) - X_1(s)\left[\frac{2}{s+3}\right] = X_2(s)$$

$$\left[U(s) - X_1(s)\right]\frac{2}{s+3} = X_2(s)$$

$$2U(s) - 2X_1(s) = sX_2(s) + 3X_2(s)$$

Applying Inverse laplace transform,

$$2u - 2x_1 = \dot{x}_2 + 3x_2$$

$$\dot{x}_2 = -2x_1 - 3x_2 + 2u \quad \text{—③}$$

(∵ from eqn ①)

from eqn ②

$$2x_2 - 2x_3 = \ddot{x}_1 + \dot{x}_1$$

$$2x_2 - 2x_3 = \dot{x}_3 + x_3$$

$$\dot{x}_3 = 2x_2 - 3x_3 \quad \text{—④}$$

State Equation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ -2 & -3 & 0\\ 0 & 2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 2\\ 0\end{bmatrix}u$$

Output Equation:

$$Y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

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Gilbert's Method:

Characteristic Equation $|\lambda I - A| = 0$

$$|\lambda I - A| = 0$$

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} = 0$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda & 0 & -1 \\ 2 & \lambda+3 & 0 \\ 0 & -2 & \lambda+3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda[(\lambda+3)(\lambda+3) - (-2)] - 0[2(\lambda+3) - 0] - 1[-4 - (\lambda+3)0] =$$

$$\Rightarrow \lambda[(\lambda+3)^2 + 4] = 0$$

$$\lambda[\lambda^2 + 9 + 6\lambda) + 4] = 0 \Rightarrow \lambda^3 + 6\lambda^2 + 9\lambda + 4 = 0$$

$$\lambda = -4, -1, -1.$$

$\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -4.$

$$[\lambda I - A] = \begin{bmatrix} \lambda_1 & 0 & -1 \\ 2 & \lambda_1+2 & 0 \\ 0 & -2 & \lambda_1+3 \end{bmatrix}$$

$C_{11} = (\lambda_1+2)(\lambda_1+3) = \lambda_1^2 + 6\lambda_1 + 9$

$C_{12} = -(2\lambda_1 + 6)$

$C_{13} = -4$

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$$x_1 = \begin{bmatrix} \lambda_1^2+6\lambda_1+9 \\ -(2\lambda_1+6) \\ -4 \end{bmatrix} \Rightarrow \begin{bmatrix} (-1)^2+6(-1)+9 \\ -[2(-1)+6] \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -4 \end{bmatrix}$$

for $\lambda = \lambda_2$

$$x_2 = \frac{d}{d\lambda}\begin{bmatrix} \lambda^2+6\lambda+9 \\ -(2\lambda+6) \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2\lambda+6 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2+6 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$$

for $\lambda_3 = -4$

$$(\lambda I - A) = \begin{bmatrix} -4 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$$

$c_{11} = (-1)(-1) = 1$

$c_{12} = 2(-1) - 0 = +2$

$c_{13} = -4 = -4$

$$x_3 = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

Now the modal matrix $M = [x_1 \; x_2 \; x_3]$

$$M = \begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -0.111 & -0.22 & -0.138 \\ 0.333 & 0.166 & 0.166 \\ 0.111 & 0.222 & -0.111 \end{bmatrix}$$

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$J = M^{-1} A M$

$$J = \begin{bmatrix} -0.111 & -0.222 & -0.138 \\ 0.33 & 0.166 & 0.166 \\ 0.111 & 0.222 & -0.111 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$ Jordon's Block.

$\bar{B} = M^{-1} B$

$$= \begin{bmatrix} -0.111 & -0.222 & -0.138 \\ 0.333 & 0.166 & 0.166 \\ 0.111 & 0.222 & -0.111 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.444 \\ 0.232 \\ 0.444 \end{bmatrix}$$

$\bar{C} = C M$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 4 & 4 & 1 \end{bmatrix}$$

∴ No Row in $\bar{B}$ and no column in $\bar{C}$ are equal to zero.

So, the given system is said to be completely controllable and completely observable.

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Kalman's method:

Controllable

$$Q_c = [B \quad AB \quad A^2B]$$

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$$

$$A^2 = A \times A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -3 \\ 6 & 9 & -2 \\ -4 & -12 & 9 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 2 & -3 \\ 6 & 9 & -2 \\ -4 & -12 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 18 \\ -24 \end{bmatrix}$$

$$|Q_c| = \begin{vmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & 24 \end{vmatrix} = 32 \neq 0.$$

observable

$$Q_o = [C^T \quad A^T C^T \quad (A^T)^2 C^T]$$

$$A^T C^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(A^T)^2 C^T = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$$Q_o = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{vmatrix} \Rightarrow Q_o = -2 \neq 0.$$

Here $|Q_c| \neq 0$ and $|Q_o| \neq 0$, so the system is said to controllability and observability.

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