DEPARTMENT OF HUMANITIES AND SCIENCES

B.Tech I Year I Semester

LINEAR ALGEBRA AND CALCULUS

Subject Code: 23HBS9901

Regulation: HM23

ANNAMACHARYA INSTITUTE OF TECHNOLOGY AND SCIENCES

(Autonomous)

(Affiliated to J.N.T.U.A, Anantapur, Approved by A.I.C.T.E, New Delhi)

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Accredited by NAAC with 'A' Grade, Bangalore.

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LINEAR ALGEBRA & CALCULUS

(Common to All Branches of Engineering)

Course Objectives:
To equip the students with standard concepts and tools at an intermediate to advanced level mathematics to develop the confidence and ability among the students to handle various real-world problems and their applications.

Course Outcomes: At the end of the course, the student will be able to

CO1: Develop and use of matrix algebra techniques that are needed by engineers for practicalapplications.
CO2: Utilize mean value theorems to real life problems.
CO3: Familiarize with functions of several variables which is useful in optimization.CO4: Learn important tools of calculus in higher dimensions.
CO5: Familiarize with double and triple integrals of functions of several variables in two dimensions using Cartesian and polar coordinates and in three dimensions using cylindricaland spherical coordinates.

UNIT I Matrices

Rank of a matrix by echelon form, normal form. Solving system of Homogeneous and Non-Homogeneous equations. Solutions of simultaneous linear equations by Gauss elimination method and Gauss-Jordan method. Iterative Methods: Jacobi's Iteration method and Gauss Seidel Iteration Method.

UNIT II Linear Transformation and Orthogonal Transformation

Eigenvalues, Eigenvectors and their properties, Diagonalization of a matrix, Cayley- Hamilton Theorem (without proof), finding inverse and powers of a matrix by Cayley- Hamilton Theorem, Quadratic forms and Nature of the Quadratic Forms, Reduction of Quadratic formto canonicalforms by Orthogonal Transformation.

UNIT III Calculus

Mean Value Theorems: Rolle's Theorem, Lagrange's mean value theorem with their geometrical interpretation, Cauchy's mean value theorem, Taylor's and Maclaurin's theorems with remainders (without proof), Problems and applications on the above theorems.

UNIT IV Partial differentiation and Applications (Multi variable calculus)

Partial derivatives, total derivatives, chain rule, change of variables, Taylor's and Maclaurin's series expansion of functions of two variables. Jacobian, maxima and minima of functionsof two variables, method of Lagrange multipliers.

UNIT V Multiple Integrals (Multi variable Calculus)

Double integrals, triple integrals, change of order of integration, change of variables to polar, cylindrical and spherical coordinates. Finding areas (by double integrals) and volumes (by double integrals and triple integrals).

Textbooks:

1. Higher Engineering Mathematics, B. S. Grewal, Khanna Publishers, 2017, 44th Edition
2. Advanced Engineering Mathematics, Erwin Kreyszig, John Wiley & Sons, 2018,10th Edition.

Reference Books:

1. Thomas Calculus, George B. Thomas, Maurice D. Weir and Joel Hass, PearsonPublishers, 2018, 14th Edition.
2. Advanced Engineering Mathematics, R. K. Jain and S. R. K. Iyengar, AlphaScienceInternational Ltd., 2021 5th Edition(9th reprint).
3. Advanced Modern Engineering Mathematics, Glyn James, Pearson publishers, 2018,5th Edition.
4. Advanced Engineering Mathematics, Micheael Greenberg, , Pearson publishers,9th edition
5. Higher Engineering Mathematics, H. K Das, Er. Rajnish Verma, S. ChandPublications,2014, Third Edition (Reprint 2021)

Subject: Linear Algebra & Calculus

Unit-I: Matrices:

Syllabus: Rank of a matrix by echelon form, normal form. Solving system of homogeneous and non-homogeneous equations linear equations.

Matrix: A set of mn numbers (Real or Complex) can be arranged in the form of m rows and n columns (each column containing m elements) is called as Matrix. The Numbers of the matrix elements. Matrices are denoted by Capital letters A, B, ..etc.

Order of the Matrix: The number of rows and columns represents the order of the matrix. It is denoted by $m\times n$, where m is number of rows and n is number of columns.

Square Matrix: A matrix in which the number of rows and number of columns are equal is said to be square matrix. It is of order $n\times n$ or a square matrix n.

Ex: $\begin{bmatrix} 1 & 2 & -3 \\ 1 & 5 & 4 \\ -1 & 4 & 3 \end{bmatrix}_{3\times 3}$ is an upper triangular matrix of order 3.

Rectangular matrix: A matrix in which the number of rows and number of columns are not equal is said to be rectangular matrix. It is of order $m\times n$.

Ex: $\begin{bmatrix} 2 & 3 & 4 \\ 4 & -5 & 6 \end{bmatrix}_{2\times 3}$ is a rectangular matrix of order 2×3.

Row Matrix: A matrix is said to be row matrix, if it contains only one row. It is denoted by It is of order $1\times n$.

Ex: $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}_{1\times 3}$ is a row matrix.

Column Matrix: A matrix is said to be column matrix, if it contains only one column. It is denoted by It is of order $n\times 1$.

Ex: $\begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix}_{3\times 1}$ is a column matrix.

Diagonal Matrix: A square matrix $A_{n\times n}$ is said to be diagonal matrix if $a_{ij} = 0, \forall i \neq j$

(Or)

A Square matrix is said to be diagonal matrix, if all the elements except principal diagonal elements are zero.

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$ a diagonal matrix.

- The elements on the diagonal are known as principle diagonal elements.
- The diagonal matrix is represented by A=Diag$\left(a_{11}, a_{22}, ..., a_{nn}\right)$.

Scalar Matrix: A Square matrix $A_{n\times n}$ is said to be a Scalar matrix if $\begin{cases} a_{ij} = 0, \forall i \neq j \\ a_{ij} = k, \forall i = j \end{cases}$

(Or)

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A diagonal matrix is said to be a Scalar matrix, if all the elements of the principle diagonal are equal. i.e. $a_{ij} = k, \forall i = j$

Ex: $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ a scalar matrices.

□□Trace of a Scalar matrix is nk

Unit Matrix (or) Identity Matrix: A Square matrix A of order is said to be a Unit (or) Identity matrix if $A = \begin{cases} a_{ij} = 0, \forall i \neq j \\ a_{ij} = 1, \forall i = j \end{cases}$

(or)

A Scalar matrix is said to be a Unit matrix if the scalar $k = 1$

Ex: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are unit matrices.

Zero or null matrix: A matrix in which all the elements are zero is called a zero or null matrix. It is of order $m \times n$ and is denoted by $O_{m\times n}$

EX: $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are zero matrices

Trace of a Matrix: Suppose is a square matrix, then the trace of is defined as the sum of its diagonal elements. It is denoted by TrA

$$i.e. Tr(A) = a_{11} + a_{22} + ... + a_{nn}$$

Properties:

(1) $Tr(A+B) = TrA + TrB.$

(2) $Tr(KA) = KTrA.$

(3) $Tr(AB) = Tr(BA)$

Transpose of a Matrix: The transpose of the given matrix is obtained by interchanging rows and columns. Then it is denoted by A^T or A^1.

Ex: If $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 5 & 4 \\ -1 & 4 & 3 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 5 & 4 \\ -3 & 4 & 3 \end{bmatrix}$

Properties: If A^T and B^T are the transpose of A and B respectively, then

(1) If A is of Order $m \times n$ then A^T is of order $n \times m.$

(2) If A is a square matrix, then $TrA = TrA^T$.

(3) $\left((A)^T\right)^T = A$

(4) $(A \pm B)^T = A^T \pm B^T$ Where A& B are of same order

(5) $(AB)^T = B^T A^T$, where A& B being conformable multiplication

(6) $(kA)^T = k A^T$, k being a constant

(7) $(I)^T = I$

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Triangular Matrix: A square matrix in which each element either above or below the principal diagonal is zero is called a Triangular Matrix

Upper Triangular Matrix: A square matrix in which all the elements below the principal diagonal are zero is called a Upper triangular matrix

Thus A square matrix is said to be an Upper Triangular matrix, if $a_{ij} = 0, \forall\, i > j$

Ex: $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ is an upper triangular matrix

Lower Triangular Matrix: A square matrix in which all the elements above the principal diagonal are zero is called a lower triangular matrix

Thus A matrix is said to be an Upper Triangular matrix, if $a_{ij} = 0, \forall\, i < j$

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 5 & 8 & 3 \end{bmatrix}$ is an lower triangular matrix

Singular Matrix: A square matrix is said to be singular if $|A| = 0$.

Ex: $\begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ are singular matrices

Non-singular Matrix: A square matrix is said to be non-singular if $|A| \neq 0$.

Ex: $\begin{bmatrix} 1 & -1 & 1 \\ -2 & 2 & 1 \\ 2 & -1 & 2 \end{bmatrix}$ is a non-singular matrix

Inverse of a Matrix: Let A square matrix of order n, then there exist another matrix B such that AB=BA=I is said to be inverse of A, and it is denoted by A^{-1} . Thus $B = A^{-1}$.

Note: (i) A square matrix A is said to be Invertible $\Leftrightarrow |A| \neq 0$ i.e. A is non singular

(ii) $A^{-1} = \dfrac{adj\, A}{|A|}$

Symmetric Matrix: A Square matrix A is said to be symmetric matrix if $a_{ij} = a_{ji}, \forall i, j$

i.e. $A^T = A$

Ex: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 6 \end{bmatrix}$ are symmetric matrices of order 3.

- Identity matrix is a symmetric matrix.
- Zero square matrix is symmetric. i.e. .
- Number of Independent elements in a symmetric matrix are , is order.

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Anti-Symmetric Matrix: A Square matrix A is said to be symmetric matrix if $a_{ij} = -a_{ji}, \forall i, j$ i.e. $A^T = -A$

Note: The diagonal elements of a skew-symmetric matrix are zero

Ex: $\begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & -5 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix}$ are skew-symmetric matrices

Minor of an Element: Let $A = (a_{ij})_{n\times n}$ be a matrix, then minor of an element a_{ij} is denoted by M_{ij} and is defined as the determinant of the sub-matrix obtained by Omitting i^{th} row and i^{th} column of the matrix.

Cofactor of an element: Let $A = (a_{ij})_{n\times m}$ be a matrix, then cofactor of an element a_{ij} is denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

Cofactor Matrix: If we find the cofactor of an element for every element in the matrix, then the resultant matrix is called as Cofactor Matrix.

Adjoint of a matrix: If A is square matrix of order n, then the transpose of the cofactor matrix of A is said to be the adjoint of a matrix A. It is denoted by **adj A.**

Thus, if A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

the cofactor matrix of A $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$\therefore \text{ adj A} = [\text{The cofactor matrix of A}]^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Note: If A is a square matrix of order n, then $\text{A}.(\text{adj A}) = (\text{adj A}).\text{A} = |\text{A}|.\text{I}$

Where I is a unit matrix of order n

Minor of order r: The determinant of a square sub matrix of the given matrix is called its minor. If the order of the square sub matrix is r, then the corresponding minor is said to be a minor of order r.

Rank of a matrix: A matrix A is said to be rank r, if

(i) It has atleast one minor of order r non zero.
(ii) Every minor of order higher than r vanishes.

Then the rank of A is denoted by $\rho(A)$ or $r(\text{A})$

Properties:-

i. The rank of a matrix is always unique.
ii. If A is a non-zero matrix, then $\rho(A) \geq 1$.
iii. The rank of a null matrix is zero i.e, $\rho(\text{O}) = 0$
iv. If A is singular matrix of order n, then $\rho(A) < n$

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v. If A is nonsingular matrix of order n, then $\rho(A) = n$

vi. If I_n is the unit matrix of order n, then $\rho(A) = n$

vii. If A is a matrix of order $m \times n$, then $\rho(A) \le \min(m, n)$

viii. The rank of the matrix is same as that of its transpose i.e., $\rho(A) = \rho(A')$

Problem 1: Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{vmatrix} = 1(0-2) - 2(3-4) + 1(-1-0) = -2 + 2 - 1 = -1 \neq 0$

i.e. minor of order 3 is non-zero.

$\therefore$ The rank of A is 3 $i.e. \rho(A) = 3$

Problem 2: Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$

Applying $C_1 \leftrightarrow C_4$, we get

$$A \sim \begin{bmatrix} -1 & 3 & 4 & 2 \\ -1 & 2 & 0 & 5 \\ -1 & 5 & 12 & -4 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$, we get

$$A \sim \begin{bmatrix} -1 & 3 & 4 & 2 \\ 0 & -1 & -4 & 3 \\ 0 & 2 & 8 & -6 \end{bmatrix}$$

Applying $R_3 \to R_3 + 2R_2$, we get

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$$A \sim \begin{bmatrix} -1 & 3 & 4 & 2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We note that all the third minors are zero but the second order minor

$$= \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} = 1 \neq 0$$

Hence rank of A is 2, $i.e. \rho(A) = 2$

Reduction of matrix A to Echelon Form:

The Echelon form of matrix A is an equivalent matrix, obtained by a finite sequence of elementary operations on A which has the following properties.

(i). The zeros, if any, are below a nonzero row.

(ii). The first non-zero element in each non-zero row is one.

(iii). The number of zeros before the first nonzero entry in a row less than the no number such zeros in the next row immediately below it.

Note (1): Condition (ii) is optimal (not compulsory)

Note 2: The rank of A is equal to the number of nonzero rows in its echelon form.

Problems:

Problem 1: Reduce the matrix A = $\begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$ to Echelon form and hence find its rank

[**Jntu(A) June, 2009**].

Solution: Given A = $\begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$

Applying $R_1 \to -R_1$, we get

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$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$, we get

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 5 & 1 & 16 \\ 0 & 8 & 4 & 31 \end{bmatrix}$$

Applying $R_3 \to 5R_3 - 8R_2$, we get

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 5 & 1 & 16 \\ 0 & 0 & 12 & 27 \end{bmatrix}$$

Thus the matrix is in the Echelon form. The number of non zero rows is 3.

Hence the rank of A = 3.

Problem 2: Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ to Echelon form and hence find its rank

[**Jntu(A) June, 2018**].

Solution: Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_2, R_4 \to R_4 - R_3$, we get

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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This is in the Echelon form. The number of non zero rows is 2.

Hence the rank of A = 2.

<u>Normal form:</u> Every $m \times n$ matrix of rank r can be reduced by a finite number of elementary transformations to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ *or* $\begin{bmatrix} I_r \end{bmatrix}$ *or* $\begin{bmatrix} I_r & 0 \end{bmatrix}$ *or* $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$, where I_r is the unit matrix of order r and 0 is the null matrix

The reduced form is known as normal form or canonical form

Problem 1: Reduce the matrix $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ to the normal form and hence find its rank.

Solution: Let the matrix be $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

Applying $C_1 \leftrightarrow C_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_1$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

Applying $R_2 \to \frac{R_2}{2}$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

Applying $C_3 \to C_3 - 2C_1, C_4 \to C_4 + 2C_1$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

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$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_2 \to \frac{C_2}{2}$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_3 \to C_3 - C_2, C_4 \to C_4 - 3C_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$

Hence rank of A = 2 i.e. $\rho(A) = 2$.

Problem 2: Reduce the matrix A to the normal form and hence find its rank. Where

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

Solution: Given $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

Applying $R_3 \to R_3 - R_1, R_4 \to R_4 - R_1$, , we get

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix}$$

Applying $C_2 \to C_2 - 2C_1, C_3 \to C_3 - 3C_1, C_4 \to C_4 - 4C_1$, , we get

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$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix}$$

Applying $R_3 \to 3R_3 - 2R_2, R_4 \to R_4 - 2R_3,$, we get

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_1 \leftrightarrow C_2$, we get

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_3 \to C_3 - 4C_2, C_4 \to C_4 - 3C_2$, we get

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_3 \leftrightarrow C_4$, we get

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_1 \to \frac{C_1}{2}, C_3 \to \frac{C_3}{-3}$, we get

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$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} I_3 & O_{3\times1} \\ O_{1\times3} & O_{1\times1} \end{bmatrix}$$

Hence the rank of A = 3.

Normal form or canonical form:

(i) A matrix of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ *or* $\begin{bmatrix} I_r \end{bmatrix}$ *or* $\begin{bmatrix} I_r & 0 \end{bmatrix}$ *or* $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$, where I_r is the unit matrix of order r and 0 is the null matrix is called the normal form or canonical form.

(ii) Every $m\times n$ matrix can be reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a series of elementary Transformations, where r is the rank of the matrix.

Problem 1: Determine the rank of the matrix A= $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ by reduced it to the normal form.

Solution: Given $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$, we get

$$A \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

Applying $C_2 \to C_2 + C_1, C_3 \to C_3 + C_1$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, we get

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$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Applying $C_3 \to C_3 - C_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Applying $C_2 \to \frac{C_2}{2}$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is of the form $\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$

Rank of A = 2.

Problem 2: Find the rank of the matrix A by reduced it to the normal form where $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 3R_1$ we get

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix}$$

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - C_1$ we get

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$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2, R_4 \to R_4 + 7R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & -30 \end{bmatrix}$$

Applying $C_3 \to C_3 - 2C_2, C_4 \to C_4 + 5C_2$ we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & -30 \end{bmatrix}$$

Applying $R_4 \to R_4 - 6R_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -18 \end{bmatrix}$$

Applying $C_4 \to C_4 + 2C_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -18 \end{bmatrix}$$

Applying $C_4 \to \frac{C_4}{-18}$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

Rank of A = 4.

Dr TSR, ATS-KADAPA

Problem 3: Find the rank of the matrix A by reduced it to the normal form where $\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

Applying $R_1 \to R_1 \leftrightarrow R_3$, we get

$$A \sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 3 & -2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - 3R_1$, we get

$$A \sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 + 2C_1, C_3 \to C_3 + 3C_1, C_4 \to C_4 - 2C_1$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Applying $R_2 \leftrightarrow R_4$ we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - 4R_2, R_4 \to R_4 - 2R_2$, we get

Dr TSR, ATS-KADAPA

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

Applying $C_3 \to C_3 - 2C_2, C_4 \to C_4 - C_2$ we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

Applying $R_4 \to R_4 + 2R_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & -23 \end{bmatrix}$$

Applying $C_4 \to C_4 + 11C_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -23 \end{bmatrix}$$

Applying $C_4 \to \frac{C_4}{-23}$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$A \sim I_4$, which is the canonical form of the matrix A

Rank of A = 4.

Normal form of the type PAQ:-

Every $m \times n$ matrix of rank r can be transformed to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$ by elementary transformations.

Dr TSR, ATS-KADAPA

Working rule:-

1. Write $A_{m\times n} = I_{m\times m} A I_{n\times n}$

2. Now apply row and column transformations on the LHS matrix A to transform its normal form carrying out every row transformation on the pre-factor $I_{m\times m}$ and every column transformation on the post factor $I_{n\times n}$ reduces to non-singular matrices P and Q such that $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$

Problem 1: Find the nonsingular matrices P and Q such that the normal form of A is PAQ where A= $\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$. Hence find its rank.

Solution: Since A is a matrix of 3×4

We write A= A = I_3AI_4

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 - 3C_1, C_3 \to C_3 - 6C_1, C_4 \to C_4 + C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Dr TSR, ATS KADAPA

Applying $C_3 \to C_3 + C_2, C_4 \to C_4 - 2C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$i.e. \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} = \text{PAQ, where } P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $A = \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$, therefore, rank of A is 2

Problem 2: Find the nonsingular matrices P and Q such that the normal form of A is PAQ where A= $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$. Hence find its rank.

Solution: Since A is a matrix of 3×4

We write A= $A = I_3 A I_4$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 - 2C_1, C_3 \to C_3 - 3C_1, C_4 \to C_4 + 2C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

Dr TSR, ATS-KADAPA

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \mathrm{A} \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_3 \to \frac{C_2}{-6}$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \mathrm{A} \begin{bmatrix} 1 & 1/3 & -3 & 2 \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_3 \to C_3 + 5C_2, C_4 \to C_4 - 7C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \mathrm{A} \begin{bmatrix} 1 & 1/3 & -4/3 & -1/3 \\ 0 & 1/6 & 5/6 & -7/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$i.e. \begin{bmatrix} \mathrm{I}_2 & O \\ O & O \end{bmatrix} = \mathrm{PAQ}, \text{ where } \mathrm{P} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \text{ and } \mathrm{Q} = \begin{bmatrix} 1 & 1/3 & -4/3 & -1/3 \\ 0 & 1/6 & 5/6 & -7/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $\mathrm{A} = \begin{bmatrix} \mathrm{I}_2 & O \\ O & O \end{bmatrix}$, therefore, rank of A is 2

Dr TSR, ATS-KADAPA

Linear system of equations:

Consider the system of 'm'linear equations in 'n'unknowns say as given below.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ...+ a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ...+ a_{2n}x_n = b_2 \\ . \quad\quad . \quad\quad\quad . \quad\quad . \\ . \quad\quad . \quad\quad\quad . \quad\quad . \\ a_{m1}x_1 + a_{m2}x_2 + ...+ a_{mn}x_n = b_m \end{array}\right\} \text{-----------------------} \quad (1)$$

The above equations can be written in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12}........a_{1n} \\ a_{21} & a_{22}........a_{2n} \\ . & . \quad\quad . \\ . & . \quad\quad . \\ a_{m1} & a_{m2}........a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix} \text{-------------------------} \quad (2)$$

$$\Rightarrow AX = B \quad \text{-------------------------} \quad (3)$$

where A is called coefficient matrix

and B is called constant matrix

The matrix $\left[A|B\right]$ is called Augumented matrix and is given by

$$\begin{bmatrix} a_{11} & a_{12}........a_{1n} & . & b_1 \\ a_{21} & a_{22}........a_{2n} & . & b_2 \\ . & . \quad\quad . & & . \\ . & . \quad\quad . & & . \\ a_{m1} & a_{m2}........a_{mn} & . & b_m \end{bmatrix}$$

The given system is said to be consistent, if the system equations posses one or more solutions. Otherwise the system is said to be inconsistent.

Gauss-elimination method:

This method is simple and general. It consists of two steps

Step 1: Reduction of the Augumented matrix to upper triangular or echelon form.

Step 2: Finding the values of the unknown variables by back substitution.

Problem 1: Solve the following system of equations by Gauss elimination method.

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$$x + y + z = 3;$$

$$3x - y + 3z = 16;$$

$$3x + y - z = -3;$$

Solution: Given system of system of equations can be written in the matrix form AX=B

With $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 16 \\ -3 \end{bmatrix}$

Consider the augumented matrix [A/B] is

$$[A/B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 16 \\ 3 & 1 & -1 & -3 \end{array}\right]$$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$ we get

$$[A/B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & 10 \\ 0 & -2 & -4 & -12 \end{array}\right]$$

Applying $R_3 \to 3R_3 - 2R_2$ we get

$$[A/B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & 10 \\ 0 & 0 & -14 & -56 \end{array}\right]$$

Applying $R_3 \to \dfrac{R_3}{-14}$ we get

$$[A/B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & 10 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

This is in the Echelon form.

This is equivalent to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 4 \end{bmatrix}$$

Which implies

Dr TSR, AITS-KADAPA

$$x+y+z=3;$$
$$-3y+z=10$$
$$z=4$$

By back substitution, we have

$$x=1, y=-2, \ z=4.$$

Problem 2: Solve the following system of equations

$$x+2y+3z=1$$
$$2x+3y+8z=2$$
$$x+y+z=3$$

Solution: Given system of system of equations can be written in the matrix form AX=B

$$\text{With} \quad A=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 8\\ 1 & 1 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, B=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

Consider the augumented matrix [A/B] is

$$[A/B]=\left[\begin{array}{ccc|c}1 & 2 & 3 & 1\\ 2 & 3 & 8 & 2\\ 1 & 1 & 1 & 3\end{array}\right]$$

Applying $R_2 \to R_2-2R_1, R_3 \to R_3-R_1$ we get

$$[A/B]\sim\left[\begin{array}{ccc|c}1 & 2 & 3 & 1\\ 0 & -1 & 2 & 0\\ 0 & -1 & -2 & 2\end{array}\right]$$

Applying $R_3 \to R_3-R_2$ we get

$$[A/B]\sim\left[\begin{array}{ccc|c}1 & 2 & 3 & 1\\ 0 & -1 & 2 & 0\\ 0 & 0 & -4 & 2\end{array}\right]$$

This is in the Echelon form.

This is equivalent to

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Which implies

$$x + 2y + 3z = 1;$$

$$-y + 2z = 0$$

$$-4z = 2$$

Solving these equations,

By back substitution, we have

$$\therefore \quad x = \frac{9}{2}, y = -1, \ z = \frac{-1}{2}.$$

Gauss-Jordan Elimination method:

Consider the system of m linear non-homogeneous equations with n unknowns.

Let the matrix form for the linear equations be AX=B.

To find the rank of A and [A/B], reduce the augmented matrix [A/B] to echelon form by elementary row operations then the matrix A automatically reduces to echelon form.

Note:

(i) The system is consistent and it has unique solution, If $\rho(A) = \rho(A/B) = n$, where n is the no. of unknowns.

(ii) The system is consistent and it has infinite solution, If $\rho(A) = \rho(A/B) \le n$.

Note: In this case, we have to give arbitrary values to $n-r$ variables and the remaining variables can be expressed in terms of these arbitrary variables.

(iii) The system is inconsistent and it has no solution, If $\rho(A) \ne \rho(A/B)$.

Problem 1: Discuss for what values of the simultaneous equations

$$x + y + z = 6; \ \ x + 2y + 3z = 10; \ x + 2y + \lambda z = \mu;$$

have (i) no solution (ii) a unique Solution (iii) an infinite number of solutions

Problem 2: Find whether the following set of equations are consistent if so, solve them

$$x + y + 2z = 4, 2x - y + 3z = 9, 3x - y - z = 2$$

Solution: The given system of equations can be written in the matrix form as

Dr TSR, ATS-KADAPA

$$\begin{bmatrix}1 & 1 & 2\\ 2 & -1 & 3\\ 3 & -1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 9\\ 2\end{bmatrix}$$

i.e., AX=B

The augmented matrix $[A/B]$ is

$$[A/B]=\begin{bmatrix}1 & 1 & 2 & 4\\ 2 & -1 & 3 & 9\\ 3 & -1 & -1 & 2\end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$[A/B]\sim\begin{bmatrix}1 & 1 & 2 & 4\\ 0 & -3 & -1 & 1\\ 0 & -4 & -7 & -10\end{bmatrix}$$

Applying $R_3 \to 3R_3 - 4R_2$,

$$[A/B]\sim\begin{bmatrix}1 & 1 & 2 & 4\\ 0 & -3 & -1 & 1\\ 0 & 0 & -17 & -34\end{bmatrix}$$

Rank of A=3 and Rank of [A/B]=3

Rank of A= Rank of [A/B]=3= no of unknowns

The given system has a unique solution.

$$\begin{bmatrix}1 & 1 & 2\\ 0 & -3 & -1\\ 0 & 0 & -17\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 1\\ -34\end{bmatrix}$$

This is equivalent to $x + y + 2z = 4$

$$-3y - z = 1$$

$$-17z = -34$$

From (3), we have $z = 2$

Substituting $z = 2$ in (2) $-3y - 2 = 1 \Rightarrow \quad y = -1$

Substituting $y = -1,\ z = 2$ in (1) $x + -1 + 4 = 4 \Rightarrow x = 1$

$\therefore x = 1, y = -1, z = 2$ is the solution.

Dr TSR, ATS-KADAPA

Problem 2: Test for consistency in the set of equations and solve them if they are consistent $x+2y+2z=2, 3x-2y-z=5, 3x-5y+3z=-4, x+4y+6z=0$

Solution: The augmented matrix $[A/B]$ is given by

$$[A/B]=\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$ and $R_4 \to R_4 - R_1$, we get

$$[A/B]\sim\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{array}\right]$$

Applying $R_2 \to \frac{R_2}{-1}$, $R_3 \to \frac{R_3}{-1}$ and $R_4 \to \frac{R_4}{2}$, we get

$$[A/B]\sim\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 8 & 7 & 1 \\ 0 & 9 & 1 & 8 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

Applying $R_2 \leftrightarrow R_4$, we get

$$[A/B]\sim\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 9 & 1 & 8 \\ 0 & 8 & 7 & 1 \end{array}\right]$$

Applying $R_3 \to R_3 - 9R_2, R_4 \to R_4 - 8R_2$, we get

$$[A/B]\sim\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -17 & 17 \\ 0 & 0 & -9 & 9 \end{array}\right]$$

Applying $R_3 \to \frac{R_3}{-17}$ and $R_4 \to \frac{R_4}{-9}$, we get

Dr TSR ATS-KADAPA

$$[A/B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Applying $R_4 \to R_4 - R_3$, we get

$$[A/B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank of A= number of non-zero rows = 3

Rank of $[A/B]$= number of non-zero rows = 3 and n=number of variables=3

$\therefore\ \rho[\mathbf{A}] = \rho[A/B] = n = 3$. So the system is consistent and has unique solution.

We have $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

$x + 2y + 2z = 2$ .... (1)

$y + 2z = -1$ .... (2)

$z = -1$ .... (3)

Solving these equations, we get

$\therefore\ x = 2, y = 1, z = -1$ is the solution.

Linear system of Homogeneous equations:

Consider the system of 'n' Homogeneous linear equations *AX=O*

Here the coefficient matrix A and augmented matrix *[A/O]* are same

Therefore $\rho(A) = \rho(A/O)$

The system of equations is consistent always.

Let r be the rank of the matrix A

Nature of the solution:

i) If $r = n$, then the system of equations have only trivial solutions
ii) If $r < n$, then the system of equations have infinite no of non-trivial solutions

Dr TSR, ATS-KADAPA

iii) If no. eqations < no. of unknowns, then the system of equations have infinite no of non-trivial solutions

Trivial solution: Zero solution is called trivial solution

Non-Trivial solution: Non-Zero solution is called non-trivial solution

Note: For the system of equations *AX=O*

(i) A is singular $\Rightarrow$ X is non-trivial solution

(ii) A is non-singular $\Rightarrow$ X is trivial solution.

Problem 1: Solve the system of equations $x+y+w=0,\ \ y+z=0,\ \ x+y+z+w=0,$

$x+y+2z=0$

Solution: The system of equations in matrix form given by

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

Applying $R_3 \to R_3 - R_1, R_4 \to R_4 - R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

Applying $R_4 \to R_4 - 2R_3$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Rank of A = r = 4 and number of variables $n = 4$

Since $r = n$

There is no non trivial solution

$\therefore x = 0, y = 0, z = 0, w = 0$ is the solution.

Dr TSR, ATS-KADAPA

Problem 2: Solve the system of equations $x_1 + 2x_2 - 2x_4 = 0$, $2x_1 - x_2 - x_4 = 0$

$x_1 + 2x_3 - x_4 = 0$, $4x_1 - x_2 + 3x_3 - x_4 = 0$.

Solution: The system of equations in matrix form given by

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 4 & -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix $A = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 4 & -1 & 3 & -1 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - 4R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -5 & 0 & 3 \\ 0 & -2 & 2 & 1 \\ 0 & -9 & 3 & 7 \end{bmatrix}$$

Applying $R_3 \to 5R_3 - 2R_2, R_4 \to 5R_4 - 9R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -5 & 0 & 3 \\ 0 & 0 & 10 & -1 \\ 0 & 0 & 15 & 8 \end{bmatrix}$$

Applying $R_4 \to 2R_4 - 3R_3$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -5 & 0 & 3 \\ 0 & 0 & 10 & -1 \\ 0 & 0 & 0 & 19 \end{bmatrix}$$

Rank of A = r = 4 and number of variables n = 4

Since $r = n$, there is no non trivial solution

$\therefore x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$ is the solution.

Problem 3: Solve the system of equations $x + y - 3z + 2w = 0, 2x - y + 2z - 3w = 0,$ $3x - 2y + z - 4w = 0, -4x + y - 3z + w = 0.$

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Solution: The system of equations can be in matrix form as

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{i.e.AX=O}$$

Consider the coefficient matrix $A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 + 4R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & -5 & 10 & -10 \\ 0 & 5 & -15 & 9 \end{bmatrix}$$

Applying $R_3 \to \frac{R_3}{-5}$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 1 & -2 & 2 \\ 0 & 5 & -15 & 9 \end{bmatrix}$$

Applying $R_3 \leftrightarrow R_4$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 5 & -15 & 9 \end{bmatrix}$$

Applying $R_3 \to R_3 + 3R_2, R_4 \to R_4 - 5R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -5 & -1 \end{bmatrix}$$

Applying $R_4 \to 2R_4 + 5R_3$, we get

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$$A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -7 \end{bmatrix},$$

This is in the Echelon form

Rank of A = Number of non zero rows = 4

Since $r = n$, there is no non-zero solution.

$\therefore x = 0, y = 0, z = 0, w = 0$ is the solution.

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Sub: Linear Algebra & Calculus
Unit-II

Syllabus: Eigen values & Eigen vectors: Eigen values and Eigenvectors and their properties, Cayley Hamilton theorem (without proof), finding inverse and power of a matrix by Cayley-Hamilton theorem, diagonalisation of a matrix.

Consider the system of 'n' linear equations in 'n' unknowns

$$\left.\begin{array}{l} (a_{11}-\lambda)x_1 + a_{12}x_2 + ...+a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + ...+a_{2n}x_n = 0 \\ . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \\ a_{n1}x_1 + a_{n2}x_2 + ...+(a_{nn}-\lambda)x_n = 0 \end{array}\right\} \text{-----------------------} \qquad (1)$$

Then these equations can be written in the form of matrix as $(A-\lambda I)X = 0$, where is a λ parameter.

These equations will have a non-trivial solution iff the matrix $(A-\lambda I)$ is singular ie., $|A-\lambda I| = 0$. This equation is known as characteristic equation.

The roots of the characteristic equation are the characteristic roots or latent values or Eigen values.

If λ is a characteristic root of a matrix A, then a non zero vector X such that $AX = \lambda X$ ia called a characteristic vector or eigen vector of A corresponding to the characteristic root λ.

OR

Let A be a square matrix of order n. A nonzero vector X is said to be a characteristic vector or eigen vector of A, if there exists a scalar λ such that $AX = \lambda X$.

Note: Eigen vector must be a non zero vector

Problem 1: Find the Eigen values and the corresponding Eigen vectors of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Sol: If X is an Eigen vector of A corresponding to the Eigen value λ of A,

We have $(A-\lambda I)X = 0$

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i.e., $\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ --------------- (1)

The Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\text{i.e., } \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

On simplying,

$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$

$\Rightarrow (\lambda+2)(\lambda-3)(\lambda-6) = 0$

$\therefore$ The Eigen values of A are $\lambda = -2, 3, 6$.

To find Eigen Vectors:-

The eigen vectors $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A corresponding to the eigen values λ are given by $(A - \lambda I)X = 0$

$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ------ (1)

Case(i): For $\lambda = -2$ from (1), we get

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing the coefficient matrix to the echelon form

Applying $R_2 \to 3R_2 - R_1$, $R_3 \to R3 - R1$

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$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since r =1, n=3 (n – r = 3 – 1=2)

Hence we have

$$x_1 + 2x_2 - 3x_3 = 0$$
$$20x_2 = 0$$

For one variables, we have to give one arbitrary constant.

$x_2 = 0,$

We taking $x_3 = k \Rightarrow x_1 = -k$

For $\lambda = -2$, the corresponding eigen vector $\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Case(ii): $\lambda = 3$, the eigen vector X is given by

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing the coefficient matrix to the echelon form

Applying $R_2 \rightarrow 3R_2 - R_1$, $R_3 \rightarrow R3 - R1$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since r =1, n=3 (n – r = 3 – 1=2)

Hence we have

$$x_1 + 2x_2 - 3x_3 = 0$$
$$20x_2 = 0$$

For one variables, we have to give one arbitrary constant.

$x_2 = 0,$

We taking $x_3 = k \Rightarrow x_1 = -k$

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Hence the Eigen vectors of A corresponding to Eigen value $\lambda = -3$ are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Case(ii): $\lambda = 5$, the eigen vector from (1) is given by

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing the coefficient matrix to Echelon form, we get

Applying $R_1 \leftrightarrow R_3$, and $R_1 \rightarrow$-R1

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 + 7R1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2/-8$ and $R3 \rightarrow R_3/16$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) < n \, (i.e\, 2 < 3)$

This implies that

$x_1 + 2x_2 + 5x_3 = 0$

$x_2 + 2x_3 = 0$

For one variable (3-2=1), we have to give one arbitrary constant

Taking $x_3 = k$, we get $x_1 = -k$; $x_2 = -2k$

$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$ is the eigen vector of A corresponding to $\lambda = 5$.

Hence the Eigen values of A are $\lambda = -3, -3, 5$ and the corresponding Eigen vectors of A are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$

Problem 2: Find the Eigen values and the corresponding Eigen vectors of $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

If X is an Eigen vector of A corresponding to the Eigen value λ of A,

We have $(A - \lambda I)X = 0$

i.e., $\begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ --------------- (1)

The Characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(1-\lambda)(-3-\lambda) - 2\right] - 2[2(-3-\lambda) + 7] = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 3)(\lambda + 4) = 0$$

$$\Rightarrow \lambda = 1, 3, -4$$

$\therefore$ The Eigen values of A are $\lambda = 1, 3, -4$

To find Eigen Vectors:

The eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A corresponding to the eigen values λ are given by $(A - \lambda I)X = 0$

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$$\Rightarrow \begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{------} \quad (1)$$

Case(i): $\lambda = 1$ from (1) , we get

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ -7 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing the coefficient matrix to the echelon form

Applying $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R3 + 7R1$, We get

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 1 \\ 0 & 16 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \rightarrow R3 + 4R2$, We get

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $n - r = 3 - 2 = 1$

For one variable, we have to give one arbitrary constant

Hence we have $x_1 + 2x_2 = 0$

$$-4x_2 + x_3 = 0$$

Let $x_2 = k$, then $x_3 = 4k$ and $x_1 = -2k$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

In particular $k = 1$, $X_1 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ is the eigen vector of A corresponding to eigen value $\lambda = 1$.

Case(ii): $\lambda = 3$ from (1) , we get

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$$\Rightarrow \begin{bmatrix} -1 & 2 & 0 \\ 2 & -2 & 1 \\ -7 & 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing the coefficient matrix to the echelon form

Applying $R_2 \to R_2 + 2R_1$, $R_3 \to R3 - 7R1$, We get

$$\Rightarrow \begin{bmatrix} -1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & -12 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \to R3 + 6R2$, We get

$$\Rightarrow \begin{bmatrix} -1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $n - r = 3 - 2 = 1$

For one variable, we have to give one arbitrary constant

Hence we have $-x_1 + 2x_2 = 0$

$$2x_2 + x_3 = 0$$

Let $x_2 = k$, then $x_3 = -2k$ and $x_1 = 2k$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

In particular $k = 1$, $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is the eigen vector of A corresponding to eigen value $\lambda = 3$.

Case(iii): $\lambda = -4$ from (1), we get

$$\Rightarrow \begin{bmatrix} 6 & 2 & 0 \\ 2 & 5 & 1 \\ -7 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing the coefficient matrix to the echelon form

Applying $R_2 \to 3R_2 - R_1$, $R_3 \to 6R3 + 7R1$, We get

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$$\Rightarrow \begin{bmatrix} 6 & 2 & 0 \\ 0 & 13 & 3 \\ 0 & 26 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \rightarrow R3 - 2R2$, We get

$$\Rightarrow \begin{bmatrix} 6 & 2 & 0 \\ 0 & 13 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $n - r = 3 - 2 = 1$

For one variable, we have to give one arbitrary constant

Hence we have $6x_1 + 2x_2 = 0$

$$13x_2 + 3x_3 = 0$$

Let $x_3 = k$, then $x_2 = \frac{-3}{13}k$ and $x_1 = \frac{-1}{13}k$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} 1/13 \\ -3/13 \\ 1 \end{bmatrix}$$

In particular $k = 13$, $X_3 = \begin{bmatrix} 1 \\ -3 \\ 13 \end{bmatrix}$ is the eigen vector of A corresponding to eigen value $\lambda = 1$.

Hence the Eigen values of A are $\lambda = 1, 3, -4$ and the corresponding Eigen vectors of A are $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -3 \\ 13 \end{bmatrix}$.

Cayley Hamilton Theorem:

State: Every square matrix satisfies its characteristic equation.

Proof: Let A be a square matrix of order n.

Then $A - \lambda I$ is also square matrix of order n.

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Let λ be a eigen value of A, then $|A-\lambda I| = 0$

Let $|A-\lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n)$ be a polynomial of order n.

Then $Adj|A-\lambda I| = B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + ... + B_1\lambda + B_0$, where B_0, B_1 ….. B_{n-1} are n-rowed matrices.

Now $(A-\lambda I)\,Adj|A-\lambda I| = |A-\lambda I|.I$ $\quad\quad \therefore |A| = A.AdjA$

$\Rightarrow (A-\lambda I)(B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + ... + B_1\lambda + B_0) = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n)$

$\Rightarrow -B_{n-1}\lambda^n + (AB_{n-1} - B_{n-2})\lambda^{n-1} + (AB_{n-2} - B_{n-3})\lambda^{n-2} + ... + (AB_1 - B_0)\lambda + AB_0$

$= (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n)$

Comparing the coefficients of like powers of λ, we get

$-B_{n-1} = (-1)^n I$

$AB_{n-1} - B_{n-2} = (-1)^n a_1 I$

$AB_{n-2} - B_{n-3} = (-1)^n a_2 I$

…………………….

…………………….

…………………….

$AB_1 - B_0 = (-1)^n a_{n-1} I$

$AB_0 = (-1)^n a_n I$.

Premultiplying the above equations successively, by $A^n, A^{n-1}, A^{n-2}........., A, I$ and then adding, we get

$-A^nB_{n-1} + A^nB_{n-1} - A^{n-1}B_{n-2} + A^{n-1}B_{n-2} + -A^{n-2}B_{n-3} + A^{n-2}B_{n-3} + ... + A^2B_1 - AB_0 + AB_0$

$= (-1)^n(A^n + a_1A^{n-1} + a_2A^{n-2} + ... + a_{n-1}A + a_n)$

$\therefore\ (-1)^n(A^n + a_1A^{n-1} + a_2A^{n-2} + ... + a_{n-1}A + a_n) = 0$ …….. (1)

Thus the matrix A satisfies its characteristic equation.

Hence the theorem.

To find Inverse matrix:

Determination of A^{-1} by Cayley Hamilton theorem

Pre-multiplying equation (1) by A^{-1} on both sides, we get

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$$A^{-1}(A^n + a_1 A^{n-1} + a_2 A^{n-2} + ... + a_{n-1}A + a_n) = 0$$
$$\Rightarrow A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + ... + a_{n-1}I + a_n A^{-1} = 0$$
$$\Rightarrow A^{-1} = \frac{-1}{a_n}\left(A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + ... + a_{n-1}I\right)$$

Problem 1: Verify Cayley Hamilton theorem for the matrix $\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$

Then the Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 1 & 5-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (3-\lambda)(5-\lambda) - 2 = 0$$
$$\Rightarrow \lambda^2 - 8\lambda + 13 = 0$$

C.H.T states that every square matrix A satisfies its characteristic equation

We have to verify that $A^2 - 8A + 13I = 0$

Now

$$A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 11 & 16 \\ 8 & 27 \end{bmatrix}$$

$$\therefore A^2 - 8A + 13I = \begin{bmatrix} 11 & 16 \\ 8 & 27 \end{bmatrix} - 8\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} + 13\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 11-24+13 & 16-16+0 \\ 8-8+0 & 27-40+13 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence Cayley Hamilton theorem is verified.

Problem 2: Find the characteristic equation for the matrix $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify that it is satisfied by A and hence obtain A^{-1}.

Solution: Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

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Then the Characteristic equation of A is $|A-\lambda I|=0$

$$i.e.\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}=0$$

Expanding, $\lambda^3-6\lambda^2+9\lambda-4=0$

To verify Cayley Hamilton theorem:

We have to show that $A^3-6A^2+9A-4I=O$

$$\text{Now } A^2=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}=\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$\text{And } A^3=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3-6A^2+9A-4=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}-6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}+9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}-4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 22-36 & -21+30 & 21-30 \\ -21+30 & 22-36 & -21+30 \\ 21-30 & -21+30 & 22-36 \end{bmatrix}+\begin{bmatrix} 18-4 & -9-0 & 9-0 \\ -9-0 & 18-4 & -9-0 \\ 9-0 & -9-0 & 18-4 \end{bmatrix}$$

$$=\begin{bmatrix} -14 & 9 & -9 \\ 9 & -14 & 9 \\ -9 & 9 & -14 \end{bmatrix}+\begin{bmatrix} 14 & -9 & 9 \\ -9 & 14 & -9 \\ 9 & -9 & 14 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A^3-6A^2+9A-4=O$

Hence CHT is verified.

To find A^{-1}

Pre-multiplying equation (1) by A^{-1}, we have

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$$4A^{-1} = A^2 - 6A + 9I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Problem 3: Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and find its A^{-1}

.

Solution: Given $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

$$\text{i.e.} \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

Expanding, $\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$

To verify Cayley Hamilton theorem:

We have to show that $A^3 - 6A^2 + 7A + 2I = O$

Now $A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$

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And $A^3 = A^2 A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$

Now $A^3 - 6A^2 + 7A + 2I = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 21-30+7-2 & 0+0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0+0+0+0 & 55-78+21+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 7A + 2I = O$

Hence CHT is verified.

To find A^{-1}

Pre-multiplying equation (1) by A^{-1}, we have

$2A^{-1} = -A^2 + 6A - 7I$

$$= -\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5+6-7 & 0-0-0 & 12-8-0 \\ -2+0-7 & -4+12-7 & -5+6+0 \\ -8+12-0 & 0+0-0 & -13+18-7 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} -6 & 0 & 4 \\ -2 & 1 & 1 \\ 4 & 0 & -2 \end{bmatrix}$$

<u>Calculation of powers of a matrix:-</u>

Diagonalization of the matrix is useful for finding power of a matrix.

Let A be the given square matrix of order n.

Then $D = P^{-1}AP$

$\therefore D^2 = (P^{-1}AP)(P^{-1}AP)$

$= P^{-1}A(P P^{-1})AP$

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$= P^{-1}AIAP$

$= P^{-1}A^2P$

Similarly, $D^3 = P^{-1}A^3P$

In general, $D^n = P^{-1}A^nP$-------------(1)

To obtain A^n Premultiplying by P and post multiplying by P^{-1} ,

$\Rightarrow A^n = P\ A^n\ P^{-1}$

Hence the power of the matrix can be obtained by $A^n = P\ A^n\ P^{-1}$

Problems:-

Problem 1: Find a matrix P which transform the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Solution: The Characteristic equation of A is $|A - \lambda I| = 0$

$$\text{i.e., } \Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\text{i.e., } \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\text{i.e., } \Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

The characteristic roots are $\lambda = 1, 2, 3$.

The eigen values are distinct. So A is diagonalizable.

To find eigen vectors for the corresponding eigen values $\lambda = 1, 2, 3$.

Case (1):- When $\lambda = 1$

Let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding $\lambda = 1$.

Then we have $[A\text{-}1.I]X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is reduce to echelon form

Applying $R_2 \leftrightarrow R_1$, we get

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$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_1$, we get

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which implies

$$\left.\begin{aligned} x_1 + x_2 + x_3 = 0 \\ -x_3 = 0 \end{aligned}\right\} \text{..................(1)}$$

The solution of the system (1) is

$x_2 = k,\ x_1 = -k, x_3 = 0,$

$\therefore\ X_1 = k \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is the eigen vector of A corresponding to $\lambda = 1$

Case (2):- When $\lambda = 2$,

Let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding $\lambda = 2$.

Then we have $[\text{A-}2.\text{I}]X = 0$

$$\Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is reduce to echelon form

Applying $R_2 \to R_2 + R_1, R_3 \to R_3 + 2R_1$ we get

$$\Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \leftrightarrow R_2$, we get

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$$\Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which implies

$$\Rightarrow \left. \begin{aligned} x_1 + x_3 &= 0 \\ 2x_2 - x_3 &= 0 \end{aligned} \right\} \text{..................(2)}$$

Let $x_2 = k,$ then we have $x_3 = 2k, x_1 = -2k$

$\therefore\ X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is the eigen vector of A corresponding to $\lambda = 3$

Case (iii): When $\lambda = 3$

Let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding $\lambda = 3$.

Then we have $[\text{A-}3.\text{I}]X = 0$

$$\Rightarrow \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is reduce to echelon form

Applying $R_2 \to R_2 + 2R_1, R_3 \to R_3 + R_1$ we get

$$\Rightarrow \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_2$, we get

$$\Rightarrow \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution of the system (3) is since $n - r = 3 - 2 = 1$

Let $x_2 = k,$ then we have $x_3 = 2k,\ x_1 = -k$

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$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$ is the eigen vector of A corresponding to $\lambda = 3$.

Writing the three eigen vectors of a matrix A as the three columns, the required transformation matrix P is $\begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

This is called the modal matrix of A.

To find P^{-1}:

The cofactors of P = $\begin{bmatrix} 0 & -2 & 2 \\ 2 & -2 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

$$AdjP = \begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

and $|P| = -1(2-2) + 2(2-0) - 1(2-0) = 0 + 4 - 2 = 2$

$$P^{-1} = \frac{AdjP}{|P|} = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

Now $P^{-1}AP = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

Hence $A^4 = PD^4P^{-1}$

$$= \frac{1}{2}\begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

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$$= \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

Problem 2: Determine the modal matrix P for $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

$$\text{i.e., } \Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\text{i.e., } \Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\text{i.e., } \Rightarrow (\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$$

The characteristic roots are $\lambda = 2, 3, 6$.

The eigen values are distinct. So A is diagonalizable.

To find eigen vectors for the corresponding eigen values $\lambda = 1, 2, 3$.

The eigen vector X to the corresponding eigen value λ is given by [A- λ I]X=0

Case (1):- When $\lambda = 2$

Let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding $\lambda = 2$.

Then we have $[\mathrm{A} - 2\,\mathrm{I}]\mathrm{X} = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is reduce to echelon form

Applying $R_2 \to R_2 + R_1, R_3 \to R_3 - R_1$, we get

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$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which implies

$$\Rightarrow \left. \begin{array}{r} x_1 - x_2 + x_3 = 0 \\ 2x_2 = 0 \end{array} \right\} \text{..................(2)}$$

The solution of the system (1) is

Let $x_3 = k$, then we have $x_2 = 0,\ x_1 = -k.$

$\therefore\ X_1 = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is the eigen vector of A corresponding to $\lambda = 2$

Case (2):- When $\lambda = 3$,

Let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding $\lambda = 3$.

Then we have $[A - 3I]X = 0$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is reduce to echelon form

Applying $R_3 \leftrightarrow R_1$, we get

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to R_2 + R_1$, we get

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

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$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which implies

$$\Rightarrow \left. \begin{array}{r} x_1 - x_2 = 0 \\ x_2 - x_3 = 0 \end{array} \right\} \text{..................(2)}$$

The solution of the system (2) is since $n - r = 3 - 2 = 1$

Let $x_3 = k$, then we have $x_1 = k,\ x_2 = k.$

$\therefore\ X_2 = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the eigen vector of A corresponding to $\lambda = 3$

Case (3):-When $\lambda = 6$

Let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding $\lambda = 6$.

Then we have $[A - 6I]X = 0$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is reduce to echelon form

Applying $R_3 \leftrightarrow R_1$, we get

$$\Rightarrow \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -1 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to R_2 + R_1, R_3 \to R_3 + 3R_1$, we get

$$\Rightarrow \begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, we get

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$$\Rightarrow \begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which implies

$$\left. \begin{aligned} x_1 - x_2 - 3x_3 = 0 \\ 2x_2 + 4x_3 = 0 \end{aligned} \right\} \text{..................(3)} \qquad \text{since } n - r = 3 - 2 = 1$$

The solution of the system (2) is

Let $x_3 = k$, then we have $x_1 = k,\ x_2 = -2k,$

$\therefore\ X_3 = k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector of A corresponding to $\lambda = 3$

Writing the three eigen vectors of a matrix A as the three columns, the required transformation matrix is

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

This matrix P is called the modal matrix.

Now to find P^{-1}:

Cofactors of $P = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & -2 \\ -3 & 2 & 1 \end{bmatrix}$

Then AdjP= $\begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$ and $|P| = 6$

$$P^{-1} = \frac{AdjP}{\det P} = \frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

Now $P^{-1}AP = \frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

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$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = D$$

Hence the given matrix A is diagonalizable.

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Subject: Linear Algebra & Calculus
Unit-III

Mean Value Theorems: Rolle's Theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, Taylor's and Maclaurin theorems with remainders (without proof) related problems.

INTRODUCTION

Continuity: A function $f(x)$ is said to be continuous at $x = a$,

if $\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = f(a)$

Let $f(x)$ be a continuous function in the closed interval $[a,b]$. This means that if $a < x < b$,

$$\lim_{x \to c} f(x) = f(c) \text{ and } \lim_{x \to a+0} f(x) = f(a), \lim_{x \to b-0} f(x) = f(b)$$

Differentiation: A function $f(x)$ is said to be differentiable at $x = c$,

if $\lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$ and $\lim_{x \to a-} \frac{f(x) - f(a)}{x - a}$ exists

Let $f(x)$ be a differentiable in the closed interval $[a,b]$. This means that if $a < x < b$,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists}$$

Further, $\lim_{x \to a+0} \frac{f(x) - f(a)}{x - a} = f(a)$ and $\lim_{x \to b-0} \frac{f(x) - f(b)}{x - b} = f(b)$ exist.

Rolle 's Theorem:

Statement: If $f(x)$ is a function such that

(i) continuous in a closed interval $[a,b]$

(ii) derivable in the open interval (a,b) and

(iii) $f(a) = f(b)$

Then, there exits at least one value of x, say c in (a,b) so that $f'(c) = 0$

Physical Interpretation: If $y = f(x)$ is continuous curve defined in $[a, b]$ and derivable in the open interval (a,b) and $f(a) = f(b)$, then there exits at least one value of c lie in (a,b) at which tangent to the curve is parallel to $x-$axis

Problem. 1: Verify Rolle's theorem for the following functions $f(x) = (x+2)^3 (x-3)^4$ in $[-2, 3]$

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Solution: We have $f(x)=(x+2)^3(x-3)^4$

(i) since every polynomial is continuous for all values of x

$f(x)$ is also continuous in $\left[-2,3\right]$

(ii) Now $f'(x)=3(x+2)^2(x-3)^4+4(x+2)^3(x-3)^3$

$$=(x+2)^2(x-3)^3[3(x-3)+4(x+2)]$$
$$=(x+2)^2(x-3)^3(7x-1)$$

which exists on $\left(-2,\ 3\right)$

(iii) $f(-2)=0,\ f(3)=0$
$\therefore f(-2)=f(3)$

Thus, all the three conditions of Rolle's theorem are satisfied.

$\therefore$ There exists $c\in\left(-2,3\right)$ such that $f'(c)=0$

$\Rightarrow (c+2)^2(c-3)^3(7c-1)=0$
$\Rightarrow c=-2\ or\ c=3\ or\ c=1/7$

Clearly $c=1/7\in(-2,3)$

$(i.e)-2<1/7<3$

Hence Rolle's Theorem is verified.

Problem 2: Verify Rolle's theorem for the following functions $f(x)=2x^3+x^2-4x-2$ in $\left[-\sqrt{2},\ \sqrt{2}\right]$

Solution:

(i) Since every polynomial is continuous for all values of x,

$f(x)$ is also continuous in $\left[-\sqrt{2},\ \sqrt{2}\right]$

(ii) Now $f'(x)=6x^2+2x-4$, which exists on $\left(-\sqrt{2},\ \sqrt{2}\right)$

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$$f\left(-\sqrt{2}\right)=2\left(-\sqrt{2}\right)^3+\left(-\sqrt{2}\right)^2-4\left(-\sqrt{2}\right)-2$$
$$=-4\sqrt{2}+2+4\sqrt{2}-2$$
$$=0$$

(iii) $$f\left(\sqrt{2}\right)=2\left(\sqrt{2}\right)^3+\left(\sqrt{2}\right)^2-4\left(\sqrt{2}\right)-2$$
$$=4\sqrt{2}+2-4\sqrt{2}-2$$
$$=0$$
$$\therefore\ f\left(-\sqrt{2}\right)=f\left(\sqrt{2}\right)$$

Thus three conditions of Rolle's Theorem are satisfied.

$\therefore$ There exists $c\in\left(-\sqrt{2},\sqrt{2}\right)$ such that $f'(c)=0$

$\Rightarrow 6c^2+2c-4=0$
$\Rightarrow c=-1\ or\ c=2/3$

$\therefore$ $c=-1$ and $c=2/3$ are in between $\left(-\sqrt{2},\sqrt{2}\right)$

Hence Rolle's Theorem is verified.

Problem 3: Verify Rolle's theorem for the following functions $f(x)=2x^3+x^2-4x-2$ in $\left[-\sqrt{3},\ \sqrt{3}\right]$

Solution: (i) Since every polynomial is continuous for all values of x,

$f(x)$ is also continuous in $\left[-\sqrt{3},\ \sqrt{3}\right]$

(ii) Now $f'(x)=6x^2+2x-4$, which exists for every $x\in\left(-\sqrt{3},\ \sqrt{3}\right)$

$\Rightarrow$ $f(x)$ is differentiable in $\left(-\sqrt{3},\ \sqrt{3}\right)$

(iii) $f\left(-\sqrt{3}\right)=1-2\sqrt{3}$ and $f\left(\sqrt{3}\right)=1+2\sqrt{3}$

$f\left(-\sqrt{3}\right)\neq f\left(\sqrt{3}\right)$

Thus, the condition (3) of the Rolle's theorem is not satisfied.

$\therefore$ Rolle's theorem is not applicable

Problem 4: Verify Roll's theorem for the function $f(x)=\log\left[\dfrac{x^2+ab}{x(a+b)}\right]$ in $[a,\ b], a>0, b>0.$

Solution: Given $f(x)=\log\left[\dfrac{x^2+ab}{x(a+b)}\right]=\log(x^2+ab)-\log x-\log(a+b)$

(i) Since $f(x)$ is a composite function of continuous functions in $[a, b]$,

So $f(x)$ is continuous in $[a, b]$

(ii) $f'(x)=\dfrac{2x}{x^2+ab}-\dfrac{1}{x}=\dfrac{x^2-ab}{x(x^2+ab)}$

$\therefore f'(x)$ exists for all $x\in(a, b)$

(iii) $f(a)=\log\left[\dfrac{a^2+ab}{a^2+ab}\right]=\log 1=0$

and $f(\mathrm{b})=\log\left[\dfrac{b^2+ab}{b^2+ab}\right]=\log 1=0$

$\therefore f(a)=f(b)$

Thus three conditions of Rolle's theorem are satisfied.

$\therefore$ There exists $c\in(a, b)$ such that $f'(c)=0$

$$\Rightarrow \frac{c^2-ab}{c(c^2+ab)}=0$$

$$\Rightarrow c^2=ab\Rightarrow c=\pm\sqrt{ab}$$

$\therefore c=\sqrt{ab}\in(a, b)$

Hence Rolle's Theorem is verified.

Lagrange's mean value theorem:

Statement: If $f(x)$ is a function defined on $[a,b]$ such that

(i) Continuous in a closed interval $[a,b]$

(ii) Derivable in the open interval (a,b)

Then, there exits atleast one value of x, say c in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Physical Interpretation: If $y=f(x)$ is continuous curve defined in $[a, b]$ and derivable in the open interval (a,b) and $f(a)=f(b)$, then there exits at least one value of c lie in (a,b) at which tangent to the curve is parallel to chord joining $A(a, f(a)$ and $B(b, f(b)$

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Problem 1: Verify Lagrange's mean value theorem for the following functions in the intervals indicated

(i) $f(x) = 2x^2 - 7x + 10;\ a = 2, b = 5$

(ii) $f(x) = x(x-1)(x-2)$ in $[0, 1/2]$

(iii) $f(x) = x^3 - 2x^2$ in $[-1,\ 1]$

Solution: (i) Given $f(x) = 2x^2 - 7x + 10$

We note that $f(x)$ is a polynomial in x

So it is continuous on [2, 5] and differentiable on (2, 5)

Thus, all the conditions of Lagrange's mean value theorem are satisfied.

By Lagrange's mean value theorem, we have

$$f'(c) = \frac{f(5) - f(2)}{5 - 2}$$

$$\Rightarrow 4c - 7 = \frac{25 - 4}{3}$$

$$\Rightarrow 4c = 14$$

$$\therefore c = \frac{7}{2} = 3.5 \in (2, 5)$$

Hence Lagrange's mean value theorem is verified.

(ii) Given $f(x) = x(x-1)(x-2)$ is defined on $[0, 1/2]$

$f(x)$ is continuous in closed interval $[0, 1/2]$

We have $f(x) = x^3 - 3x^2 + 2x$

$$f'(x) = 3x^2 - 6x + 2$$

Which exits on $(0, 1/2)$

$\therefore f(x)$ is differentiable on $(0, 1/2)$

Thus, all the conditions of Lagrange's mean value theorem are satisfied in $[0, 1/2]$.

By Lagrange's mean value theorem, we have

$$f'(c) = \frac{f(1/2) - f(0)}{1/2 - 0}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{3/8 - 0}{1/2 - 0}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{3}{4}$$

$$\Rightarrow 12c^2 - 24c - 5 = 0$$

$$\therefore c = 1 \pm \frac{\sqrt{51}}{6} = 1 \pm 0.764 = 1.764; 0.236$$

But only the value of $c = 0.236$ lies between 0 and 1/2.

Hence Lagrange's mean value theorem is verified.

(iii) Given $f(x) = x^3 - 2x^2$

We note that $f(x)$ is a polynomial in x

So it is continuous on [-1, 1]

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and $f'(x) = 3x^2 - 4x$ which exists on (-1, 1)

it is differentiable on (-1, 1)

Thus, all the conditions of Lagrange's mean value theorem are satisfied.

By Lagrange's mean value theorem, we have

$$f'(c) = \frac{f(1) - f(-1)}{5-2}$$

$$\Rightarrow 4c^2 - 4c = \frac{-1+7}{2} = 3$$

$$\Rightarrow 4c^2 - 4c - 3 = 0$$

$$\Rightarrow c = \frac{4 \pm \sqrt{16+48}}{8} = \frac{4 \pm 8}{8}$$

$$\therefore c = \frac{1}{2} \pm 1 = \frac{3}{2}, \frac{-1}{2} = 1.5, -0.5$$

Hence Lagrange's mean value theorem is verified.

But only the value of $c = -0.5$ lies between -1 and 1.

Hence Lagrange's mean value theorem is verified.

Problem 2: Verify Lagrange's mean value theorem for $f(x) = \log x$ in $[1, e]$

Solution: Given $f(x) = \log x$

Since $f(x)$ is continuous and derivable for all $x > 0$.

Also, $f'(x) = \frac{1}{x}$

Taking $a = 1, b = e$

$f(1) = \log 1 = 0$ and $f(\text{e}) = \log \text{e} = 1$

Thus, all the conditions of Lagrange's mean value theorem are satisfied.

By Lagrange's mean value theorem, we have

$$\frac{1}{c} = \frac{1-0}{e-1} \Rightarrow c = e - 1 \in (1, e)$$

Hence Lagrange's mean value theorem is verified.

Problem 3: Verify Lagrange's mean value theorem for $f(x) = e^x$ in [0, 1]

Solution: Given function is $f(x) = e^x$

(i) Since e^x is continuous for all x.

$\therefore$ $f(x) = e^x$ is continuous in [0, 1]

(ii) $f'(x) = e^x$, which exists for all in (0, 1)

$\therefore$ $f(x) = e^x$ is derivable in (0, 1)

Taking $a = 0, b = 1$

(iii) $f(0) = e^0 = 1$ and $f(1) = \text{e}$

Thus, all the conditions of Lagrange's mean value theorem are satisfied.

By Lagrange's mean value theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Dr FSR ANTS-KADAPA

$$\Rightarrow e^c = \frac{e-1}{1-0} \Rightarrow c = \log(e-1) \qquad \therefore \ [\log(e)=1]$$

Since $2 < \mathrm{e} < 3 \Rightarrow 1 < e-1 < 2 \Rightarrow 0 < \log(e-1) < 1$

$\therefore c = \log(e-1)$ lies in $(0, 1)$

Hence Lagrange's mean value theorem is verified.

Problem 4: Calculate approximately $\sqrt[5]{245}$ by using Lagrange's mean value theorem

Solution: Let $f(x) = \sqrt[5]{x} = x^{1/5},\ x \in (243, 245)$

Taking $a = 243, b = 245$

$$f'(x) = \frac{1}{5}x^{-4/5} \Rightarrow f'(c) = \frac{1}{5}c^{-4/5}$$

By Lagrange's mean value theorem, we have

$$\frac{1}{5}c^{-4/5} = \frac{f(245)-f(243)}{245-243}$$

i.e., c lies between (243, 245)

$$\Rightarrow \frac{2}{5}c^{-4/5} = f(245) - f(243)$$

$$\Rightarrow \frac{2}{5}c^{-4/5} = \sqrt[5]{245} - (243)^{1/5}$$

$$\Rightarrow \sqrt[5]{245} = 3 + \frac{2}{5}(244)^{-4/5} \quad [\text{put } c = 244]$$

$$\Rightarrow \sqrt[5]{245} = 3 + \frac{2}{5}.\frac{1}{81.26} = 3.0049$$

Cauchy Mean Value Theorem:

Statement: If f and g be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $g'(x) \neq 0$ for all $x \in (a,b)$. Then there exists $x \in (a,b)$ such that $\dfrac{f(b)-f(a)}{\mathrm{g}(b)-g(a)} = \dfrac{f'(\mathrm{c})}{\mathrm{g}'(\mathrm{c})}$.

Problem 1: Apply Cauchy's mean value theorem for $f(x) = x^2 + 9$ and $\mathrm{g}(x) = x^3 - 2$ in $[1,\ 2]$.

Solution: Given $f(x) = x^2 + 9$ and $\mathrm{g}(x) = x^3 - 2$

(i) Since $f(x)$ and $\mathrm{g}(x)$ are continuous and derivable for all x

$f(x)$ and $\mathrm{g}(x)$ are continuous on $[1,\ 2]$.

(ii) We have $f'(x) = 2x$ and $\mathrm{g}'(x) = 3x^2$ which are exists on $(1,\ 2)$.

Thus, all the conditions of Cauchy's mean value theorem are satisfied.

By Cauchy's mean value theorem $c \in (1,\ 2)$ such that

$$\frac{f(2)-f(1)}{\mathrm{g}(2)-g(1)} = \frac{f'(\mathrm{c})}{\mathrm{g}'(\mathrm{c})}$$

$$\Rightarrow \frac{13-10}{6+1} = \frac{2c}{3c^2}$$

$$\Rightarrow c = \frac{14}{9} \in (1,\ 2)$$

This means that Cauchy's mean value theorem is verified.

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Problem 2: If $f(x)=e^{x}$ and $g(x)=e^{-x}$ for in$[a,\ b],\ 0<a<b$, show that c is the average of a and b by Cauchy's mean value theorem

Solution: Given $f(x)=e^{x}$ and $g(x)=e^{-x}$

Since $f(x)$ and $g(x)$ are continuous and derivable for all x

$f(x)$ and $g(x)$ are continuous on $[a,\ b]$.

Also, we have $f'(x)=e^{x}$ and $g'(x)=-e^{-x}$ which are exists on $(a,\ b)$.

Thus, all the conditions of Cauchy's mean value theorem are satisfied.

By Cauchy's mean value theorem $c\in(a,\ b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{e^{b}-e^{a}}{e^{-b}-e^{-a}}=\frac{e^{c}}{-e^{-c}}\Rightarrow -e^{a+b}=-e^{2c}$$

$\Rightarrow 2c=a+b\,(\text{or})\,c=\dfrac{a+b}{2}\in(a,\ b)$ ($\therefore$ It is the arithmetic mean between a and b)

This verifies Cauchy's mean value theorem.

Problem 3: Verify Cauchy's mean value theorem for $f(x)=\sqrt{x}$ and $g(x)=\dfrac{1}{\sqrt{x}}$ in $[a,\ b],\ 0<a<b.$

Solution: Given $f(x)=\sqrt{x}$ and $g(x)=\dfrac{1}{\sqrt{x}}$

Since $f(x)$ and $g(x)$ are continuous and derivable for all $x>0$.

$f(x)$ and $g(x)$ are continuous on $[a,\ b]$

Also, we have $f'(x)=\dfrac{1}{2\sqrt{x}}$ and $g'(x)=\dfrac{-1}{2x\sqrt{x}}$ which are exists on $(a,\ b)$.

Thus, all the conditions of Cauchy's mean value theorem are satisfied.

By Cauchy's mean value theorem $c\in(a,\ b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}.$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}=\frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}}\Rightarrow \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}\Rightarrow -c=-\sqrt{ab}$$

$\Rightarrow c=\sqrt{ab}\in(a,\ b)$ ($\therefore$ It is the geometric mean between a and b)

Hence Cauchy's mean value theorem is verified.

Taylor's and Maclaurin's theorem

Definition: A series of the form

$$f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^{2}}{2!}f''(a)+\frac{(x-a)^{3}}{3!}f'''(a)+\frac{(x-a)^{4}}{4!}f^{iv}(a)+\frac{(x-a)^{5}}{5!}f^{v}(a)+\ldots\ldots\ldots$$

Is called as Taylor's series expansion of $f(x)$ about $x=a.$

Dr. TSRATS-KADAPA

Definition: A series of the form

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + ...$$

Is called as Maclaurin's series expansion of $f(x)$.

Problem 1: Show that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

Solution: Maclaurin's series expansion of $f(x)$ is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + ... \quad \text{--------} (1)$$

Let $f(x) = \sin x$ At $x = 0$, $f(0) = \sin 0 = 0$

Then we have

$f'(x) = \cos x$	$f'(0) = \cos 0 = 1$
$f''(x) = -\sin x$	$f''(0) = -\sin 0 = 0$
$f'''(x) = -\cos x$	$f'''(0) = -\cos 0 = -1$
$f^{iv}(x) = \sin x$	$f^{iv}(0) = \sin 0 = 0$
$f^{v}(x) = \cos x$	$f^{v}(0) = \cos 0 = 1$

Substituting these values in (1)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

Problem 2: Show that $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$

Solution: Maclaurin's series expansion of $f(x)$ is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + ... \quad \text{--------} (1)$$

Let $f(x) = \cosh x$ At $x = 0$, $f(0) = \cosh 0 = 1$

Then we have

$f'(x) = \sinh x$	$f'(0) = \sinh 0 = 0$
$f''(x) = \cosh x$	$f''(0) = \cosh 0 = 1$
$f'''(x) = \sinh x$	$f'''(0) = \sinh 0 = 0$
$f^{iv}(x) = \cosh x$	$f^{iv}(0) = \cosh 0 = 1$
$f^{v}(x) = \sinh x$	$f^{v}(0) = \sinh 0 = 0$

Substituting these values in (1)

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$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$$

Problem 3: Expand $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - ....$

Solution: Maclaurin's series expansion of $f(x)$ is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + ... \quad \text{--------} \ (1)$$

Let $f(x) = \tan^{-1} x$ At $x = 0$, $f(0) = \tan^{-1} 0 = 0$

Then we have

$f'(x) = \frac{1}{1+x^2}$	$f'(0) = 1$
$f''(x) = \frac{-2x}{\left(1+x^2\right)^2}$	$f''(0) = 0$
$f'''(x) = \frac{6x^2 - 2}{\left(1+x^2\right)^3}$	$f'''(0) = -2$
$f^{iv}(x) = \frac{24x - 24x^3}{\left(1+x^2\right)^4}$	$f^{iv}(0) = 0$
$f^{v}(x) = \frac{24(1 - 10x^2 - 26x^3)}{\left(1+x^2\right)^5}$	$f^{v}(0) = 24$

Substituting these values in (1)

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - ....$$

Problem 4: Obtain the Maclaurins series expansion for $\log(1+x)$

Solution: Maclaurin's series expansion of $f(x)$ is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + ... \quad \text{--------} \ (1)$$

Let $f(x) = \log(1+x)$ At $x = 0$, $f(0) = \log(1+0) = 0$

Then we have

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$$f'(x)=\frac{1}{1+x} \qquad f'(0)=\frac{1}{1+0}=1$$

$$f''(x)=\frac{-1}{(1+x)^2} \qquad f''(0)=\frac{-1}{(1+0)^2}=-1$$

$$f'''(x)=\frac{2}{(1+x)^3} \qquad f'''(0)=\frac{2}{(1+0)^3}=2$$

$$f^{iv}(x)=\frac{-6}{(1+x)^3} \qquad f^{iv}(0)=\frac{-6}{(1+0)^3}=-6$$

$$f^{v}(x)=\frac{24}{(1+x)^3} \qquad f^{v}(0)=\frac{24}{(1+0)^3}=24$$

Substituting these values in (1)

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}+.........$$

Dr TSR AITS-KADAPA

Subject: Linear Algebra & Calculus

Unit-IV

Multivariable Calculus: Partial derivatives, total derivatives, chain rule, change of variables, Jacobians, maxima and minima of functions of two variables, method of Lagrange multipliers.

PARTIAL DERIVATIVES,

Let $z = f(x, y)$ be the function of two variables x and y. If we keep y constant and varies then z becomes a function of a variable x only. The derivative of z with respect to x, keeping y as constant is called partial derivative with respect to x and is denoted by the symbols $\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y)$ etc.

Then $\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$

Similarly, the partial derivative of 'z' with respect to 'y' keeping x as constant is denoted by $\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y)$ etc.

$$\frac{\partial z}{\partial y} = \lim_{\delta x \to 0} \frac{f(x,\ y+\delta y) - f(x, y)}{\delta y}$$

Standard notation:

$$p = \frac{\partial z}{\partial x} = z_x,\ q = \frac{\partial z}{\partial y} z_y,\ r = \frac{\partial^2 z}{\partial x^2} = z_{xx},\ s = \frac{\partial^2 z}{\partial x \partial y} = z_{xy},\ t = \frac{\partial^2 z}{\partial y^2} = z_{yy}$$

Problem 1: Find the first and second order partial derivatives of $f(x, y) = x^3 + y^3 - 3axy$ and verify $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

Solution: We have $f(x, y) = x^3 + y^3 - 3axy$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay,\ \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial^2 f}{\partial x^2} = 6x,\ \frac{\partial^2 f}{\partial y \partial x} = -3a,\ \frac{\partial^2 f}{\partial x \partial y} = -3a\ \frac{\partial^2 f}{\partial x^2} = 6y$$

We observe that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

Problem 2: Find the first and second order partial derivatives of $x^2 + y^2 + 2hxy$ and verify $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

Solution: Let $f(x, y) = x^2 + y^2 + 2hxy$

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$$\frac{\partial f}{\partial x} = 2x + 2hy, \ \frac{\partial f}{\partial y} = 2y + 2hx,$$

$$\frac{\partial^2 f}{\partial x^2} = 2, \ \frac{\partial^2 f}{\partial y \partial x} = 2h, \ \frac{\partial^2 f}{\partial x \partial y} = 2h, \ \frac{\partial^2 f}{\partial x^2} = 2.$$

Thus $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ is verified.

Problem 3: Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial x}$, if $z = \log\left(x^2 + y^2\right)$

Solution: Given $f(x, y) = \log\left(x^2 + y^2\right)$

Then we have $\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$ and $\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$

Problem 4: If $u = \tan^{-1}\left(\frac{x}{y}\right)$, Prove that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Solution: Given that $u = \tan^{-1}\left(\frac{x}{y}\right)$,

Now $\frac{\partial u}{\partial x} = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{y^2}{x^2 + y^2} \frac{1}{y} = \frac{y}{x^2 + y^2}$

$$\frac{\partial u}{\partial y} = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{y^2}{x^2 + y^2} \frac{-x}{y^2} = \frac{-x}{x^2 + y^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{x^2 + y^2 .1 - y2y}{\left(x^2 + y^2\right)^2} = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{x^2 + y^2 .1 - x2x}{\left(x^2 + y^2\right)^2} = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}$$

Hence $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

Problem 5: If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ Prove that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Solution: Given that $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Now $\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$

$$= \frac{y}{\sqrt{y^2 - x^2}} \frac{1}{y} + \frac{x^2}{x^2 + y^2} \frac{-y}{x^2} = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2} \qquad \text{……. (1)}$$

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$$\frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial y}\left(\frac{x}{y}\right)+\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$=\frac{y}{\sqrt{y^2-x^2}}\frac{-x}{y^2}+\frac{x^2}{x^2+y^2}\frac{1}{x}=\frac{-x}{y\sqrt{y^2-x^2}}+\frac{x}{x^2+y^2} \qquad \ldots\ldots\ (2)$$

Multiplying (1) by x and (2) by y and then on adding, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

EULER'S THEOREM ON HOMOGENEOUS FUNCTION:

Definition 1: A function $f(x,y)$ is said to be homogeneous function of degree n, if the degree or power of each term in $f(x,y)$ is n, where n is the real number

Problem 1: $f(x,y)=a_0x^n+a_1x^{n-1}y++a_2x^{n-2}y^2+.........+a_ny^n$.

Note: A function $f(x,y)$ is said to be homogeneous function of degree or power n in x,y can be expressed in the form as $y^n f\left(\frac{x}{y}\right)$ or $x^n f\left(\frac{y}{x}\right)$.

Problem 1: Let $f(x,y)=x^3+2x^2y+y^3$

Then $f(x,y)=x^3\left(1+2\frac{y}{x}+\frac{y^3}{x^3}\right)=x^3 f\left(\frac{y}{x}\right)$

$\therefore$ $f(x,y)$ is a homogeneous function of degree 3.

Definition 2: A function $f(x,y)$ is said to be homogeneous function of degree n, if $f(kx,ky)=k^n f(x,y)$ where n is the real number

Problem 1: Let $f(x,y)=\frac{x^4+y^4}{x+y}$

Then $f(kx,ky)=\frac{(kx)^4+(ky)^4}{kx+ky}=\frac{k^4}{k}\frac{x^4+y^4}{x+y}$

$$=k^3\frac{x^4+y^4}{x+y}$$

$$=k^3 f(x,y)$$

$\therefore$ $f(x,y)$ is a homogeneous function of degree 3.

Problem 2: Let $f(x,y)=\tan^{-1}\left(\frac{x^2+y^2}{2xy}\right)$

Then $f(kx,ky)=\tan^{-1}\left(\frac{(kx)^2+(ky)^2}{2kxky}\right)$

$$=\tan^{-1}\left(\frac{k^2(x+y)}{k^2 2xy}\right)$$

$$=\tan^{-1}\left(\frac{x^2+y^2}{2xy}\right)=k^0 f(x,y)$$

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$\therefore\ f(x, y)$ is a homogeneous function of degree zero.

CHAIN RULE OF PARTIAL DIFFERENTIATION:

(1) Let $z = f(x, y)$ where $x = \phi_1(t)$ and $y = \phi_2(t)$ are functions of t. Then z is called a composite function of a variable t.

(2) Let $z = f(u, v)$ where $u = \phi_1(x, y)$ and $v = \phi_2(x, y)$ are functions of x, y. Then z is called a composite function of a variable x and y.

Theorem: Let $z = f(u, v)$ where $u = \phi_1(x, y)$ and $v = \phi_2(x, y)$ are functions of x, y.

Then $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}$

These equations referred to as the chain rule of partial differentiation. The above rule can be extended to functions of more than two independent variables.

TOTAL DERIVATIVE (or) TOTAL DEFFERENTIAL COEFFICIENT:

Let $z = f(x, y)$ where $x = \phi_1(t)$ and $y = \phi_2(t)$ are functions of t. Then the derivative of z w.r.to t. i.e., $\dfrac{dz}{dt}$ is called the total differential coefficient or total derivative of z.

$$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Similarly, if $u = f(x, y, z)$ where x, y and z are functions of t. Then the chain rule is

$$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Problem 1: If $z = u^2 + v^2$, where $u = at^2$ and $v = 2at$, find $\dfrac{dz}{dt}$.

Solution: Given $z = u^2 + v^2$, where $u = at^2$ and $v = 2at$

We have $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial u}\dfrac{du}{dt} + \dfrac{\partial z}{\partial v}\dfrac{dv}{dt}$

$= 2u.2at + 2v.2a$

$= 4at^2.2at + 4at.2a$

$$\therefore \frac{dz}{dt} = 4a^2t\left(t^2 + 2\right)$$

Problem 2: If $z = y^2 - 4ax$, where $x = at^2$ and $y = 2at$, find $\dfrac{dz}{dt}$.

Solution: Given $z = y^2 - 4ax$, where $x = at^2$ and $y = 2at$

We have $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

$= -4a.2at + 2y.2a$

$= -8a^2t + 8a^2t = 0$

$$\therefore \frac{dz}{dt} = 0$$

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Problem 3: If $z=\sin\left(\frac{x}{y}\right)$, where $x=e^t$ and $y=t^2$, find $\frac{dz}{dt}$ as a function of t. Verify your result by direct substitution.

Solution: Given $z=\sin\left(\frac{x}{y}\right)$, where $x=e^t$ and $y=t^2$

We have $\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$

$$=\cos\left(\frac{x}{y}\right).\frac{1}{y}e^t+\cos\left(\frac{x}{y}\right).\frac{-x}{y^2}.2t$$

$$=\cos\left(\frac{e^t}{t^2}\right).\frac{e^t}{t^2}+\cos\left(\frac{e^t}{t^2}\right).\frac{-e^t}{t^4}.2t$$

$$=\cos\left(\frac{e^t}{t^2}\right).\frac{e^t}{t^2}\left(1-\frac{2}{t}\right)$$

$$=\cos\left(\frac{e^t}{t^2}\right).\frac{e^t}{t^3}(t-2)$$

Also, $\frac{dz}{dt}=\cos\left(\frac{e^t}{t^2}\right)\left(\frac{t^2e^t-e^t2t}{t^4}\right)=\cos\left(\frac{e^t}{t^2}\right)\frac{e^t}{t^3}(t-2)$ as before.

CHANGE OF VARIABLES:

Change of two independent variables x and y by any other variable t.

Let $z=f(x,y)$, where $x=\phi_1(t)$ and $y=\phi_2(t)$ are functions of single variable t.

Then $\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$ is called the total differential coefficient of z.

Change of two independent variables x and y by other two variables u and v.

Let $z=f(x,y)$, where $x=\phi_1(s,\ t)$ and $y=\phi_2(s,\ t)$

Then we have z is composite function of t.

Then $\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$ and $\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

Corollary: Let $u=f(x,y,z)$, where $x=\phi_1(s,\ t)$, $y=\phi_2(s,\ t)$ and $z=\phi_3(s,\ t)$

Here u is composite function of t.

Then $\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$

and $\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$

Problem 1: If $u=F(x-y,\ y-z,\ z-x)$ Prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.

Solution: Given $u=F(x-y,\ y-z,\ z-x)$

Put $r=x-y,\ s=y-z,\ t=z-x$

$u=F(r,s,t)$

DrTSRAITS-KADAPA

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$=\frac{\partial u}{\partial r}.1+\frac{\partial u}{\partial s}.0+\frac{\partial u}{\partial t}(-1)=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$=\frac{\partial u}{\partial r}.(-1)+\frac{\partial u}{\partial s}.1+\frac{\partial u}{\partial t}.0=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}$$

and $\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$

$$=\frac{\partial u}{\partial r}.(0)+\frac{\partial u}{\partial s}.(-1)+\frac{\partial u}{\partial t}.1=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}$$

Therefore, $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}=0$

Hence the results.

Problem 2: If $u=f(x,y)$ where $x=e^r\cos\theta$ and $y=e^r\sin\theta$, Prove that $x\frac{\partial u}{\partial r}+y\frac{\partial u}{\partial \theta}=e^{2r}\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2\right]$.

Solution: Given $u=f(x,y)$ where $x=e^r\cos\theta$ and $y=e^r\sin\theta$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}e^r\cos\theta+\frac{\partial u}{\partial y}e^r\sin\theta$$

$$=e^r\left(\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta\right) \qquad \ldots\ldots (1)$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial u}{\partial x}-e^r\sin\theta+\frac{\partial u}{\partial y}e^r\cos\theta$$

$$=e^r\left(-\frac{\partial u}{\partial x}\sin\theta+\frac{\partial u}{\partial y}\cos\theta\right) \qquad \ldots\ldots (2)$$

Now $y\frac{\partial u}{\partial r}+x\frac{\partial u}{\partial \theta}=e^r\sin\theta.e^r\left(\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta\right)+e^r\cos\theta.e^r\left(-\frac{\partial u}{\partial x}\sin\theta+\frac{\partial u}{\partial y}\cos\theta\right)$

$$=e^{2r}\left(\frac{\partial u}{\partial x}\sin\theta\cos\theta+\frac{\partial u}{\partial y}\sin^2\theta-\frac{\partial u}{\partial x}\sin\theta\cos\theta+\frac{\partial u}{\partial y}\cos^2\theta\right)$$

$$\therefore\ y\frac{\partial u}{\partial r}+x\frac{\partial u}{\partial \theta}=e^{2r}\frac{\partial u}{\partial y}$$

Squaring (1) and (2) and then adding, we get

Dr TSR AITS-KADAPA

$$\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2=e^{2r}\left(\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta\right)^2+e^{2r}\left(-\frac{\partial u}{\partial x}\sin\theta+\frac{\partial u}{\partial y}\cos\theta\right)^2$$

$$=e^{2r}\left[\left(\frac{\partial u}{\partial x}\right)^2\left(\cos^2\theta+\sin^2\theta\right)+\left(\frac{\partial u}{\partial y}\right)^2\left(\cos^2\theta+\sin^2\theta\right)\right]$$

$$\therefore\ \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2\right]$$

3.1 Jacobian:

Definition: If u and v are functions of two independent variables x and y. Then the determinant $\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$ is called the Jacobian of with respect to x, y and is denoted by $\frac{\partial(u,v)}{\partial(x,\mathrm{y})}$

or $J\left(\frac{u,v}{x,\mathrm{y}}\right)$

Similarly, the Jacobian of u, v, w with respect to x, y, w is

$$\frac{\partial(u,v,w)}{\partial(x,\mathrm{y},\mathrm{z})}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

3.2 PROPERTIES OF JACOBIANS:

Properties 1: If u and v are functions of x and y and x and y are functions of r and θ. Then

$$\frac{\partial(u,v)}{\partial(r,\theta)}=\frac{\partial(u,v)}{\partial(x,\mathrm{y})}.\frac{\partial(x,\mathrm{y})}{\partial(r,\theta)}$$

Properties 2: If u and v are functions of x and y and x and y are functions of u and v. Then

$\frac{\partial(u,v)}{\partial(x,\mathrm{y})}.\frac{\partial(x,\mathrm{y})}{\partial(u,v)}=1$ (or) If $J=\frac{\partial(u,v)}{\partial(x,\mathrm{y})}$ and $J'=\frac{\partial(x,\mathrm{y})}{\partial(u,v)}$ then $JJ'=1$

Problem 1: If $u=e^{x+y}, v=e^{-(x+y)}$; find $J\left(\frac{u,\ v}{x,\ y}\right)$

Solution : Given $u=e^{x+y}, v=e^{-x+y}$;

$$\frac{\partial u}{\partial x}=e^{x+y},\qquad \frac{\partial v}{\partial x}=-e^{-(x+y)}$$

$$\frac{\partial u}{\partial y}=e^{x+y},\qquad \frac{\partial v}{\partial y}=-e^{-(x+y)}$$

Dr PSR AITS-KADAPA

Now $J\left(\frac{u,v}{x,y}\right)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}e^{x+y} & e^{x+y}\\ -e^{-(x+y)} & -e^{-(x+y)}\end{vmatrix}$

$=-e^{x+y}.e^{-(x+y)}+e^{x+y}.e^{-(x+y)}$

$=-1+1=0$

Problem 2: If $u=3x+5y, v=4x-3y$; find $J\left(\frac{u,\ v}{x,\ y}\right)$

Solution : Given $u=3x+5y, v=4x-3y$;

$$\frac{\partial u}{\partial x}=3, \qquad \frac{\partial v}{\partial x}=4$$

$$\frac{\partial u}{\partial y}=5, \qquad \frac{\partial v}{\partial y}=-3$$

$$\therefore\ J\left(\frac{u,v}{x,y}\right)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}3 & 5\\ 4 & -3\end{vmatrix}=-29$$

Problem 3: If $u=x^2-2y^2, v=2x^2-y^2$; find $J\left(\frac{u,\ v}{x,\ y}\right)$

Solution : Given $u=x^2-2y^2, v=2x^2-y^2$;

$$\frac{\partial u}{\partial x}=2x, \qquad \frac{\partial v}{\partial x}=4x$$

$$\frac{\partial u}{\partial y}=-4y, \qquad \frac{\partial v}{\partial y}=-2y$$

$$\therefore\ J\left(\frac{u,v}{x,y}\right)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}2x & -4y\\ 4x & -2y\end{vmatrix}$$

$=-4xy+16xy$

$=12xy$

Problem 4: If $x=r\cos\theta, y=r\sin\theta$; find $J\left(\frac{x,y}{r,\ \theta}\right)$ and $J'\left(\frac{r,\theta}{x,y}\right)$. Also show that $JJ'=1$

Solution: We have $x=r\cos\theta,\qquad y=r\sin\theta$

$$\frac{\partial x}{\partial r}=\cos\theta, \qquad \frac{\partial y}{\partial r}=\sin\theta$$

$$\frac{\partial x}{\partial \theta}=-r\sin\theta \qquad \frac{\partial y}{\partial \theta}=r\cos\theta$$

Dr.TSR AITS KADAPA

$$\frac{\partial(x,\mathrm{y})}{\partial(r,\,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta=r(\cos^2\theta+\sin^2\theta)=r$$

Also, we have $r^2=x^2+y^2,\qquad \theta=\tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial r}{\partial x}=\frac{x}{r}=,\qquad \frac{\partial\theta}{\partial x}=\frac{-y}{x^2+y^2}=\frac{-y}{r^2}$$

$$\frac{\partial r}{\partial y}=\frac{y}{r},\qquad \frac{\partial\theta}{\partial y}=\frac{x}{x^2+y^2}=\frac{x}{r^2}$$

$$\frac{\partial(r,\,\theta)}{\partial(x,\mathrm{y})}=\begin{vmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{x}{r} & \frac{y}{r}\\ \frac{-y}{r^2} & \frac{x}{r^2}\end{vmatrix}=\frac{x^2}{r^3}+\frac{y^2}{r^3}$$

$$=\frac{x^2+y^2}{r^3}=\frac{r^2}{r^3}=\frac{1}{r}$$

$$\therefore\ J\left(\frac{x,y}{r,\,\theta}\right)\times J\left(\frac{r,\theta}{x,y}\right)=r.\frac{1}{r}=1$$

.3 FUNCTIONALLY DEPENDENT AND FUNCTIONALLY INDEPENDENT:

Let $u=f(x,y),\ v=g(x,y)$ be two given differentiable functions of the two independent variables x and y. Suppose these functions u and v are connected by a relation $F(u,v)=0,$ where F is differentiable. Then these functions u and v are said to be functionally dependent on one another(i.e., one function say u is a function of the second function v) if the partial derivatives u_x,u_y,v_x and v_y are not all zero simultaneously)

Necessary and Sufficient condition for the two functions $u(x,y)$ and $v(x,y)$ are functionally dependent:

Problem 1: If $u=\frac{x}{y},\ v=\frac{x+y}{x-y}$. Find $J\left(\frac{u,\ v}{x,\ y}\right)$. Hence prove that u and v are functionally dependent. Find the functionally relation between them.

Solution: Given $u=\frac{x}{y},\qquad v=\frac{x+y}{x-y}$

$$\frac{\partial u}{\partial x}=\frac{x}{y},\ \frac{\partial u}{\partial y}=\frac{-x}{y^2}$$

$$\frac{\partial v}{\partial x}=\frac{(x-y).1-(x+y).1}{(x-y)^2}=\frac{-2y}{(x-y)^2}$$

$$\frac{\partial v}{\partial y}=\frac{(x-y).1-(x+y).(-1)}{(x-y)^2}=\frac{2x}{(x-y)^2}$$

Dr TSR ATIS-KADAPA

$$\therefore \quad \frac{\partial(u,\, v)}{\partial(x,\, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{y} & \frac{-x}{y^2} \\ \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \end{vmatrix}$$

$$= \frac{2x}{y(x-y)^2} - \frac{2x}{y(x-y)^2} = 0$$

$\therefore$ u and v are functionally dependent.

Now $v = \dfrac{y\left(\frac{x}{y}+1\right)}{y\left(\frac{x}{y}-1\right)} = \dfrac{u+1}{u-1}$

$\therefore$ $v = \dfrac{u+1}{u-1}$ is the functionally relation between u and v.

Problem 1: If $u = x^2 + y^2 + 2xy + 2x + 2y, v = e^{(x+y)}$; find $J\left(\dfrac{u,\, v}{x,\, y}\right)$

Solution : Given $u = x^2 + y^2 + 2xy + 2x + 2y, v = e^{(x+y)}$;

$$\frac{\partial u}{\partial x} = 2x + 2y + 2, \qquad \frac{\partial v}{\partial x} = e^{x+y}$$

$$\frac{\partial u}{\partial y} = 2y + 2x + 2, \qquad \frac{\partial v}{\partial y} = e^{x+y}$$

$$\therefore J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x+2y+2 & 2y+2x+2 \\ e^{(x+y)} & e^{(x+y)} \end{vmatrix}$$

$$= (2x+2y+2)e^{(x+y)} - (2x+2y+2)e^{(x+y)}$$

$$= 0$$

Hence u and v are functionally dependent.

Now $v = e^{x+y} \Rightarrow \log v = x + y$

$\therefore u = (\log v)^2 + 2\log v$ is the functionally relation between u and v.

Problem 2: Show that the functions $u = x + y + z,\ v = xy + yz + zx,\ w = x^2 + y^2 + z^2$ are functionally related and find the functionally relation between them.

Solution: Given $u = x + y + z,\ v = xy + yz + zx,\ w = x^2 + y^2 + z^2$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = y + z \qquad \frac{\partial w}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial y} = x + z \qquad \frac{\partial w}{\partial y} = 2y$$

$$\frac{\partial u}{\partial z} = 1 \qquad \frac{\partial v}{\partial z} = x + y \qquad \frac{\partial w}{\partial z} = 2z$$

Dr TSR AITS-KADAPA

Now $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \\ 2x & 2y & 2z \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \\ x & y & z \end{vmatrix}$$

Applying $R_2 \to R_2 + R_3$, we get

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ x & y & z \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} =$$

$$= 2(x+y+z).0 \quad [\text{ since } R_1 \text{ and } R_2 \text{ are identical}]$$

$$= 0$$

Hence u, v and w are functionally dependent. So that functionally relation exists between u, v and w.

Now $u = (x+y+z)^2$

$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

$\therefore u = w + 2v$ is the functionally relation between u, v and w.

Problem 3: Prove that the functions $u = x - y + 3z,\ v = 2x - y - z$ and $w = 2x - y + z$ are functionally related and find the functionally relation between them.

Solution: Given $u = x - y + 3z,\ v = 2x - y - z$ and $w = 2x - y + z$

$$\frac{\partial u}{\partial x} = 2 \qquad \frac{\partial v}{\partial x} = 2 \qquad \frac{\partial w}{\partial x} = 2$$

$$\frac{\partial u}{\partial y} = -1 \qquad \frac{\partial v}{\partial y} = -1 \qquad \frac{\partial w}{\partial y} = -1$$

$$\frac{\partial u}{\partial z} = 3 \qquad \frac{\partial v}{\partial z} = -1 \qquad \frac{\partial w}{\partial z} = 1$$

Dr TSR AITS-KADAPA

Now $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix}$

$$= 2(-1)\begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = (-2).(0)$$

$\therefore$ u, v and w are functionally dependent.

Now $u + v = 4x - 2y + 2z = 2(2x - y + z)$

Hence $u + v = 2w$ is the functionally relation between u, v and w.

TAYLOR'S EXPANSION FOR A FUNCTION OF TWO VARIABLES:

$$f(x,\ y) = f(a,\ b) + (x-a)\, f_x(a,\ b) + (y-b)\, f_y(a,\ b)$$

$$+\frac{1}{2!}\Big((x-a)^2 f_{xx}(a,\ b) + 2(x-a)(y-b) f_{xy}(a,\ b) + (y-b)^2 f_{yy}(a,\ b)\Big) + ...$$

Problem 1: Expand $e^x \sin y$ in powers of x and y by Taylor's theorem.

Solution: Let $f(x, y) = e^x \sin y$

Then $f_x(x, y) = e^x \sin y$

$f_y(x, y) = e^x \cos y$

$f_{xx}(x, y) = e^x \sin y$

$f_{yx}(x, y)$ or $f_{xy}(x, y) = e^x \cos y$

$f_{yy}(x, y) = -e^x \sin y$

$f_{xxx}(x, y) = e^x \sin y$

$f_{xxy}(x, y) = e^x \cos y$

$f_{xyy}(x, y) = -e^x \sin y$

$f_{yyy}(x, y) = -e^x \cos y$

At $(0,\ 0)$, $f(0,\ 0) = 0$

$f_x(0,\ 0) = 0$

$f_y(0,\ 0) = 1$

$f_{xx}(0,\ 0) = 0$

$f_{yx}(0,\ 0) = 1$

$f_{yy}(0,\ 0) = 0$

$f_{xxx}(0,\ 0) = 0$

$f_{xxy}(0,\ 0) = 1$

$f_{xyy}(0,\ 0) = 0$

Dr TSP AITS-KADAPA

$$f_{yyy}(0,\ 0) = -1$$

By Taylor's theorem, we have

$$f(x,\ y) = f(0,\ 0) + xf_x(0,\ 0) + yf_y(0,\ 0) + \frac{1}{2!}\left[x^2 f_{xx}(0,\ 0) + 2xyf_{xy}(0,\ 0) + y^2 f_{yy}(0,\ 0)\right]$$

$$+\frac{1}{3!}\left[x^3 f_{xxx}(0,\ 0) + 3x^2 y f_{xxy}(0,\ 0) + 3xy^2 f_{xyy}(0,\ 0) + y^3 f_{yyy}(0,\ 0)\right] + ...$$

i.e., $e^x \sin y = y + xy + \dfrac{x^2 y}{2} - \dfrac{y^3}{6} + ...$

Problem 2: Expand $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$ in powers of $(x-1)$ and $(y-1)$

(or) Expand $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$ in the neighborhood of $(1, 1)$.

Solution: Let $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$

Then $f_x(x, y) = \dfrac{-y}{x^2 + y^2}$

$$f_y(x, y) = \frac{x}{x^2 + y^2}$$

$$f_{xx}(x, y) = \frac{2xy}{\left(x^2 + y^2\right)^2}$$

$$f_{yx}(x, y) = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$f_{yy}(x, y) = \frac{-2xy}{\left(x^2 + y^2\right)^2}$$

$$f_{xxx}(x, y) = \frac{2y^3 - 6x^2 y}{\left(x^2 + y^2\right)^3}$$

$$f_{xxy}(x, y) = \frac{2x^3 - 6xy^2}{\left(x^2 + y^2\right)^3}$$

$$f_{xyy}(x, y) = \frac{6x^2 y - 2y^3}{\left(x^2 + y^2\right)^3}$$

$$f_{yyy}(x, y) = \frac{6xy^2 - 2x^3}{\left(x^2 + y^2\right)^3}$$

At (1, 1), $f(1,1) = \tan^{-1}(1) = \dfrac{\pi}{4}$

$$f_x(1,\ 1) = -\frac{1}{2}$$

Dr TSR AITS-KADAPA

$$f_y(1,1)=\frac{1}{2}$$

$$f_{xx}(1,1)=\frac{1}{2}$$

$$f_{xy}(1,1)=0$$

$$f_{yy}(1,1)=-\frac{1}{2}$$

$$f_{xxx}(1,1)=-\frac{1}{2}$$

$$f_{xxy}(1,1)=-\frac{1}{2}$$

$$f_{xyy}(1,1)=\frac{1}{2}$$

$$f_{yyy}(1,1)=\frac{1}{2}$$

By Taylor's theorem for $f(x,y)$ in powers of $(x-1)$ and $(y-1)$, we have

$$f(x,y)=f(1,1)+(x-1)f_x(1,1)+(y-1)f_y(1,1)$$
$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1)+2(x-1)(y-1)f_{xy}(1,1)+(y-1)^2 f_{yy}(1,1)\right]$$
$$+\frac{1}{3!}\left[(x-1)^3 f_{xxx}(1,1)+3(x-1)^2(y-1)f_{xxy}(1,1)+3(x-1)(y-1)^2 f_{xyy}(1,1)+(y-1)^3 f_{yyy}(1,1)\right]+\ldots..$$

$$\therefore \tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{4}+\frac{1}{2}\left[(x-1)-(y-1)\right]+\frac{1}{4}\left[(x-1)^2-(y-1)^2\right]$$
$$-\frac{1}{12}\left[(x-1)^3+3(x-1)^2(y-1)-3(x-1)(y-1)^2-(y-1)^3\right]+\ldots..$$

Problem 3: Expand e^{xy} in powers of $(x-1)$ and $(y-1)$

Solution: Let $f(x,y)=e^{xy}$

Then $f_x(x,y)=ye^{xy}$

$$f_y(x,y)=xe^{xy}$$

$$f_{xx}(x,y)=y^2e^{xy}$$

$$f_{yx}(x,y)=xye^{xy}+e^{xy}$$

$$f_{yy}(x,y)=x^2e^{xy}$$

At (1, 1), $f(1,1)=e$

$$f_x(1,1)=e$$

$$f_y(1,1)=e$$

$$f_{xx}(1,1)=e$$

$$f_{yx}(1,1)=e+e=2e$$

$$f_{yy}(1,1)=e$$

By Taylor's theorem, we have

Dr TSR ATS-KADAPA

$$f(x,\ y) = f(1,\ 1) + (x-1)\, f_x(1,\ 1) + (y-1)\, f_y(1,\ 1)$$
$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,\ 1) + 2(x-1)(y-1)\, f_{xy}(1,\ 1) + (y-1)^2 f_{yy}(1,\ 1)\right] + ...$$

i.e., $$e^{xy} = e + (x-1)e + (y-1)e + \frac{1}{2!}\left[e(x-1)^2 + 4e(x-1)(y-1) + e(y-1)^2\right] + ...$$
$$= e\left[1 + (x-1) + (y-1) + \frac{(x-1)^2}{2!} + 2(x-1)(y-1) + \frac{(y-1)^2}{2!} + ...\right]$$

MAXIMUM AND MINIMUM OF FUNCTIONS OF TWO VARIABLES:

Working rule:

1. Find $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, solving these equations for x and y.

   Let (a_1, b_1) and (a_2, b_2) be the pairs of values.

2. Find $r = \frac{\partial^2 z}{\partial x^2},\ s = \frac{\partial^2 z}{\partial x \partial y},\ t = \frac{\partial^2 z}{\partial y^2}$ for each airs of values obtained in step (1).

3. (i) If $rt - s^2 > 0$ and $r < 0$ at (a_1, b_1), then $f(a_1, \mathrm{b}_1)$ has maximum at (a_1, b_1).

   (ii) If $rt - s^2 > 0$ and $r > 0$ at (a_1, b_1), then $f(a_1, \mathrm{b}_1)$ has maximum at (a_1, b_1).

   (iii) If $rt - s^2 < 0$ and $r > 0$ at (a_1, b_1), then $f(a_1, \mathrm{b}_1)$ is not an extreme at (a_1, b_1), i.e., there is neither a maximum nor minimum at (a_1, b_1). In this case at (a_1, b_1) is said to be saddle point.

   (iv) If $rt - s^2 = 0$ at (a_1, b_1), then there is no conclusion can be drawn about maximum or minimum and we needs further investigation.

   Similarly, examine the other pairs of points $(a_2, \mathrm{b}_2), (a_3, \mathrm{b}_3), ...$ one by one.

Problem 1: Find the maximum and minimum value of $x^3 + y^3 - 3axy,\ a > 0$

Solution: Let $z = x^3 + y^3 - 3axy$ .... (1)

For f to be maxima or minima

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

We have $\frac{\partial z}{\partial x} = 3(x^2 - ay) = 0$ .... (2)

and $\frac{\partial z}{\partial y} = 3(y^2 - ax) = 0$ .... (3)

Solving (2) and (3), we get

$x = 0, x = a$

Corresponding values of y are $y = 0, y = a$

The stationary points are $(0,\ 0)$ and $(a,\ a)$

Now $r = \frac{\partial^2 z}{\partial x^2} = 6x$

$$s = \frac{\partial^2 z}{\partial x \partial y} = -3a$$

$$t = \frac{\partial^2 z}{\partial y^2} = 6y$$

At the point (0, 0), $rt - s^2 = 36xy - 9a^2 = -9a^2 < 0$

$\therefore$ The function does not have extreme value at (0, 0).

At the point (a, a), $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$ and $r = 6a > 0$

$\therefore$ The given function is minimum at (a, a).

The minimum value is $z(a, a) = -a^3$.

Problem 2: Find the maximum and minimum value of $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

Solution: Given $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ ……. (1)

For f to be maxima or minima

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

We have $\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x = 0$ ……. (2)

and $\frac{\partial z}{\partial y} = 6xy - 6y = 0$ ……. (3)

Solving (2) and (3), we get

$x = 0, 1, 2$ and $y = 0, \pm 1$

Hence $(0,0), (2,0), (1,\pm 1)$ are the stationary points of f.

Now $r = \frac{\partial^2 f}{\partial x^2} = 6x - 6$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 6$$

At (0, 0), $rt - s^2 = (6x - 6)^2 - 36y^2 = 36 > 0$ and $r = 6x - 6 = -6 < 0$

$\therefore$ $f(0, 0) = 4$ is the maximum value

At (2, 0), $rt - s^2 = (6x - 6)^2 - 36y^2 = 36 > 0$ and $r = 6x - 6 = 6 > 0$

$\therefore$ $f(2, 0) = 0$ is the minimum value

At $(1, \pm 1)$, $rt - s^2 = (6x - 6)^2 - 36y^2 = -36 < 0$

$\therefore$ $f(1, \pm 1)$ is not an extreme value.

Problem 3: Find the maximum and minimum value of $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$, $(x > 0, y > 0)$.

Solution: Given $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ ……. (1)

For f to be maxima or minima

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

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We have $\frac{\partial f}{\partial x} = 4(x^3 - x + y) = 0$ ……. (2)

and $\frac{\partial f}{\partial y} = 4(y^3 + x - y) = 0$ ……. (3)

Solving (2) and (3), we get

$x = 0, -\sqrt{2}, \sqrt{2}$ and the corresponding for are $y = 0, \sqrt{2}, -\sqrt{2}$

Hence $(0,0), \left(-\sqrt{2}, \sqrt{2}\right)$ and $\left(\sqrt{2}, -\sqrt{2}\right)$ are the stationary points of f.

Now $r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$

$s = \frac{\partial^2 f}{\partial x \partial y} = 4$

$t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$

At the point (0, 0), $rt - s^2 = \left(12x^2 - 4\right)\left(12y^2 - 4\right) - 16 = 0.$

Therefore, we cannot say anything. It needs further investigation.

At the points $\left(-\sqrt{2}, \sqrt{2}\right), \left(\sqrt{2}, -\sqrt{2}\right)$

$rt - s^2 = 20 \times 20 - 16 = 384 > 0$ and $r = 20 > 0$

$\therefore$ The function f attains minimum value at $\left(-\sqrt{2}, \sqrt{2}\right)$ and $\left(\sqrt{2}, -\sqrt{2}\right)$.

Problem 4: The sum of three numbers is constant. Prove that product is maximum when they are equal.

Solution: Let the three numbers be x, y, z

Given $x + y + z = a.$

$\therefore z = a - x - y$

Let the product of three numbers be

$P = xyz = xy(a - x - y)$ ……. (1)

The product is maximum or minimum if

For maxima or minima

$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

Now $\frac{\partial P}{\partial x} = ay - 2xy - y^2 = 0 \Rightarrow y(a - 2x - y) = 0$

$\Rightarrow 2x + y = a$ ……. (2)

and $\frac{\partial P}{\partial y} = ax - 2xy - x^2 = 0 \Rightarrow x(a - 2y - x) = 0$

$\Rightarrow x + 2y = a$ ……. (3)

Solving (2) and (3), we get

$x = y = z = \frac{a}{3}$

Now $r = \frac{\partial^2 P}{\partial x^2} = -2y$

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$$s=\frac{\partial^2 P}{\partial x\partial y}=a-2x-2y$$

$$t=\frac{\partial^2 P}{\partial y^2}=-2x$$

At the point $\left(\frac{a}{3},\frac{a}{3},\frac{a}{3}\right)$,

$$rt-s^2=4xy-(a-2x-2y)^2$$

$$=\frac{4a^2}{9}-\frac{a^2}{9}=\frac{a^2}{3}>0$$

and $r=\frac{-2a}{3}<0$

$\therefore$ P is maximum at $\left(\frac{a}{3},\frac{a}{3},\frac{a}{3}\right)$

The product is maximum, if the numbers are equal.

Problem 5: Discuss the maxima and minima of $u(x,y)=\sin x\sin y\sin(x+y)$, where $0<x<\pi$ and $0<y<\pi$.

Solution: Given function is $u(x,y)=\sin x\sin y\sin(x+y)$

$$\therefore \frac{\partial u}{\partial x}=\sin y\left[\sin x\cos(x+y)+\cos x\sin(x+y)\right]$$

$$=\sin y\sin(2x+y)$$

$$r=\frac{\partial^2 u}{\partial x^2}=2\sin y\cos(2x+y)$$

$$\frac{\partial u}{\partial y}=\sin x\left[\sin y\cos(x+y)+\cos y\sin(x+y)\right]$$

$$=\sin x\sin(x+2y)$$

$$s=\frac{\partial^2 u}{\partial x\partial y}=\sin x\cos(x+2y)+\cos x\sin(x+2\mathrm{y})$$

$$=\sin(2x+2\mathrm{y})$$

$$t=\frac{\partial^2 u}{\partial y^2}=2\sin x\cos(x+2y)$$

Now $\frac{\partial u}{\partial x}=0\Rightarrow \sin y\sin(2x+y)=0$

where $0<x<\pi$, $\sin y\neq 0$

Hence $\sin(2x+y)=0$ ……. (1)

Similarly, $\frac{\partial u}{\partial y}=0\Rightarrow\sin(x+2y)=0$ ……. (2)

From (1) and (2), we get

$$2x+y=\pi \text{ and } x+2y=\pi$$

Solving these equations, we get

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$$x = \frac{\pi}{3},\ y = \frac{\pi}{3}$$

Now At the point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$,

$$rt - s^2 = 2\sin(\pi/3)\cos(\pi).2\cos(\pi/3)\sin\pi - \sin(4\pi/3)$$

$$= \left(-\sqrt{3}\right)\left(-\sqrt{3}\right) - \left(\frac{-\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

and $r = -\sqrt{3} < 0$

Hence $u(x, y)$ is maximum at $(\pi/3, \pi/3)$

Maximum value of $u(x, y) = \sin(\pi/3)\sin(\pi/3)\sin(2\pi/3) = \left(3\sqrt{3}\right)/8.$

LAGRANGE'S METHOD OF UNDETERMINED MULTIFLIERS:

Note: To find the maxima or minima for a function $f(x, y, z) = 0$ subject to the conditions $\phi_1(x, y, z) = 0$ and $\phi_2(x, y, z) = 0$, form the Lagrange's function as

$$\text{F}(x,\ y) = f(x, y, z) + \lambda\phi_1(x, y, z) + \mu\phi_2(x, y, z)$$

where λ and μ are the Lagrange's multipliers and proceed as above.

Problem 1: Find the points on the plane $ax + by + cz = d$ which is nearest to the origin.

Solution: Let $\text{P}(x,\ y,\ z)$ be any point on the given plane.

Then $OP = \sqrt{x^2 + y^2 + z^2}$

Let $f = x^2 + y^2 + z^2$ …… (1)

Now we have to minimize (1) subject to the condition

$$\phi(x, y, z) = ax + by + cz - d = 0 \qquad \text{…… (2)}$$

Consider the Lagrangian function

$$i.e.,\ F(x, y, z) = x^2 + y^2 + z^2 + \lambda(ax + by + cz - d)$$

For F to be minima or maxima, $\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda a = 0 \qquad \therefore\ x = -\frac{a\lambda}{2} \qquad \text{…… (3)}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y + \lambda b = 0 \qquad \therefore\ y = -\frac{b\lambda}{2} \qquad \text{…… (4)}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda c = 0 \qquad \therefore\ z = -\frac{c\lambda}{2} \qquad \text{…… (5)}$$

Substituting (3), (4) and (5) in (2), we get

$$-\frac{a^2\lambda}{2} - \frac{b^2\lambda}{2} - \frac{c^2\lambda}{2} - d = 0 \Rightarrow \lambda = \frac{-2d}{a^2 + b^2 + c^2} = \frac{-2d}{p},\ \text{where } p = a^2 + b^2 + c^2$$

Putting this value of λ in (3), (4), (5), we get

$$x = \frac{ad}{p}, \text{y} = \frac{bd}{p}, z = \frac{cd}{p}$$

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Hence $\left(\frac{ad}{p},\frac{bd}{p},\frac{cd}{p}\right)$ is the point on the given plane which nearest to the origin.

Problem 2: Find the minimum value of $x^2+y^2+z^2$ subject to the condition $xyz=a^3$

Solution: Let $f=x^2+y^2+z^2$ …… (1)

and $\phi(x,y,z)=xyz-a^3=0$ …… (2)

Consider the Lagrangian function:

$$i.e.,\ F(x,y,z)=x^2+y^2+z^2+\lambda(xyz-a^3)$$

For F to be minima, $\frac{\partial F}{\partial x}=0,\ \frac{\partial F}{\partial y}=0,\ \frac{\partial F}{\partial z}=0$

$$\frac{\partial F}{\partial x}=0\Rightarrow 2x+\lambda yz=0 \qquad \therefore\ \frac{x}{yz}=-\frac{\lambda}{2} \qquad \ldots\ldots\ (3)$$

$$\frac{\partial f}{\partial y}=0\Rightarrow 2y+\lambda xz=0 \qquad \therefore\ \frac{y}{xz}=-\frac{\lambda}{2} \qquad \ldots\ldots\ (4)$$

$$\frac{\partial F}{\partial z}=0\Rightarrow 2z+\lambda xy=0 \qquad \therefore\ \frac{z}{xy}=-\frac{\lambda}{2} \qquad \ldots\ldots\ (5)$$

Substituting (3), (4) and (5) in (2), we get

$$\frac{x}{yz}=\frac{y}{zx}=\frac{z}{xy}=-\frac{\lambda}{2} \qquad \ldots\ldots\ (6)$$

From the first two members , we have

$$\frac{x}{yz}=\frac{y}{zx}\Rightarrow x^2=y^2 \qquad \ldots\ldots\ (7)$$

From the last two members , we have

$$\frac{y}{zx}=\frac{z}{xy}\Rightarrow y^2=z^2 \qquad \ldots\ldots\ (8)$$

From (7) and (8), we have

$$x^2=y^2=z^2\Rightarrow x=y=z \qquad \ldots\ldots\ (9)$$

Solving (2) and (9), we get

$$x=y=z=a$$

Minimum value of $f=a^2+a^2+a^2=3a^2$

Problem 3: Find the minimum value of $x^2+y^2+z^2$ given $x+y+z=3a$

Solution: Let $f=x^2+y^2+z^2$ …… (1)

and $\phi(x,y,z)=x+y+z-3a=0$ …… (2)

Consider the Lagrangian function:

$$i.e.,\ F(x,y,z)=x^2+y^2+z^2+\lambda(x+y+z-3a)$$

For F to be minima, $\frac{\partial F}{\partial x}=0,\ \frac{\partial F}{\partial y}=0,\ \frac{\partial F}{\partial z}=0$

$$\frac{\partial F}{\partial x}=0\Rightarrow 2x+\lambda=0 \qquad \therefore\ x=-\frac{\lambda}{2} \qquad \ldots\ldots\ (3)$$

$$\frac{\partial f}{\partial y}=0\Rightarrow 2y+\lambda=0 \qquad \therefore\ y=-\frac{\lambda}{2} \qquad \ldots\ldots\ (4)$$

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$$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda = 0 \qquad \therefore\ z = -\frac{\lambda}{2} \qquad \ldots\ldots \quad (5)$$

Substituting (3), (4) and (5) in (2), we get

$$-\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} = 3a \Rightarrow -\frac{3\lambda}{2} = 3a \text{ or } \lambda = -2a \qquad \ldots\ldots \quad (6)$$

Using this value $\lambda = -2a$ in (3), (4) and (5), we have

$$\therefore\ x = a,\ y = a,\ z = a \qquad \ldots\ldots \quad (7)$$

The possible extreme point is (a, a, a)

Hence the minimum value of $f = a^2 + a^2 + a^2 = 3a^2$

Problem 4: Find the maximum value of $x^m y^n z^p$ subject to $x + y + z = a$

Solution: Let $f = x^m y^n z^p$ $\qquad \ldots\ldots \quad (1)$

and $\phi(x, y, z) = x + y + z - a = 0$ $\qquad \ldots\ldots \quad (2)$

Consider the Lagrangian function:

$$i.e.,\ F(x, y, z) = x^m y^n z^p + \lambda(x + y + z - a)$$

For F to be maxima, $\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow m x^{m-1} y^n z^p + \lambda = 0 \qquad \therefore\ x = -\frac{mf}{\lambda} \qquad \ldots\ldots \quad (3)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow n x^m y^{n-1} z^p + \lambda = 0 \qquad \therefore\ y = -\frac{nf}{\lambda} \qquad \ldots\ldots \quad (4)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow p x^m y^n z^{p-1} + \lambda = 0 \qquad \therefore\ z = -\frac{pf}{\lambda} \qquad \ldots\ldots \quad (5)$$

Substituting (3), (4) and (5) in (2), we get

$$-\frac{mf}{\lambda} - \frac{nf}{\lambda} - \frac{pf}{\lambda} = a \Rightarrow \lambda = \frac{-(m + n + p) f}{a} \qquad \ldots\ldots \quad (6)$$

Substituting this value of λ in (3), (4) and (5), we have

$$\therefore\ x = \frac{am}{m + n + p},\ y = \frac{an}{m + n + p},\ z = \frac{ap}{m + n + p} \qquad \ldots\ldots \quad (7)$$

Hence the maximum value of $f = \left(\frac{am}{m + n + p}\right)^m \left(\frac{an}{m + n + p}\right)^n \left(\frac{ap}{m + n + p}\right)^p$

$$= \frac{a^{m+n+p} . m^m n^n p^p}{(m + n + p)^{m+n+p}}$$

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Unit-IV

Multiple Integrals

Double integrals, change of order of integration, change of variables. Evaluation of triple integrals, change of variables between Cartesian, cylindrical and spherical polar co-ordinates. Finding areas and volumes using double and triple integrals.

Double Integrals:

Problem 1: Evaluate $\int_0^1\int_0^{x^2} e^{\frac{y}{x}}\,dydx$

Solution: $I=\int_0^1\int_0^{x^2} e^{\frac{y}{x}}\,dydx=\int_0^1\left(\frac{e^{y/x}}{1/x}\right)_0^{x^2}dx$

$$=\int_0^1 x\left(e^x-1\right)dx=\int_0^1\left(xe^x-x\right)dx$$

$$=\left[e^x(x-1)-\frac{x^2}{2}\right]_0^1$$

$$=0-\frac{1}{2}-(-1)=\frac{1}{2}$$

Problem 2: Evaluate $\int_{x=0}^{a}\int_{y=0}^{b}\left(x^2+y^2\right)dy\,dx$

Solution: Given integral $I=\int_{x=0}^{a}\int_{y=0}^{b}\left(x^2+y^2\right)dy\,dx$

$$=\int_0^a\left(x^2y+\frac{y^3}{3}\right)_0^b dx=\int_0^a\left(x^2b+\frac{b^3}{3}\right)_0^b dx$$

$$=\left[b\frac{x^3}{3}+\frac{b^3}{3}x\right]_0^a=\frac{ba^3}{3}+\frac{ab^3}{3}$$

$$=\frac{ab}{3}\left(a^2+b^2\right)$$

Problem 3: Evaluate $\int_0^1\int_0^1\frac{1}{\sqrt{\left(1-x^2\right)\left(1-y^2\right)}}\,dx\,dy$

Solution: Given integral $I=\int_{x=0}^{1}\int_{y=0}^{1}\frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}\,dx\,dy$

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$$= \int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}} dx \int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}} \, dy$$

$$= \left[\sin^{-1} x\right]_{x=0}^{1} \left[\sin^{-1} y\right]_{y=0}^{1}$$

$$= \frac{\pi}{2} . \frac{\pi}{2} = \frac{\pi^2}{4}$$

Problem 4: Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} dy\, dx$

Solution: Given integral $I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} dy\, dx$

$$= \int_0^1 \int_0^a \frac{1}{a^2+y^2} dy\, dx \quad \text{put } \sqrt{1+x^2} = a$$

$$= \int_0^1 \left(\frac{1}{a} \tan^{-1} \frac{y}{a} \right)_0^a dx$$

$$= \int_0^1 \frac{1}{a} \left(\tan^{-1} 1 - \tan^{-1} 0 \right) dx$$

$$= \int_0^1 \frac{1}{a} \left(\pi / 4 - 0 \right) dx = \frac{\pi}{4} \int_0^1 \sqrt{1+x^2}\, dx$$

$$= \frac{\pi}{4} \left[\log\left(x + \sqrt{1+x^2} \right) \right]_0^1$$

$$= \frac{\pi}{4} \log\left(1+\sqrt{2}\right) \; (or) \; \frac{\pi}{4} \sinh^{-1} 1$$

Problem 5: Evaluate $\int_0^1 \int_y^{y^2+1} x^2 y \; dx\, dy$

Solution: Given integral $I = \int_{y=0}^{1} \int_{x=y}^{y^2+1} x^2 y \; dx\, dy = \int_0^3 y \left(\frac{x^3}{3} \right)_{x=y}^{x=y^2+1} dx$

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$$= \int_0^3 y\left(\frac{(y^2+1)^3}{3} - \frac{y^3}{3}\right)_{x=y}^{x=} dx$$

$$= \int_0^3 y\left(\frac{y^6+3y^4+3y^2+1}{3} - \frac{y^3}{3}\right)dx$$

$$= \frac{1}{3}\int_0^3 \left(y^7+3y^5-y^4+3y^3+y\right)dx$$

$$= \frac{1}{3}\left(\frac{y^8}{8}+3\frac{y^6}{6}-\frac{y^5}{5}+3\frac{y^4}{4}+\frac{y^2}{2}\right)_{y=0}^{3}$$

$$= \frac{1}{3}\left(\frac{y^8}{8}+3\frac{y^6}{6}-\frac{y^5}{5}+3\frac{y^4}{4}+\frac{y^2}{2}\right)_{y=0}^{3}$$

$$= \frac{1}{3}\left(\frac{6561}{8}+\frac{2187}{6}-\frac{243}{5}+\frac{243}{4}+\frac{9}{2}\right) = \frac{67}{120}$$

Problem 6: Evaluate $\iint_R \left(x^2+y^2\right)dx\,dy$ in the positive quadrant for which $x+y \le 1$.

Solution: Given integral $I = \iint_R \left(x^2+y^2\right)dx\,dy$

$$= \int_{x=0}^{1}\int_{y=0}^{y=1-x} \left(x^2+y^2\right)dy\,dx$$

$$= \int_{x=0}^{1}\left(x^2y+\frac{y^3}{3}\right)_{y=0}^{y=1-x} dx$$

$$= \int_{x=0}^{1}\left(x^2-x^3+\frac{(1-x)^3}{3}\right)dx$$

$$= \left(\frac{x^3}{3}-\frac{x^4}{4}-\frac{(1-x)^4}{12}\right)_{x=0}^{x=1}$$

$$= \frac{1}{3}-\frac{1}{4}-\frac{1}{12} = \frac{1}{6}$$

Problem 7: Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by x –axis, ordinate $x = 2a$ and the curve $x^2 = 4ay.$

Solution: Given integral $I = \iint_R y\,dx\,dy = \int_{y=0}^{a}\int_{x=2\sqrt{ay}}^{x=2a} \left(x^2+y^2\right)dy\,dx$

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$$= \int_{y=0}^{a} \int_{x=2\sqrt{ay}}^{x=2a} xy\,dy\,dx$$

$$= \int_{y=0}^{a} y\left[\frac{x^2}{2}\right]_{x=2\sqrt{ay}}^{x=2a} dy$$

$$= \int_{y=0}^{a} y\left(2a^2 - 2ay\right)dy$$

$$= \left(2a^2\frac{y^2}{2} - 2a\frac{y^3}{3}\right)_{y=0}^{a}$$

$$= a^4 - \frac{2a^4}{3} = \frac{a^4}{3}$$

Problem 8: Find the value of $\iint xy\,dx\,dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Solution: Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

The region of integration can be expressed as

$$0 \le x \le a,\ 0 \le y \le \frac{b}{a}\sqrt{a^2 - x^2},$$

Given integral $I = \iint_R xy\,dx\,dy = \int_{x=0}^{a} \int_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} x\,y\,dy\,dx$

$$= \int_{x=0}^{a} \int_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} x\,y\,dy\,dx$$

$$= \int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{b^2}{2a^2}\int_{x=0}^{a}(xa^2 - x^3)dx = \frac{b^2}{2a^2}\left(a^2\frac{x^2}{2} - \frac{x^4}{4}\right)_{x=0}^{a}$$

$$= \frac{b^2}{2a^2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{b^2}{2a^2}\frac{a^4}{4} = \frac{a^2b^2}{8}$$

Home Work:

Problem 1: Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dy\,dx$

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Problem 2: Evaluate $(i)\ \int_0^2\int_0^x e^{x+y}dy\,dx \qquad (ii)\ \int_0^1\int_0^x e^{x+y}dy\,dx$

Problem 3: Evaluate $\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$

Problem 4: Find the value of $\iint xy\,dx\,dy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$.

Problem 5: Find the area included between the parabolas $y^2=4x$ and $x^2=4y$.

DOUBLE INTEGRALS IN POLAR COORDINATES:

To evaluate $\int_{\theta=\theta_1}^{\theta=\theta_2}\int_{r=r_1}^{r=r_2} f(r,\theta)\,dr\,d\theta$ over the region bounded by the lines $\theta=\theta_1$ and $\theta=\theta_2$ the curves $r=r_1, r=r_2$. We first integrate w.r.to r between the limits $r=r_1$ and $r=r_2$. Keeping θ fixed and then integrate w.r.to θ from θ_1 and θ_2 . In this integral r_1 are r_2 functions of θ and θ_1 and θ_2 are constants.

Problem 1: Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

Solution: Given integral $I=\int_{\theta=0}^{\pi}\int_{r=0}^{a\sin\theta} r\,dr\,d\theta=\int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_{r=0}^{r=a\sin\theta} d\theta$

$$=\frac{1}{2}\int_{\theta=0}^{\pi}\left(a^2\sin^2\theta\right)d\theta=\frac{a^2}{2}\int_{\theta=0}^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$=\frac{a^2}{4}\int_{\theta=0}^{\pi}(1-\cos 2\theta)d\theta$$

$$=\frac{a^2}{4}\left(\theta-\frac{\sin 2\theta}{2}\right)_{\theta=0}^{\theta=\pi}=\frac{\pi a^2}{4}$$

Problem 2: Evaluate $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r\,dr\,d\theta$

Solution: Given integral $I=\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r\,dr\,d\theta=-\frac{1}{2}\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}-2r\,dr\,d\theta$

$$=-\frac{1}{2}\int_0^{\pi/2}\int_0^{\infty} d\left(e^{-r^2}\right)d\theta$$

$$=-\frac{1}{2}\int_{\theta=0}^{\pi/2}\left[e^{-r^2}\right]_{r=0}^{\infty} d\theta$$

$$=-\frac{1}{2}\int_{\theta=0}^{\pi/2}\left[0-1\right]d\theta$$

$$=\frac{1}{2}\int_{\theta=0}^{\pi/2} 1.d\theta=\frac{1}{2}[\theta]_{\theta=0}^{\pi/2}=\frac{\pi}{4}$$

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Problem 3: Evaluate $\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sin\theta}\frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta$

Solution: Given integral $I=\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sin\theta}\frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta=-\frac{1}{2}\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sin\theta}\frac{-2r}{\sqrt{a^2-r^2}}\,dr\,d\theta$

$$=-\frac{1}{2}\int_{\theta=0}^{\pi/4}\left(2\sqrt{a^2-r^2}\right)_{r=0}^{r=a\sin\theta}d\theta$$

$$=-\int_{\theta=0}^{\pi/4}\left(\sqrt{a^2-a^2\sin^2\theta}-\sqrt{a^2}\right)d\theta$$

$$=-a\int_{\theta=0}^{\pi/4}\left(\sqrt{\cos^2\theta}-1\right)d\theta=a\int_{\theta=0}^{\pi/4}\left(1-\cos\theta\right)d\theta$$

$$=a\left(\theta-\sin\theta\right)_{\theta=0}^{\pi/4}=a\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right)$$

Problem 4: Evaluate $\int_{\theta=0}^{\pi/2}\int_{r=a}^{a(1+\cos\theta)}r\,dr\,d\theta$

Solution: Given integral $I=\int_{\theta=0}^{\pi/2}\int_{r=a}^{a(1+\cos\theta)}r\,dr\,d\theta=\int_{\theta=0}^{\pi/2}\left[\frac{r^2}{2}\right]_{r=a}^{r=a(1+\cos\theta)}d\theta$

$$=\int_{\theta=0}^{\pi/2}\left[\frac{a^2(1+\cos\theta)^2}{2}-\frac{a^2}{2}\right]d\theta$$

$$=a^2\int_{\theta=0}^{\pi/2}\left[\frac{\cos^2\theta+2\cos\theta}{2}\right]d\theta$$

$$=a^2\int_{\theta=0}^{\pi/2}\left[\frac{1+\cos2\theta}{4}+\cos\theta\right]d\theta$$

$$=a^2\left[\frac{\theta}{4}+\frac{\sin2\theta}{8}+\sin\theta\right]_{\theta=0}^{\pi/2}=a^2\left(\frac{\pi}{8}+1\right)$$

TO FIND THE POLAR LIMITS OF DOUBLE INTEGRALS:

Consider the double integral $\iint_R f(r,\,\theta)\,dr\,d\theta$ over a region R, where the limits of integration of the region are not specified.

Hence $\iint_R f(r,\,\theta)\,dr\,d\theta=\int_{\theta=\theta_1}^{\theta=\theta_2}\int_{r=f(\theta_1)}^{r=f(\theta_2)}f(r,\theta)\,dr\,d\theta$

Problem 1: Evaluate $\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta}r\sqrt{a^2-r^2}\,dr\,d\theta$

Solution: Given integral $I=\frac{-1}{2}\int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{a\cos\theta}(-2r)\sqrt{a^2-r^2}\,dr\right]d\theta$

$$= \frac{-1}{2} \int_{\theta=0}^{\pi/2} \left[\frac{\left(a^2 - r^2\right)^{3/2}}{3/2} \right]_{r=0}^{r=a\cos\theta} d\theta$$

$$= \frac{-1}{3} \int_{\theta=0}^{\pi/2} \left[\left(a^2 - a^2\cos^2\theta\right)^{3/2} - \left(a^2\right)^{3/2} \right] d\theta$$

$$= \frac{-1}{3} \int_{\theta=0}^{\pi/2} \left[\left(a^2\sin^2\theta\right)^{3/2} - \left(a^2\right)^{3/2} \right] d\theta$$

$$= \frac{-1}{3} \int_{\theta=0}^{\pi/2} \left[a^3\sin^3\theta - a^3 \right] d\theta$$

$$= \frac{-a^3}{3}\left[\frac{2}{3}.1 - \frac{\pi}{2}\right] = \frac{a^3}{18}[3\pi - 4].$$

Problem 2: Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the cardiod $r = a(1-\cos\theta)$ above the initial line

Solution: The cardiod $r = a(1-\cos\theta)$ is symmetrical about the initial line $\theta = 0.$ The region of integration R above the initial line is r is Various from $r = 0$ to $r = a(1-\cos\theta)$ and θ various from $\theta = 0$ to $\theta = \pi$

Given integral $I = \iint_R r\sin\theta\, dr\, d\theta = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1-\cos\theta)} r\, dr \sin\theta d\theta$

$$= \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_{r=0}^{a(1-\cos\theta)} d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi} (1-\cos\theta)^2 \sin\theta\, d\theta$$

$$= \frac{a^2}{2}\left[\frac{(1-\cos\theta)^3}{3}\right]_{\theta=0}^{\theta=\pi}$$

$$= \frac{a^2}{2}\left(\frac{2^3}{3} - \frac{0^3}{3}\right)$$

$$= \frac{4a^2}{3}$$

Problem 3: Evaluate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta.$

Solution: The region of integration R is shown shaded.

Here r is various from $r = 2\sin\theta$ to $r = 4\sin\theta$ and θ various from $\theta = 0$ to $\theta = \pi$

Given integral $I = \iint_R r^3\, dr\, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=2\sin\theta}^{r=4\sin\theta} r^3\, dr \sin\theta d\theta$

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$$= \int_{\theta=0}^{\pi} \left[\frac{r^3}{3} \right]_{r=2\sin\theta}^{r=4\sin\theta} d\theta$$

$$= \int_{\theta=0}^{\pi} \left(\frac{(4\sin\theta)^4}{4} - \frac{(2\sin\theta)^4}{4} \right) d\theta$$

$$= \int_{\theta=0}^{\pi} \left(\frac{256\sin^4\theta}{4} - \frac{16\sin^4\theta}{4} \right) d\theta$$

$$= \frac{240}{4} \int_{\theta=0}^{\pi} \sin^4\theta d\theta = 60 \int_{\theta=0}^{\pi} \sin^4\theta d\theta$$

$$= 120 \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^4\theta d\theta = 120 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{45\pi}{2}$$

Change of order of integration:

In double integral with variable limits, the change of order of integration requires the change of limits also. While doing so, sometimes it is required to spit up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always to draw a strip (rough sketch) of the region of integration.

The change of order of integration quite often facilities the evaluation of a double integral. The following Problems will makes these ideas clear.

Problem 1: Change the order of integration for the integral

$$\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$$

Solution: Given limits are

$x = -a, x = a$ and $y = 0, y = \sqrt{a^2 - x^2}$

i.e., $x = -a, x = a$ and $y = 0, x^2 + y^2 = a^2$

With these limits, the region of integration as shown in the figure. This is the region of semi-circular area.

To change the order of integration, we take a strip parallel to x-axis. This strip moves on $x = -\sqrt{a^2 - y^2}$ and $x = \sqrt{a^2 - y^2}$ from $y = 0$ *to* $y = a$.

Thus, $I = \int_{y=0}^{a} \int_{x=\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

$$= \int_{y=0}^{a} \int_{x=-\sqrt{a^2-y^2}}^{x=\sqrt{a^2-y^2}} f(x,y)dx\,dy$$

Problem 2: To the change of order of integration and evaluate the integral

$$\int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} y^2\,dx\,dy$$

Solution: Given limits are

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$x = 0, x = a$ and $y = 0, y = \sqrt{a^2 - x^2}$

i.e., $x = 0, x = a$ and $y = 0, x^2 + y^2 = a^2$

The region of integration is the region bounded by $x^2 + y^2 = a^2$ in the first quadrant.

To change the order of integration, we take a strip parallel to *x*-axis. This strip moves on $x = 0$ and $x = \sqrt{a^2 - y^2}$ from $y = 0 \; to \; y = a$.

Thus, $I = \int\limits_{y=0}^{a} \int\limits_{x=0}^{\sqrt{a^2-y^2}} y^2 \, dx \, dy$

$$= \int\limits_{y=0}^{a} y^2 [y]_{x=0}^{\sqrt{a^2-y^2}} dx = \int\limits_{y=0}^{a} y^2 \sqrt{a^2 - y^2} \, dx$$

$$= \int\limits_{x=0}^{\pi/2} a^2 \sin^2\theta \, a^2 \cos^2\theta d\theta \qquad put \; y = a\sin\theta \Rightarrow dy = a\cos\theta d\theta$$

$$= \frac{a^4}{4} \int\limits_{x=0}^{\pi/2} 4\sin^2\theta \cos^2\theta d\theta \qquad 4\sin^2\theta\cos^2\theta = \sin^2 2\theta$$

$$= \frac{a^4}{4} \int\limits_{x=0}^{\pi/2} \frac{(1-\cos 4\theta)}{2} d\theta$$

$$= \frac{a^4}{8} \left[\theta - \frac{\sin 4\theta}{2}\right]_{\theta=0}^{\theta=\pi/2} = \frac{\pi a^4}{16}$$

Problem 3: Change the order of integration and evaluate

$$\int\limits_{0}^{4a} \int\limits_{x^2/4a}^{2\sqrt{ax}} dy \, dx$$

Solution: Given limits are

$x = 0, x = 4a$ and $y = \frac{x^2}{4a}, y = 2\sqrt{ax}$

i.e., $x = 0, x = 4a$ and $x^2 = 4ay, y^2 = 4ax$

The region of integration is the shaded region in the figure.

To change the order of integration, we take a strip parallel to *x*-axis. This strip moves on $x = \frac{y^2}{4a}$ *and* $x = 2\sqrt{ay}$ from $y = 0 \; to \; y = 4a$.

Thus, $I = \int\limits_{y=0}^{4a} \left[\int\limits_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx \right] dy = \int\limits_{y=0}^{4a} [x]_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dy$

$$= \int\limits_{y=0}^{4a} \left[2\sqrt{ay} - \frac{y^2}{4a}\right] dy = \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_{y=0}^{4a}$$

$$= \frac{2\sqrt{a}.4a\sqrt{4a}}{3/2} - \frac{64a^3}{12a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

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Problem 4: By changing the order of integration, evaluate $\int_0^1 \int_0^{2-x} xy\,dx\,dy$

Solution: Given limits are

$x = 0, x = 1$ and $y = 0, y = 2 - x$

i.e., $x = 0, x = 1$ and $y = 0,\ x + y = 2$

The region of integration is as shown in the figure.

To change the order of integration, we take a strip parallel to x-axis. This strip moves on $x = 0$ and $x = 2 - y$ from $y = 1$ *to* $y = 2$.

Thus, $I = \int_{y=1}^{2} \int_{x=0}^{2-y} xy\,dx\,dy$

$$= \int_{y=1}^{2} y\left[\int_{x=0}^{2-y} x\,dx\right]dy = \int_{y=1}^{2} y\left[\frac{x^2}{2}\right]_{x=0}^{x=2-y} dy$$

$$= \int_{y=1}^{2} y\left[\frac{(2-y)^2}{2}\right]dy = \frac{1}{2}\int_{y=1}^{2} y\left[\frac{4-4y+y^2}{2}\right]dy$$

$$= \frac{1}{2}\int_{y=1}^{2}\left(4y - 4y^2 + y^3\right)dy = \frac{1}{2}\left[\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right]_{y=1}^{y=2}$$

$$= \frac{1}{2}\left(\frac{4}{3} - \frac{11}{12}\right) = \frac{5}{24}$$

Problem 12: Change the order of integration and evaluate $\int_0^3 \int_{y^2/9}^{\sqrt{10-y^2}} dy\,dx$

Solution: Given limits are

$y = 0, \mathrm{y} = 3$ and $x = y^2/9, x = \sqrt{10 - y^2}$

i.e., $y = 0, \mathrm{y} = 3$ and $y^2 = 9, x^2 + y^2 = 10$

The region of integration is OABC as shown in the figure. This region OABC divided into two parts by drawing a line parallel to y-axis at the point of intersection D.

To change the order of integration, For the region ABD, we take a strip parallel to y-axis. This strip moves on $y = 0$ and $y = 3\sqrt{x}$ from $x = 0$ to $x = 1$. For the region ADC, we take a strip parallel to x-axis. This strip moves on $y = 0$ and $y = \sqrt{10 - x^2}$ from $x = 1$ to $x = \sqrt{10}$ .Hence,

$$\int_0^3 \int_{y^2/9}^{\sqrt{10-y^2}} dy\,dx = \int_0^1 \int_{y=0}^{3\sqrt{x}} dydx + \int_1^{\sqrt{10}} \int_0^{\sqrt{10-x^2}} dx\,dy = I_1 + I_2 (say)$$

Now $I_1 = \int_0^1 \int_{y=0}^{3\sqrt{x}} dydx$

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$$= \int_0^1 [y]_{y=0}^{3\sqrt{x}} dx = \int_0^1 3\sqrt{x}\,dx = 3.\left[\frac{x^{3/2}}{3/2}\right]_0^1 = 2$$

and $$I_2 = \int_1^{\sqrt{10}} \int_0^{\sqrt{10-x^2}} dx\,dy = \int_1^{\sqrt{10}} [y]_{y=0}^{\sqrt{10-x^2}}\, dx$$

$$= \int_1^{\sqrt{10}} \sqrt{10-x^2}\,dx = \left[\frac{x}{2}\sqrt{10-x^2} + \frac{10}{2}\sin^{-1}\left(\frac{x}{\sqrt{10}}\right)\right]_{x=1}^{\sqrt{10}}$$

$$= 5\sin^{-1}(1) - \frac{3}{2} - 5\sin^{-1}\left(\frac{1}{\sqrt{10}}\right) = \frac{5\pi}{2} - \frac{3}{2} - 5\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)$$

Hence, $$\int_0^3 \int_{y^2/9}^{\sqrt{10-y^2}} dy\,dx = 2 + \frac{5\pi}{2} - \frac{3}{2} - 5\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)$$

$$= \frac{1}{2}(1+5\pi) - 5\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)$$

Problem 13: By changing the order of integration, evaluate $\int_0^1 \int_{x^2}^{2+x} xy\,dx\,dy$

Solution: The given integral can be written as $I = \int_{x=0}^{1} \int_{y=x^2}^{2+x} xy\,dy\,dx$

The region of integration is given by

$x = 0, x = 1$ and $y = x^2, y = 2 - x$

i.e., $x = 0, x = 1$ and $y = x^2, x + y = 2$ …… (1)

The region of integration is OAB as shown in the figure. This region OAB divided into two parts
by drawing a line parallel to *x*-axis at the point of intersection C. The point of intersection is given by solving the equation (1).

To change the order of integration, For the region OAC, we take a strip parallel to *x*-axis. This strip moves on $x = 0$ and $x = \sqrt{y}$ from $y = 0$ to $y = 1$. For the region CAB, we take a strip parallel to *x*-axis. This strip moves on $x = 0$ and $x = 2 - y$ from $y = 1$ to $y = 2$.

Hence, $$I = \int_{x=0}^{1} \int_{y=x^2}^{2+x} xy\,dy\,dx$$

$$= \int_{y=0}^{1} y\left[\int_{x=0}^{\sqrt{y}} x\,dx\right]dy + \int_{y=1}^{2} y\left[\int_{x=0}^{2-y} x\,dx\right]dy$$

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$$= \int_{y=0}^{1} y\left[\frac{x^2}{2}\right]_{x=0}^{\sqrt{y}} dy + \int_{y=1}^{2} y\left[\frac{x^2}{2}\right]_{x=0}^{2-y} dy$$

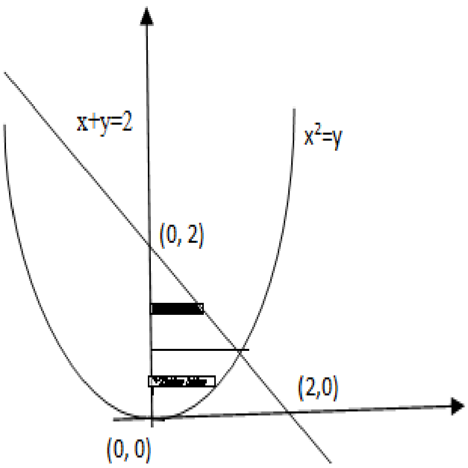
$$= \int_{y=0}^{1} y\left[\frac{y}{2}\right] dy + \int_{y=1}^{2} y\left[\frac{(2-y)^2}{2}\right] dy$$

$$= \frac{1}{2}\int_{y=0}^{1} y^2 dy + \frac{1}{2}\int_{y=1}^{2} \left(4y - 4y^2 + y^3\right) dy$$

$$= \frac{1}{2}.\left[\frac{y^3}{3}\right]_{y=0}^{1} + \frac{1}{2}.\left[4\frac{y^2}{2} - 4\frac{y^3}{3} + \frac{y^4}{4}\right]_{y=1}^{2}$$

$$= \frac{1}{2}.\frac{1}{3} + \frac{1}{2}\left[8 - \frac{32}{3} + 4 - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{9}{24} = \frac{3}{8}$$



Problem 14: By changing the order of integration, evaluate $\int_0^1 \int_x^{1/x} \frac{y}{(1+xy)^2\left(1+y^2\right)} dydx$

Solution: Let $I = \int_{x=0}^{1} \int_{y=x}^{1/x} \frac{y}{(1+xy)^2\left(1+y^2\right)} dydx$

The region of integration is given by

$x = 0, x = 1$ and $y = x, y = 1/x$

i.e., $x = 0, x = 1$ and $y = x,\ xy = 1$ …… (1)

The region of integration is OAB as shown in the figure. This region OAB divided into two parts

by drawing a line parallel to *x*-axis at the point of intersection A(1, 1). The point of intersection is given by solving the equation (1).

To change the order of integration, For the region OCD, we take a strip parallel to *x*-axis. This strip moves on $x = 0$ and $x = y$ from $y = 0$ to $y = 1$. For the region CAB, we take a strip parallel to *x*-axis. This strip moves on $x = 0$ and $x = 1/y$ from $y = 1$ to $y \to \infty$.

Hence, $I = \int_{x=0}^{1}\int_{y=x}^{1/x} \frac{y}{(1+xy)^2(1+y^2)}\,dydx$

$$= \int_{y=0}^{1} \frac{y}{(1+y^2)}\left[\int_{x=0}^{y}\frac{1}{(1+xy)^2}\,dx\right]dy + \int_{y=1}^{\infty}\frac{y}{(1+y^2)}\left[\int_{x=0}^{1/y}\frac{1}{(1+xy)^2}\,dx\right]dy$$

$= I_1 + I_2 \ (say)$

Now $I_1 = \int_{y=0}^{1} \frac{y}{(1+y^2)}\left[\int_{x=0}^{y}\frac{1}{(1+xy)^2}\,dx\right]dy$

$$= \int_{y=0}^{1} \frac{y}{(1+y^2)}\left[\frac{-1}{y(1+xy)}\right]_{x=0}^{y} dy = -\int_{y=0}^{1}\frac{1}{(1+y^2)}\left[\frac{1}{(1+y^2)}-1\right]dy$$

$$= -\int_{y=0}^{1}\left[\frac{1}{(1+y^2)^2}-\frac{1}{(1+y^2)}\right]dy$$

put $y = \tan\theta$ in the first term of the integral $dy = \sec^2\theta d\theta$

When $y = 0 \Rightarrow \theta = 0$ and when $y = 1 \Rightarrow \theta = \pi/4$

$$= -\int_{\theta=0}^{\pi/4}\left[\frac{\sec^2\theta d\theta}{\sec^4\theta}\right] + \left[\tan^{-1}(y)\right]_{y=0}^{1}$$

$$= -\int_{\theta=0}^{\pi/4}\left[\frac{1+\cos 2\theta}{2}\right]d\theta + \left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= -\frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\theta=0}^{\pi/4} + \frac{\pi}{4} - 0$$

$$= -\frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right] + \frac{\pi}{4}$$

$$= -\frac{\pi}{8} - \frac{1}{4} + \frac{\pi}{4} = \frac{\pi}{8} - \frac{1}{4}$$

and $I_2 = \int_{y=0}^{1} \frac{y}{(1+y^2)}\left[\int_{x=0}^{1/y}\frac{1}{(1+xy)^2}\,dx\right]dy$

$$= \int_{y=1}^{\infty} \frac{y}{(1+y^2)}\left[\frac{-1}{y(1+xy)}\right]_{x=0}^{1/y} dy$$

$$= -\int_{y=1}^{\infty} \frac{1}{(1+y^2)}\left[\frac{1}{2}-1\right]dy$$

$$= \frac{1}{2}\int_{y=1}^{\infty}\frac{1}{1+y^2}\,dy = \frac{1}{2}\left[\tan^{-1}(y)\right]_{y=1}^{\infty}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{8}$$

$\therefore I = \frac{\pi}{8} - \frac{1}{4} + \frac{\pi}{8} = \frac{\pi - 1}{4}$

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Triple Integrals:

Let $f(x,y,z)$ be a function defined over a three dimensional finite region *V*. Divide the region V into n elementary volumes $\delta V_1, \delta V_2, ...\delta V_n$. Let (x_r, y_r, z_r) be any point within the r[th] sub-division δV_r. The limit of the sum $\sum_{r=1}^{\infty} f\left(x_r, y_r, z_r\right)\delta V_r,$ as $n \to \infty$ and $\delta V_r \to 0$ is known as triple integral of $f(x,y,z)$ over the region V.

Symbolically, it is denoted by $\iiint_V f(x,y,z)dV$.

Evaluation of Triple Integral:

Problem 1: Evaluate $\int_0^a\int_0^b\int_0^c \left(x^2+y^2+z^2\right)dxdydz$

Solution: Since the limits are constants. So, the order of integration is immaterial.
Integrating first w.r.to *x* keeping *y* and *z* are constants, we have

Given integral $I = \int_0^a\int_0^b \left[\frac{x^3}{3} + xy^2 + xz^2\right]_{x=0}^{c} dydz$

$$= \int_0^a\int_0^b \left[\frac{c^3}{3} + cy^2 + cz^2\right]_{x=0}^{c} dydz$$

Now Integrating first w.r.to *y* keeping *z* is constant, we have

$$= \int_0^a \left[\frac{c^3}{3}y + c\frac{y^3}{3} + cyz^2\right]_{y=0}^{b} dz$$

$$= \int_0^a \left[\frac{bc^3}{3} + \frac{cb^3}{3} + cbz^2\right]dz$$

Finally, Integrating first w.r.to *z*, we get

$$= \left[\frac{bc^3}{3}z + \frac{cb^3}{3}z + cb\frac{z^3}{3}\right]_{z=0}^{z=a}$$

$$= \frac{abc^3}{3} + \frac{acb^3}{3} + cb\frac{a^3}{3}$$

$$\therefore I = \frac{abc}{3}\left(a^2+b^2+c^2\right)$$

Problem 2: Evaluate $\int_0^1\int_1^2\int_2^3 xyz\, dxdydz$

Solution: Given integral $I = \int_{z=0}^{1}\int_{y=1}^{2}\int_{x=2}^{3} xyz\, dxdydz$

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$$= \int_{z=0}^{1} \int_{y=1}^{2} yz \left[\frac{x^2}{2} \right]_{x=2}^{x=3} dydz$$

$$= \frac{5}{2} \int_{z=0}^{1} z \left[\frac{y^2}{2} \right]_{y=1}^{y=2} dz$$

$$= \frac{15}{4} \int_{z=0}^{1} zdz = \frac{15}{4} \left[\frac{z^2}{2} \right]_{z=0}^{z=1} = \frac{15}{8}$$

Alternative Method

Given integral $I = \int_{z=0}^{1} \int_{y=1}^{2} \int_{x=2}^{3} xyz\, dxdydz$

$$= \left(\int_{z=0}^{1} zdz \right) \left(\int_{y=1}^{2} ydy \right) \left(\int_{x=2}^{3} x\, dx \right)$$

$$= \left[\frac{z^2}{2} \right]_{z=0}^{1} \left[\frac{y^2}{2} \right]_{y=1}^{2} \left[\frac{x^2}{2} \right]_{x=2}^{3}$$

$$= \left(\frac{1}{2} - 0 \right) \left(\frac{4}{2} - \frac{1}{2} \right) \left(\frac{9}{2} - \frac{4}{2} \right) = \frac{1}{2} . \frac{3}{2} . \frac{5}{2} = \frac{15}{8}$$

Problem 3: Evaluate $\int_0^2 \int_1^z \int_0^{yz} xyz\, dxdydz$

Solution: Given integral $I = \int_0^2 \int_1^z \int_0^{yz} xyz\, dxdydz$

$$= \int_{z=0}^{2} \int_{y=1}^{z} yz \left[\frac{x^2}{2} \right]_{x=0}^{yz} dydz$$

$$= \frac{1}{2} \int_{z=0}^{2} \int_{y=1}^{z} yz \left[(yz)^2 \right] dydz$$

$$= \frac{1}{2} \int_{z=0}^{2} z^3 \int_{y=1}^{z} y^3\, dydz = \frac{1}{2} \int_{z=0}^{2} z^3 \left[\frac{y^4}{4} \right]_{y=1}^{z} dz$$

$$= \frac{1}{8} \int_{z=0}^{2} z^3 \left[z^4 - 1 \right] dz = \frac{1}{8} \left[\frac{z^8}{8} - \frac{z^4}{4} \right]_{z=0}^{2}$$

$$= \frac{1}{8} \left[\frac{2^8}{8} - \frac{2^4}{4} \right] = \frac{1}{8} (32 - 4) = \frac{7}{2}$$

Problem 4: Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dzdydx$

Solution: Given integral $I = \int_{y=1}^{e} \int_{x=1}^{\log y} \left[\int_{z=1}^{e^x} \log z\, dz \right] dxdy$

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$$= \int_{y=1}^{e} \int_{x=1}^{\log y} \left[z \log z - z \right]_{z=1}^{e^x} dxdy$$

$$= \int_{y=1}^{e} \int_{x=1}^{\log y} \left[xe^x - e^x + 1 \right] dxdy$$

$$= \int_{y=1}^{e} \left[xe^x - e^x - e^x + x \right]_{x=1}^{\log y} dy$$

$$= \int_{y=1}^{e} \left[y \log y - 2y + \log y + (e-1) \right] dy$$

$$= \int_{y=1}^{e} \left[(y+1) \log y - 2y + (e-1) \right] dy$$

$$= \left[\left(\frac{y^2}{2} + y \right) \log y - \left(\frac{y^2}{4} + y \right) - y^2 + (e-1)y \right]_{y=1}^{e}$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4} = \frac{1}{4}\left(e^2 - 8e + 13 \right)$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} \left[xy + \frac{y^2}{2} + zy \right]_{y=x-z}^{y=x+z} dxdz$$

$$= \int_{z=1}^{1} \int_{x=0}^{z} \left[xy + \frac{y^2}{2} + zy \right]_{y=x-z}^{y=x+z} dxdz$$

$$= \int_{z=-1}^{-1} \int_{x=0}^{z} \left[x(x+z-x+z) + \frac{(x+z)^2 - (x+z)^2}{2} + z(x+z-x+z) \right] dxdz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} \left[2xz + \frac{4xz}{2} + 2z^2 \right] dxdz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} \left[4xz + 2z^2 \right] dxdz = 2 \int_{z=-1}^{1} \left[2z \frac{x^2}{2} + z^2 x \right]_{x=0}^{x=z} dz$$

$$= 2 \int_{z=-1}^{1} \left[z^3 + z^3 \right]_{x=0}^{x=z} dz$$

$$= 4 \int_{z=-1}^{1} z^3 dz = 4 \left[\frac{z^4}{4} \right]_{z=-1}^{z=1}$$

$$= 1 - 1 = 0$$

Problem 7: Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} xdzdxdy$

Solution: Given integral I $= \int_{y=0}^{1} \int_{x=y2}^{1} \int_{z=0}^{1-x} xdzdxdy$

$$= \int_{y=0}^{1} \int_{x=y^2}^{1} x \left[\int_{z=0}^{1-x} 1.dz \right] dxdy$$

$$= \int_{y=0}^{1} \int_{x=y^2}^{1} x \left[z \right]_{z=0}^{1-x} dxdy$$

$$= \int_{y=0}^{1} \int_{x=y^2}^{1} x(1-x)\,dxdy = \int_{y=0}^{1} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{x=y^2}^{1} dy$$

$$= \int_{y=0}^{1} \left[\left(\frac{1}{2} - \frac{1}{3} \right) - \left(\frac{x^4}{2} - \frac{y^6}{3} \right) \right] dy$$

$$= \left[\frac{y}{6} - \frac{y^5}{10} + \frac{y^7}{21} \right]_{x=0}^{1} = \frac{4}{35}$$

Problem 8: Evaluate $\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} dzdydx$

Solution: Given integral $I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} dzdydx$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left[\int_{z=0}^{p} \frac{1}{\sqrt{p^2-z^2}} dz \right] dydx, \quad \text{where } p = \sqrt{1-x^2-y^2}$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left[\int_{z=0}^{p} \frac{1}{\sqrt{p^2-z^2}} dz \right] dydx$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{p} \right) \right]_{z=0}^{p} dydx$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \frac{\pi}{2} dydx = \frac{\pi}{2} \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} 1\,dydx$$

$$= \frac{\pi}{2} \int_{x=0}^{1} [y]_{y=0}^{\sqrt{1-x^2}} dx = \frac{\pi}{2} \int_{x=0}^{1} \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2} \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x) \right]_{x=0}^{1} = \frac{\pi}{2}\left(0 + \frac{1}{2}.\frac{\pi}{2} \right) = \frac{\pi^2}{8}$$

Problem 9: Evaluate the triple integral $\iiint xy^2 z\, dxdydz$ take through the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$

Solution: Equation of the sphere $x^2 + y^2 + z^2 = a^2$

The limits of the integration are

$z = 0, z = \sqrt{a^2 - x^2 - y^2}, y = 0, y = \sqrt{a^2 - x^2}$ and $x = 0, x = a$

Given integral $I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} xy^2 z\, dzdydx$

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$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy^2 \left[\int_{z=0}^{\sqrt{a^2-x^2-y^2}} z\,dz \right] dydx$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy^2 \left[\frac{z^2}{2} \right]_{z=0}^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= \frac{1}{2} \int_{x=0}^{a} x \int_{y=0}^{\sqrt{a^2-x^2}} y^2 \left[a^2 - x^2 - y^2 \right] dydx$$

$$= \frac{1}{2} \int_{x=0}^{a} x \int_{y=0}^{\sqrt{a^2-x^2}} \left[y^2 (a^2 - x^2) - y^4 \right] dydx$$

$$= \frac{1}{2} \int_{x=0}^{a} x \left[(a^2 - x^2)^{5/2} \left(\frac{1}{3} - \frac{1}{5} \right) \right] dx$$

$$= \frac{1}{2} \frac{2}{15} \int_{x=0}^{a} x(a^2 - x^2)^{5/2}\, dx$$

$$= \frac{-1}{15} \int_{x=0}^{a} -2x(a^2 - x^2)^{5/2}\, dx$$

$$= \frac{-1}{15} \left[\frac{(a^2 - x^2)^{7/2}}{7/2} \right]_{x=0}^{a} = \frac{a^7}{105}$$

CHANGE OF VARIABLES IN A TRIPLE INTEGRAL:

Let the variables x, y, z be changed to new variables u, v, w by the transformation.

$$x = \phi_1(u, v, w), \mathrm{y} = \phi_2(u, v, w), \mathrm{z} = \phi_3(u, v, w)$$

Where $\phi_1(u, v, w), \phi_2(u, v, w), \phi_3(u, v, w)$ are continuous and continuous first order derivatives in some region V' in the uvw-plane which corresponds to the region V in the xyz-plane. Then

$$\iiint_V f(x, y, z)\, dxdydz = \iiint_{V'} f(\phi_1, \phi_2, \phi_3) \left| J \right| dudvdw$$

Where $J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = (\neq 0)$ is the Jacobian transformation of x, y, z w.r.to u, v, w.

(a) To change rectangular coordinates to spherical coordinates:

We have $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

and $J = \frac{\partial(x, y, z)}{\partial(r, \theta, \mathrm{z})} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2 \sin\theta$$

Thus $\iiint_V f(x, y, z)\, dxdydz = \iiint_{V'} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2 \sin\theta drd\theta d\phi$

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(b) To change rectangular coordinates to cylindrical coordinates:

We have $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

and $J = \dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$

Thus $\iiint\limits_V f(x,y,z)\,dxdydz = \iiint\limits_{V'} f(r\cos\theta, r\sin\theta, z)\,rdrd\theta dz$

Problem 1: Evaluate the triple integral $\iiint (x^2+y^2+z^2)\,dxdydz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=a^2$ by transforming into spherical polar coordinates.

Solution: Equation of the sphere $x^2+y^2+z^2=a^2$

Introducing spherical polar coordinates

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

we have $dxdydz = r^2\sin\theta drd\theta d\phi$

and $x^2+y^2+z^2=a^2$

$\Rightarrow r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta = a^2$

$\Rightarrow r^2 = a^2 \Rightarrow r = a$

The limits of the integration are

$\theta = 0, \theta = \pi, \phi = 0, \phi = 2\pi, r = 0, r = a$

Given integral $I = \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} a^2 r^2 \sin\theta drd\theta d\phi$

$$= a^2\int_{r=0}^{a} r^4 \int_{\theta=0}^{\pi}\sin\theta\left[\int_{\phi=0}^{2\pi} 1.d\phi\right]d\theta dr$$

$$= a^2\int_{r=0}^{a} r^4 \int_{\theta=0}^{\pi}\sin\theta\left[\phi\right]_{\phi=0}^{2\pi} d\theta dr$$

$$= a^2\int_{r=0}^{a} r^4 \int_{\theta=0}^{\pi}\sin\theta.2\pi.d\theta dr$$

$$= 2\pi a^2\int_{r=0}^{a} r^4 \int_{\theta=0}^{\pi}\sin\theta d\theta dr$$

$$= 2\pi a^2\int_{r=0}^{a} r^4 \left[-\cos\theta\right]_{\theta=0}^{\pi} dr$$

$$= 2\pi a^2\int_{r=0}^{a} r^4 \left[1+1\right] dr$$

$$= 4\pi a^2\int_{r=0}^{a} r^4 dr = 4\pi a^2\left[\frac{r^5}{5}\right]_{r=0}^{a}$$

$$= \frac{4\pi a^7}{5}$$

Problem 2: Evaluate $\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} \dfrac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}}$ taken over the positive octant of the sphere $x^2+y^2+z^2=a^2$ by changing to spherical polar coordinates.

Solution: By changing to spherical polar coordinates

Putting $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

we have $dxdydz = r^2 \sin\theta dr d\theta d\phi$

and $x^2 + y^2 + z^2 = r^2$

$\Rightarrow r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + r^2 \cos^2\theta = a^2$

$\Rightarrow r^2 = a^2 \Rightarrow r = a$

The limits of the integration are

$\theta = 0, \theta = \pi/2, \phi = 0, \phi = \pi/2, r = 0, r = a$

$$\text{Given integral } I = \int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} \frac{1}{\sqrt{1-r^2}} r^2 \sin\theta dr d\theta d\phi$$

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} \frac{1-(1-r^2)}{\sqrt{1-r^2}} \sin\theta dr d\theta d\phi$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} \sin\theta \left[\int_{r=0}^{a} \frac{1-(1-r^2)}{\sqrt{1-r^2}} dr\right] d\theta d\phi$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} \sin\theta \left[\int_{r=0}^{a} \left(\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2}\right) dr\right] d\theta d\phi d\phi$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} \sin\theta \left[\sin^{-1}\left(\frac{r}{a}\right) - \left\{\frac{r}{2}\sqrt{a^2-r^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{r}{a}\right)\right\}\right]_{r=0}^{r=a} d\theta$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} \sin\theta \left[\frac{\pi}{2} - \frac{a^2}{2}\frac{\pi}{2}\right] d\theta d\phi$$

$$= \frac{\pi}{2}\left(1 - \frac{a^2}{2}\right)\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} \sin\theta \, d\theta d\phi$$

$$= \frac{\pi}{2}\left(1 - \frac{a^2}{2}\right)\int_{\phi=0}^{\pi/2} \left[-\cos\theta\right]_{\theta=0}^{\pi/2} d\phi$$

$$= \frac{\pi}{2}\left(1 - \frac{a^2}{2}\right)\int_{\phi=0}^{\pi/2} \left[0+1\right] d\phi$$

$$= \frac{\pi}{2}\left(1 - \frac{a^2}{2}\right)\left[\phi\right]_{\theta=0}^{\pi/2}$$

$$= \frac{\pi^2}{4}\left(1 - \frac{a^2}{2}\right)$$

Problem 3: By changing to spherical polar coordinates, find the volume the sphere $x^2 + y^2 + z^2 = a^2$

Solution: The region of integration is $\left\{(x, y, z) : 0 \le x^2 + y^2 + z^2 \le a^2\right\}$

By changing to spherical polar coordinates

Putting $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

we have $dxdydz = r^2 \sin\theta dr d\theta d\phi$

and $x^2 + y^2 + z^2 = r^2$

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$$\Rightarrow r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta = a^2$$

$$\Rightarrow r^2 = a^2 \Rightarrow r = a$$

Using this transformation, The limits of the integration are

$\theta = 0, \theta = \pi, \phi = 0, \phi = 2\pi, r = 0, r = a$

$$\text{Given integral } I = \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \sin\theta dr d\theta d\phi$$

$$= \left(\int_{r=0}^{a} r^2 dr\right)\left(\int_{\theta=0}^{\pi} \sin\theta d\theta\right)\left(\int_{\phi=0}^{2\pi} 1.d\phi\right)$$

$$= \left[\frac{r^3}{3}\right]_{r=0}^{a} \left[-\cos\theta\right]_{\theta=0}^{\pi} \left[\phi\right]_{\theta=0}^{2\pi}$$

$$= \frac{a^3}{3}(1+1)(2\pi - 0) = \frac{4\pi a^3}{3}$$

Problem 4: By changing to cylindrical polar coordinates, find the volume the cylinder with base radius a and hight h.

Solution: The region of integration is bounded by $\left\{x^2 + y^2 \le a^2, 0 \le z \le h\right\}$

By changing to cylindrical polar coordinates

Putting $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

we have $dxdydz = rdrd\theta d\phi$

Using this transformation, The limits of the integration are

$\theta = 0, \theta = \pi,\ r = 0, r = a, z = 0, z = h$

$$\text{Given integral } I = \int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{h} r\, dr d\theta d\phi$$

$$= \left(\int_{r=0}^{a} r\, dr\right)\left(\int_{\theta=0}^{2\pi}\int_{z=0}^{h} 1.d\theta\right)\left(\int_{z=0}^{h} 1.d\phi\right)$$

$$= \left[\frac{r^2}{2}\right]_{r=0}^{a} \left[\theta\right]_{\theta=0}^{2\pi} \left[z\right]_{z=0}^{h} = \pi a^2 h$$

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UNIT -5

Beta and Gamma functions

Beta and Gamma functions and their properties, relation between beta and gamma functions, evaluation of definite integrals using beta and gamma functions.

GAMMA FUNCTION:

Definition: If n is a positive number, Then the definite integral $\int_0^\infty e^{-x}x^{n-1}dx, n>0,$ is called the Gamma function and is denoted by $\Gamma(n)$. i.e., $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx.$

Gamma function is also called Eulerian integral of second kind.

PROPERTIES OF GAMMA FUNCTION:

(i) $\Gamma(1)=1$

Proof: By the definition of Gamma function, we have

$$\therefore \qquad \Gamma(1)=\int_0^\infty e^{-x}x^0dx=\int_0^\infty e^{-x}.1dx=\left(\frac{e^{-x}}{-1}\right)_0^\infty=-(0-1)=1$$

(ii) $\Gamma(n+1)=n\Gamma(n)$

By the definition of Gamma function, we have

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx \qquad \text{…………..(1)}$$

Changing n to $n+1$ in (1)

$$\Gamma(n+1)=\int_0^\infty e^{-x}x^{n}dx=\left(x^n\frac{e^{-x}}{-1}\right)_0^\infty-\int_0^\infty nx^{n-1}\frac{e^{-x}}{-1}dx$$

$$=0+n\int_0^\infty e^{-x}x^{n-1}dx$$

$$=n\Gamma(n)$$

For Problem, $\Gamma\left(\frac{5}{2}\right)=\Gamma\left(\frac{3}{2}+1\right)$

$$=\frac{3}{2}.\Gamma\left(\frac{1}{2}+1\right)$$

$$=\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{3}{4}\sqrt{\pi}$$

This is called recurrence formula for $\Gamma(n)$.

Let us discuss the following cases when

(a) n is positive integer
(b) n is positive fraction
(c) n is negative fraction

Case (a): When n is positive integer

$$\therefore\ \Gamma(n+1)=n\Gamma(n)$$

$$=n(n-1)\Gamma(n-1)$$

$$=n(n-1)(n-2)\Gamma(n-2)$$

$$=n(n-1)(n-2)......3.2.1.\Gamma(1)$$

$$=n(n-1)(n-2)......3.2.1.1 \text{ Since } \Gamma(1)=1$$

$$\therefore\ \Gamma(n+1)=n!$$

For Problem, $\Gamma(8)=7.6.5.4.3.2.1=7!$

Case (b): When n is positive fraction

$$\therefore\ \Gamma(n)=(n-1)\Gamma(n-1)$$

$$=(n-1)(n-2)\Gamma(n-2)$$

$$=(n-1)(n-2)(n-3)\Gamma(n-3) \text{ and so on}.$$

For Problem, $\Gamma\left(\frac{7}{2}\right)=\left(\frac{7}{2}-1\right)\left(\frac{5}{2}-1\right)\left(\frac{3}{2}-1\right)\Gamma\left(\frac{1}{2}\right)$

$$=\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{15}{8}\sqrt{\pi}$$

Case (c): When n is negative fraction

We have $\Gamma(n+1)=n\Gamma(n)$

$$\Gamma(n)=\frac{\Gamma(n+1)}{n}$$

$$=\frac{1}{n}\Gamma(n+1)$$

On using (1), we have

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$$= \frac{1}{n}\frac{\Gamma(n+2)}{(n+1)}$$

$$= \frac{1}{n}.\frac{1}{(n+1)}.\frac{\Gamma(n+3)}{(n+2)}$$

$$= \frac{1}{n}.\frac{1}{(n+1)}........\frac{\Gamma(n+k+1)}{(n+k)}$$

$$= \frac{\Gamma(n+k+1)}{n(n+1)........(n+k)}$$

Problem (1): $\Gamma\left(\frac{-1}{2}\right) = \frac{\Gamma\left(\frac{-1}{2}+1\right)}{\frac{-1}{2}} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$

Problem (2): $\Gamma\left(\frac{-1}{5}\right) = \frac{\Gamma\left(\frac{-1}{5}+1\right)}{\frac{-1}{5}} = -5\Gamma\left(\frac{4}{5}\right)$

Note: 1. $\Gamma(n)$ is defined when $n > 0$

2. $\Gamma(n)$ is defined when 'n'is a negative fraction.

3. $\Gamma(n)$ is undefined when $n = 0$ and 'n' is a negative integer.

OTHER FORMS OF GAMMA FUNCTION:

(i) Prove that $\int_0^\infty e^{-kx}x^{n-1}dx = \frac{\Gamma(n)}{k^n}$

Put $y = kx$

So that $dy = kdx$

Also $x = 0; y = 0$ and $x \to \infty; y \to \infty$

$$\int_0^\infty e^{-kx}x^{n-1}dx = \int_0^\infty e^{-y}(ky)^{n-1}kdy$$

$$= k^n \int_0^\infty e^{-y}y^{n-1}dy$$

$$\therefore \int_0^\infty e^{-kx}x^{n-1}dx = k^n\Gamma(n)$$

(ii) Prove that $\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-y^{1/n}}dy$

By the definition of Gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx \qquad \text{…………..(1)}$$

Put $y = x^n$

So that $dy = nx^{n-1}dx$

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Also $x=0; y=0$ and $x\to\infty; y\to\infty$

$$\Gamma(n)=\int_0^\infty e^{-y^{1/n}}\frac{dy}{n}=\frac{1}{n}\int_0^\infty e^{-y^{1/n}}dy$$

Hence $\Gamma(n)=\frac{1}{n}\int_0^\infty e^{-y^{1/n}}dy$

(iii) Prove that $\Gamma(n)=\int_0^1 \log\left(\frac{1}{y}\right)^{n-1}dy$

We have $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx$ …………..(1)

Put $e^{-x}=y \Rightarrow x=\log\left(\frac{1}{y}\right)$

So that $-e^{-x}dx=dy$

Also when $x=0; y=1$ and $x\to\infty; y\to 0$

Substituting, $\Gamma(n)=-\int_1^0 \log\left(\frac{1}{y}\right)^{n-1}dy$

$\therefore\ \Gamma(n)=\int_0^1 \log\left(\frac{1}{y}\right)^{n-1}dy$

(iv) Prove that $\Gamma(n)=2\int_0^\infty e^{-y^2}y^{2n-1}dy$

We have $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx$ …………..(1)

Put $x=y^2$

So that $dx=2ydy$

Also $x=0; y=0$ and $x\to\infty; y\to\infty$

Substituting, $\Gamma(n)=2\int_0^\infty e^{-y^2}y^{2(n-1)}2ydy$

$\therefore\ \Gamma(n)=2\int_0^\infty e^{-y^2}y^{2n-1}ydy$

SOLVED PROBLEMS:

Problem (1): Prove that $\int_0^\infty e^{-x^2}dx=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$. Hence show that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

Solution: We have $\Gamma(n)=\int_0^\infty e^{-t}t^{n-1}dt$

Put $n=\frac{1}{2}$

$\therefore\ \Gamma\left(\frac{1}{2}\right)=\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt$ ……………………… (1)

Let $t=x^2$

Dr TSR, AITS-KADAPA

$$\Rightarrow dt = 2xdx$$

Also $x = 0; y = 0$ and $x \to \infty; y \to \infty$

$$\therefore \ \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x^2} . \frac{1}{x} .2xdx$$

$$= 2\int_0^\infty e^{-x^2} dx$$

Hence $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$ ………………………… (2)

Deduction:

By changing x to y, we have

$$\int_0^\infty e^{-y^2} dy = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \quad \text{………………………… (3)}$$

Multiplying (2) and (3), weget

$$\left[\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^\infty e^{-x^2} dx \times \int_0^\infty e^{-y^2} dy$$

$$= \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dxdy \quad \text{By property of multiple integrals}$$

$$\therefore \ \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dxdy$$

The region of integration is the first quadrant of the xy – plane.

Change Cartesian coordinates to polar coordinates

by putting $x = r\cos\theta, y = r\sin\theta$

and $dxdy = rdrd\theta$

From this region, r varies from 0 to ∞ while θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \ \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_{\theta=0}^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta = 4\int_{\theta=0}^{\frac{\pi}{2}} d\theta \int_0^\infty e^{-r^2} rdr$$

$$= \frac{4}{-2}\int_{\theta=0}^{\frac{\pi}{2}} d\theta \int_0^\infty d(e^{-r^2}) = -2\int_{\theta=0}^{\frac{\pi}{2}} d\theta \left(e^{-r^2}\right)_0^\infty$$

$$= -2\int_{\theta=0}^{\frac{\pi}{2}} d\theta \left(e^{-\infty} - 1\right) = 2\left[\theta\right]_0^{\pi/2}$$

i.e., $\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$

Hence, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Problem 2: Evaluate the following integrals

(i) $\int_0^\infty e^{-a^2x^2} dx$ (ii) $\int_0^\infty \sqrt{x} e^{-x^3} dx$ (iii) $\int_0^\infty x^2 e^{-x^2} dx$

Dr TSR, AITS-KADAPA

(iv) $\int_0^\infty e^{-x^3}dx$ (v) $\int_0^\infty x^4 e^{-x^2}dx$ (vi) $\int_0^\infty e^{-x^4}dx$

(i) **Solution:** Given integral, $\mathrm{I}=\int_0^\infty e^{-a^2x^2}dx$

Put $a^2x^2 = y \Rightarrow x=\frac{\sqrt{y}}{a}$

So that $dx=\frac{1}{2a\sqrt{y}}dy$

And the limits are same.

$$\therefore\quad \mathrm{I}=\int_0^\infty e^{-y}\frac{1}{2a\sqrt{y}}dy=\frac{1}{2a}\int_0^\infty e^{-y}y^{-\frac{1}{2}}dy$$

$$=\frac{1}{2a}\int_0^\infty e^{-y}y^{\left(-\frac{1}{2}+1\right)-1}dy=\frac{1}{2a}\int_0^\infty e^{-y}y^{\frac{1}{2}-1}dy$$

$$=\frac{1}{2a}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2a}.$$

(ii) **Solution:**Given integral, $\mathrm{I}=\int_0^\infty \sqrt{x}e^{-x^3}dx$

Put $x^3=y\Rightarrow x=y^{\frac{1}{3}}$ so that $dx=\frac{1}{3}y^{-\frac{2}{3}}dy$

$$\therefore\quad \mathrm{I}=\int_0^\infty y^{\frac{1}{6}}e^{-y}\frac{1}{3}y^{-\frac{2}{3}}dy=\frac{1}{3}\int_0^\infty e^{-y}y^{-\frac{1}{2}}dy$$

$$=\frac{1}{3}\int_0^\infty e^{-y}y^{\frac{1}{2}-1}dy=\frac{1}{3}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{3}.$$

(iii) **Solution:** Given integral, $\mathrm{I}=\int_0^\infty x^2e^{-x^2}dx$

Put $x^2=y\Rightarrow x=\sqrt{y}$

So that $dx=\frac{1}{2\sqrt{y}}dy$

$$\therefore\quad \mathrm{I}=\int_0^\infty y\,e^{-y}\frac{1}{2\sqrt{y}}dy=\frac{1}{2}\int_0^\infty e^{-y}y^{\frac{1}{2}}dy$$

$$=\frac{1}{2}\int_0^\infty e^{-y}y^{\left(\frac{1}{2}+1\right)-1}dy=\frac{1}{2}\int_0^\infty e^{-y}y^{\frac{3}{2}-1}dy$$

$$=\frac{1}{2}\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\therefore\quad \int_0^\infty x^2e^{-x^2}dx=\frac{\sqrt{\pi}}{4}$$

Dr TSR, AITS-KADAPA

(iv) **Solution:** Given integral, $\text{I} = \int_0^\infty e^{-x^3} dx$

Put $x^3 = y \Rightarrow x = y^{\frac{1}{3}}$ so that $dx = \frac{1}{3} y^{-\frac{2}{3}} dy$

$$\therefore \quad \text{I} = \int_0^\infty e^{-y} \frac{1}{3} y^{-\frac{2}{3}} dy = \frac{1}{3} \int_0^\infty e^{-y} y^{\frac{1}{3}-1} dy$$

$$= \frac{1}{3} \Gamma\left(\frac{1}{3}\right).$$

(v) **Solution:** Given integral, $\text{I} = \int_0^\infty x^4 e^{-x^2} dx$

Put $x^2 = y \Rightarrow x = \sqrt{y}$

So that $dx = \frac{1}{2\sqrt{y}} dy$

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