DEPARTMENT OF HUMANITIES AND SCIENCES

B.Tech I Year II Semester

DIFFERENTIAL EQUATIONS AND VECTOR CALCULUS

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ANNAMACHARYA INSTITUTE OF TECHNOLOGY AND SCIENCES

(Autonomous)

(Affiliated to J.N.T.U.A, Anantapur, Approved by A.I.C.T.E, New Delhi)

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Accredited by NAAC with 'A' Grade, Bangalore.

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DIFFERENTIAL EQUATIONS AND VECTOR CALCULUS

(Common to All Branches of Engineering)

Course Objectives:

- To enlighten the learners in the concept of differential equations and multivariablecalculus.
- To furnish the learners with basic concepts and techniques at plus two level to lead theminto advanced level by handling various real-world applications.

Course Outcomes: At the end of the course, the student will be able to

CO1: Solve the differential equations related to various engineering fields.

CO2: Identify solution methods for partial differential equations that model physical processes.

CO3: Interpret the physical meaning of different operators such as gradient, curl and divergence.

CO4: Estimate the work done against a field, circulation and flux using vector calculus.

UNIT I Differential equations of first order and first degree

Linear differential equations – Bernoulli's equations- Exact equations and equations reducibleto exact form. Applications: Newton's Law of cooling – Law of natural growth and decay- Electrical circuits.

UNIT II Linear differential equations of higher order (Constant Coefficients)

Definitions, homogenous and non-homogenous, complimentary function, general solution, particular integral, Wronskian, Method of variation of parameters. Simultaneous linear equations, Applications to L-C-R Circuit problems and Simple Harmonic motion.

UNIT III Partial Differential Equations

Introduction and formation of Partial Differential Equations by elimination of arbitrary constants and arbitrary functions, solutions of first order linear equations using Lagrange's method. Homogeneous Linear Partial differential equations with constant coefficients.

UNIT IV Vector differentiation

Scalar and vector point functions, vector operator Del, Del applies to scalar point functions- Gradient, Directional derivative, del applied to vector point functions-Divergence and Curl, vector identities.

UNIT V Vector integration

Line integral-circulation-work done, surface integral-flux, Green's theorem in the plane (without proof), Stoke's theorem (without proof), volume integral, Divergence theorem (without proof) and applications of these theorems.

Textbooks:

1. Higher Engineering Mathematics, B. S. Grewal, Khanna Publishers, 2017, 44thEdition
2. Advanced Engineering Mathematics, Erwin Kreyszig, John Wiley & Sons, 2018,10thEdition.

Reference Books:

1. Thomas Calculus, George B. Thomas, Maurice D. Weir and Joel Hass, PearsonPublishers, 2018, 14th Edition.
2. Advanced Engineering Mathematics, Dennis G. Zill and Warren S. Wright, JonesandBartlett, 2018.
3. Advanced Modern Engineering Mathematics, Glyn James, Pearson publishers,2018,5th Edition.
4. Advanced Engineering Mathematics, R. K. Jain and S. R. K. Iyengar, AlphaScienceInternational Ltd., 2021 5th Edition (9th reprint).
5. Higher Engineering Mathematics, B. V. Ramana, , McGraw Hill Education, 2017

UNIT-I
DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

Definition: An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a Differential Equation.

Types of Differential Equations: there are two types of differential equations
1. Ordinary differential equations 2. Partial differential equations

Ordinary Differential Equation: A differential equation is said to be ordinary, if the derivatives in the equation are ordinary derivatives.

Ex: 1. $\left(\frac{dy}{dx}\right)^3 - \left(\frac{dy}{dx}\right)^2 + 7y = \cos x$

2. $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \tan y$

3. $(x^2 + y^2 - x)dy + (ye^y - 2xy)dx = 0$

4. $x\frac{d^2y}{dx^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}$

The general form of an ordinary differential equation is

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0$$

Partial Differential Equation: A differential equation is said to be partial, if the derivatives in the equation have reference to two or more independent variables.

Ex: 1. $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ (One-dimensional wave equation)

2. $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial y}{\partial t}$ (One-dimensional heat equation)

3. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Two-dimensional Laplace's equation)

These equations can studied in detail later.
We now discuss only ordinary differential equations.

Order of a Differential Equation: The order of the highest order derivative in a differential equation is called the order of the differential equation (Or) A differential equation is said to be of order n, if the n^{th} order derivative is the highest derivative in that equation.

Ex: 1. $(x^2 + 1)\frac{dy}{dx} + 2xy = 4x^2$

The first order derivative $\frac{dy}{dx}$ is the highest derivative in the above equation.

$\therefore$ The order of above differential equation is 1.

2. $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x$

$\frac{d^2y}{dx^2}$ is the highest derivative in the above equation.

$\therefore$ The order of above differential equation is 2.

Degree of a Differential Equation: The degree of a differential equation is the highest degree of the highest order derivative which occurs in it, after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned.

Let $f\left(x, y, y', y'', ..., y^{(n)}\right) = 0$ be a differential equation of order n which is free from radicals and fractions as far as the derivatives are concerned. If the given differential equation is a polynomial in $y^{(n)}$, then the highest degree of $y^{(n)}$ is defined as the degree of the differential equation.

Ex: 1. $y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$$\Rightarrow \left(y - x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \left(1 - x^2\right)\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + \left(1 - y^2\right) = 0$$

This is a differential equation of order 1. The highest degree of $\frac{dy}{dx}$ is 2.

Hence the degree of the above differential equation is 2.

2. $a\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Rightarrow a^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$

This is a differential equation of order 2. The highest degree of $\frac{d^2y}{dx^2}$ is 2.

Hence the degree of the above differential equation is 2.

Solution of Differential Equation: Any relation between the dependent and independent variables not containing their derivatives, which satisfies the given differential equation is called a solution or integral of the differential equation.

For example, $y = A\cos x + B\sin x$ is a solution of $\frac{d^2y}{dx^2} + y = 0$.

Observe that $y = A\cos x + B\sin x$ is a solution of the given differential equation for any real constants A and B which are called arbitrary constants.

General solution: A solution containing the number of independent arbitrary constants which is equal to the order of the differential equation is called the general solution or complete solution of the equation.

For example, $y = c_1e^x + c_2e^{2x}$ is the general solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$, as it contains two independent arbitrary constants.

Particular solution: A solution obtained from the general solution of a differential equation by giving particular values to the independent arbitrary constants is called a particular solution to the given differential equation.

For example, some particular solutions of $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$ are given by $y=e^x+e^{2x}, y=e^x-2e^{2x}$ etc.

Singular solution: A solution which cannot be obtained from any general solution of a differential equation by any choice of the independent arbitrary constants is called a singular solution of the given differential equation.

For example, $y=(x+c)^2$ (1)

is the general solution of $y_1^2-4y=0$ (2)

$y=0$ is also a solution of (2). Moreover $y=0$ cannot be obtained by any choice of c in (1).

Hence $y=0$ is a singular solution of (2).

Formation of differential equation:

In general an ordinary differential equation is obtained by eliminating the arbitrary constants $c_1, c_2, ..., c_n$ from a relation like $\phi(x, y, c_1, c_2, ..., c_n)=0$ or from a physical problem.

Consider $\phi(x, y, c_1, c_2, ..., c_n)=0$ (1)

Where $c_1, c_2, ..., c_n$ are arbitrary constants. Differentiating (1) successively with respect to x, n times and eliminating the n arbitrary constants $c_1, c_2, ..., c_n$ from the above $n+1$ equations, we obtain the differential equation $f\left(x, y, y^{'}, y^{''}, ..., y^{(n)}\right)=0$. Its general solution is given by the relation (1) itself.

Examples

1. By eliminating A and B, form the differential equation of which $y=Ae^{-2x}+Be^{5x}$ is a solution.

Solution: Given $y=Ae^{-2x}+Be^{5x}$ **(1)**

Differentiating (1) with respect to x successively two times, we get

$$y^{'}=\frac{dy}{dx}=-2Ae^{-2x}+5Be^{5x} \quad (2)$$

$$y^{''}=\frac{d^2y}{dx^2}=4Ae^{-2x}+25Be^{5x} \quad (3)$$

Eliminating A and B from (1), (2), and (3), we get

$$\begin{vmatrix} e^{-2x} & e^{5x} & -y \\ -2e^{-2x} & 5e^{5x} & -y^{'} \\ 4e^{-2x} & 25e^{5x} & -y^{''} \end{vmatrix}=0 \Rightarrow \begin{vmatrix} 1 & 1 & y \\ -2 & 5 & y^{'} \\ 4 & 25 & y^{''} \end{vmatrix}=0 \Rightarrow y^{''}-3y^{'}-10y=0$$

This is the required differential equation obtained by eliminating the arbitrary constants A and B from $y=Ae^{-2x}+Be^{5x}$.

2. Find the differential equation corresponding to $y=ae^x+be^{2x}+ce^{3x}$ where a, b, c are arbitrary constants.

Solution: Given $y = ae^x + be^{2x} + ce^{3x}$ **(1)**

Differentiating (1) with respect to x, we get

$$y' = \frac{dy}{dx} = ae^x + 2be^{2x} + 3ce^{3x}$$
$$= \left(ae^x + be^{2x} + ce^{3x}\right) + \left(be^{2x} + 2ce^{3x}\right)$$
$$= y + \left(be^{2x} + 2ce^{3x}\right), \text{ using (1)}$$

Or $y' - y = be^{2x} + 2ce^{3x}$ (2)

Differentiating (2) with respect to x, we get

$$y'' - y' = 2be^{2x} + 6ce^{3x}$$
$$= 2\left(be^{2x} + 2ce^{3x}\right) + 2ce^{3x}$$
$$= 2\left(y' - y\right) + 2ce^{3x}, \text{ using (2)}$$

Or $y'' - 3y' + 2y = 2ce^{3x}$ (3)

Differentiating (3) with respect to x, we get

$$y''' - 3y'' + 2y' = 6ce^{3x} = 3\left(2ce^{3x}\right)$$

$\Rightarrow y''' - 3y'' + 2y' = 3\left(y'' - 3y' + 2y\right)$, using (3)

$\Rightarrow y''' - 6y'' + 11y' - 6y = 0$

This is the required differential equation.

3. Form the differential equation by eliminating the arbitrary constants A and B from the equation $y = e^x\left(A\cos x + B\sin x\right)$.

Solution: Given $y = e^x\left(A\cos x + B\sin x\right)$ **(1)**

Differentiating (1) with respect to x, we get

$$y' = \frac{dy}{dx} = e^x\left(A\cos x + B\sin x\right) + e^x\left(-A\sin x + B\cos x\right)$$

$= y + e^x\left(-A\sin x + B\cos x\right)$, using (1) (2)

Again differentiating with respect to x, we get

$$y'' = \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x\left(-A\sin x + B\cos x\right) + e^x\left(-A\cos x - B\sin x\right)$$

$= \frac{dy}{dx} - y + \left(\frac{dy}{dx} - y\right)$, using (1) and (2)

$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ is the required differential equation.

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND OF THE FIRST DEGREE

Definition: An equation of the form $\frac{dy}{dx} = f(x, y)$ is called a differential equation of first order and of first degree.

LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER

An equation of the form $\frac{dy}{dx} + P(x)\,y = Q(x)$ (1)

where P and Q are either constants or functions of x only is called a linear differential equation of first order in y.

Working rule: To solve the linear equation $\frac{dy}{dx} + P(x)\,y = Q(x)$

(i) Write the integrating factor (I.F.) $= e^{\int P(x)dx}$

(ii) Solution is given by $y \times (\text{I.F.}) = \int Q(x) \times (\text{I.F.})\, dx + c$

Note 1: Given $\frac{dy}{dx} + P(x)\,y = Q(x)$, we may directly proceed as above and solve. Sometimes it may be convenient to put the differential equation in the form $\frac{dx}{dy} + P(y)\,x = Q(y)$ and treat x as the dependent variable and y as the independent variable. In this case , the general solution is given by $x \times (\text{I.F.}) = \int Q(y) \times (\text{I.F.})\, dy + c$ where $\text{I.F.} = e^{\int P(y)dy}$.

Note 2: Remember the following results which are useful in evaluating some integrals directly

(i) $\int t\, e^{t} dt = (t-1)e^{t} + c$ (ii) $\int t\, e^{-t} dt = (-t-1)e^{-t} + c$

EXAMPLES

1. Solve $x\frac{dy}{dx} + y = \log x$.

Solution: Given differential equation is $x\frac{dy}{dx} + y = \log x$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x}y = \frac{\log x}{x} \qquad (1)$$

This is of the form $\frac{dy}{dx} + P(x)\,y = Q(x)$, where P and Q are functions of x only.

Here $P = \frac{1}{x}, Q = \frac{\log x}{x}$

$\therefore$ $\text{I.F.} == e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

General Solution is given by $y \times (\text{I.F.}) = \int Q(x) \times (\text{I.F.})\, dx + c$

$$\Rightarrow y \times x = \int \frac{\log x}{x} \times x\, dx + c = \int \log x\, dx + c$$

$$\Rightarrow xy = x(\log x - 1) + c$$

2. Solve $\left(1-x^2\right)\dfrac{dy}{dx}+xy=ax$.

Solution: Given differential equation is $\left(1-x^2\right)\dfrac{dy}{dx}+xy=ax$

$$\Rightarrow \frac{dy}{dx}+\frac{x}{1-x^2}\,y=\frac{ax}{1-x^2} \qquad (1)$$

This is a linear equation of first order in y .

Comparing it with $\dfrac{dy}{dx}+P(x)\,y=Q(x)$, we have $P=\dfrac{x}{1-x^2}, Q=\dfrac{ax}{1-x^2}$

$\therefore \quad$ I.F. $==e^{\int P(x)dx}=e^{\int \frac{x}{1-x^2}dx}=e^{-\frac{1}{2}\log\left(1-x^2\right)}=\dfrac{1}{\sqrt{1-x^2}}$

General Solution is given by $y\times(\text{I.F.})=\int Q(x)\times(\text{I.F.})\,dx+c$

$$\Rightarrow y\times\frac{1}{\sqrt{1-x^2}}=\int\frac{ax}{1-x^2}\times\frac{1}{\sqrt{1-x^2}}\,dx+c=a\int\frac{x}{\left(1-x^2\right)^{3/2}}\,dx+c$$

$$\Rightarrow \frac{y}{\sqrt{1-x^2}}=-\frac{a}{2}\int(-2x)\left(1-x^2\right)^{-3/2}dx+c$$

$$\Rightarrow \frac{y}{\sqrt{1-x^2}}=-\frac{a}{2}\frac{\left(1-x^2\right)^{-3/2+1}}{\frac{-3}{2}+1}+c\text{, where }\int\left[f(x)\right]^n f^{'}(x)\,dx=\frac{\left[f(x)\right]^{n+1}}{n+1}$$

$$\Rightarrow \frac{y}{\sqrt{1-x^2}}=\frac{a}{\sqrt{1-x^2}}+c$$

$\Rightarrow y==a+c\sqrt{1-x^2}$ is the required general solution.

3. Solve $\left(1-x^2\right)\dfrac{dy}{dx}+2xy=x\sqrt{1-x^2}$.

Solution: Given differential equation is $\left(1-x^2\right)\dfrac{dy}{dx}+2xy=x\sqrt{1-x^2}$

$$\Rightarrow \frac{dy}{dx}+\frac{2x}{1-x^2}\,y=\frac{x}{\sqrt{1-x^2}} \qquad (1)$$

This is a linear equation of first order in y .

Comparing it with $\dfrac{dy}{dx}+P(x)\,y=Q(x)$, we have $P=\dfrac{2x}{1-x^2}, Q=\dfrac{x}{\sqrt{1-x^2}}$

$\therefore \quad$ I.F. $==e^{\int P(x)dx}=e^{\int \frac{2x}{1-x^2}dx}=e^{-\log\left(1-x^2\right)}=\dfrac{1}{1-x^2}$

General Solution is given by $y\times(\text{I.F.})=\int Q(x)\times(\text{I.F.})\,dx+c$

$$\Rightarrow y\times\frac{1}{1-x^2}=\int\frac{x}{\sqrt{1-x^2}}\times\frac{1}{1-x^2}\,dx+c=\int\frac{x}{\left(1-x^2\right)^{3/2}}\,dx+c$$

$$\Rightarrow \frac{y}{1-x^2}=-\frac{1}{2}\int(-2x)\left(1-x^2\right)^{-3/2}dx+c$$

$$\Rightarrow \frac{y}{1-x^2} = -\frac{1}{2}\frac{\left(1-x^2\right)^{-3/2+1}}{\frac{-3}{2}+1} + c \text{, where } \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$\Rightarrow \frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + c$$

$\Rightarrow y = \sqrt{1-x^2} + c\left(1-x^2\right)$ is the required general solution.

4. Solve $\frac{dy}{dx} + 2xy = 2e^{-x^2}$.

Solution: Given equation is $\frac{dy}{dx} + 2xy = 2e^{-x^2}$

$$\Rightarrow \frac{dy}{dx} + (2x)y = 2e^{-x^2}$$

This is a linear differential equation

Here $P = 2x$ and $Q = 2e^{-x^2}$

$$I.F. = e^{\int P\,dx} = e^{\int 2x dx} = e^{2\int x dx} = e^{2\frac{x^2}{2}} = e^{x^2}$$

Its solution is $y(I.F.) = \int Q(I.F.)dx + c$

$$ye^{x^2} = 2\int dx + c \qquad \Rightarrow ye^{x^2} = 2x + c$$

This is the required solution.

5. **Solve** $\frac{dy}{dx} + y\tan x = \cos^3 x$.

Solution: Given differential equation is $\frac{dy}{dx} + y\tan x = \cos^3 x$

It is a linear differential equation in y.

$$\text{So Integrating Factor} = \text{I.F.} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$$

Therefore the general solution of given differential equation is

$$y(\text{I.F.}) = \int \cos^3 x\,.(\text{I.F.})\,dx + c$$

$$y\sec x = \int \cos^3 x\,.\sec x\,dx + c$$

$$= \int \cos^2 x\ dx + c = \frac{1}{2}\int (1+\cos 2x)\,dx + c$$

$$= \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c$$

6. Solve $(1+y^2) + \left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = 0$.

Solution: Given differential equation is $(1+y^2) + \left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dx}{dy}+\frac{1}{1+y^2}x=\frac{e^{tan^{-1}y}}{1+y^2} \qquad (1)$$

It is a linear differential equation in x.

$$\text{So Integrating Factor} = e^{\int \frac{1}{1+y^2}dy}=e^{tan^{-1}y}$$

Therefore the general solution of given differential equation is

$$x(I.F.)=\int \frac{e^{tan^{-1}y}}{1+y^2}.(I.F.)\,dy+c$$

$$x\,e^{tan^{-1}y}=\int \frac{e^{tan^{-1}y}}{1+y^2}.e^{tan^{-1}y}\,dy+c$$

$$=\int \frac{1}{1+y^2}e^{2tan^{-1}y}\,dy+c$$

$$=\int e^{2u}\,du+c, \text{Put} tan^{-1}y=u$$

$$=\frac{e^{2u}}{2}+c=\frac{1}{2}e^{2tan^{-1}y}+c$$

Hence the required solution of (1) is

$$x\,e^{tan^{-1}y}=\frac{1}{2}e^{2tan^{-1}y}+c$$

7. Solve $x(x-1)\frac{dy}{dx}-y=x^2(x-1)^3$**.**

Solution: Given differential equation is

$$x(x-1)\frac{dy}{dx}-y=x^2(x-1)^3$$

$$\Rightarrow \frac{dy}{dx}-\frac{1}{x(x-1)}y=\frac{x^2(x-1)^3}{x(x-1)}$$

$$\Rightarrow \frac{dy}{dx}-\frac{1}{x(x-1)}y=x(x-1)^2 \qquad (1)$$

It is a first order linear differential equation of the form

$$\frac{dy}{dx}+P(x)\,y=Q(x)$$

here $P(x)=-\frac{1}{x(x-1)}$ and $Q(x)=x(x-1)^2$

So $\text{I.F.}=e^{\int P(x)\,dx}=e^{\int -\frac{1}{x(x-1)}dx}=e^{\int\left(\frac{1}{x}-\frac{1}{x-1}\right)dx}=e^{\log x-\log(x-1)}$

$$=e^{\log\frac{x}{x-1}}=\frac{x}{x-1}$$

Therefore the general solution of (1) is

$$y(\text{I.F.})=\int Q(x)\,(\text{I.F.})\,dx+c$$

$$y\left(\frac{x}{x-1}\right)=\int x(x-1)^2\left(\frac{x}{x-1}\right)dx+c$$

$$=\int x^2(x-1)\,dx+c$$

$$=\frac{x^4}{4}-\frac{x^3}{3}+c$$

Hence the required solution of (1) is

$$y\left(\frac{x}{x-1}\right) == \frac{x^4}{4} - \frac{x^3}{3} + c$$

8. Solve $x^3 \sec^2 y \dfrac{dy}{dx} + 3x^2 \tan y = \cos x$.

Solution: Given differential equation is

$$x^3 \sec^2 y \frac{dy}{dx} + 3x^2 \tan y = \cos x$$

$$\text{i.e.,} \qquad \sec^2 y \frac{dy}{dx} + \frac{3}{x} \tan y = \frac{\cos x}{x^3} \qquad \textbf{(1)}$$

Put $\tan y = u$ then $\sec^2 y \dfrac{dy}{dx} = \dfrac{du}{dx}$

$$\therefore (1) \Rightarrow \frac{du}{dx} + \frac{3}{x} u = \frac{\cos x}{x^3} \qquad (2)$$

It is a first order linear differential equation of the form $\dfrac{du}{dx} + P(x)u = Q(x)$, we have

$$P = \frac{3}{x}, Q = \frac{\cos x}{x^3}$$

$$\therefore \quad \text{I.F.} == e^{\int P(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\log x} = x^3$$

Therefore the general solution of (1) is

$$u \times (\text{I.F.}) = \int Q(x) \times (\text{I.F.})\, dx + c$$

$$\Rightarrow u \times x^3 = \int \frac{\cos x}{x^3} \times x^3 \, dx + c$$

$$\Rightarrow \tan y \times x^3 = \int \cos x \, dx + c$$

$$\Rightarrow x^3 \tan y = \sin x + c$$

It is the required solution of (1).

9. Solve $\cos^2 x \dfrac{dy}{dx} + y = \tan x$.

Solution: Given equation $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

$$\Rightarrow \frac{dy}{dx} + \left(\sec^2 x\right) y = \tan x \sec^2 x$$

This is a linear equation in y.

Here $P = \sec^2 x$ and $Q = \tan x \sec^2 x$

$$I.F. = e^{\int P dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

The solution of given differential equation is

$$y(IF) = \int Q(IF)dx + c$$

$$ye^{\tan x} = \int \tan x \ \sec^2 x \, e^{\tan x} dx + c$$

$$ye^t = \int te^t \, dt + c = e^t(t-1) + c \text{ (Put } \tan x = t, \ \sec^2 x \, dx = dt \text{)}$$

$$ye^{\tan x} = e^{\tan x}(\tan x - 1) + c$$

$$y = (\tan x - 1) + ce^{-\tan x}$$

which is the required solution.

10. Solve $dx + xdy = e^{-y} \sec^2 y \, dy.$

Solution: Given equation can be written as $\dfrac{dx}{dy} + x = e^{-y} \sec^2 y$

Here $P = 1$ and $Q = e^{-y} \sec^2 y$

$$I.F. = e^{\int P dy} = e^{\int 1 dy} = e^y$$

Solution is given by

$$x(IF) = \int Q(IF)dy + c$$

$$xe^y = \int e^{-y} \sec^2 e^y dy + c = \int \sec^2 y + c$$

$xe^y = \tan y + c$, which is the required solution.

BERNOULLI'S EQUATION

A first order and first degree differential equation of the form

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n \qquad (1)$$

is called Bernoulli's equation if P and Q are constants or functions of x alone and n is a real constant.

Case 1: If $n = 1$ then the equation (1) can be written as

$$\frac{dy}{dx} + (P - Q)\, y = 0 \qquad (2)$$

Here the variables are separable. The general solution is

$$\int \frac{dy}{y} + \int (P - Q)\, dx = 0$$

Case 2: If $n \neq 1$, multiplying (1) with y^{-n}, we get

$$y^{-n} \frac{dy}{dx} + P(x) y^{1-n} = Q(x) \qquad (3)$$

Now, putting $z = y^{1-n}$ and $\dfrac{dz}{dx} = (1-n) y^{-n} \dfrac{dy}{dx}$ in equation (3), we get

$$\frac{1}{1-n} \frac{dz}{dx} + P(x)\, z = Q(x)$$

$$\Rightarrow \frac{dz}{dx} + (1-n)P(x)\, z = (1-n)Q(x) \qquad (4)$$

This is a first order linear differential equation in z.

$$\therefore \quad I.F. = e^{\int (1-n)P(x) \mathrm{dx}}$$

Hence the general solution of (3) is

$$z(I.F.)=\int (1-n)Q(x)(I.F.)\,dx+c$$

$$z\left(e^{\int (1-n)P(x)\mathrm{dx}}\right)=\int (1-n)Q(x)\left(e^{\int (1-n)P(x)\mathrm{dx}}\right)dx+c \qquad (4)$$

Substituting $z=y^{1-n}$ in (4), we get the general solution of (1).

Examples

1. **Solve** $x\frac{dy}{dx}+y=x^3y^6$.

Solution: Given equation is $x\frac{dy}{dx}+y=x^3y^6$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{x}=x^2y^6 \qquad (1)$$

It is of the form $\frac{dy}{dx}+Py=Qy^n$ we have

$$P=\frac{1}{x},\quad Q=x^2 \text{ and } y^n=y^6$$

Multiplying on both sides of (1) by y^{-6}, we get

$$y^{-6}\frac{dy}{dx}+y^{-6}\frac{y}{x}=y^{-6}x^2y^6$$

$$\Rightarrow y^{-6}\frac{dy}{dx}+\frac{1}{x}y^{-5}=x^2 \qquad (2)$$

Put $u=y^{-5}$ then $\frac{du}{dx}=-5y^{-6}\frac{dy}{dx}$

$$\Rightarrow -\frac{1}{5}\frac{du}{dx}=y^{-6}\frac{dy}{dx} \qquad (3)$$

Using (3) in (2), we get

$$-\frac{1}{5}\frac{du}{dx}+\frac{1}{x}u=x^2 \quad \Rightarrow \frac{du}{dx}-\frac{5}{x}u=-5x^2 \qquad (4)$$

It is a linear differential equation in *u*.

Here $P=\frac{-5}{x}$ and $Q=-5x^2$

$$I.F.=e^{\int Pdx}=e^{\int\left(\frac{-5}{x}\right)dx}=e^{-5\int\frac{1}{x}dx}=e^{-5\log x}=e^{\log x^{-5}}=x^{-5}=\frac{1}{x^5}$$

Its solution is $u\,(I.F.)=\int Q\,(I.F.)dx+c$

$$u\frac{1}{x^5}=\int\left(-5x^2\right)\frac{1}{x^5}dx+c=-5\int\frac{1}{x^3}dx+c$$

$$=-5\frac{x^{-3+1}}{-3+1}+c=\frac{5}{2x^2}+C$$

Since $u=y^{-5}$ then the general solution of (1) is

$$\frac{1}{y^5x^5}=\frac{5}{2x^2}+c\Rightarrow\frac{1}{y^5}=\frac{5x^3}{2}+c\,x^5$$

This is the required solution.

2. Solve $\frac{dy}{dx}\left(x^2y^3+xy\right)=1.$

Solution: Given equation is $\frac{dy}{dx}\left(x^2y^3+xy\right)=1$

$$\Rightarrow\frac{dx}{dy}-xy=x^2y^3 \qquad (1)$$

This is a Bernoulli's equation in 'x'

Multiplying (1) with x^{-2}, we get

$$x^{-2}\frac{dx}{dy}-x^{-2}xy=x^{-2}x^2y^3\Rightarrow x^{-2}\frac{dx}{dy}-x^{-1}y=y^3 \qquad (2)$$

Put $u=x^{-1}$ then $\frac{du}{dy}=-x^{-2}\frac{dx}{dy}\Rightarrow-\frac{du}{dy}=x^{-2}\frac{dx}{dy}$ (3)

Using (3) in (2), we get

$$-\frac{du}{dy}-uy=y^3 \quad \Rightarrow\frac{du}{dx}+y\,u=-y^3 \qquad (4)$$

It is a linear differential equation in *u*.

Here $P=y$ and $Q=-y^3$

$$I.F.=e^{\int P\,dy}=e^{\int y\,dy}=e^{\frac{y^2}{2}}$$

Its solution is $u(I.F.)=\int Q\,(I.F.)dy+c$

$$ue^{\frac{y^2}{2}}=\int\left(-y^3\right)e^{\frac{y^2}{2}}dy+c=-\int y^3\,e^{\frac{y^2}{2}}dy+c$$

$$=-\int yy^2\,e^{\frac{y^2}{2}}dy+c=-\int 2te^t dt+c \qquad \left(\text{Put } \frac{y^2}{2}=t \text{ then } y\,dy=dt\right)$$

$$=-2\int te^t dt+c=-2e^t\left(t-1\right)+c$$

$$u\,e^{\frac{y^2}{2}}=-2e^{\frac{y^2}{2}}\left(\frac{y^2}{2}-1\right)+c$$

$$\frac{1}{x}e^{\frac{y^2}{2}}=-e^{\frac{y^2}{2}}y^2+2e^{\frac{y^2}{2}}+c \quad \text{Since } u=x^{-1}=\frac{1}{x}$$

$$\frac{1}{x}e^{\frac{y^2}{2}}=e^{\frac{y^2}{2}}\left(2-y^2\right)+c \qquad \text{or } \left(\frac{1}{x}-2+y^2\right)e^{\frac{y^2}{2}}=c$$

This is the required solution.

3. Solve $\frac{dy}{dx}+y\tan x=y^2\sec x$.

Solution: Given equation is $\frac{dy}{dx}+y\tan x=y^2\sec x$ (1)

Multiplying on both sides of (1) by y^{-2} we get

$$y^{-2}\frac{dy}{dx}+y^{-1}\tan x=\sec x \quad (2)$$

Put $y^{-1}=u$, then $-y^{-2}\frac{dy}{dx}=\frac{du}{dx}$ (3)

Using (3) in (2), we get

$$-\frac{du}{dx}+u\tan x=\sec x \text{ or } \frac{du}{dx}-u\tan x=-\sec x \quad (4)$$

This is a linear differential equation in u.

$$I.F.=e^{\int P\,dx}=e^{-\int \tan x\,dx}=e^{\log(\cos x)}=\cos x$$

Therefore the general solution of (4) is

$$u\,(I.F.)=\int Q\,(I.F.)dx+c$$

$$u\cos x=-\int \sec x\cos x\,dx+c=-\int dx+c=-x+c$$

$$\text{or } \frac{1}{y}\cos x=-x+c \text{ since } u=y^{-1}=\frac{1}{y}$$

this is the required general solution of (1).

4. Solve $(1-x^2)\frac{dy}{dx}+xy=y^3\sin^{-1}x$.

Solution: Given equation is $(1-x^2)\frac{dy}{dx}+xy=y^3\sin^{-1}x$

Dividing throughout by $(1-x^2)$, $\frac{dy}{dx}+\frac{x}{1-x^2}y=y^3\frac{\sin^{-1}x}{1-x^2}$ (1)

Multiplying on both sides of (1) by y^{-3} we get

$$y^{-3}\frac{dy}{dx}+\frac{x}{1-x^2}y^{-2}=\frac{\sin^{-1}x}{1-x^2} \quad (2)$$

Put $y^{-2}=u$, then $-2y^{-3}\frac{dy}{dx}=\frac{du}{dx}\Rightarrow y^{-3}\frac{dy}{dx}=-\frac{1}{2}\frac{du}{dx}$ (3)

Using (3) in (2), we get

$$-\frac{1}{2}\frac{du}{dx}+\frac{x}{1-x^2}u=\frac{\sin^{-1}x}{1-x^2} \text{ or } \frac{du}{dx}-\frac{2x}{1-x^2}u=-\frac{2\sin^{-1}x}{1-x^2} \quad (4)$$

This is a linear differential equation in u.

$$I.F.=e^{\int P\,dx}=e^{\int \frac{-2x}{1-x^2}dx}=e^{\log(1-x^2)}=1-x^2$$

Therefore the general solution of (4) is

$$u\,(I.F.)=\int Q\,(I.F.)dx+c$$

$$u\left(1-x^2\right)=-2\int\frac{\sin^{-1}x}{1-x^2}\left(1-x^2\right)dx+c=-2\int\sin^{-1}x\,dx+c$$

$$u\left(1-x^2\right)=-2\left[x\sin^{-1}x+\sqrt{1-x^2}\right]+c$$, integration by parts

since $u=y^{-2}=\frac{1}{y^2}$ we get the general solution of (1) is

$$\frac{1-x^2}{y^2}=-2\left[x\sin^{-1}x+\sqrt{1-x^2}\right]+c$$

5. **Solve** $e^x\frac{dy}{dx}=2xy^2+ye^x$**.**

Solution: Given equation is $e^x\frac{dy}{dx}=2xy^2+ye^x$

Dividing throughout by e^x, $\frac{dy}{dx}-y=2xe^{-x}y^2$ (1)

This is Bernoulli's equation. Multiplying on both sides of (1) by y^{-2} we get

$$y^{-2}\frac{dy}{dx}-y^{-1}=2xe^{-x} \quad (2)$$

Put $-y^{-1}=u$, then $y^{-2}\frac{dy}{dx}=\frac{du}{dx}$ (3)

Using (3) in (2), we get

$$\frac{du}{dx}+u=2xe^{-x} \quad (4)$$

This is a linear differential equation in u.

$$I.F.=e^{\int P\,dx}=e^{\int dx}=e^x$$

Therefore the general solution of (4) is

$$u\left(I.F.\right)=\int Q\left(I.F.\right)dx+c$$

$$u\,e^x=2\int xe^{-x}\,e^x\,dx+c=2\int x\,dx+c=x^2+c$$

since $u=-y^{-1}=-\frac{1}{y}$ we get the general solution of (1) is

$$\frac{e^x}{y}=-x^2+c$$

DIFFERENTIAL EQUATIONS REDUCIBLE TO LINEAR EQUATION BY SUBSTITUTION

1. Solve $\sec^2y\frac{dy}{dx}+2x\tan y=x^3$**.**

Solution: Given equation is $\sec^2y\frac{dy}{dx}+2x\tan y=x^3$ (1)

Put $\tan y=u$ so that $\sec^2y\frac{dy}{dx}=\frac{du}{dx}$

Substituting these values in (1), we get

$$\frac{du}{dx}+2x\,u=x^3 \quad (2)$$

This is a linear equation in u. Here $P = 2x$ and $Q = x^3$

$$I.F. = e^{\int P\,dx} = e^{\int 2x\,dx} = e^{x^2}$$

Therefore the general solution of (2) is

$$u\left(I.F.\right) = \int Q\left(I.F.\right)dx + c$$

$$u\,e^{x^2} = \int x^3 e^{x^2}\,dx + c = \frac{1}{2}\left(x^2 - 1\right)e^{x^2} + c \text{ (Put } x^2 = t \text{ so that } x\,dx = \frac{1}{2}dt\text{)}$$

Substituting $u = \tan y$, we get the general solution of (1) is

$$e^{x^2}\tan y = \frac{1}{2}\left(x^2 - 1\right)e^{x^2} + c$$

2. Solve $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$**.**

Solution: Given equation is $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$ (1)

This can be written as $\frac{1}{\cos^2 y}\frac{dy}{dx} + x\frac{2\sin y\cos y}{\cos^2 y} = x^3$

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3 \qquad (2)$$

Put $\tan y = u$ so that $\sec^2 y\frac{dy}{dx} = \frac{du}{dx}$

Substituting these values in (2), we get

$$\frac{du}{dx} + 2x\,u = x^3 \qquad (3)$$

This is a linear equation in u. Here $P = 2x$ and $Q = x^3$

$$I.F. = e^{\int P\,dx} = e^{\int 2x\,dx} = e^{x^2}$$

Therefore the general solution of (3) is

$$u\left(I.F.\right) = \int Q\left(I.F.\right)dx + c$$

$$u\,e^{x^2} = \int x^3 e^{x^2}\,dx + c = \frac{1}{2}\left(x^2 - 1\right)e^{x^2} + c \text{ (Put } x^2 = t \text{ so that } x\,dx = \frac{1}{2}dt\text{)}$$

Substituting $u = \tan y$, we get the general solution of (2) is

$$e^{x^2}\tan y = \frac{1}{2}\left(x^2 - 1\right)e^{x^2} + c$$

3. Solve $2y\cos y^2\frac{dy}{dx} - \frac{2}{x+1}\sin y^2 = \left(x+1\right)^3$**.**

Solution: Given equation is $2y\cos y^2\frac{dy}{dx} - \frac{2}{x+1}\sin y^2 = \left(x+1\right)^3$ (1)

Put $\sin y^2 = u$ then $2y\cos y^2\frac{dy}{dx} = \frac{du}{dx}$

Substituting these values in (1), we get

$$\frac{du}{dx} - \frac{2}{x+1}u = \left(x+1\right)^3 \qquad (2)$$

This is a linear equation in u. Here $P=-\dfrac{2}{x+1}$ and $Q=(x+1)^3$

$$I.F.=e^{\int P\,dx}=e^{\int\frac{-2}{x+1}\,dx}=e^{-2\log(x+1)}=\frac{1}{(x+1)^2}$$

Therefore the general solution of (2) is $u\,(I.F.)=\int Q\,(I.F.)dx+c$

$$u\,\frac{1}{(x+1)^2}=\int (x+1)^3\,\frac{1}{(x+1)^2}\,dx+c=\int (x+1)dx+c$$

$$u\,\frac{1}{(x+1)^2}=\frac{(x+1)^2}{2}+c \qquad \text{or} \qquad u=\frac{(x+1)^4}{2}+c(x+1)^2$$

Substituting $u=\sin\,y^2$, we get the general solution of (1) is

$$\sin\,y^2=\frac{(x+1)^4}{2}+c(x+1)^2$$

EXACT DIFFERENTIAL EQUATIONS

The differential of a function $f(x,y)$ is denoted by df and is given by

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy \qquad (1)$$

Consider $M(x,y)dx+N(x,\ y)dy=0$ (2)

Suppose $\dfrac{\partial f}{\partial x}=M(x,y)$ (3)

and $\dfrac{\partial f}{\partial y}=N(x,y)$ (4)

Using equations (3) and (4), then the equation (1) becomes

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=M(x\,y)dx+N(x,y)dy=0$$

i.e., $df=0$

On integration, $f(x,y)=c$, arbitrary constant.

Therefore the expression of (2), $M\,dx+N\,dy=0$ is said to be an exact differential equation if there exists a function $f(x,y)$ such that $M=\dfrac{\partial f}{\partial x}$ and $N=\dfrac{\partial f}{\partial y}$.

Ex: 1. $2xydx+x^2dy=0$

2. $ydx+xdy=0$

Condition for Exactness

If $M(x,y)$ and $N(x,y)$ are two real valued functions which have continuous partial derivatives, then a necessary and sufficient condition for the differential equation $M\,dx+N\,dy=0$ to be exact is $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$.

Working rule to solve an Exact Differential Equation

Step 1: Let the differential equation be of the form $M(x,y)dx+N(x,\ y)dy=0$.

Check the condition for exactness $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, if exact proceed to step 2.

Step 2: The solution of the given equation is $\int M\, dx + \int N\; dy = c$

In the first integral treating y as constant and in second integral take only those terms in N which do not contain x.

(OR) the solution the given equation is $\int\limits_{(y\text{ constant})} M\; dx + \int (\text{terms independent of } x \text{ in } N)dy = c$

EXAMPLES

1. Solve $(hx + by + f)dy + (ax + hy + g)dx = 0$.

Solution: Given differential equation is

$$(hx + by + f)dy + (ax + hy + g)dx = 0 \qquad (1)$$

This is of the form $M\, dx + N\, dy = 0$, where

$$M = ax + hy + g \text{ and } N = hx + by + f$$

Now $\frac{\partial M}{\partial y} = h, \quad \frac{\partial N}{\partial x} = h \qquad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the given differential equation is exact.

$\therefore$ The general solution is given by

$$\int\limits_{(y\text{ constant})} M\; dx + \int (\text{terms independent of } x \text{ in } N)dy = c$$

$$\Rightarrow \int\limits_{(y\text{ constant})} (ax + hy + g)\, dx + \int (by + f)dy = c$$

$$\Rightarrow a\frac{x^2}{2} + hyx + gx + b\frac{y^2}{2} + fy = c$$

$$\Rightarrow ax^2 + 2hyx + 2gx + 2fy + by^2 = c$$

This is the required general solution of (1).

2. Solve $(2x - y + 1)dx + (2y - x - 1)dy = 0$.

Solution: Given differential equation is

$$(2x - y + 1)dx + (2y - x - 1)dy = 0 \qquad (1)$$

This is of the form $M\, dx + N\, dy = 0$, where

$$M = 2x - y + 1 \text{ and } N = 2y - x - 1$$

Now $\frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = -1 \qquad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the given differential equation is exact.

$\therefore$ The general solution is given by

$$\int\limits_{(y\text{ constant})} M\; dx + \int (\text{terms independent of } x \text{ in } N)dy = c$$

$$\Rightarrow \int_{(y\text{ constant})} (2x - y + 1)\, dx + \int (2y-1) dy = c$$

$$\Rightarrow x^2 - yx + x + y^2 - y = c$$

$$\Rightarrow x^2 - xy + x - y + y^2 = c$$

This is the required general solution of (1).

3. Solve $(1 + e^{x/y}) dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$.

Solution: α) Given differential equation is

$$(1 + e^{x/y}) dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0 \qquad (1)$$

It is of the form $M(x,\ y)\ dx + N(x,\ y)\ dy = 0$, we have

$M(x,\ y) = 1 + e^{x/y}$ and $N(x,\ y) = e^{x/y}\left(1 - \frac{x}{y}\right)$

Then $\frac{\partial M}{\partial y} = e^{x/y}.\frac{-x}{y^2}$ and $\frac{\partial N}{\partial x} = e^{\frac{x}{y}}.\frac{1}{y}.\left(1 - \frac{x}{y}\right) - e^{\frac{x}{y}}.\frac{1}{y} = e^{x/y}.\frac{-x}{y^2}$

$i.e.,$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$i.e.,$ (1) is an Exact differential equation.

So the general solution of (1) is

$$\int_{y=\text{constant}} M\ dx + \int (\text{terms of } N \text{ independent of } x)\ dy = c$$

$$\int_{y=\text{constant}} (1 + e^{x/y}) dx + \int (0)\ dy = c$$

$$x + y\ e^{x/y} = c$$

4. Solve $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$.

Solution: Given differential equation is

$$(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0 \qquad (1)$$

This is of the form $M\ dx + N\ dy = 0$, where

$$M = (e^y + 1)\cos x \text{ and } N = e^y \sin x$$

Now $\frac{\partial M}{\partial y} = e^y \cos x, \quad \frac{\partial N}{\partial x} = e^y \cos x \ \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the given differential equation is exact.

$\therefore$ The general solution is given by

$$\int\limits_{(y\text{ constant})} M\,dx + \int (\text{terms independent of } x \text{ in } N)\,dy = c$$

$$\Rightarrow \int\limits_{(y\text{ constant})} (e^y+1)\cos x\,dx + \int (0)\,dy = c$$

$$\Rightarrow (e^y+1)\int \cos x\,dx + 0 = c$$

$$\Rightarrow (e^y+1)\sin x = c$$

This is the required general solution of (1).

5. Solve $(y^2-2xy)dx = (x^2-2xy)dy$ **.**

Solution: Given differential equation is

$$(y^2-2xy)dx = (x^2-2xy)dy$$

$$\Rightarrow (y^2-2xy)dx + (2xy-x^2)dy = 0 \qquad (1)$$

This is of the form $M\,dx + N\,dy = 0$, where

$$M = y^2-2xy \quad \text{and} \quad N = 2xy - x^2$$

Now $\frac{\partial M}{\partial y} = 2y-2x, \quad \frac{\partial N}{\partial x} = 2y-2x \;\; \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the given differential equation is exact.

$\therefore$ The general solution is given by

$$\int\limits_{(y\text{ constant})} M\,dx + \int (\text{terms independent of } x \text{ in } N)\,dy = c$$

$$\Rightarrow \int\limits_{(y\text{ constant})} (y^2-2xy)\,dx + \int (0)\,dy = c$$

$$\Rightarrow y^2x - yx^2 = c$$

This is the required general solution of (1).

EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Integrating Factor:

Let $M\,dx + N\,dy = 0$ be not an exact differential equation.

If $M\,dx + N\,dy = 0$ can be made exact by multiplying it with a suitable factor $u(x,y) \neq 0$ called an integrating factor.

Example: Let $ydx - xdy = 0$ (1)

Here $M = y, N = -x$

Then $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -1$

$i.e.,\quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So that (1) is not an exact differential equation.

Multiplying (1) with $1/x^2$, we get

$\frac{y}{x^2}dx - \frac{1}{x}dy = 0$ (2)

Here $M = \frac{y}{x^2}, N = -\frac{1}{x}$

Since $\frac{\partial M}{\partial y} = \frac{1}{x^2} = \frac{\partial N}{\partial x}$

So (2) is an exact differential equation.

Hence $1/x^2$ is an integrating factor of $ydx - xdy = 0$.

Note: Also since $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$, $d\left(\log\frac{x}{y}\right) = \frac{ydx - xdy}{xy}$, $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$

The functions $\frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2 + y^2}$ are also integrating factors of $ydx - xdy = 0$.

From the above example we observe that a differential equation can have more than one integrating factor.

Methods to find integrating factor of $\boldsymbol{Mdx + Ndy = 0}$

Method 1: with some experience integrating factors can be **found by inspection**. For this purpose the student should keep in mind the following differentials.

1. $d(xy) = x\,dy + y\,dx$
2. $d\left(\frac{x^2 + y^2}{2}\right) = x\,dx + y\,dy$
3. $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$
4. $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$
5. $d\left(\log\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{xy}$
6. $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{xy}$
7. $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}$
8. $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{x^2 + y^2}$
9. $d(\log(xy)) = \frac{y\,dx + x\,dy}{xy}$
10. $d\left[\log\left(x^2 + y^2\right)\right] = \frac{2(xdx + ydy)}{x^2 + y^2}$

Examples

1. Solve $x\,dy - y\,dx + a\left(x^2 + y^2\right)dx = 0$

Solution: Given equation is $x\,dy - y\,dx + a\left(x^2 + y^2\right)dx = 0$

$$\frac{x\,dy - y\,dx}{x^2 + y^2} + a\,dx = 0$$

Integrating $\int \frac{xdy - ydx}{x^2 + y^2} + a\int dx = c$

$$\tan^{-1}\left(\frac{y}{x}\right) + ax = c$$

Which is the required solution.

2. Solve $x\,dx + y\,dy = \frac{x\,dy - y\,dx}{x^2 + y^2}$

Solution: Given equation is $x\,dx + y\,dy = \frac{x\,dy - y\,dx}{x^2 + y^2}$

$$d\left(\frac{x^2+y^2}{2}\right)=d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

Integrating $$\int d\left(\frac{x^2+y^2}{2}\right)=\int d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)+c$$

$$\Rightarrow \frac{x^2+y^2}{2}=\tan^{-1}\left(\frac{y}{x}\right)+c$$

$$\Rightarrow x^2+y^2=2\tan^{-1}\left(\frac{y}{x}\right)+2c$$

which is the required solution.

3. **Solve** $ydx-xdy=a(x^2+y^2)dx$

Solution: Given equation is $ydx-xdy=a(x^2+y^2)dx$

$$\Rightarrow \frac{ydx-xdy}{x^2+y^2}=adx$$

$$\Rightarrow d\left(\tan^{-1}\left(\frac{x}{y}\right)\right)=adx$$

Integrating, we get $\tan^{-1}\left(\frac{x}{y}\right)=ax+c$, which is the required solution.

4. **Solve** $xdy-ydx=xy^2dx$

Solution: Given equation is $xdy-ydx=xy^2dx$

$$\Rightarrow \frac{xdy-ydx}{y^2}=xdx \Rightarrow xdx-\frac{xdy-ydx}{y^2}=0$$

$$\Rightarrow xdx+\frac{ydx-xdy}{y^2}=0 \Rightarrow xdx+d\left(\frac{x}{y}\right)=0$$

Integrating, we get $\frac{x^2}{2}+\frac{x}{y}=c$

This is the required solution.

5. Solve $\frac{y(xy+e^x)dx-e^xdy}{y^2}=0$.

Solution: Given equation is $\frac{y(xy+e^x)dx-e^xdy}{y^2}=0$

$$\Rightarrow \frac{(xy^2+ye^x)dx-e^xdy}{y^2}=0 \Rightarrow xdx+\frac{ye^xdx-e^xdy}{y^2}=0$$

$$\Rightarrow xdx+d\left(\frac{e^x}{y}\right)=0$$

Integrating, we get $\frac{x^2}{2}+\frac{e^x}{y}=c$

Method 2: If $Mdx+Ndy=0$ is a homogeneous differential equation and $Mx+Ny\neq 0$ then $\frac{1}{Mx+Ny}$ is an integrating factor of $Mdx+Ndy=0$.

EXAMPLES

1. Solve $x^2y\,dx-(x^3+y^3)dy=0$.

Solution: Given differential equation is

$$x^2y\,dx-(x^3+y^3)dy=0 \qquad (1)$$

This is of the form $M\,dx+N\,dy=0$, we have

$$M=x^2y \text{ and } N=x^3+y^3$$

Then $\frac{\partial M}{\partial y}=x^2$ and $\frac{\partial N}{\partial x}=3x^2$

i.e., $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

So that (1) is not an exact differential equation.

But (1) is homogeneous differential equation and

$Mx+Ny=x^3y+(-x^3-y^3)y=-y^4\neq 0$

$\therefore\quad \text{I.F.}=\frac{1}{Mx+Ny}=-\frac{1}{y^4}$

Multiplying (1) with $-\frac{1}{y^4}$, we get

$$-\frac{x^2}{y^3}dx+\left(\frac{x^3+y^3}{y^4}\right)dy=0$$

$$-\frac{x^2}{y^3}dx+\left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy=0 \qquad (2)$$

Again it is of the form $M_1\,dx+N_1\,dy=0$, we have

$M_1=-\frac{x^2}{y^3}$ and $N_1=\frac{x^3}{y^4}+\frac{1}{y}$

Then $\frac{\partial M_1}{\partial y}=\frac{3x^2}{y^4}$ and $\frac{\partial N_1}{\partial x}=\frac{3x^2}{y^4}$

i.e., $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$

So that (2) is an exact differential equation.

Therefore the general solution of (2) is

$$\int_{y=constant} M_1\;dx+\int(terms\ of\ N_1\ not\ cotaining\ x)\,dy=c$$

$$\Rightarrow \int_{y=constant}\left(-\frac{x^2}{y^3}\right)dx+\int\frac{1}{y}dy=c \;\Rightarrow\; -\frac{1}{y^3}\int x^2\;dx+\int\frac{1}{y}dy=c$$

$$\Rightarrow\; -\frac{1}{y^3}\left(\frac{x^3}{3}\right)+\log y=c \;\Rightarrow\; -\frac{x^3}{3y^3}+\log y=c$$

It is the required general solution of (1).

2. Solve $y^2 dx + (x^2 - xy - y^2)dy = 0$.

Solution: Given differential equation is

$$y^2dx + (x^2 - xy - y^2)dy = 0 \qquad (1)$$

This is of the form $M\,dx + N\,dy = 0$, we have

$$M = y^2 \text{and } N = x^2 - xy - y^2$$

Then $\frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = 2x - y$

$i.e.,\quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So that (1) is not an exact differential equation.

But (1) is homogeneous differential equation and

$Mx + Ny = y^2x + (x^2 - xy - y^2)y = y(x^2 - y^2) \neq 0$

$\therefore\quad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{y(x^2 - y^2)}$

Multiplying (1) with $\frac{1}{y(x^2-y^2)}$, we get

$$\frac{y^2}{y(x^2 - y^2)}dx + \left[\frac{x^2 - xy - y^2}{y(x^2 - y^2)}\right]dy = 0$$

$$\frac{y}{x^2 - y^2}dx + \left(\frac{1}{y} - \frac{x}{x^2 - y^2}\right)dy = 0 \qquad (2)$$

Again it is of the form $M_1\,dx + N_1\,dy = 0$, we have

$M_1 = \frac{y}{x^2 - y^2}$ and $N_1 = \frac{1}{y} - \frac{x}{x^2 - y^2}$

Then $\frac{\partial M_1}{\partial y} = \frac{(x^2 - y^2).1 - y.(0 - 2y)}{(x^2 - y^2)^2} = \frac{x^2 + y^2}{(x^2 - y^2)^2}$

and $\frac{\partial N_1}{\partial x} = -\frac{(x^2 - y^2).1 - x.(0 - 2x)}{(x^2 - y^2)^2} = \frac{x^2 + y^2}{(x^2 - y^2)^2}$

$i.e.,\quad \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

So that (2) is an exact differential equation.

Therefore the general solution of (2) is

$$\int_{y=constant} M_1\ dx + \int (\text{terms of} N_1 \text{not cotaining } x)\,dy = c$$

$$\Rightarrow \int_{y=constant} \frac{y}{x^2 - y^2}\ dx + \int \frac{1}{y}dy = c$$

$$\Rightarrow \frac{1}{2}log\left(\frac{x - y}{x + y}\right) + log\ y = c$$

It is the required general solution of (1).

3. Solve $y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$.

Solution: Given differential equation is

$$y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$$

$i.e.,\ (x - y)dx + (x + y)dy = 0 \qquad (1)$

This is of the form $M\,dx + N\,dy = 0$, we have

$$M = x - y \text{ and } N = x + y$$

Then $\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = 1$

i.e., $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So that (1) is not an exact differential equation.

But (1) is a homogeneous differential equation and

$Mx + Ny = (x - y)x + (x + y)y = x^2 + y^2 \neq 0$

$\therefore$ $\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x^2 + y^2}$

Multiplying (1) with $\frac{1}{x^2+y^2}$, we get

$$\left(\frac{x - y}{x^2 + y^2}\right) dx + \left(\frac{x + y}{x^2 + y^2}\right) dy = 0 \qquad (2)$$

Again it is of the form $M_1\, dx + N_1\, dy = 0$, we have

$M_1 = \frac{x - y}{x^2 + y^2}$ and $N_1 = \frac{x + y}{x^2 + y^2}$

Then $\frac{\partial M_1}{\partial y} = \frac{(x^2 + y^2).(-1) - (x - y).(0 + 2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2}$

and $\frac{\partial N_1}{\partial x} = \frac{(x^2 + y^2).1 - (x + y).(0 + 2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2}$

i.e., $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

So that (2) is an exact differential equation.

Therefore the general solution of (2) is

$$\int_{y=constant} M_1\ dx + \int (terms\ of\ N_1\ not\ cotaining\ x)\, dy = c$$

$$\Rightarrow \int_{y=constant} \frac{x - y}{x^2 + y^2}\ dx + \int (0)\, dy = c$$

$$\Rightarrow \int_{y=constant} \frac{x}{x^2 + y^2}\ dx - \int_{y=constant} \frac{y}{x^2 + y^2}\ dx + 0 = c$$

$$\Rightarrow \frac{1}{2} log(x^2 + y^2) - tan^{-1}\left(\frac{x}{y}\right) = c$$

It is the required general solution of (1).

4. Solve $xy\, dx - (x^2 + y^2)dy = 0$.

Solution: Given differential equation is

$$xy\, dx - (x^2 + y^2)dy = 0 \qquad (1)$$

This is of the form $M\, dx + N\, dy = 0$, we have

$$M = xy \text{ and } N = -x^2 - y^2$$

Then $\frac{\partial M}{\partial y} = x$ and $\frac{\partial N}{\partial x} = -2x$

i.e., $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So that (1) is not an exact differential equation.

But (1) is a homogeneous differential equation and

$Mx + Ny = (xy)x + (-x^2 - y^2)y = -y^3 \neq 0$

$\therefore \quad \text{I.F.} = \frac{1}{Mx + Ny} = -\frac{1}{y^3}$

Multiplying (1) with $\frac{-1}{y^3}$, we get

$$\left(\frac{xy}{-y^3}\right)dx - \left(\frac{x^2 + y^2}{-y^3}\right)dy = 0$$

$$\frac{x}{-y^2}dx + \left(\frac{x^2}{y^3} + \frac{1}{y}\right)dy = 0 \qquad (2)$$

Again it is of the form $M_1\, dx + N_1\, dy = 0$, we have

$M_1 = \frac{x}{-y^2}$ and $N_1 = \frac{x^2}{y^3} + \frac{1}{y}$

Then $\frac{\partial M_1}{\partial y} = \frac{2x}{y^3}$ and $\frac{\partial N_1}{\partial x} = \frac{2x}{y^3}$

$i.e., \quad \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

So that (2) is an exact differential equation.

Therefore the general solution of (2) is

$$\int_{y=constant} M_1\ dx + \int (terms\ of\ N_1\ not\ cotaining\ x)\ dy = c$$

$$\Rightarrow \int_{y=constant} \left(\frac{x}{-y^2}\right) dx + \int \left(\frac{1}{y}\right) dy = c$$

$$\Rightarrow \frac{1}{y^2}\int x\ dx + \log y = c$$

$$\Rightarrow \frac{x}{y^2} + \log y = c$$

It is the required general solution of (1).

5. Solve $(3xy^2 - y^3)dx - (2x^2y - xy^2)dy = 0.$

Solution: Given equation is $(3xy^2 - y^3)dx - (2x^2y - xy^2)dy = 0$

Here $M = 3xy^2 - y^3 \quad N = -(2x^2y - xy^2)$

We have $\quad \frac{\partial M}{\partial y} = 6xy - 3y^2 \qquad \frac{\partial N}{\partial x} = -4xy + y^2$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Hence it is not exact. It is a homogeneous equation

$$I.F. = \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}$$

Multiplying with I.F. we get

$$\frac{(3xy^2 - y^3)}{x^2y^2}dx - \frac{(2x^2y - xy^2)}{x^2y^2}dy = 0$$

$$\left(\frac{3}{x}-\frac{y}{x^2}\right)dx-\left(\frac{2}{y}-\frac{1}{x}\right)dy=0 \qquad (2)$$

This is an exact equation, its solution is

$$\int\left(\frac{3}{x}-\frac{y}{x^2}\right)dx-\int\left(\frac{2}{y}-\frac{1}{x}\right)dy=0$$

$$\Rightarrow 3\log x-y\left(\frac{-1}{x}\right)-2\log y=c$$

$\Rightarrow 3\log x+\frac{y}{x}-2\log y=c$, which is the required solution.

6. Find an integrating factor so that $\frac{dy}{dx}=\frac{y}{x}+\frac{x^2+y^2}{x^2}$.

Solution: Given differential equation is

$$\frac{dy}{dx}=\frac{y}{x}+\frac{x^2+y^2}{x^2} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{xy+x^2+y^2}{x^2}$$

$$(xy+x^2+y^2)dx-x^2dy=0$$

It is of the form $M\,dx+N\,dy=0$ and it is a homogeneous differential equation.

So integrating factor $=\frac{1}{Mx+Ny}=\frac{1}{(xy+x^2+y^2)x-x^2y}=\frac{1}{x(x^2+y^2)}$

Method 3: If the differential equation $Mdx+Ndy=0$ is of the form

$y\,f_1(xy)dx+x\,f_2(xy)dy=0$, then I.F.$=\frac{1}{Mx-Ny}$ provided $Mx-Ny\neq 0$.

1. Solve $y(x^2y^2+2)\,dx+x(2-2x^2y^2)dy=0$.

Solution: Given differential equation is

$$y(x^2y^2+2)\,dx+x(2-2x^2y^2)dy=0 \qquad (1)$$

This is of the form $M\,dx+N\,dy=0$, we have

$$M=y(x^2y^2+2)\text{and } N=x(2-2x^2y^2)$$

Then $\frac{\partial M}{\partial y}=3x^2y^2+2$ and $\frac{\partial N}{\partial x}=2-6x^2y^2$

$i.e.,\quad \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

So that (1) is not an exact differential equation.

But (1) is of the form $y\,f_1(xy)dx+x\,f_2(xy)dy=0$ and

$Mx-Ny=y(x^2y^2+2)x-x(2-2x^2y^2)y=3x^3y^3\neq 0$

$\therefore\quad \text{I.F.}=\frac{1}{Mx-Ny}=\frac{1}{3x^3y^3}$

Multiplying (1) with $\frac{1}{3x^3y^3}$, we get

$$\frac{y(x^2y^2+2)}{3x^3y^3}\,dx+\frac{x(2-2x^2y^2)}{3x^3y^3}dy=0$$

$$\left(\frac{1}{3x}+\frac{2}{3x^3y^2}\right)dx+\left(\frac{2}{3x^2y^3}-\frac{2}{3y}\right)dy=0 \qquad (2)$$

Again it is of the form $M_1\ dx + N_1\ dy = 0$, we have

$M_1 = \frac{1}{3x}+\frac{2}{3x^3y^2}$ and $N_1 = \frac{2}{3x^2y^3}-\frac{2}{3y}$

Then $\frac{\partial M_1}{\partial y} = \frac{-4}{3x^3y^3}$ and $\frac{\partial N_1}{\partial x} = \frac{-4}{3x^3y^3}$

i.e., $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

So that (2) is an exact differential equation.

Therefore the general solution of (2) is

$$\int_{y=constant} M_1\ dx + \int (terms\ of\ N_1\ not\ cotaining\ x)\ dy = c$$

$$\Rightarrow \int_{y=constant}\left(\frac{1}{3x}+\frac{2}{3x^3y^2}\right)dx+\int\left(-\frac{2}{3y}\right)dy=c$$

$$\Rightarrow \frac{1}{3}\int\frac{1}{x}dx+\frac{2}{3y^2}\int\frac{1}{x^3}\ dx-\frac{2}{3}\int\frac{1}{y}dy=c$$

$$\Rightarrow \frac{1}{3}\log x+\frac{2}{3y^2}\left(-\frac{1}{2x^2}\right)-\frac{2}{3}\log y=c$$

$$\Rightarrow \frac{1}{3}\log x-\frac{1}{3x^2y^2}-\frac{2}{3}\log y=c$$

It is the required general solution of (1).

2. Solve $y(xy\ sin\ xy + cos\ xy)dx + x(xy\ sin\ xy - cos\ xy)dy = 0.$

Solution: Given differential equation is

$$y(xy\ sin\ xy + cos\ xy)dx + x(xy\ sin\ xy - cos\ xy)dy = 0 \qquad (1)$$

This is of the form $M\ dx + N\ dy = 0$, we have

$$M = y(xy\ sin\ xy + cos\ xy) \text{ and } N = x(xy\ sin\ xy - cos\ xy)$$

Then $\frac{\partial M}{\partial y} = (x^2y^2+1)cos\ xy + xy\ sin\ xy$

and $\frac{\partial N}{\partial x} = (x^2y^2-1)cos\ xy + 3xy\ sin\ xy$

i.e., $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So that (1) is not an exact differential equation.

But (1) is of the form $y\ f_1(xy)dx + x\ f_2(xy)dy = 0$ and

$$Mx - Ny = y(xy\ sin\ xy + cos\ xy)x - x(xy\ sin\ xy - cos\ xy)y$$
$$= 2xy\ cos\ xy \neq 0$$

$\therefore$ I. F. $= \frac{1}{Mx-Ny} = \frac{1}{2xy\ cos\ xy}$

Multiplying (1) with $\frac{1}{2xy\ cos\ xy}$, we get

$$\frac{y(xy\ sin\ xy + cos\ xy)}{2xy\ cos\ xy}\ dx + \frac{x(xy\ sin\ xy - cos\ xy)}{2xy\ cos\ xy}dy = 0$$

$$\left(\frac{y}{2}\frac{\sin xy}{\cos xy}+\frac{1}{2x}\right)dx+\left(\frac{x}{2}\frac{\sin xy}{\cos xy}-\frac{1}{2y}\right)dy=0$$

$$\left(\frac{y}{2}\tan xy+\frac{1}{2x}\right)dx+\left(\frac{x}{2}\tan xy-\frac{1}{2y}\right)dy=0 \qquad (2)$$

Again it is of the form $M_1\,dx+N_1\,dy=0$, we have

$M_1=\frac{y}{2}\tan xy+\frac{1}{2x}$ and $N_1=\frac{x}{2}\tan xy-\frac{1}{2y}$

Then $\frac{\partial M_1}{\partial y}=\frac{1}{2}(\tan xy+xy\sec^2 xy)$ and $\frac{\partial N_1}{\partial x}=\frac{1}{2}(\tan xy+xy\sec^2 xy)$

i.e., $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$

So that (2) is an exact differential equation.

Therefore the general solution of (2) is

$$\int_{y=constant} M_1\,dx+\int(\text{terms of } N_1 \text{ not cotaining } x)\,dy=c$$

$$\Rightarrow \int_{y=constant}\left(\frac{y}{2}\tan xy+\frac{1}{2x}\right)dx+\int\left(-\frac{1}{2y}\right)dy=c$$

$$\Rightarrow \frac{y}{2}\int \tan xy\,dx+\frac{1}{2}\int\frac{1}{x}\,dx-\frac{1}{2}\int\frac{1}{y}dy=c$$

$$\Rightarrow \frac{y}{2}\frac{\log(\sec xy)}{y}+\frac{1}{2}\log x-\frac{1}{2}\log y=c$$

$$\Rightarrow \frac{1}{2}\log\left(\frac{x}{y}\sec xy\right)=c$$

$$\Rightarrow \frac{x}{y}\sec xy=e^{2c}=c_1$$

It is the required general solution of (1).

3. Solve $y(1+xy)dx+x(1-xy)dy=0$.

Solution: Given differential equation is

$$y(1+xy)dx+x(1-xy)dy=0 \qquad (1)$$

This is of the form $M\,dx+N\,dy=0$, we have

$$M=y(1+xy) \text{ and } N=x(1-xy)$$

Then $\frac{\partial M}{\partial y}=1+2xy$ and $\frac{\partial N}{\partial x}=1-2xy$

i.e., $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

So that (1) is not an exact differential equation.

But (1) is of the form $y\,f_1(xy)dx+x\,f_2(xy)dy=0$ and

$$Mx-Ny=y(1+xy)x-x(1-xy)y$$
$$=2x^2y^2\neq 0$$

$\therefore$ I.F. $=\frac{1}{Mx-Ny}=\frac{1}{2x^2y^2}$

Multiplying (1) with $\frac{1}{2x^2y^2}$, we get

$$\frac{y(1+xy)}{2x^2y^2}\ dx + \frac{x(1-xy)}{2x^2y^2} dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right) dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right) dy = 0 \qquad (2)$$

Again it is of the form $M_1\ dx + N_1\ dy = 0$, we have

$M_1 = \frac{1}{2x^2y} + \frac{1}{2x}$ and $N_1 = \frac{1}{2xy^2} - \frac{1}{2y}$

Then $\frac{\partial M_1}{\partial y} = -\frac{1}{2x^2y^2}$ and $\frac{\partial N_1}{\partial x} = -\frac{1}{2x^2y^2}$

i.e., $\quad \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

So that (2) is an exact differential equation.

Therefore the general solution of (2) is

$$\int_{y=constant} M_1\ dx + \int (terms\ of\ N_1\ not\ cotaining\ x)\ dy = c$$

$$\Rightarrow \int_{y=constant} \left(\frac{1}{2x^2y} + \frac{1}{2x}\right) dx + \int \left(-\frac{1}{2y}\right) dy = c$$

$$\Rightarrow \frac{1}{2y}\int \frac{1}{x^2}\ dx + \frac{1}{2}\int \frac{1}{x}\ dx - \frac{1}{2}\int \frac{1}{y} dy = c$$

$$\Rightarrow \frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2} log\ x - \frac{1}{2} log\ y = c$$

$$\Rightarrow -\frac{1}{2xy} + \frac{1}{2}\log\left(\frac{x}{y}\right) = c \quad \text{or} \quad \frac{1}{2}\log\left(\frac{x}{y}\right) - \frac{1}{2xy} = c$$

It is the required general solution of (1).

4. Solve $\boldsymbol{y(xy+2x^2y^2)dx + x(xy-x^2y^2)dy = 0}$.

Solution: Given differential equation is

$$y(xy+2x^2y^2)dx + x(xy-x^2y^2)dy = 0 \qquad (1)$$

This is of the form $M\ dx + N\ dy = 0$, we have

$$M = y(xy+2x^2y^2) \text{ and } N = x(xy-x^2y^2)$$

Then $\frac{\partial M}{\partial y} = 2xy + 6x^2y^2$ and $\frac{\partial N}{\partial x} = 2xy - 3x^2y^2$

i.e., $\quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So that (1) is not an exact differential equation.

But (1) is of the form $y\ f_1(x\,y)dx + x\ f_2(x\,y)dy = 0$ and

$$Mx - Ny = y(xy+2x^2y^2)x - x(xy-x^2y^2)y$$
$$= 3x^3y^3 \neq 0$$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx-Ny} = \frac{1}{3x^3y^3}$$

Multiplying (1) with $\frac{1}{3x^3y^3}$, we get

$$\frac{y(xy+2x^2y^2)}{3x^3y^3}\ dx + \frac{x(xy-x^2y^2)}{3x^3y^3} dy = 0$$

$$\left(\frac{1}{3x^2y}+\frac{2}{3}\frac{1}{x}\right)dx+\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy=0 \qquad (2)$$

Again it is of the form $M_1\,dx+N_1\,dy=0$, we have

$M_1=\frac{1}{3x^2y}+\frac{2}{3}\frac{1}{x}$ and $N_1=\frac{1}{3xy^2}-\frac{1}{3y}$

Then $\frac{\partial M_1}{\partial y}=-\frac{1}{3x^2y^2}$ and $\frac{\partial N_1}{\partial x}=-\frac{1}{3x^2y^2}$

i.e., $\quad \frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$

So that (2) is an exact differential equation.

Therefore the general solution of (2) is

$$\int_{y=constant} M_1\,dx+\int(terms\ of\ N_1\ not\ cotaining\ x)\,dy=c$$

$$\Rightarrow \int_{y=constant}\left(\frac{1}{3x^2y}+\frac{2}{3}\frac{1}{x}\right)dx+\int\left(-\frac{1}{3y}\right)dy=c$$

$$\Rightarrow \frac{1}{3y}\int\frac{1}{x^2}\,dx+\frac{2}{3}\int\frac{1}{x}\,dx-\frac{1}{3}\int\frac{1}{y}dy=c$$

$$\Rightarrow \frac{1}{3y}\left(-\frac{1}{x}\right)+\frac{2}{3}log\,x-\frac{1}{3}log\,y=c$$

$$\Rightarrow -\frac{1}{3xy}+\frac{1}{3}\log\left(\frac{x^2}{y}\right)=c \quad \text{or} \quad \frac{1}{3}\log\left(\frac{x^2}{y}\right)-\frac{1}{3xy}=c$$

It is the required general solution of (1).

Method 4: If there exists a continuous single valued function $f(x)$ such that

$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=f(x)$, then $e^{\int f(x)dx}$ is an integrating factor of $M\,dx+N\,dy=0$.

1. Solve $\boldsymbol{2xy\,dy-(x^2+y^2+1)dx=0}$.

Solution: Given differential equation is

$2xy\,dy-(x^2+y^2+1)dx=0 \qquad (1)$

This is of the form $M\,dx+N\,dy=0$, we have

$$M=-x^2-y^2-1 \text{ and } N=2xy$$

Then $\frac{\partial M}{\partial y}=-2y$ and $\frac{\partial N}{\partial x}=2y$

i.e., $\quad \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

So that (1) is not an exact differential equation.

But (1) is a homogeneous differential equation and

$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2xy}[-2y-2y]=-\frac{2}{x}=f(x)$

$\therefore \quad \text{I.F.}=e^{\int f(x)\,dx}=e^{\int\left(-\frac{2}{x}\right)dx}=e^{-2\,log\,x}=\frac{1}{x^2}$

Multiplying (1) with $\frac{1}{x^2}$, we get

$$\left(\frac{2xy}{x^2}\right)dy - \left(\frac{x^2+y^2+1}{x^2}\right)dx = 0$$

$$\frac{2y}{x}dy - \left(1+\frac{1}{x^2}+\frac{y^2}{x^2}\right)dx = 0 \qquad (2)$$

Again it is of the form $M_1\,dx + N_1\,dy = 0$, we have

$M_1 = -\left(1+\frac{1}{x^2}+\frac{y^2}{x^2}\right)$ and $N_1 = \frac{2y}{x}$

Then $\frac{\partial M_1}{\partial y} = -\frac{2y}{x^2}$ and $\frac{\partial N_1}{\partial x} = -\frac{2y}{x^2}$

$i.e.,\quad \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

So that (2) is an exact differential equation.

Therefore the general solution of (2) is

$$\int_{y=constant} M_1\ dx + \int (terms\ of\ N_1\ not\ cotaining\ x)\ dy = c$$

$$\Rightarrow -\int_{y=constant}\left(1+\frac{1}{x^2}+\frac{y^2}{x^2}\right)dx + \int (0)\,dy = c$$

$$\Rightarrow -\left(x-\frac{1}{x}-\frac{y^2}{x}\right) = c$$

$$\Rightarrow\ 1+y^2-x^2 = cx$$

It is the required general solution of (1).

2. Solve $\boldsymbol{(x^2+y^2+2x)\,dx + 2y\,dy = 0}$.

Solution: Given differential equation is

$$(x^2+y^2+2x)\,dx + 2y\,dy = 0 \qquad (1)$$

This is of the form $M\,dx + N\,dy = 0$, we have

$M = x^2+y^2+2x$ and $N = 2y$

Then $\frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = 0$

$i.e.,\quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So that (1) is not an exact differential equation.

But (1) is a homogeneous differential equation and

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{1}{2y}[2y-0] = 1 = f(x)$$

$\therefore\quad \text{I.F.} = e^{\int f(x)\,dx} = e^{\int 1.dx} = e^x$

Multiplying (1) with e^x, we get

$$e^x(x^2+y^2+2x)\,dx + 2ye^x\,dy = 0 \qquad (2)$$

Again it is of the form $M_1\,dx + N_1\,dy = 0$, we have

$M_1 = e^x(x^2+y^2+2x)$ and $N_1 = 2ye^x$

Then $\frac{\partial M_1}{\partial y} = 2ye^x$ and $\frac{\partial N_1}{\partial x} = 2ye^x$

$i.e.,\quad \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

So that (2) is an exact differential equation.
Therefore the general solution of (2) is

$$\int_{y=constant} M_1 \ dx + \int (terms\ of\ N_1\ not\ cotaining\ x)\ dy = c$$

$$\Rightarrow \int_{y=constant} e^x(x^2+y^2+2x)\ dx + \int (0)\ dy = c$$

$$\Rightarrow \int e^x(x^2+2x)\,dx + y^2\int e^x\ dx = c$$

$$\Rightarrow\ e^x x^2 + e^x y^2 = c \Rightarrow e^x(x^2+y^2) = c$$

3. Solve $(x^3-2y^2)dx + 2xydy = 0.$

Solution: Given equation is $(x^3-2y^2)dx + 2xydy = 0$

Here $M = x^3 - 2y^2$ $\quad\quad N = 2xy$

We have $\quad \frac{\partial M}{\partial y} = -4y \quad\quad \frac{\partial N}{\partial x} = 2y$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Hence the equation is not exact

But $\quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-4y-2y}{2xy} = \frac{-6y}{2xy} = \frac{-3}{x} = f(x)$

$$I.F. = e^{\int f(x)dx} = e^{\int \frac{-3}{x}dx} = e^{-3\log x} = e^{\log x^{-3}} = x^{-3}$$

Multiplying the equation with $\frac{1}{x^3}$ we get

$$\frac{(x^3-2y^2)dx}{x^3} + \frac{2xy}{x^3}dy = 0$$

$\left(1 - \frac{2y^2}{x^3}\right)dx + \frac{2y}{x^2}dy = 0$. It is an exact equation

It solution is $\int_{y=constant}\left(1-\frac{2y^2}{x^3}\right)dx + \int (0)dy = c$

$x - 2y^2\frac{x^{-2}}{-2} = c \Rightarrow x + \frac{y^2}{x^2} = cx^2$, which is the required solution.

Method 5: If there exists a continuous single valued function $g(y)$ such that $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$, then $e^{\int g(y)dy}$ is an integrating factor of $M\ dx + N\ dy = 0$.

1. Solve $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0.$

Solution: Given equation is $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$ (1)

Here $\quad M = y^4 + 2y; \quad\quad N = xy^3 + 2y^4 - 4x$

We have $\dfrac{\partial M}{\partial y}=4y^3+2;\qquad \dfrac{\partial N}{\partial x}=y^3-4$

$\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$. Hence it is not exact

But $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}\right)=\dfrac{y^3-4-4y^3-2}{y^4+2y}=\dfrac{-6-3y^3}{y\left(y^3+2\right)}$

$=\dfrac{-3\left(y^3+2\right)}{y\left(y^3+2\right)}=\dfrac{-3}{y}=g(y)$

I.F. $=e^{\int g(y)dy}=e^{\int\frac{-3}{y}dy}=e^{-3\int\frac{1}{y}dy}=e^{-3\log y}=e^{\log y^{-3}}=\dfrac{1}{y^3}$

Equation (1) multiplied by I.F., we get

$$\frac{\left(y^4+2y\right)}{y^3}dx+\frac{\left(xy^3+2y^4-4x\right)}{y^3}dy=0$$

$$\Rightarrow\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0 \qquad (2)$$

(2) is an exact differential equation. So its solution is

$$\int\limits_{y=constant}\left(y+\frac{2}{y^2}\right)dx+\int 2y\,dy=c$$

$$\Rightarrow yx+\frac{2}{y^2}x+2\frac{y^2}{2}=c$$

$$\Rightarrow\left(y+\frac{2}{y^2}\right)x+y^2=c$$

this is the required solution.

2. Solve $\left(y+y^2\right)dx+xy\,dy=0.$

Solution: Given equation is $\left(y+y^2\right)dx+xy\,dy=0$

Here $M=y+y^2\qquad N=xy$

We have $\dfrac{\partial M}{\partial y}=1+2y\qquad \dfrac{\partial N}{\partial x}=y$

$\therefore\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$. Hence it is not exact

But $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}\right)=\dfrac{y-(1+2y)}{y+y^2}=\dfrac{y-1-2y}{y(1+y)}$

$=\dfrac{-1-y}{y(1+y)}=\dfrac{-(1+y)}{y(1+y)}=\dfrac{-1}{y}=g(y)$

$$I.F.=e^{\int g(y)dy}=e^{\int\frac{-1}{y}dy}=e^{-\log y}=e^{\log y^{-1}}=y^{-1}=\frac{1}{y}$$

Multiplying the equation with $\frac{1}{y}$ we get

$$\frac{\left(y+y^2\right)}{y}dx+\frac{xy}{x}dy=0 \qquad (2)$$

It is an exact equation. So its solution is

$$\int_{y=constant}(1+y)dx+\int 0.\,dy=c \Rightarrow x+yx=c$$

which is the required solution.

3. Solve $\left(xy^2-x^2\right)dx+\left(3x^2y^2+x^2y-2x^3+y^2\right)dy=0.$

Solution: Given equation is $\left(xy^2-x^2\right)dx+\left(3x^2y^2+x^2y-2x^3+y^2\right)dy=0$

Here $M=xy^2-x^2$ $\qquad N=3x^2y^2+xh2y-2x^3+y^2$

We have $\frac{\partial M}{\partial y}=2xy$ $\qquad \frac{\partial N}{\partial x}=3y^2(2x)+y(2x)-6x^2=6xy^2+2xy-6x^2$

$\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. Hence it is not exact

But $\quad \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{6xy^2+2xy-6x^2-2xy}{xy^2-x^2}=\frac{6x\left(y^2-x\right)}{x\left(y^2-x\right)}=6=g(y)$

$$I.F.=e^{\int g(y)dy}=e^{\int 6dy}=e^{6y}$$

Multiplying the equation with e^{6y} we get

$$e^{6y}\left(xy^2-x^2\right)dx+e^{6y}\left(3x^2y^2+x^2y-2x^3+y^2\right)dy=0 \qquad (2)$$

It is an exact equation, its solution is

$$\int_{y=constant}\left(e^{6y}xy^2-6e^{6y}x^2\right)+\int e^{6y}y^2dy=c$$

$$e^{6y}y^2\frac{x^2}{2}-e^{6y}\frac{x^3}{3}+\left[\frac{y^2e^{6y}}{6}-\frac{2ye^{6y}}{36}+\frac{2e^{6y}}{216}\right]=c$$, using integration by parts

$$e^{6y}\frac{x^2y^2}{2}-e^{6y}\frac{x^3}{3}+\frac{ye^{6y}}{6}-\frac{ye^{6y}}{18}+\frac{e^{6y}}{108}=c$$

$e^{6y}\left[\frac{x^2y^2}{2}-\frac{x^3}{3}+\frac{y^2}{6}-\frac{y}{18}+\frac{1}{108}\right]=c$, which is required the solution.

Newton's law of Cooling

Statement: The rate of change of temperature of a body is proportional to the difference of the temperature of the body and its surrounding medium.

Let θ be the temperature of the body at time t and θ_0 be the temperature of its surrounding medium (usually air). By the Newton' law of cooling, we have

$$\frac{d\theta}{dt}\ \alpha\ \theta-\theta_0 \text{ or } \frac{d\theta}{dt}=-k\ (\theta-\theta_0), \text{where } k \text{ is a positive constant}$$

Examples

1. A body is originally at 80^{0}C and cools down to 60^{0}C in 20 minutes. If the temperature of the air is 40^{0}C, find the temperature of the body after 40 minutes.

Solution: Let θ be the temperature of the body at time t.

By Newton's law of cooling, we have

$$\frac{d\theta}{dt}=-k(\theta-\theta_0), \text{ where } \theta_0 \text{ is the temperature of the air}$$

$$\Rightarrow \frac{d\theta}{dt}=-k(\theta-40), \ (\text{since } \theta_0=40 \text{ be given})$$

or $\quad \dfrac{d\theta}{\theta-40}=-k\,dt$ (variables separable) (1)

Integrating on both sides, we get

$$\int\frac{d\theta}{\theta-40}=-k\int dt$$

$$\Rightarrow \log(\theta-40)=-kt+\log c, \ c \text{ is an integrating constant}$$

$$\Rightarrow \log\left(\frac{\theta-40}{c}\right)=-kt \Rightarrow \frac{\theta-40}{c}=e^{-kt}$$

$$\Rightarrow \theta-40=ce^{-kt} \Rightarrow \theta=40+ce^{-kt} \qquad (2)$$

Given that when $t=0, \theta=80^0$ and when $t=20, \theta=60^0$

Substituting this in (2), we get $c=40$ and $60=40+ce^{-20k}$

$$\Rightarrow 20=40e^{-20k} \Rightarrow e^{-20k}=\frac{1}{2} \Rightarrow e^{20k}=2$$

$$\Rightarrow 20k=\log 2 \Rightarrow k=\frac{1}{20}\log 2$$

$\therefore$ (2) becomes $\theta=40+40e^{-\left(\frac{1}{20}\log 2\right)t}$ (3)

When $t=30, \ \theta=?$

$\therefore (3) \Rightarrow \theta=40+40e^{-\left(\frac{1}{20}\log 2\right)40}$, from (3)

$$=40+40e^{-2\log 2}=40+40e^{\log\left(\frac{1}{4}\right)}$$

$$=40+40\left(\frac{1}{4}\right)=50^0\text{C}$$

2. If the temperature of a body is changing from 100^0C to 70^0C in 15 minutes, find the time when the temperature will be 50^0C, if the temperature of the air is 30^0C.

Solution: Let θ be the temperature of the body at time t.

By Newton's law of cooling, we have

$$\frac{d\theta}{dt} = -k(\theta - \theta_0), \text{ where } \theta_0 \text{ is the temperature of the air}$$

$$\Rightarrow \frac{d\theta}{dt} = -k(\theta - 30), \ (\text{since } \theta_0 = 30 \text{ be given })$$

$$\text{or} \quad \frac{d\theta}{\theta - 30} = -k\, dt \text{ (variables separable)} \qquad (1)$$

Integrating on both sides, we get

$$\int \frac{d\theta}{\theta - 30} = -k \int dt$$

$$\Rightarrow \log(\theta - 30) = -kt + \log c, \ c \text{ is an integrating constant}$$

$$\Rightarrow \log\left(\frac{\theta - 30}{c}\right) = -kt \Rightarrow \frac{\theta - 30}{c} = e^{-kt}$$

$$\Rightarrow \theta - 30 = ce^{-kt} \Rightarrow \theta = 30 + ce^{-kt} \qquad (2)$$

Given that when $t = 0, \theta = 100^0$ and when $t = 15, \theta = 70^0$

Substituting this in (2), we get $c = 70$ and $70 = 30 + ce^{-15k}$

$$\Rightarrow 40 = 70e^{-15k} \Rightarrow e^{-15k} = \frac{4}{7} \Rightarrow e^{15k} = \frac{7}{4}$$

$$\Rightarrow 15k = \log\left(\frac{7}{4}\right) \Rightarrow k = \frac{1}{15}\log\left(\frac{7}{4}\right)$$

$$\therefore \text{(2) becomes } \theta = 30 + 70e^{-\left(\frac{1}{15}\log\left(\frac{7}{4}\right)\right)t} \qquad (3)$$

When $\theta = 40^0$, $t = ?$

$$\therefore (3) \Rightarrow 40 = 30 + 70e^{-\left(\frac{1}{15}\log\left(\frac{7}{4}\right)\right)t}$$

$$\Rightarrow 10 = 70e^{-\left(\frac{1}{15}\log\left(\frac{7}{4}\right)\right)t} \Rightarrow \frac{1}{7} = e^{-\left(\frac{1}{15}\log\left(\frac{7}{4}\right)\right)t}$$

$$\Rightarrow 7 = e^{\left(\frac{1}{15}\log\left(\frac{7}{4}\right)\right)t} \Rightarrow \log 7 = \left(\frac{1}{15}\log\left(\frac{7}{4}\right)\right)t$$

$$\Rightarrow t = 15 \times \frac{\log 7}{\log\left(\frac{7}{4}\right)} = 15 \times 3.48$$

$$\Rightarrow t = 52.16 \text{ Minutes}$$

3. If the air is maintained at $15^\circ C$ and the temperature of the body drops from $70^\circ C$ to $40^\circ C$ in 10 minutes. What will be its temperature after 30 minutes.

Solution: Let θ be the temperature of the body at time t.

By Newton's law of cooling, we have

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$, where θ_0 is the temperature of the air

or $$\frac{d\theta}{\theta - \theta_0} = -k\,dt$$ (variables separable)

Integrating on both sides, we get

$$\int \frac{d\theta}{\theta - \theta_0} = -k \int dt$$

$\Rightarrow \log(\theta - \theta_0) = -kt + \log c$, c is an integrating constant

$$\Rightarrow \log\left(\frac{\theta - \theta_0}{c}\right) = -kt \Rightarrow \frac{\theta - \theta_0}{c} = e^{-kt}$$

$$\Rightarrow \theta - \theta_0 = ce^{-kt} \Rightarrow \theta = \theta_0 + ce^{-kt} \qquad (1)$$

Given $\theta_0 = 15^0\text{C}$ so that (1) becomes

$$\theta = 15 + ce^{-kt} \qquad (2)$$

Given that when $t = 0, \theta = 70^0$ and when $t = 10, \theta = 40^0$

Substituting this in (2), we get $c = 55$ and $40 = 15 + 55e^{-20k}$

$$\Rightarrow 25 = 55e^{-10k} \;\Rightarrow e^{-10k} = \frac{25}{55} \qquad (3)$$

When $t = 30,\ \theta = ?$

$\therefore$ (2) $\Rightarrow \theta = 15 + 55e^{-30k}$

$$= 15 + 55\left(e^{-10k}\right)^3 = 15 + 55\left(\frac{25}{55}\right)^3$$

$$= 20.1653 \approx 20^0\text{C}$$

4. A body kept in air with temperature 25^0 C cools from 140^0C to 80^0C in 20 minutes, find the time when the body cools down to 35^0C.

Solution: Let θ be the temperature of the body at time t.

By Newton's law of cooling, we have

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$, where θ_0 is the temperature of the air

$$\Rightarrow \frac{d\theta}{dt} = -k(\theta - 25),$$ (since $\theta_0 = 25$ be given)

or $$\frac{d\theta}{\theta - 25} = -k\,dt$$ (variables separable) (1)

Integrating on both sides, we get

$$\int \frac{d\theta}{\theta - 25} = -k \int dt$$

$\Rightarrow \log(\theta - 25) = -kt + \log c$, c is an integrating constant

$$\Rightarrow \log\left(\frac{\theta - 25}{c}\right) = -kt \Rightarrow \frac{\theta - 25}{c} = e^{-kt}$$

$$\Rightarrow \theta - 25 = ce^{-kt} \Rightarrow \theta = 25 + ce^{-kt} \qquad (2)$$

Given that when $t=0, \theta=140^0$ and when $t=20, \theta=80^0$

Substituting this in (2), we get $c=115$ and $80=25+ce^{-20k}$

$$\Rightarrow 55=115e^{-20k} \Rightarrow e^{-20k}=\frac{55}{115} \Rightarrow e^{20k}=\frac{115}{55}$$

$$\Rightarrow 20k=\log\left(\frac{115}{55}\right) \Rightarrow k=\frac{1}{20}\log\left(\frac{115}{55}\right)$$

$\therefore$ (2) becomes $\theta=25+115e^{-\left(\frac{1}{20}\log\left(\frac{115}{55}\right)\right)t}$ (3)

When $\theta=35^0\text{C}, \ t=?$

$$\therefore (3) \Rightarrow 35=25+115e^{-\left(\frac{1}{20}\log\left(\frac{115}{55}\right)\right)t}$$

$$\Rightarrow 10=115e^{-\left(\frac{1}{20}\log\left(\frac{115}{55}\right)\right)t} \Rightarrow \frac{10}{115}=e^{-\left(\frac{1}{20}\log\left(\frac{115}{55}\right)\right)t}$$

$$\Rightarrow \frac{115}{10}=e^{\left(\frac{1}{20}\log\left(\frac{115}{55}\right)\right)t} \Rightarrow \log\left(\frac{115}{10}\right)=\left(\frac{1}{20}\log\left(\frac{115}{55}\right)\right)t$$

$$\Rightarrow t=20\times\frac{\log\left(\frac{115}{10}\right)}{\log\left(\frac{115}{55}\right)}=20\times 3.31$$

$$\Rightarrow t=66.2 \text{ Minutes}$$

Law of Natural Growth or Decay

Let $x(t)$ be the amount of a substance at time t and let the substance be getting converted chemically. A law of chemical conversion states that the rate of change of amount $x(t)$ of a chemically changing substance is proportional to the amount of the substance available at that time, i.e., $\frac{dx}{dt} \alpha x$.

If as t increases, x increases, we can take $\frac{dx}{dt}=kx(k>0)$ and if x decreases as t increases we can take $\frac{dx}{dt}=-kx(k>0)$.

Examples

1. The number N of bacteria in culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What was the value of N after $1\frac{1}{2}$ hours.

Solution: According to law of natural growth, we have

$$\frac{dN}{dt} \alpha N \quad i.e., \quad \frac{dN}{dt}=kN \qquad (1)$$

Separating the variables, we get $\frac{dN}{N}=k\,dt$

Integrating, $\log N = kt + \log c \Rightarrow \frac{N}{c} = e^{kt} \Rightarrow N = ce^{kt}$ (2)

When $t = 0$, we have $N = 100$ so that $c = 100$

$\therefore$ (2) $\Rightarrow$ $N = 100e^{kt}$ (3)

When $t = 1$ hour, $N = 332$ so that from (3), we have

$332 = 100e^{k}$ (4)

When $t = 1\frac{1}{2}$ hours $= \frac{3}{2}$ hours, $N = 100e^{3k/2}$

$$\Rightarrow N = 100\left(e^{k}\right)^{3/2} = 100\left(\frac{332}{100}\right)^{3/2}, \text{from (4)}$$

$$\therefore \; N = 604.5 = 605$$

2. In a certain chemical reaction the rate of conversion of a substance at time t is proportional to the quantity of the substance still untransformed at that instant. At the end of one hour 60 grams remain and at the end of four hours 21 grams. How many grams of the first substance was there initially?

Solution: According to law of natural decay, we have

$\frac{dy}{dt} \; \alpha \; y \quad i.e., \quad \frac{dy}{dt} = -ky$ (1)

Separating the variables, we get $\frac{dy}{y} = -k\,dt$

Integrating, $\log y = -kt + \log c \Rightarrow \frac{y}{c} = e^{-kt} \Rightarrow y = ce^{-kt}$ (2)

Let $y = y_0$ at $t = 0$, then $y = y_0 e^{-kt}$ (3)

When $t = 1$ hour, $y = 60$ grams

$\therefore (3) \Rightarrow \quad 60 = y_0 e^{-k} \; or \; e^{-k} = 60/y_0$ (4)

When $t = 4$ hours, $y = 21$ grams, so that from (3), we have

$21 = y_0 e^{-4k}$ (5)

Using (4) in (5), we get

$$21 = y_0(60/y_0)^4 \Rightarrow y_0^3 = \frac{60^4}{21}$$

$$\therefore y_0 = \left(\frac{60^4}{21}\right)^{1/3} = 85.13 \text{ grams}$$

3. In a chemical reaction a given substance is being converted into another at a rate proportional to the amount of substance unconverted. If $\left(\frac{1}{5}\right)^{th}$ of the original amount has been transformed in 4 minutes, how much time will be required to transform one half.

Solution: Let x grams be the amount of the remaining substance after 't' minutes.

$\therefore$ The differential equation is $\frac{dx}{dt} = -kx, k > 0 \Rightarrow x = ce^{-kt}$ (1)

Let the original amount of substance be 'm' grams.

Given when $t=0, x=m, \quad \therefore (1) \Rightarrow c=m$

andwhen $t=4, x=m-\dfrac{m}{5}=\dfrac{4m}{5}$

$\therefore (1) \Rightarrow \dfrac{4m}{5}=me^{-4k} \quad \Rightarrow e^{-4k}=\dfrac{4}{5}$

$\Rightarrow -4k=\log\left(\dfrac{4}{5}\right) \quad \Rightarrow k=\dfrac{1}{4}(\log 5-\log 4)$ (2)

We have to find t when $x=\dfrac{m}{2}$

$\therefore (1) \Rightarrow \dfrac{m}{2}=me^{-kt} \quad \Rightarrow kt=\log 2$

$\Rightarrow t=\dfrac{1}{k}\log 2 \quad \Rightarrow \text{t}=\dfrac{4\log 2}{\log 5-\log 4}=12.42=13\,\text{minutes}$

4. A bacterial culture, growing exponentially, increases from 200 to 500 grams in the period from 6 a.m. to 9 a.m. how many grams will be present at noon.

Solution: Let N be the number of bacteria in a culture at any time $t>0$.

Then according law of natural growth $N=ce^{kt}$ (1)

Where c is a constant and k, the rate constant.

Given that $N=200$ grams when $t=0$

$\therefore (1) \Rightarrow c=200$

Thus we have $N=200ce^{kt}$ (2)

But when $t=3$ hours (from 6 a.m. to 9 a.m.), $N=500$ grams

Using these in (2) we get

$500=200ce^{3k} \quad \Rightarrow e^{3k}=\dfrac{5}{2}=2.5$

$\Rightarrow 3k=\log(2.5) \Rightarrow k=\dfrac{1}{3}\log(2.5)=0.3054$

Hence the number of bacteria in the culture at any instant of time $t>0$ is given by

$N=200ce^{(0.3054)t}$.

To know N when $t=6$ hours (from 6 a.m. to 12 noon)

$N=200ce^{(0.3054)6}=1249.8$ grams

ELECTRICAL CIRCUITS

We will consider circuits made up of

(i) voltage source which may be a battery or a generator

(ii) Resistance, Inductance and Capacitance

The formation of differential equation for an electric circuit depends upon the following laws.

Let i be the current and q the charge in the condenser plate at any time t. Then

(i) $i = \frac{dq}{dt}$ or $q = \int i\, dt$

(ii) Voltage drop across resistance $R = Ri = R\frac{dq}{dt}$

(iii) Voltage drop across inductance $L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$

(iv) Voltage drop across capacitance $C = \frac{q}{C}$

Kirchoff's law:

1. Voltage law: The algebraic sum of the voltage drops in each part of any closed electrical circuit is equal to the resultant electromotive force (e.m.f.) in that circuit.

2. Current law: At a junction or node, current coming is equal to current going.

Examples

1. If a voltage of $20\cos 5t$ is applied to a series circuit consisting of 10 ohm resistor and 2 henry inductor, determine the current at any time t.

Solution: Let i be the current flowing in the circuit containing resistance R and inductance L in series, with voltage source E at any time t.

Given $E = 20\cos 5t,\ R = 10 \text{ ohm},\ L = 2 \text{ henry}$

By voltage law, we have

$$L\frac{di}{dt} + Ri = E \quad \Rightarrow \frac{di}{dt} + \frac{R}{L}i = E$$

$$\Rightarrow \frac{di}{dt} + \frac{10}{2}i = 20\cos 5t$$

$$\Rightarrow \frac{di}{dt} + 5i = 20\cos 5t \qquad (1)$$

This is a linear differential equation is of the form $\frac{di}{dt} + Pi = Q$, where $P = 5, Q = 20\cos 5t$

Now $I.F. = e^{\int P dt} = e^{5\int dt} = e^{5t}$

$\therefore$ The general solution of (1) is

$$i \times (\text{I.F.}) = \int Q \times (\text{I.F.}) dt + c$$

$$\Rightarrow i \times e^{5t} = \int 20\cos 5t \times e^{5t} dt + c$$

$$= 20\frac{e^{5t}}{25+25}(5\cos 5t + 5\sin 5t) + c$$

$$= 2e^{5t}(\cos 5t + \sin 5t) + c$$

$$\Rightarrow i = 2(\cos 5t + \sin 5t) + ce^{-5t} \qquad (2)$$

At $t=0$, $i=0 \Rightarrow 0=2+c \Rightarrow c=-2$

Thus (2) becomes, $i=2(\cos 5t+\sin 5t)-2e^{-5t}$

2. A circuit has in series on electromotive force given by $E=100\sin(40t)$V a resistor of 10Ω and an inductor of 0.5H. if the initial current is 0, find the current at time $t>0$.

Solution: Let i denote the current in amperes at time t

The total electric magnetic force if $E=100\sin(40t)$

Then by the laws of electric circuits, we have

the voltage drop across the resistor $=Ri=10i$

voltage drop across the inductor $=L\dfrac{di}{dt}=\dfrac{1}{2}\dfrac{di}{dt}$

Applying Kirchoff's law, we have

$$\frac{1}{2}\frac{di}{dt}+10i=100\sin(40t)$$

$$\Rightarrow \frac{di}{dt}+20i=200\sin(40t) \qquad (1)$$

This is a linear differential equation is of the form $\dfrac{di}{dt}+Pi=Q$, where $P=20, Q=200\sin(40t)$

Now $I.F.=e^{\int P dt}=e^{20\int dt}=e^{20t}$

$\therefore$ The general solution of (1) is

$$i\times(\text{I.F.})=\int Q\times(\text{I.F.})dt+c$$

$$\Rightarrow i\times e^{20t}=\int 200\sin(40t)\times e^{20t}dt+c$$

$$=200\frac{e^{20t}}{20^2+40^2}(20\sin 40t-40\cos 40t)+c$$

$$=2e^{20t}(\sin 40t-2\cos 40t)+c$$

$$\Rightarrow i=2(\sin 40t-2\cos 40t)+ce^{-20t} \qquad (2)$$

At $t=0$, $i=0 \Rightarrow 0=-4+c \Rightarrow c=4$

Thus (2) becomes, $i=2(\sin 40t-2\cos 40t)+4e^{-20t}$

Unit-II
LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

Linear Differential Equation with constant coefficients:

Definition: An equation of the form $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = Q(x)$

Where $a_1, a_2, \ldots, a_n$ are real constants and $Q(x)$ is a continuous function of x is called an ordinary linear equation of order n with constant coefficients. We now state a theorem without proof.

Theorem: If y_1 and y_2 are two solutions of the equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = Q(x) \qquad (1)$$

then $y = c_1 y_1 + c_2 y_2$ is also its solution, where c_1 and c_2 are constants.

The general solution of a n^{th} order contains n arbitrary constants. If $y_1, y_2, \ldots, y_n$ are n independent solutions of (1) then $y = c_1 y_1 + c_2 y_2 + \ldots + c_{n1} y_n$ is the most general solution of (1). Let us denote this with u.

If $y = v$ is any particular solution of (1) then $y = u + v$ is the most general solution of (1). The part 'u' is called the "Complementary Function" (C.F.) and the part 'v' is called the "Particular Integral" (P.I.) of (1). The complete solution of (1) is given by

$$y = C.F. + P.I.$$

Operator D:

Let us denote $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots, \frac{d^n}{dx^n}$ with $D, D^2, D^3, \ldots, D^n$ so that

$$Dy = \frac{dy}{dx}, D^2 y = \frac{d^2 y}{dx^2}, D^3 y = \frac{d^3 y}{dx^3}, \ldots, D^n y = \frac{d^n y}{dx^n}$$

Now equation (1) can be written in symbolic form as

$$\left(D^n + a_1 D^{n-1} + a_2 D^{n-2} \ldots + a_{n-1} D + a_n\right) y = Q(x)$$

$$i.e., \quad f(D) y = Q(x)$$

Where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} \ldots + a_{n-1} D + a_n$ is a polynomial in D. The symbol D stands for the operation of differentiation.

To find the General solution (Complementary Function) of $f(D)y = 0$

The algebraic equation $f(m)y = 0$**,** $m^n + a_1 m^{n-1} + a_2 m^{n-2} \ldots + a_{n-1} m + a_n = 0$ where $a_1, a_2, \ldots, a_n$ are real constants, is called the auxiliary equation (A.E.) of $f(D)y = 0$**. Since the A.E., $f(m) = 0$** is a polynomial equation of degree n**, it will have n roots, say** $m_1, m_2, \ldots, m_n$.

S.No.	Roots of A.E. $f(m) = 0$	C.F. (Complementary Function)
1	$m_1, m_2, \ldots, m_n$, i.e., all roots are real and distinct	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$
2	$m_1, m_1, m_3, \ldots, m_n$ (i.e.,	$(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$

	two roots are real and equal and remaining are all real and different)	
3	$m_1, m_1, m_1, m_4, ..., m_n$ (i.e., three roots are real and equal and remaining are all real and different)	$(c_1 + c_2 x + c_3 x^2)e^{m_1 x} + c_4 e^{m_4 x} + ... + c_n e^{m_n x}$
4	Two roots of A.E. are complex say $\alpha + i\beta$ and $\alpha - i\beta$ and the remaining roots are real and different.	$e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$
5	A pair of conjugate complex roots $\alpha \pm i\beta$ are repeated twice and the remaining roots are real and different.	$e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + c_5 e^{m_5 x} + ... + c_n e^{m_n x}$
6	A pair of conjugate complex roots $\alpha \pm i\beta$ are repeated thrice and the remaining roots are real and different.	$e^{\alpha x}[(c_1 + c_2 x + c_3 x^2)\cos\beta x + (c_4 + c_5 x + c_6 x^2)\sin\beta x]$ $+ c_7 e^{m_7 x} + ... + c_n e^{m_n x}$

Note: If $\alpha + \sqrt{\beta}$ is a real irrational root of $f(m) = 0$, $\alpha - \sqrt{\beta}$ is also a root of the equation. The part of the complementary function corresponding to these roots can also be put in the form

$$e^{\alpha x}(c_1 \cosh\beta x + c_2 \sinh\beta x)$$

Examples

1. Solve $\frac{d^2 y}{dx^2} - a^2 y = 0, a \neq 0$.

Solution: Given Differential equation is $\frac{d^2 y}{dx^2} - a^2 y = 0$ (1)

Its operator form is $(D^2 - a^2)y = 0$

$i.e., f(D)y = 0$, where $f(D) = D^2 - a^2$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 - a^2 = 0 \Rightarrow m = \pm a$$

The roots are real and different

$\therefore$ The general solution of (1) is $y = c_1 e^{ax} + c_2 e^{-ax}$

where c_1 and c_2 are arbitrary constants.

2. Solve $\frac{d^2 y}{dx^2} + 1.5\frac{dy}{dx} + 0.5y = 0$.

Solution: Given Differential equation is $\frac{d^2y}{dx^2} + 1.5\frac{dy}{dx} + 0.5y = 0$ (1)

Its operator form is $(D^2 + 1.5D + 0.5)y = 0$

i.e., $f(D)y = 0$, where $f(D) = D^2 + 1.5D + 0.5$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 + 1.5m + 0.5 = 0$$
$$\Rightarrow 2m^2 + 3m + 1 = 0$$
$$\Rightarrow (m+1)(2m+1) = 0$$
$$\Rightarrow m = -1, -\frac{1}{2}$$

The roots are real and different

$\therefore$ The general solution of (1) is

$y = c_1 e^{-x} + c_2 e^{-\frac{x}{2}}$, where c_1 and c_2 are arbitrary constants.

3. Solve $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$.

Solution: Given Differential equation is $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$ (1)

Its operator form is $(D^3 - 9D^2 + 23D - 15)y = 0$

i.e., $f(D)y = 0$, where $f(D) = D^3 - 9D^2 + 23D - 15$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^3 - 9m^2 + 23m - 15 = 0$$
$$\Rightarrow (m-1)(m-3)(m-5) = 0$$
$$\Rightarrow m = 1, 3, 5$$

The roots are real and different

$\therefore$ The general solution of (1) is

$y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{5x}$, where c_1, c_2 and c_3 are arbitrary constants.

4. **Solve** $\frac{d^3x}{dt^3} - 2\frac{d^2x}{dt^2} - 3\frac{dx}{dt} = 0$.

Solution: Given Differential equation is $\frac{d^3x}{dt^3} - 2\frac{d^2x}{dt^2} - 3\frac{dx}{dt} = 0$ (1)

Its operator form is $(D^3 - 2D^2 - 3D)x = 0$

i.e., $f(D)y = 0$, where $f(D) = D^3 - 2D^2 - 3D$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^3 - 2m^2 - 3m = 0$$
$$\Rightarrow m(m-3)(m+1) = 0$$
$$\Rightarrow m = 0, 3, -1$$

The roots are real and different

$\therefore$ The general solution of (1) is

$$x = c_1 + c_2 e^{3t} + c_3 e^{-t}$$, where c_1, c_2 and c_3 are arbitrary constants.

5. Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$.

Solution: Given Differential equation is $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$ (1)

Its operator form is $(D^3 - 3D + 2)y = 0$

i.e., $f(D)y = 0$, where $f(D) = D^3 - 3D + 2$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^3 - 3m + 2 = 0$$
$$\Rightarrow (m-1)(m^2 + m - 2) = 0$$
$$\Rightarrow (m-1)(m-1)(m+2) = 0$$
$$\Rightarrow m = 1, 1, -2$$

Sine two roots of $f(m) = 0$ are equal

$\therefore$ The general solution of (1) is

$$y = (c_1 + c_2 x)e^{x} + c_3 e^{-2x}$$, where c_1, c_2 and c_3 are arbitrary constants.

6. Solve $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$.

Solution: Given Differential equation is $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$ (1)

i.e., $f(D)y = 0$, where $f(D) = D^4 - 2D^3 - 3D^2 + 4D + 4$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^4 - 2m^3 - 3m^2 + 4m + 4 = 0$$
$$\Rightarrow (m+1)(m^3 - 3m^2 + 4) = 0$$
$$\Rightarrow (m+1)(m+1)(m^2 - 4m + 4) = 0$$
$$\Rightarrow (m+1)(m+1)(m-2)(m-2) = 0$$
$$\Rightarrow m = -1, -1, 2, 2$$

$\therefore$ The general solution of (1) is

$$y = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^{2x}$$, where c_1, c_2, c_3 and c_4 are arbitrary constants.

7. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$.

Solution: Given Differential equation is $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ (1)

Its operator form is $(D^2 + D + 1)y = 0$

i.e., $f(D)y = 0$, where $f(D) = D^2 + D + 1$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 + m + 1 = 0$$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

The roots are complex.

$\therefore$ The general solution of (1) is

$$y = e^{\frac{-x}{2}}\left(c_1 \cos\frac{\sqrt{3}\,x}{2} + c_2 \sin\frac{\sqrt{3}\,x}{2}\right)$$, where c_1 and c_2 are arbitrary constants.

8. Solve $(D^4 + 8D^2 + 16)y = 0$.

Solution: Given Differential equation is $(D^4 + 8D^2 + 16)y = 0$ (1)

i.e., $f(D)y = 0$, where $f(D) = D^4 + 8D^2 + 16$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^4 + 8m^2 + 16 = 0$$

$$\Rightarrow (m^2 + 4)^2 = 0$$

$$\Rightarrow (m - 2i)^2 (m + 2i)^2 = 0$$

$$\Rightarrow m = 2i, 2i, -2i, -2i$$

$\therefore$ The general solution of (1) is

$$y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

9. Solve $(D^3 - 14D + 8)y = 0$.

Solution: Given Differential equation is $(D^3 - 14D + 8)y = 0$ (1)

i.e., $f(D)y = 0$, where $f(D) = D^3 - 14D + 8$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^3 - 14m + 8 = 0$$

$$\Rightarrow (m + 4)(m^2 - 4m + 2) = 0$$

$$\Rightarrow m = -4 \text{ and } m = 2 \pm \sqrt{2}$$

$\therefore$ The general solution of (1) is

$$y = c_1 e^{-4x} + e^{2x}\left(c_2 \cosh\sqrt{2}\,x + c_3 \sinh\sqrt{2}\,x\right)$$

where c_1, c_2 and c_3 are arbitrary constants.

10. Solve $y'' + 6y' + 9y = 0,\ y(0) = -4,\ y'(0) = 14$.

Solution: Solution: Given Differential equation is $y'' + 6y' + 9y = 0$ (1)

Its operator form is $(D^2 + 6D + 9)y = 0$

i.e., $f(D)y = 0$, where $f(D) = D^2 + 6D + 9$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 + 6m + 9 = 0$$

$$\Rightarrow (m + 3)^2 = 0$$

$$\Rightarrow m = -3, -3$$

$\therefore$ The general solution of (1) is

$$y = (c_1 + c_2 x)e^{-3x} \qquad (2)$$

where c_1 and c_2 are arbitrary constants.

Differentiating (2) with respect to 'x', we get

$$y' = (c_1 + c_2 x)(-3e^{-3x}) + c_2 e^{-3x} \qquad (3)$$

Given $y(0) = -4$, then from (2), we have $c_1 = -4$ (4)

and $y'(0) = 14$, then from (3), we have $14 = -3c_1 + c_2$

$$c_2 = 14 + 3c_1 = 14 - 12 = 2 \qquad (5)$$

Using (4) and (5) in (2), we get the required solution of (1) is

$$y = (-4 + 2x)e^{-3x} = (2x - 4)e^{-3x}$$

11. Solve $y'' + y' - 2y = 0, y(0) = 4, y'(0) = 1$**.**

Solution: Solution: Given Differential equation is $y'' + y' - 2y = 0$ (1)

Its operator form is $(D^2 + D - 2)y = 0$

$i.e.,$ $f(D)y = 0$, where $f(D) = D^2 + D - 2$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 + m - 2 = 0$$

$$\Rightarrow (m-1)(m+2) = 0$$

$$\Rightarrow m = 1, -2$$

$\therefore$ The general solution of (1) is

$$y = c_1 e^{x} + c_2 e^{-2x} \qquad (2)$$

where c_1 and c_2 are arbitrary constants.

Differentiating (2) with respect to 'x', we get

$$y' = c_1 e^{x} - 2c_2 e^{-2x} \qquad (3)$$

Given $y(0) = 4$, then from (2), we have $c_1 + c_2 = 4$ (4)

and $y'(0) = 1$, then from (3), we have $c_1 - 2c_2 = 1$ (5)

solve (4) and (5), we get $c_1 = 3, c_2 = 1$

Using these values in (2), we get the required solution of (1) is

$$y = 3e^{x} + e^{-2x}$$

12. Solve $(\boldsymbol{D^3 - 1})\boldsymbol{y = 0}$**.**

Solution: Given differential equation is

$$(D^3 - 1)y = 0 \quad i.e., [f(D)]y = 0 \qquad (1)$$

Where $f(D) = D^3 - 1$

Now the auxiliary equation of the given D.E. is

$$f(D) = 0 \qquad i.e., \qquad D^3 - 1 = 0$$

$$(D - 1)(D^2 + D + 1) = 0$$

$$i.e., \qquad D - 1 = 0 \text{ and} D^2 + D + 1 = 0$$

$$D = 1 \text{ and } D = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

Therefore the general solution of (1) is

$$y = c_1 e^x + e^{-x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$$

13. Roots of the auxiliary equation for $\left(LD^2 + RD + \frac{1}{c}\right)q = E\sin pt$.

Solution: Given differential equation is

$$\left(LD^2 + RD + \frac{1}{c}\right)q = E\sin pt \quad i.e., \ [f(D)]q = E\sin pt \qquad (1)$$

where $f(D) = LD^2 + RD + \frac{1}{c}$

Now the auxiliary equation is

$$f(D) = 0$$

$$LD^2 + RD + \frac{1}{c} = 0$$

$$\therefore \text{ The roots are } D = \frac{-R \pm \sqrt{R^2 - \frac{4L}{c}}}{2L}$$

Inverse operator:

The operator D^{-1} or $\frac{1}{D}$ is called inverse of the differential operator D.

Definition: If Q is any function of x then $D^{-1}Q$ or $\frac{1}{D}Q$ is called the integral of Q.

We write $\frac{1}{D}Q = \psi \Rightarrow D\psi = Q$

Ex: $\frac{1}{D}\cos 3x = \int \cos 3x\, dx = \frac{\sin 3x}{3}$, Since $D\left(\frac{\sin 3x}{3}\right) = \cos 3x$

Definition: If $f(D)$ is differential operator defined earlier. Let $Q(x)$ be any function of x,

then we write $\frac{1}{f(D)}Q(x) = \psi(x)$ or $[f(D)]\psi(x) = Q(x)$

Ex: $\frac{1}{D^2 + 3D + 2}e^{4x} = \frac{e^{4x}}{30}$

Since $\left(D^2 + 3D + 2\right)\frac{e^{4x}}{30} = \frac{16e^{4x}}{30} + \frac{12e^{4x}}{30} + \frac{2e^{4x}}{30} = e^{4x}$

Ex: $\frac{1}{D+2}\cos 3x = \sin 3x$ is incorrect, Since $(D+2)(\sin 3x) = 3\cos 3x + 2\sin 3x$

Theorem:If $Q(x)$ is any function of x and α is a constant, then the particular value of $\frac{1}{D-\alpha}Q(x)$ is equal to $e^{\alpha x}\int Q(x)e^{-\alpha x}dx$.

i.e., P.I. of $\frac{1}{D-\alpha}Q(x) = e^{\alpha x}\int Q(x)e^{-\alpha x}dx$

Also P.I. of $\frac{1}{D+\alpha}Q(x) = e^{-\alpha x}\int Q(x)e^{\alpha x}dx$

Reason: Let $\frac{1}{D-\alpha}Q(x) = y \Rightarrow (D-\alpha)y = Q(x)$

It is a first order linear differential equation, so its particular solution is given by

$$ye^{-\alpha x} = \int Q(x)e^{-\alpha x}dx \text{ or } y = e^{\alpha x}\int Q(x)e^{-\alpha x}dx$$

Definition: If $\frac{1}{D-\beta}, \frac{1}{D-\alpha}$ are two inverse operators, then we define

$$\frac{1}{(D-\beta)(D-\alpha)}Q(x) = \frac{1}{(D-\beta)}\left[\frac{1}{D-\alpha}Q(x)\right]$$

where α, β are constants and Q is a function of x.

i.e., $\frac{1}{(D-\beta)(D-\alpha)}Q(x) = \frac{1}{(D-\beta)}\left[e^{\alpha x}\int Q(x)e^{-\alpha x}dx\right] = e^{\beta x}\int\left[e^{\alpha x}\int Q(x)e^{-\alpha x}dx\right]e^{-\beta x}dx$

Examples

1. Find $\frac{1}{D}x^2$.

Solution: Now $\frac{1}{D}x^2 = \int x^2 dx = \frac{x^3}{3}$

2. Find $\frac{1}{D^3}\cos x$.

Solution: Now $\frac{1}{D^3}\cos x = \frac{1}{D^2}\left(\frac{1}{D}\cos x\right) = \frac{1}{D^2}\left(\int \cos x\, dx\right) = \frac{1}{D^2}(\sin x)$

$$= \frac{1}{D}\left(\frac{1}{D}\sin x\right) = \frac{1}{D}\left(\int \sin x\, dx\right) = \frac{1}{D}(-\cos x)$$

$$= -\int \cos x\, dx = -\sin x$$

3. Find the particular value of $\frac{1}{D+1}x$.

Solution: Now $\frac{1}{D+1}x = e^{-x}\int xe^x dx = e^{-x}\left(xe^x - e^x\right) = x-1$

4. Find the particular value of $\frac{1}{(D-2)(D-3)}e^{2x}$.

Solution: Now $\frac{1}{(D-2)(D-3)}e^{2x} = \frac{1}{(D-2)}\left[\frac{1}{D-3}e^{2x}\right]$

Since $\frac{1}{D-3}e^{2x} = e^{3x}\int e^{2x}e^{-3x}dx = e^{3x}\left(-e^{-x}\right) = -e^{2x}$

$$\therefore \frac{1}{(D-2)}\left[\frac{1}{D-3}e^{2x}\right] = \frac{1}{D-2}\left(-e^{2x}\right) = -e^{2x}\int e^{2x}e^{-2x}dx = -e^{2x}\int dx = -xe^{2x}$$

Note: The above method to find particular integral (P.I.) is a general method and it will be more useful when $Q(x)$ is of the form $\tan ax, \cot ax, \sec ax, \text{cosec } ax$.

<u>General solution of</u> $f(D)y = Q(x)$:

We know that if $y = y_p$ is a particular solution of $f(D)y = Q(x)$ containing no arbitrary **constants and** $y = y_c$ is the general solution of $f(D)y = 0$ then $y = y_c + y_p$ is the general solution of $f(D)y = Q(x)$.

We have previously discussed the methods to find the general solution of $f(D)y = 0$.

Now we will discuss methods to find P.I. of $f(D)y = Q(x)$.

<u>Particular Integral of</u> $f(D)y = Q(x)$:

Given equation is $f(D)y = Q(x)$ (1)

Operating (1) by $\frac{1}{f(D)}$, we get $\frac{1}{f(D)}[f(D)y] = \frac{1}{f(D)}Q(x)$

$$\Rightarrow y = \frac{1}{f(D)}Q(x)$$

Clearly (1) is satisfied, if we take $y = \frac{1}{f(D)}Q(x)$

Thus particular integral $=$ P.I. $= \frac{1}{f(D)}Q(x)$

Note 1: To find the P.I. of $f(D)y = Q(x)$, we find the value of $\frac{1}{f(D)}Q(x)$.

Note 2: P.I. of $f(D)y = Q(x)$ contains no arbitrary constants.

Note 3: P.I. of $f(D)y = Q(x)$ when $\frac{1}{f(D)}$ is expressed as partial fractions.

Let $f(D) = (D-\alpha_1)(D-\alpha_2)...(D-\alpha_n)$, then

$$\text{P.I.} = \frac{1}{f(D)}Q(x) = \frac{1}{(D-\alpha_1)(D-\alpha_2)...(D-\alpha_n)}Q(x)$$

$$= \left[\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + ... + \frac{A_{n1}}{D-\alpha_n}\right]Q(x), \text{ resolving into partial fractions}$$

$$= A_1 e^{\alpha_1 x}\int Q(x)e^{-\alpha_1 x}dx + A_2 e^{\alpha_2 x}\int Q(x)e^{-\alpha_2 x}dx + ... + A_n e^{\alpha_n x}\int Q(x)e^{-\alpha_n x}dx$$

Examples

1. Solve $(D^2 - 5D + 6)y = xe^{4x}$.

Solution: Given differential equation is $(D^2 - 5D + 6)y = xe^{4x}$

$$i.e., f(D)y = Q(x) \quad (1)$$

where $f(D) = D^2 - 5D + 6$ and $Q(x) = xe^{4x}$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 - 5m + 6 = 0$$

$$\Rightarrow (m-2)(m-3)=0$$

$$\Rightarrow m = 2, 3$$

$\therefore \quad y_c = C.F. = c_1 e^{2x} + c_2 e^{3x}$ (2)

Now $y_p = P.I. = \dfrac{1}{f(D)} Q(x) = \dfrac{1}{D^2 - 5D + 6} x e^{4x}$

$= \dfrac{1}{(D-2)(D-3)} x e^{4x} = \left[\dfrac{1}{D-3} - \dfrac{1}{D-2}\right] x e^{4x}$, using partial fractions

$= \dfrac{1}{D-3} x e^{4x} - \dfrac{1}{D-2} x e^{4x}$

$= e^{3x}\int x e^{4x} e^{-3x} dx - e^{2x}\int x e^{4x} e^{-2x} dx$

$= e^{3x}\int x e^{x} dx - e^{2x}\int x e^{2x} dx$

$= e^{3x}\left(x e^{x} - e^{x}\right) - e^{2x}\left(x\dfrac{e^{2x}}{2} - \dfrac{e^{2x}}{4}\right)$, integration by parts

$= e^{4x}\dfrac{2x-3}{4}$

$\therefore$ The general solution (1) is

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{4} e^{4x}(2x-3)$$

2.Solve $\left(D^2 + a^2\right) y = \sec ax$.

Solution: Given differential equation is $\left(D^2 + a^2\right) y = \sec ax$

i.e., $f(D) y = Q(x)$ (1)

where $f(D) = D^2 + a^2$ and $Q(x) = \sec ax$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 + a^2 = 0$$

$$\Rightarrow m = \pm ia$$

$\therefore \quad y_c = C.F. = c_1 \cos ax + c_2 \sin ax$ (2)

Now $y_p = P.I. = \dfrac{1}{f(D)} Q(x) = \dfrac{1}{D^2 + a^2} \sec ax$

$= \dfrac{1}{(D-ia)(D+ia)} \sec ax = \dfrac{1}{2ai}\left[\dfrac{1}{D-ia} - \dfrac{1}{D+ia}\right] \sec ax$, using partial fractions

$= \dfrac{1}{2ai}\left[\dfrac{1}{D-ia}\sec ax - \dfrac{1}{D+ia}\sec ax\right]$ (3)

Now $\dfrac{1}{D-ia}\sec ax = e^{iax}\int \sec ax\, e^{-iax} dx = e^{iax}\int \dfrac{\cos ax - i \sin ax}{\cos ax} dx$

$= e^{iax}\int (1 - i \tan ax) dx = e^{iax}\left[x + \dfrac{i}{a}\log \cos ax\right]$ (4)

Similarly, $\dfrac{1}{D-ia}\sec ax = e^{-iax}\left[x - \dfrac{i}{a}\log\cos ax\right]$ (5)

Using (4) and (5) in (3), we get

$$y_p = \frac{1}{2ai}\left\{e^{iax}\left[x + \frac{i}{a}\log\cos ax\right] - e^{-iax}\left[x - \frac{i}{a}\log\cos ax\right]\right\}$$

$$= \frac{x}{a}\left(\frac{e^{iax} - e^{-iax}}{2i}\right) + \frac{1}{a^2}\left(\frac{e^{iax} + e^{-iax}}{2}\right)\log\cos ax$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\log(\cos ax) \quad (6)$$

$\therefore$ The general solution (1) is

$$y = y_c + y_p = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\log(\cos ax)$$

3.Solve $(D^2 + a^2)y = \tan ax$.

Solution: Given differential equation is $(D^2 + a^2)y = \tan ax$

$$i.e.,\ f(D)y = Q(x) \quad (1)$$

where $f(D) = D^2 + a^2$ and $Q(x) = \tan ax$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 + a^2 = 0$$

$$\Rightarrow m = \pm ia$$

$$\therefore \quad y_c = C.F. = c_1\cos ax + c_2\sin ax \quad (2)$$

Now $y_p = P.I. = \dfrac{1}{f(D)}Q(x) = \dfrac{1}{D^2 + a^2}\tan ax$

$= \dfrac{1}{(D-ia)(D+ia)}\tan ax = \dfrac{1}{2ai}\left[\dfrac{1}{D-ia} - \dfrac{1}{D+ia}\right]\tan ax$, using partial fractions

$$= \frac{1}{2ai}\left[\frac{1}{D-ia}\tan ax - \frac{1}{D+ia}\tan ax\right] \quad (3)$$

Now $\dfrac{1}{D-ia}\tan ax = e^{iax}\int \tan ax\, e^{-iax}dx = e^{iax}\int(\cos ax - i\sin ax)\dfrac{\sin ax}{\cos ax}\,dx$

$$= e^{iax}\int\left(\sin ax - i\frac{1-\cos^2 ax}{\cos ax}\right)dx$$

$$= e^{iax}\int\left(\sin ax - i\frac{1-\cos^2 ax}{\cos ax}\right)dx$$

$$= e^{iax}\int\sin ax\,dx - i\int\sec ax\,dx + i\int\cos ax\,dx$$

$$= e^{iax}\left[-\frac{\cos ax}{a} - \frac{i}{a}\log(\sec ax + \tan ax) + \frac{i}{a}\sin ax\right]$$

$$= -\frac{e^{iax}}{a}\left[(\cos ax - i\sin ax) + i\log(\sec ax + \tan ax)\right]$$

$$=-\frac{e^{iax}}{a}\left[e^{-iax}+i\log(\sec ax+\tan ax)\right]$$

$$=-\frac{1}{a}-\frac{i}{a}e^{iax}\log(\sec ax+\tan ax) \qquad (4)$$

Replace i by $-i$ in (3), we get

$$\frac{1}{D+ia}\tan ax=-\frac{1}{a}+\frac{i}{a}e^{-iax}\log(\sec ax+\tan ax) \qquad (5)$$

Using (4) and (5) in (3), we get

$$y_p=\frac{1}{2ai}\left\{\left[-\frac{1}{a}-\frac{i}{a}e^{iax}\log(\sec ax+\tan ax)\right]-\left[-\frac{1}{a}+\frac{i}{a}e^{-iax}\log(\sec ax+\tan ax)\right]\right\}$$

$$=-\frac{1}{a^2}\left(\frac{e^{iax}+e^{-iax}}{2}\right)\log(\sec ax+\tan ax)$$

$$=-\frac{1}{a^2}\cos ax\log(\sec ax+\tan ax) \qquad (6)$$

$\therefore$ The general solution (1) is

$$y=y_c+y_p=c_1\cos ax+c_2\sin ax-\frac{1}{a^2}\cos ax\log(\sec ax+\tan ax)$$

4. Solve $(D^2+4D+3)y=e^{e^x}$.

Solution: Given differential equation is $(D^2+4D+3)y=e^{e^x}$

$$i.e.,\ f(D)y=Q(x) \qquad (1)$$

where $f(D)=D^2+4D+3$ and $Q(x)=e^{e^x}$

Now the auxiliary equation of (1) is $f(m)=0$

$$\Rightarrow m^2+4m+3=0$$
$$\Rightarrow (m+1)(m+3)=0$$
$$\Rightarrow m=-1,-3$$

$$\therefore\quad y_c=C.F.=c_1e^{-x}+c_2e^{-3x} \qquad (2)$$

Now $y_p=P.I.=\dfrac{1}{f(D)}Q(x)=\dfrac{1}{D^2+4D+3}xe^{4x}$

$$=\frac{1}{(D+1)(D+3)}e^{e^x}=\frac{1}{2}\left[\frac{1}{D+1}-\frac{1}{D+3}\right]e^{e^x}$$, using partial fractions

$$=\frac{1}{2}\left[\frac{1}{D+1}e^{e^x}-\frac{1}{D+3}e^{e^x}\right] \qquad (3)$$

Now $\dfrac{1}{D+1}e^{e^x}=e^{-x}\int e^{e^x}e^xdx=e^{-x}\int e^tdt$, [Put $e^x=t\Rightarrow e^xdx=dt$]

$$=e^{-x}e^t=e^{-x}e^{e^x} \qquad (4)$$

and $\dfrac{1}{D+3}e^{e^x}=e^{-3x}\int e^{e^x}e^{3x}dx=e^{-3x}\int t^2e^tdt$, [Put $e^x=t\Rightarrow e^xdx=dt$]

$$=e^{-3x}e^t(t^2-2t+2)=e^{-3x}e^{e^x}(e^{2x}-2e^x+2) \qquad (5)$$

Using (4) and (5) in (3), we get

$$y_p = \frac{1}{2} e^{e^x}\left(e^{-x} - e^{-x} + 2e^{-2x} - 2e^{-3x}\right)$$

$$= e^{e^x}\left(e^{-2x} - e^{-3x}\right) \qquad (6)$$

$\therefore$ The general solution (1) is

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{-3x} + e^{e^x}\left(e^{-2x} - e^{-3x}\right)$$

RULES FOR FINDING PARTICULAR INTEGRAL IN SOME SPECIAL CASES

<u>Method 1</u>: P.I. of $f(D)y = Q(x)$ when $Q(x) = e^{ax}$, where 'a' is constant.

Case I: Let $f(D)y = e^{ax}$, then

$$y_p = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}, \text{ if } f(a) \neq 0$$

Case II: If $f(a) = 0$, then $(D - a)$ is a factor of $f(D)$. If 'a' is a root repeated k times for $f(a) = 0$, then $f(D) = (D-a)^k \phi(D)$ where $\phi(a) \neq 0$, then we have

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^k \phi(D)} e^{ax} = \frac{1}{\phi(a)} \frac{e^{ax}}{(D-a)^k} = \frac{1}{\phi(a)} e^{ax} \frac{x^k}{k!}$$

Hence $y_p = \dfrac{1}{f(D)} e^{ax} = \dfrac{e^{ax}}{\phi(a)} \dfrac{x^k}{k!}$, if $f(a) = 0$ and $\phi(a) \neq 0$

Note: In order to find the P.I. of $\sinh ax$ or $\cosh ax$ express them as $\dfrac{e^{ax} - e^{-ax}}{2}$ and $\dfrac{e^{ax} + e^{-ax}}{2}$ respectively.

Examples

1. Solve $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = e^{2x}$**.**

Solution: Given differential equation is

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x}$$

$$i.e., \left(D^2 + 4D + 3\right)y = e^{2x}$$

$$i.e., f(D)y = Q(x) \qquad (1)$$

where $f(D) = D^2 + 4D + 3$ and $Q(x) = e^{2x}$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 + 4m + 3 = 0$$

$$\Rightarrow (m+1)(m+3) = 0$$

$$\Rightarrow m = -1, -3$$

$$\therefore \quad y_c = C.F. = c_1 e^{-x} + c_2 e^{-3x} \qquad (2)$$

Now $y_p = P.I. = \dfrac{1}{f(D)} Q(x) = \dfrac{1}{D^2 + 4D + 3} e^{2x}$

$$= \frac{e^{2x}}{2^2 + 4(2) + 3}, \text{ put } D = 2 \text{ since } f(2) \neq 0$$

$$= \frac{e^{2x}}{16} \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{-3x} + \frac{e^{2x}}{16}$$

where c_1 and c_2 are constants.

2. Solve $(D^2 - 3D + 2)y = e^{5x}$.

Solution: Given differential equation is

$$(D^2 - 3D + 2)y = e^{5x}$$

$$i.e., f(D)y = Q(x) \qquad (1)$$

where $f(D) = D^2 - 3D + 2$ and $Q(x) = e^{5x}$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 - 3m + 2 = 0$$

$$\Rightarrow (m-1)(m-2) = 0$$

$$\Rightarrow m = 1, 2$$

$$\therefore \quad y_c = C.F. = c_1 e^{x} + c_2 e^{2x} \qquad (2)$$

Now $y_p = P.I. = \frac{1}{f(D)} Q(x) = \frac{1}{D^2 - 3D + 2} e^{5x}$

$$= \frac{e^{5x}}{5^2 - 3(5) + 2}, \text{ put } D = 5 \text{ since } f(5) \neq 0$$

$$= \frac{e^{5x}}{12} \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{2x} + \frac{e^{5x}}{12}$$

where c_1 and c_2 are constants.

3. Solve $(D^2 - 4D + 13)y = e^{2x}$.

Solution: Given differential equation is

$$(D^2 - 4D + 13)y = e^{2x}$$

$$i.e., f(D)y = Q(x) \qquad (1)$$

where $f(D) = D^2 - 4D + 13$ and $Q(x) = e^{2x}$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 - 4m + 13 = 0$$

$$\Rightarrow m = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm i6}{2}$$

$$\Rightarrow m = 2 \pm i3$$

$\therefore \quad y_c = C.F. = e^{2x}\left(c_1 \cos 3x + c_2 \sin 3x\right)$ (2)

Now $y_p = P.I. = \dfrac{1}{f(D)} Q(x) = \dfrac{1}{D^2 - 4D + 13} e^{2x}$

$= \dfrac{e^{2x}}{2^2 - 4(2) + 13}$, put $D = 2$ since $f(2) \neq 0$

$= \dfrac{e^{2x}}{9}$ (3)

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = e^{2x}\left(c_1 \cos 3x + c_2 \sin 3x\right) + \frac{e^{2x}}{9}$$

where c_1 and c_2 are constants.

4. Solve $\left(D^2 + 16\right)y = e^{-4x}$.

Solution: Given differential equation is

$$\left(D^2 + 16\right)y = e^{-4x}$$

i.e., $f(D)y = Q(x)$ (1)

where $f(D) = D^2 + 16$ and $Q(x) = e^{-4x}$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 + 16 = 0$$

$$\Rightarrow m = \pm i4$$

$\therefore \quad y_c = C.F. = c_1 \cos 4x + c_2 \sin 4x$ (2)

Now $y_p = P.I. = \dfrac{1}{f(D)} Q(x) = \dfrac{1}{D^2 + 16} e^{-4x}$

$= \dfrac{e^{-4x}}{(-4)^2 + 16}$, put $D = -4$ since $f(-4) \neq 0$

$= \dfrac{e^{-4x}}{32}$ (3)

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1 \cos 4x + c_2 \sin 4x + \frac{e^{-4x}}{32}$$

where c_1 and c_2 are constants.

5. Solve $\left(D^2 - 5D + 6\right)y = 4e^x + 5$.

Solution: Given differential equation is

$$\left(D^2 - 5D + 6\right)y = 4e^x + 5$$

i.e., $f(D)y = Q(x)$ (1)

where $f(D) = D^2 - 5D + 6$ and $Q(x) = 4e^x + 5$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2-5m+6=0$$
$$\Rightarrow m=2,3$$

$\therefore \quad y_c = C.F. = c_1e^{2x}+c_2e^{3x}$ (2)

Now $y_p = P.I. = \frac{1}{f(D)}Q(x) = \frac{1}{D^2-5D+6}\left(4e^x+5\right)$

$$=4\frac{1}{D^2-5D+6}e^x+5\frac{1}{D^2-5D+6}e^{(0)x}$$
$$=4\frac{e^x}{1^2-5(1)+6}+5\frac{e^{(0)x}}{0^2-5(0)+6}$$

$=2e^x+\frac{5}{6}$ (3)

$\therefore$ The general solution of (1) is

$$y=y_c+y_p=c_1e^{2x}+c_2e^{3x}+2e^x+\frac{5}{6}$$

where c_1 and c_2 are constants.

6. Solve $\left(D^3-5D^2+8D-4\right)y=e^{2x}$.

Solution: Given differential equation is

$$\left(D^3-5D^2+8D-4\right)y=e^{2x}$$

i.e., $f(D)y=Q(x)$ (1)

where $f(D)=D^3-5D^2+8D-4$ and $Q(x)=e^{2x}$

Now the auxiliary equation of (1) is $f(m)=0$

$$\Rightarrow m^3-5m^2+8m-4=0$$
$$\Rightarrow (m-1)(m-2)^2=0$$
$$\Rightarrow m=1,2,2$$

$\therefore \quad y_c = C.F. = c_1e^x+\left(c_2+c_3x\right)e^{2x}$ (2)

Now $y_p = P.I. = \frac{1}{f(D)}Q(x) = \frac{1}{(D-1)(D-2)^2}e^{2x}$

Here $f(2)=0$. Let $\phi(D)=D-1$, then $\phi(2)=2-1=1\neq 0$

$\therefore \quad y_p=\frac{1}{(2-1)}\frac{1}{(D-2)^2}e^{2x}=\frac{x^2}{2!}e^{2x}$ (3)

$\therefore$ The general solution of (1) is

$$y=y_c+y_p=c_1e^x+\left(c_2+c_3x\right)e^{2x}+\frac{x^2}{2!}e^{2x}$$

where c_1, c_2 and c_3 are constants.

7. Solve $\left(D^2-3D+2\right)y=\cosh x$.

Solution: Given differential equation is

$$\left(D^2-3D+2\right)y=\cosh x$$
$$i.e.,\ f(D)y=Q(x) \qquad (1)$$

where $f(D)=D^2-3D+2$ and $Q(x)=\cosh x$

Now the auxiliary equation of (1) is $f(m)=0$

$$\Rightarrow m^2-3m+2=0$$
$$\Rightarrow m=1,2$$

$$\therefore\quad y_c=C.F.=c_1e^x+c_2e^{2x} \qquad (2)$$

Now $y_p=P.I.=\dfrac{1}{f(D)}Q(x)=\dfrac{1}{D^2-3D+2}\cosh x$

$$=\frac{1}{(D-1)(D-2)}\left[\frac{e^x+e^{-x}}{2}\right]$$
$$=\frac{1}{2}\left[\frac{1}{(D-1)(D-2)}e^x+\frac{1}{(D-1)(D-2)}e^{-x}\right]$$
$$=\frac{1}{2}\left[\frac{1}{(D-1)(1-2)}e^x+\frac{1}{(-1-1)(-1-2)}e^{-x}\right]$$
$$=\frac{1}{2}\left[-xe^x+\frac{1}{6}e^{-x}\right] \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y=y_c+y_p=c_1e^x+c_2e^{2x}+\frac{1}{2}\left[-xe^x+\frac{1}{6}e^{-x}\right]$$

where c_1 and c_2 are constants.

8. Solve $(D+2)(D-1)^2y=e^{-2x}+2\sinh x$.

Solution: Given differential equation is

$$(D+2)(D-1)^2y=e^{-2x}+2\sinh x$$
$$i.e.,\ f(D)y=Q(x) \qquad (1)$$

where $f(D)=(D+2)(D-1)^2$ and $Q(x)=e^{-2x}+2\sinh x$

Now the auxiliary equation of (1) is $f(m)=0$

$$\Rightarrow (m+2)(m-1)^2=0$$
$$\Rightarrow m=1,1,2$$

$$\therefore\quad y_c=C.F.=(c_1+c_2x)e^x+c_3e^{2x} \qquad (2)$$

Now $y_p=P.I.=\dfrac{1}{f(D)}Q(x)=\dfrac{1}{(D+2)(D-1)^2}\left(e^{-2x}+2\sinh x\right)$

$$=\frac{1}{(D+2)(D-1)^2}e^{-2x}+\frac{1}{(D+2)(D-1)^2}2\sinh x$$
$$=\frac{1}{(D+2)(-2-1)^2}e^{-2x}+\frac{1}{(D+2)(D-1)^2}\left(e^x-e^{-x}\right)$$

$$=\frac{1}{9}xe^{-2x}+\frac{1}{(1+2)(D-1)^2}e^{x}-\frac{1}{(-1+2)(-1-1)^2}e^{-x}$$

$$=\frac{1}{9}xe^{-2x}+\frac{x^2}{6}e^{x}-\frac{1}{4}e^{-x} \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y=y_c+y_p=(c_1+c_2x)e^{x}+c_3e^{2x}+\frac{1}{9}xe^{-2x}+\frac{x^2}{6}e^{x}-\frac{1}{4}e^{-x}$$

where c_1, c_2 and c_3 are constants.

9. Solve the differential equation $(D^3-1)y=(e^x+1)^2$.

Solution: Given differential equation is

$$(D^3-1)y=(e^x+1)^2$$

$$i.e., f(D)y=Q(x) \qquad (1)$$

where $f(D)=D^3-1$ and $Q(x)=(e^x+1)^2$

Now the auxiliary equation of (1) is $f(m)=0$

$$\Rightarrow m^3-1=0$$

$$\Rightarrow (m-1)(m^2+m+1)=0$$

$$\Rightarrow m=1 \text{ and } m=\frac{-1\pm i\sqrt{3}}{2}$$

$$\therefore \quad y_c=C.F.=c_1e^{x}+e^{-x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right) \qquad (2)$$

Now $y_p=P.I.=\frac{1}{f(D)}Q(x)=\frac{1}{(D^3-1)}(e^x+1)^2$

$$=\frac{1}{(D-1)(D^2+D+1)}(e^{2x}+2e^{x}+1)$$

$$=\frac{1}{(D-1)(D^2+D+1)}e^{2x}+\frac{2}{(D-1)(D^2+D+1)}e^{x}+\frac{1}{(D-1)(D^2+D+1)}e^{0x}$$

$$=\frac{1}{(2-1)(2^2+2+1)}e^{2x}+\frac{2}{(D-1)(1^2+1+1)}e^{x}+\frac{1}{(0-1)(0^2+0+1)}e^{0x}$$

$$=\frac{e^{2x}}{7}+\frac{2}{3}xe^{x}-1 \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y=y_c+y_p=c_1e^{x}+e^{-x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)+\frac{e^{2x}}{7}+\frac{2}{3}xe^{x}-1$$

where c_1, c_2 and c_3 are constants.

10. Solve the differential equation $(D^3-3D^2+4)y=(1+e^{-x})^3$.

Solution: Given differential equation is

$$(D^3-3D^2+4)y=(1+e^{-x})^3$$
$$i.e.,\ f(D)y=Q(x) \qquad (1)$$

where $f(D)=D^3-3D^2+4$ and $Q(x)=(1+e^{-x})^3$

Now the auxiliary equation of (1) is $f(m)=0$

$$\Rightarrow m^3-3m^2+4=0$$
$$\Rightarrow (m+1)(m^2-4m+4)=0$$
$$\Rightarrow (m+1)(m-2)^2=0$$
$$\Rightarrow m=-1,2,2$$

$$\therefore \quad y_c=C.F.=c_1e^{-x}+(c_2+c_3x)e^{2x} \qquad (2)$$

Now $y_p=P.I.=\dfrac{1}{f(D)}Q(x)=\dfrac{1}{(D^3-3D^2+4)}(1+e^{-x})^3$

$$=\frac{1}{(D+1)(D^2-4D+4)}(1+e^{-3x}+3e^{-2x}+3e^{-x})$$

$$=\frac{1}{D^3-3D^2+4}+\frac{e^{-3x}}{D^3-3D^2+4}+\frac{3e^{-2x}}{D^3-3D^2+4}+\frac{3e^{-x}}{(D+1)(D^2-4D+4)}$$

$$=\frac{1}{0^3-3(0^2)+4}+\frac{e^{-3x}}{(-3)^3-3(-3)^2+4}+\frac{3e^{-2x}}{(-2)^3-3(-2)^2+4}+\frac{3e^{-x}}{(D+1)[(-1)^2-4(-1)+4]}$$

$$=\frac{1}{4}+\frac{e^{-3x}}{4}+\frac{3e^{-2x}}{-16}+\frac{xe^{-x}}{3} \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y=y_c+y_p=c_1e^{-x}+(c_2+c_3x)e^{2x}+\frac{1}{4}+\frac{e^{-3x}}{4}+\frac{3e^{-2x}}{-16}+\frac{xe^{-x}}{3}$$

where c_1, c_2 and c_3 are constants.

11. Find particular integral of $(D^2+1)y=\cosh 2x$.

Solution: Given differential equation is

$$(D^2+1)y=\cosh 2x$$

Now $\quad$ P.I. $=\dfrac{1}{D^2+1}\cosh 2x=\dfrac{1}{D^2+1}\left(\dfrac{e^{2x}+e^{-2x}}{2}\right)$

$$=\frac{1}{2}\left[\frac{1}{D^2+1}e^{2x}+\frac{1}{D^2+1}e^{-2x}\right]$$

$$=\frac{1}{2}\left[\frac{1}{2^2+1}e^{2x}+\frac{1}{(-2)^2+1}e^{-2x}\right], \qquad \text{since } \frac{1}{f(D)}e^{ax}=\frac{e^{ax}}{f(a)}, \text{ if } f(a)\neq 0$$

$$=\frac{1}{5}\left(\frac{e^{2x}+e^{-2x}}{2}\right)=\frac{1}{5}\cosh 2x$$

12. Find the particular integral of $(D^2+a^2)y=\cos ax$.

Solution: Given differential equation is

$$(D^2+a^2)y = cos\ ax$$

$[f(D)]y = Q(x)$ (1)

where $f(D) = D^2 + a^2$ and $Q(x) = cos\ ax$

Now P.I. $= \frac{1}{f(D)} Q(x) = \frac{1}{D^2+a^2} cos\ ax$ (2)

Since $\frac{1}{D^2+a^2} e^{iax} = \frac{1}{(D+ia)(D-ia)} e^{iax} = \frac{1}{(D-ia)}\left[\frac{1}{D+ia} e^{iax}\right]$

$= \frac{1}{(D-ia)}\left[\frac{1}{2ia} e^{iax}\right], \quad \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$ if $f(a) \neq 0$

$= \frac{x}{2ia} e^{iax}, \quad \frac{1}{D-a} e^{ax} = x\, e^{ax}$

$= \frac{x}{2ia}(cos\ ax + i\ sin\ ax)$

i.e., $\frac{1}{D^2+a^2}(cos\ ax + i\ sin\ ax) = -i\frac{x}{2a} cos\ ax + \frac{x}{2a} sin\ ax$

Equating real and imaginary parts, we get

$\frac{1}{D^2+a^2} cos\ ax = \frac{x}{2a} sin\ ax$

$\frac{1}{D^2+a^2} sin\ ax = -\frac{x}{2a} sin\ ax$

<u>Method 2:</u> P.I. of $f(D)y = Q(x)$ when $Q(x) = \sin ax$ or $\cos ax$, where 'a' is constant.

Case I: $\frac{1}{f(D)} \sin ax = \frac{1}{\phi(D^2)} \sin ax = \frac{\sin ax}{\phi(-a^2)}$, if $\phi(-a^2) \neq 0$

Similarly, $\frac{1}{f(D)} \cos ax = \frac{1}{\phi(D^2)} \cos ax = \frac{\cos ax}{\phi(-a^2)}$, if $\phi(-a^2) \neq 0$

Case II: Let $\phi(-a^2) \neq 0$. Then $D^2 + a^2$ is a factor of $\phi(D^2)$ and hence it is a factor of $f(D)$.

Let $f(D) = (D^2+a^2)g(D^2)$, where $g(-a^2) \neq 0$. It can be shown that

$$\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax, \quad \frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$$

Examples

1. Solve $(D^2+3D+2)y = \sin 3x$.

Solution: Given differential equation is

$$(D^2+3D+2)y = \sin 3x$$

i.e., $f(D)y = Q(x)$ (1)

where $f(D) = D^2+3D+2$ and $Q(x) = \sin 3x$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 + 3m + 2 = 0$$

$$\Rightarrow m = -1, -2$$

$\therefore \quad y_c = C.F. = c_1 e^{-x} + c_2 e^{-2x}$ (2)

Now $y_p = P.I. = \frac{1}{f(D)}Q(x) = \frac{1}{D^2+3D+2}\sin 3x$

$$= \frac{1}{-9+3D+2}\sin 3x \ [\text{Put } D^2 = -3^2 = -9]$$

$$= \frac{1}{3D-7}\sin 3x = \frac{3D+7}{9D^2-49}\sin 3x$$

$$= \frac{3D+7}{9(-9)-49}\sin 3x \ [\text{Put } D^2 = -3^2 = -9]$$

$$= -\frac{1}{130}\left[3\frac{d}{dx}\sin 3x + 7\sin 3x\right]$$

$$= -\frac{1}{130}[9\cos 3x + 7\sin 3x] \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1e^{-x} + c_2e^{-2x} - \frac{1}{130}[9\cos 3x + 7\sin 3x]$$

where c_1 and c_2 are constants.

2. Solve $(D^2 - 3D + 2)y = \cos 3x$ **.**

Solution: Given differential equation is

$$(D^2 - 3D + 2)y = \cos 3x$$

$$i.e., \ f(D)y = Q(x) \qquad (1)$$

where $f(D) = D^2 - 3D + 2$ and $Q(x) = \cos 3x$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 - 3m + 2 = 0$$

$$\Rightarrow m = 1, 2$$

$\therefore \quad y_c = C.F. = c_1e^x + c_2e^{2x} \qquad (2)$

Now $y_p = P.I. = \frac{1}{f(D)}Q(x) = \frac{1}{D^2-3D+2}\cos 3x$

$$= \frac{1}{-9-3D+2}\cos 3x \ [\text{Put } D^2 = -3^2 = -9]$$

$$= \frac{1}{-3D-7}\cos 3x = \frac{-1}{3D+7}\cos 3x = -\frac{3D-7}{9D^2-49}\cos 3x$$

$$= -\frac{3D-7}{9(-9)-49}\cos 3x \ [\text{Put } D^2 = -3^2 = -9]$$

$$= \frac{1}{130}\left[3\frac{d}{dx}\cos 3x - 7\cos 3x\right]$$

$$= \frac{1}{130}[-9\sin 3x - 7\cos 3x] \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{2x} - \frac{1}{130}[9\sin 3x + 7\cos 3x]$$

where c_1 and c_2 are constants.

3. Solve $(D^2 - 4)y = 2\cos^2 x$**.**

Solution: Given differential equation is

$$(D^2 - 4)y = 2\cos^2 x$$

$$i.e., f(D)y = Q(x) \quad (1)$$

where $f(D) = D^2 - 4$ and $Q(x) = 2\cos^2 x$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 - 4 = 0$$

$$\Rightarrow m = -2, 2$$

$$\therefore \quad y_c = C.F. = c_1 e^{-2x} + c_2 e^{2x} \quad (2)$$

Now $y_p = P.I. = \frac{1}{f(D)} Q(x) = \frac{1}{D^2 - 4} 2\cos^2 x = \frac{1}{D^2 - 4}(1 + \cos 2x)$

$$= \frac{e^{0x}}{D^2 - 4} + \frac{1}{D^2 - 4}\cos 2x \quad (3)$$

Since $\frac{e^{0x}}{D^2 - 4} = \frac{e^{0x}}{0^2 - 4}$, $[\text{Put } D = 0]$

$$= \frac{e^{0x}}{0 - 4} = -\frac{1}{4}$$

and $\frac{1}{D^2 - 4}\cos 2x = \frac{1}{-4 - 4}\cos 2x \; [\text{Put } D^2 = -2^2 = -4]$

$$= -\frac{1}{8}\cos 2x$$

$$\therefore \; (3) \; \Rightarrow \; y_p = -\frac{1}{4} - \frac{1}{8}\cos 2x \quad (4)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1 e^{-2x} + c_2 e^{2x} - \frac{1}{4} - \frac{1}{8}\cos 2x$$

where c_1 and c_2 are constants.

4. Solve $(D^2 + 4)y = e^{x} + \sin 2x + \cos 2x$**.**

Solution: Given differential equation is

$$(D^2 + 4)y = e^{x} + \sin 2x + \cos 2x$$

$$i.e., f(D)y = Q(x) \quad (1)$$

where $f(D) = D^2 + 4$ and $Q(x) = e^{x} + \sin 2x + \cos 2x$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 + 4 = 0$$

$$\Rightarrow m = \pm i2$$

$\therefore \quad y_c = C.F. = c_1 \cos 2x + c_2 \sin 2x \qquad (2)$

Now $y_p = P.I. = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+4}\left(e^x + \sin 2x + \cos 2x\right)$

$$= \frac{e^x}{D^2+4} + \frac{1}{D^2+4}\sin 2x + \frac{1}{D^2+4}\cos 2x \qquad (3)$$

Since $\frac{e^x}{D^2+4} = \frac{e^x}{1^2+4}$, $[\text{Put } D=1]$

$$= \frac{e^x}{1+4} = \frac{e^x}{5}$$

$$\frac{1}{D^2+4}\sin 2x = -\frac{x}{2(2)}\cos 2x = -\frac{x}{4}\cos 2x$$

$$\left[\text{Case of failure } f(-a^2)=0, \text{ using } \frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax\right]$$

and $\frac{1}{D^2+4}\cos 2x = \frac{x}{2(2)}\sin 2x = \frac{x}{4}\sin 2x$

$$\left[\text{Case of failure } f(-a^2)=0, \text{ using } \frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax\right]$$

$\therefore$ (3) $\Rightarrow y_p = \frac{e^x}{5} - \frac{x}{4}\cos 2x + \frac{x}{4}\sin 2x \qquad (4)$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{e^x}{5} - \frac{x}{4}\cos 2x + \frac{x}{4}\sin 2x$$

where c_1 and c_2 are constants.

5. Solve $(D^2+1)y = \sin x \sin 2x$.

Solution: Given differential equation is

$$(D^2+1)y = \sin x \sin 2x$$

$$i.e., f(D)y = Q(x) \qquad (1)$$

where $f(D) = D^2+1$ and $Q(x) = \sin x \sin 2x$

Now the auxiliary equation of (1) is $f(m)=0$

$$\Rightarrow m^2+1=0$$

$$\Rightarrow m = \pm i$$

$\therefore \quad y_c = C.F. = c_1 \cos x + c_2 \sin x \qquad (2)$

Now $y_p = P.I. = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+1}\sin x \sin 2x$

$$= \frac{1}{2}\frac{1}{D^2+1}[\cos x - \cos 3x]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+1}\cos x - \frac{1}{D^2+1}\cos 3x\right] \qquad (3)$$

since $\frac{1}{D^2+1}\cos x = \frac{x}{2(1)}\sin x = \frac{x}{2}\sin x$

$\left[\text{Case of failure } f(-a^2)=0, \text{ using } \frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax\right]$

and $\frac{1}{D^2+1}\cos 3x = \frac{\cos 3x}{-9+1}$, $[\text{Put } D^2=-3^2=-9]$

$$= -\frac{\cos 3x}{8}$$

$\therefore$ (3) $\Rightarrow$ $y_p = \frac{x}{4}\sin x + \frac{x}{16}\cos 3x$ (4)

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1\cos x + c_2\sin x + \frac{x}{4}\sin x + \frac{x}{16}\cos 3x$$

where c_1 and c_2 are constants.

6. Solve $\boldsymbol{(D^2-4D)y = e^x + \sin 3x\cos 2x}$.

Solution: Given differential equation is

$(D^2-4D)y = e^x + \sin 3x\cos 2x$ (1)

$i.e.,$ $[f(D)]y = Q(x)$

where $f(D) = D^2-4D$, and $Q(x) = e^x + \sin 3x\cos 2x$

Now the auxiliary equation is $f(m)=0, i.e.,$ $m^2-4m=0$

$i.e.,$ $m = 0,$ 4

The roots are real and different.

$\therefore$ $C.F. = c_1 + c_2e^{4x}$ (2)

Now $P.I. = \frac{1}{f(D)}Q(x) = \frac{1}{f(D)}[e^x + \sin 3x\cos 2x]$

$$= \frac{1}{D^2-4D}e^x + \frac{1}{2}\frac{1}{D^2-4D}2\sin 3x\cos 2x$$

$$= -\frac{e^x}{3} + \frac{1}{2}\frac{1}{D^2-4D}(\sin 5x + \sin x)$$

$$= -\frac{e^x}{3} + \frac{1}{2}\frac{1}{D^2-4D}\sin 5x + \frac{1}{2}\frac{1}{D^2-4D}\sin x \quad (3)$$

Since $\frac{1}{D^2-4D}\sin 5x = \frac{1}{-25-4D}\sin 5x = -\frac{25-4D}{(25+4D)(25-4D)}\sin 5x$

$$= -\frac{25-4D}{625-16D^2}\sin 5x = \frac{4D-25}{1025}\sin 5x$$

$$= \frac{1}{1025}(20\cos 5x - 25\sin 5x)$$

$$= \frac{1}{205}(4\cos 5x - 5\sin 5x) \quad (4)$$

and $\frac{1}{D^2-4D}\sin x = \frac{1}{-1-4D}\sin x = -\frac{1-4D}{(1+4D)(1-4D)}\sin x$

$$= -\frac{1-4D}{1-16D^2}\sin x = \frac{4D-1}{17}\sin x$$

$$= \frac{1}{17}(4\cos x - \sin x) \quad (5)$$

Substituting (4) and (5) in (3), we get

$$P.I. = -\frac{e^x}{3} + \frac{1}{410}(4\cos 5x - 5\sin 5x) + \frac{1}{34}(4\cos x - \sin x) \quad (6)$$

Therefore the general solution of (1) is

$$y = C.F. + P.I.$$

$$y = c_1 + c_2 e^{4x} - \frac{e^x}{3} + \frac{1}{410}(4\cos 5x - 5\sin 5x) + \frac{1}{34}(4\cos x - \sin x)$$

<u>Method 3:</u> P.I. of $f(D)y = Q(x)$ when $Q(x) = x^k$ where k is a positive integer:

Let $f(D)y = x^k$, operating by $\frac{1}{f(D)}$, we get $y = \frac{1}{f(D)}x^k$

$\therefore \quad$ P.I. $= \frac{1}{f(D)}x^k$

To evaluate P.I., reduce $\frac{1}{f(D)}$ to the form $\frac{1}{1 \pm \phi(D)}$ by taking out the lowest degree term from $f(D)$. Now write $\frac{1}{f(D)}$ as $[1 \pm \phi(D)]^{-1}$ and expand it in ascending powers of D using Binomial theorem upto the term containing D^k. Then operate x^k with the terms of the expansion of $[1 \pm \phi(D)]^{-1}$.

If $f(D)$ is resolvable into factors then split up $\frac{1}{f(D)}$ into partial fractions and proceed.

We frequently use the following rules:

(i) $\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + \ldots$

(ii) $\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + \ldots$

(iii) $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \ldots$

(iv) $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \ldots$

(v) $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \ldots$

(vi) $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \ldots$

Examples

1. Solve $(D^2 + D + 1)y = x^3$.

Solution: Given differential equation is

$$(D^2 + D + 1)y = x^3$$

$$i.e., f(D)y = Q(x) \quad (1)$$

where $f(D) = D^2 + D + 1$ and $Q(x) = x^3$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 + m + 1 = 0$$

$$\Rightarrow m = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\therefore \quad y_c = C.F. = e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right] \qquad (2)$$

Now $y_p = P.I. = \frac{1}{f(D)}Q(x) = \frac{1}{D^2 + D + 1}x^3$

$$= \left[1 + \left(D + D^2\right)\right]^{-1} x^3$$

$$= \left[1 - \left(D + D^2\right) + \left(D + D^2\right)^2 - \left(D + D^2\right)^3 + \ldots\right]x^3$$

$= \left(1 - D + D^3\right)x^3$, since $D^4\left(x^3\right) = D^5\left(x^3\right) = \ldots.. = 0$

$$= x^3 - D\left(x^3\right) + D^3\left(x^3\right) = x^3 - 3x^2 + 6 \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right] + x^3 - 3x^2 + 6$$

where c_1 and c_2 are constants.

2. Solve $\left(D^3 + 2D^2 + D\right)y = x^3$.

Solution: Given differential equation is

$$\left(D^3 + 2D^2 + D\right)y = x^3$$

$$i.e., \; f(D)y = Q(x) \qquad (1)$$

where $f(D) = D^3 + 2D^2 + D$ and $Q(x) = x^3$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^3 + 2m^2 + m = 0$$

$$\Rightarrow m(m+1)^2 = 0$$

$$\Rightarrow m = 0, -1, -1$$

$$\therefore \quad y_c = C.F. = c_1 + \left(c_2 + c_3 x\right)e^{-x} \qquad (2)$$

Now $y_p = P.I. = \frac{1}{f(D)}Q(x) = \frac{1}{D^3 + 2D^2 + D}x^3 = \frac{1}{D(D+1)^2}x^3$

$$= \frac{1}{(D+1)^2}\int x^3 dx = \frac{1}{(D+1)^2}\frac{x^4}{4}$$

$$= \frac{1}{4}[1 + D]^{-2} x^4$$

$$= \frac{1}{4}\left[1 - 2D + 3D^2 - 4D^3 + 5D^4 - \ldots\right]x^4$$

$$= \frac{1}{4}\left(x^4 - 8x^3 + 36x^2 - 96x + 120\right) \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1 + (c_2 + c_3 x)e^{-x} + \frac{1}{4}\left(x^4 - 8x^3 + 36x^2 - 96x + 120\right)$$

where c_1 and c_2 are constants.

3. Solve $D^2(D^2+4)y = 320(x^3 + 2x^2)$**.**

Solution: Given differential equation is

$$D^2(D^2+4)y = 320(x^3+2x^2)$$

$$i.e., f(D)y = Q(x) \qquad (1)$$

where $f(D) = D^2(D^2+4)$ and $Q(x) = 320(x^3+2x^2)$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2(m^2+4) = 0$$

$$\Rightarrow m = 0, 0, \pm i2$$

$$\therefore \quad y_c = C.F. = c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x \qquad (2)$$

Now $y_p = P.I. = \frac{1}{f(D)}Q(x) = \frac{1}{D^2(D^2+4)}320(x^3+2x^2)$

$$= \frac{1}{4D^2\left(1+\frac{D^2}{4}\right)}320(x^3+2x^2) = \frac{1}{4D^2}\left(1+\frac{D^2}{4}\right)^{-1}320(x^3+2x^2)$$

$$= \frac{1}{4D^2}\left[1 - \frac{D^2}{4} + \frac{D^4}{16} - \frac{D^6}{64} + \dots\right]320(x^3+2x^2)$$

$$= \frac{320}{4}\left[\frac{1}{D^2} - \frac{1}{4} + \frac{D^2}{16} - \frac{D^4}{64} + \dots\right](x^3+2x^2)$$

$$= 80\left[\left(\frac{x^5}{20} + \frac{x^3}{6}\right) - \frac{1}{4}(x^3+2x^2) + \frac{1}{16}(6x+4)\right]$$

$$= 4x^5 + \frac{40}{3}x^4 - 20x^3 - 40x^2 + 30x + 20 \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x + 4x^5 + \frac{40}{3}x^4 - 20x^3 - 40x^2 + 30x + 20$$

where c_1, c_2, c_3 and c_4 are constants.

4. Solve $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x + \sin 2x$**.**

Solution: Given differential equation is

$$(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x + \sin 2x$$

$$i.e., f(D)y = Q(x) \qquad (1)$$

where $f(D) = D^3 + 2D^2 + D$ and $Q(x) = e^{2x} + x^2 + x + \sin 2x$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^3 + 2m^2 + m = 0 \Rightarrow m(m^2 + 2m + 1) = 0$$

$$\Rightarrow m(m+1)^2 = 0 \Rightarrow m = 0, -1, -1$$

$\therefore \quad y_c = C.F. = c_1 + (c_2 + c_3 x)e^{-x}$ (2)

Now $y_p = P.I. = \dfrac{1}{f(D)} Q(x) = \dfrac{1}{D^3 + 2D^2 + D}\left(e^{2x} + x^2 + x + \sin 2x\right)$

$$= \frac{e^{2x}}{D^3 + 2D^2 + D} + \frac{x^2 + x}{D^3 + 2D^2 + D} + \frac{\sin 2x}{D^3 + 2D^2 + D}$$

$$= \frac{e^{2x}}{2^3 + 2(2^2) + 2} + \frac{x^2 + x}{D[1 + (D^2 + 2D)]} + \frac{\sin 2x}{D(-4) + 2(-4) + D}$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}[1 + (D^2 + 2D)]^{-1}(x^2 + x) - \frac{\sin 2x}{3D + 8}$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\left[1 - (D^2 + 2D) + (D^2 + 2D)^2 - \ldots\right](x^2 + x) - \frac{(3D - 8)\sin 2x}{9D^2 - 64}$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\left[1 - (D^2 + 2D) + 4D^2\right](x^2 + x) - \frac{(3D - 8)\sin 2x}{9(-4) - 64}$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\left[1 - 2D + 3D^2\right](x^2 + x) + \frac{(3D - 8)\sin 2x}{100}$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\left[(x^2 + x) - 2(2x + 1) + 3(2)\right] + \frac{[3(2\cos 2x) - 8\sin 2x]}{100}$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\left[x^2 - 3x + 4\right] + \frac{6\cos 2x - 8\sin 2x}{100}$$

$$= \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x + \frac{3\cos 2x - 4\sin 2x}{100} \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1 + (c_2 + c_3 x)e^{-x} + \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x + \frac{3\cos 2x - 4\sin 2x}{100}$$

where c_1, c_2 and c_3 are constants.

<u>Method 4:</u>P.I. of $f(D)y = Q(x)$ when $Q(x) = e^{ax}V$ **where** a **is constant and** V **is a function of** x**:**

We will use this method to find P.I. when V is $\sin ax$ or $\cos ax$ or x^k or a polynomial of degree k.

In this case, $\quad \text{P.I.} = \dfrac{1}{f(D)}\left(e^{ax}V\right) = e^{ax}\dfrac{1}{f(D+a)}V$

Working Rule: To find P.I. for $e^{ax}V$, take out e^{ax} to the left of $f(D)$ and replace every D with $D + a$ so that $f(D)$ becomes $f(D+a)$ and now operate $\dfrac{1}{f(D+a)}$ with V alone by the previous methods.

Examples

1. Solve $\boldsymbol{(D^3 + 2D^2 - 3D)y = xe^{3x}}$**.**

Solution: Given differential equation is

$(D^3 + 2D^2 - 3D)y = xe^{3x}$ (1)

$i.e., \quad [f(D)]y = xe^{3x}$

where $f(D) = D^3 + 2D^2 - 3D$

Now the auxiliary equation is $f(m) = 0$,

$i.e., \ m^3 + 2m^2 - 3m = 0$

$i.e., \quad m(m-1)(m+3) = 0$

$i.e., \quad m = 0, 1, -3$

The roots are real and different.

$\therefore \quad C.F. = c_1 + c_2 e^x + c_3 e^{-3x}$ (2)

Now $P.I. = \dfrac{1}{f(D)}[xe^{3x}] = \dfrac{1}{D^3 + 2D^2 - 3D}[xe^{3x}]$

$= e^{3x}\dfrac{1}{(D+3)^3 + 2(D+3)^2 - 3(D+3)}x,$

since $\dfrac{1}{f(D)}[e^{ax}V(x)] = e^{ax}\dfrac{1}{f(D+a)}V(x)$

$= e^{3x}\dfrac{1}{D^3 + 11D^2 + 36D + 36}x$

$= \dfrac{e^{3x}}{36}\left[1 + \dfrac{D^3 + 11D^2 + 36D}{36}\right]x$

$= \dfrac{e^{3x}}{36}\left[x + \dfrac{36}{36}\right] = \dfrac{e^{3x}}{36}[x+1]$

Therefore the general solution of (1) is

$$y = C.F. + P.I. = c_1 + c_2 e^x + c_3 e^{-3x} + \frac{e^{3x}}{36}(x+1)$$

2. Solve $(D^2 - 7D + 6)y = e^{2x}(1+x)$**.**

Solution: Given differential equation is

$$(D^2 - 7D + 6)y = e^{2x}(1+x)$$

$i.e.,\ f(D)y = Q(x)$ (1)

where $f(D) = D^2 - 7D + 6$ and $Q(x) = e^{2x}(1+x)$

Now the auxiliary equation of (1) is $f(m) = 0$

$\Rightarrow m^2 - 7m + 6 = 0$

$\Rightarrow (m-1)(m-6) = 0$

$\Rightarrow m = 1, 6$

$\therefore \quad y_c = C.F. = c_1 e^x + c_2 e^{6x}$ (2)

Now $y_p = P.I. = \dfrac{1}{f(D)}Q(x) = \dfrac{1}{D^2 - 7D + 6}e^{2x}(1+x)$

$= e^{2x}\dfrac{1}{(D+2)^2 - 7(D+2) + 6}(1+x)$, since $\dfrac{1}{f(D)}(e^{ax}V) = e^{ax}\dfrac{1}{f(D+a)}V$

$= e^{2x}\dfrac{1}{D^2 - 3D - 4}(1+x) = \dfrac{e^{2x}}{-4}\dfrac{1}{\left(1 - \dfrac{D^2 - 3D}{4}\right)}(1+x)$

$$= \frac{e^{2x}}{-4}\left(1 - \frac{D^2 - 3D}{4}\right)^{-1}(1+x)$$

$$= \frac{e^{2x}}{-4}\left(1 + \frac{D^2 - 3D}{4} + \ldots\right)(1+x)$$

$$= \frac{e^{2x}}{-4}\left[(1+x) + \frac{D^2 - 3D}{4}(1+x)\right]$$

$$= \frac{e^{2x}}{-4}\left[(1+x) - \frac{3}{4}\right] = -\frac{e^{2x}}{16}(4x+1) \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1 e^x + c_2 e^{6x} - \frac{e^{2x}}{16}(4x+1)$$

where c_1 and c_2 are constants.

3. Solve $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$.

Solution: Given differential equation is

$$(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$$

$$i.e.,\ f(D)y = Q(x) \qquad (1)$$

where $f(D) = D^2 - 3D + 2$ and $Q(x) = xe^{3x} + \sin 2x$

Now the auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow m^2 - 3m + 2 = 0$$

$$\Rightarrow (m-1)(m-2) = 0$$

$$\Rightarrow m = 1, 2$$

$$\therefore \quad y_c = C.F. = c_1 e^x + c_2 e^{2x} \qquad (2)$$

Now $y_p = P.I. = \frac{1}{f(D)}Q(x) = \frac{1}{D^2 - 3D + 2}\left(xe^{3x} + \sin 2x\right)$

$$= \frac{1}{D^2 - 3D + 2}xe^{3x} + \frac{1}{D^2 - 3D + 2}\sin 2x$$

$$= e^{3x}\frac{1}{(D+3)^2 - 3(D+3) + 2}x + \frac{1}{-4 - 3D + 2}\sin 2x$$

since $\frac{1}{f(D)}\left(e^{ax}V\right) = e^{ax}\frac{1}{f(D+a)}V$ and $\frac{1}{f(D^2)}\sin ax = \frac{\sin ax}{f(-a^2)}$, if $f(-a^2) \neq 0$

$$= e^{3x}\frac{1}{D^2 + 3D + 2}x - \frac{1}{3D + 2}\sin 2x$$

$$= \frac{e^{3x}}{2}\left(1 + \frac{D^2 + 3D}{2}\right)^{-1}x - \frac{3D - 2}{9D^2 - 4}\sin 2x$$

$$= \frac{e^{3x}}{2}\left(1 - \frac{D^2 + 3D}{2} + \ldots\right)x - \frac{3D - 2}{9(-4) - 4}\sin 2x$$

$$=\frac{e^{3x}}{2}\left[x-\frac{3}{2}\right]+\frac{3D-2}{40}\sin 2x$$

$$=\frac{e^{3x}}{2}\left(x-\frac{3}{2}\right)+\frac{1}{40}(6\cos 2x-2\sin 2x)$$

$$=\frac{e^{3x}}{2}\left(x-\frac{3}{2}\right)+\frac{1}{20}(3\cos 2x-\sin 2x) \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y=y_c+y_p=c_1e^{x}+c_2e^{2x}+\frac{e^{3x}}{2}\left(x-\frac{3}{2}\right)+\frac{1}{20}(3\cos 2x-\sin 2x)$$

where c_1 and c_2 are constants.

4. Solve $(D^2+1)y=e^{-x}+x^3+e^x\sin x$.

Solution: Given differential equation is

$$(D^2+1)y=e^{-x}+x^3+e^x\sin x$$

$$i.e.,\ f(D)y=Q(x) \qquad (1)$$

where $f(D)=D^2+1$ and $Q(x)=e^{-x}+x^3+e^x\sin x$

Now the auxiliary equation of (1) is $f(m)=0$

$$\Rightarrow m^2+1=0$$

$$\Rightarrow m^2=-1=i^2$$

$$\Rightarrow m=\pm i$$

$$\therefore \quad y_c=C.F.=c_1\cos x+c_2\sin x \qquad (2)$$

Now $y_p=P.I.=\frac{1}{f(D)}Q(x)=\frac{1}{D^2+1}\left(e^{-x}+x^3+e^x\sin x\right)$

$$=\frac{e^{-x}}{D^2+1}+\frac{1}{D^2+1}x^3+\frac{1}{D^2+1}e^x\sin x$$

$$=\frac{e^{-x}}{(-1)^2+1}+\left(1+D^2\right)^{-1}x^3+e^x\frac{1}{(D+1)^2+1}\sin x$$

$$=\frac{e^{-x}}{2}+\left(1-D^2+D^4-...\right)x^3+e^x\frac{1}{D^2+2D+2}\sin x$$

$$=\frac{e^{-x}}{2}+\left(x^3-6x\right)+e^x\frac{1}{-1+2D+2}\sin x$$

$$=\frac{e^{-x}}{2}+\left(x^3-6x\right)+e^x\frac{1}{2D+1}\sin x$$

$$=\frac{e^{-x}}{2}+\left(x^3-6x\right)+e^x\frac{2D-1}{4D^2-1}\sin x$$

$$=\frac{e^{-x}}{2}+\left(x^3-6x\right)+e^x\frac{2D-1}{4(-1)-1}\sin x$$

$$=\frac{e^{-x}}{2}+\left(x^3-6x\right)-\frac{e^x}{5}(2\cos x-\sin x) \qquad (3)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + \frac{e^{-x}}{2} + \left(x^3 - 6x\right) - \frac{e^x}{5}\left(2\cos x - \sin x\right)$$

where c_1 and c_2 are constants.

<u>Method 5</u>: P.I. of $f(D)y = Q(x)$ when $Q(x) = x^m \mathrm{V}$, m being a positive integer and V is any function of x:

Here V is either $\sin ax$ or $\cos ax$ only. It should not be of the form x^n or e^{ax}.

If V is x^n then $x^m\ \mathrm{V} = x^{m+n}$ and P.I. can be evaluated by the short method discussed in **Method 3**.

If V is e^{ax} then $x^m\ \mathrm{V} = x^m e^{ax}$ and P.I. can be evaluated by the short method discussed in **Method 4**.

But V is of the form $\sin ax$ or $\cos ax$, P.I. can be evaluated as follows.

Working Rule for finding P.I. of $f(D)y = x^m \sin ax$ or $x^m \cos ax$:

(i) $\text{P.I.} = \frac{1}{f(D)} x^m \sin ax = \text{Imaginary Part (I.P.) of } \frac{1}{f(D)} x^m \left(\cos ax + i \sin ax\right)$

$= \text{I.P. of } \frac{1}{f(D)} x^m e^{iax}$

(ii) $\text{P.I.} = \frac{1}{f(D)} x^m \cos ax = \text{Real Part (R.P.) of } \frac{1}{f(D)} x^m e^{iax}$

Now P.I. can be evaluated by the short method discussed in Method 4.

<u>Method6:</u> Alternative method for finding P.I. of $f(D)y = Q(x)$ when $Q(x) = x^m \mathrm{V}$ (when $m = 1$) where V is any function of x:

Let $f(D)y = x\mathrm{V}$ where V is a function of x. Operating with $\frac{1}{f(D)}$, we get $y = \frac{1}{f(D)}(x\mathrm{V})$.

$\therefore \quad \text{P.I.} = \frac{1}{f(D)}(x\mathrm{V})$

Consider $D(x\mathrm{V}) = x\,D\mathrm{V} + \mathrm{V}$; $D^2(x\mathrm{V}) = x\,D^2\mathrm{V} + 2D\mathrm{V}$

Similarly $D^n(x\mathrm{V}) = x\,D^n\mathrm{V} + nD^{n-1}\mathrm{V}$

$$\therefore \left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1}D + a_n\right)(x\mathrm{V}) = x\left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1}D + a_n\right)\mathrm{V}$$
$$+ \left(nD^{n-1} + a_1(n-1)D^{n-2} + \ldots + a_{n-1}\right)\mathrm{V}$$

$$\Rightarrow [f(D)](x\mathrm{V}) = x[f(D)]\mathrm{V} + [f'(D)]\mathrm{V} \qquad \textbf{(1)}$$

Let $[f(D)]\mathrm{V} = \mathrm{V}_1 \Rightarrow \mathrm{V} = \frac{1}{f(D)}\mathrm{V}_1$ (2)

$\therefore [f(D)]\left(x\frac{1}{f(D)}\mathrm{V}_1\right) = x\,\mathrm{V}_1 + f'(D)\frac{1}{f(D)}\mathrm{V}_1$, from (1) and (2)

Operating with $\frac{1}{f(D)}$ on both sides, we get

$$x\frac{1}{f(D)}V_1 = \frac{1}{f(D)}(x V_1) + \frac{1}{f(D)} f'(D) \frac{1}{f(D)} V_1$$

$$\Rightarrow \frac{1}{f(D)}(xV_1) = x\frac{1}{f(D)}V_1 - \frac{1}{f(D)} f'(D)\frac{1}{f(D)}V_1$$

$$\Rightarrow \frac{1}{f(D)}(xV_1) = \left[x - \frac{1}{f(D)} f'(D)\right]\frac{1}{f(D)}V_1$$

$$\therefore \quad \frac{1}{f(D)}(xV) = \left[x - \frac{1}{f(D)} f'(D)\right]\frac{1}{f(D)}V$$

Examples

1. Solve $(\boldsymbol{D^2-1})\boldsymbol{y = xe^x \sin x.}$

Solution: Given differential equation is

$(D^2-1)y = xe^x \sin x$ (1)

$i.e.,\quad [f(D)]y = xe^x \sin x$

where $f(D) = D^2 - 1$

Now the auxiliary equation is $f(m) = 0, i.e.,\ m^2 - 1=0$

$i.e.,\quad m = -1,\quad 1$

The roots are real and different.

$\therefore\quad C.F. = c_1 e^{-x} + c_2 e^x$ (2)

Now $P.I. = \frac{1}{f(D)}[xe^x \sin x] = \frac{1}{D^2-1}[xe^x \sin x]$

$$= e^x \frac{1}{(D+1)^2-1}[x\sin x], \qquad \text{since} \frac{1}{f(D)}[e^{ax}V(x)] = e^{ax}\frac{1}{f(D+a)}V(x)$$

$$= e^x \frac{1}{D^2+2D}[x\sin x]$$

$$= e^x\left[x - \frac{2D+2}{D^2+2D}\right]\frac{1}{D^2+2D}\sin x, \qquad \text{since}\frac{1}{f(D)}[xV(x)] = \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}V(x)$$

$$= e^x\left[x - \frac{2D+2}{D^2+2D}\right]\frac{1}{(-1+2D)}\sin x, \qquad \text{since}\frac{1}{f(D^2)}\sin ax = \frac{\sin ax}{f(-a^2)}$$

$$= e^x\left[x - \frac{2D+2}{D^2+2D}\right]\frac{2D+1}{4D^2-1}\sin x$$

$$= -\frac{e^x}{5}\left[x - \frac{2D+2}{D^2+2D}\right](2\cos x + \sin x)$$

$$= -\frac{e^x}{5}\left[x(2\cos x + \sin x) - \frac{2D+2}{-1+2D}(2\cos x + \sin x)\right]$$

$$= -\frac{e^x}{5}\left[x(2\cos x + \sin x) - \frac{(2D+2)(2D+1)}{4D^2-1}(2\cos x + \sin x)\right]$$

$$= -\frac{e^x}{5}\left[x(2\cos x + \sin x) + \frac{1}{5}(4D^2+6D+2)(2\cos x + \sin x)\right]$$

$$= -\frac{e^x}{5}\left[x(2\cos x + \sin x) + \frac{1}{5}(2\cos x - 14\ \sin x)\right]$$

Therefore the general solution of (1) is

$$y = C.F. + P.I. = c_1 e^{-x} + c_2 e^x - \frac{e^x}{5}\left[x(2\cos x + \sin x) + \frac{1}{5}(2\cos x - 14\ \sin x)\right]$$

2. Solve $\boldsymbol{(D^2-4D+4)y = 8x^2e^{2x}sin\,2x.}$

Solution: Given differential equation is

$(D^2-4D+4)y = 8x^2e^{2x}sin\,2x$ (1)

$i.e.,\quad [f(D)]y = Q(x)$

where $f(D) = D^2-1$ and $Q(x) = 8x^2e^{2x}sin\,2x$

Now the auxiliary equation is $f(m) = 0 \Rightarrow m^2-4m+4$=0

$\Rightarrow (m-2)(m-2) = 0 \Rightarrow\ m = 2,\quad 2$

The roots are real and equal.

$\therefore\quad C.F. = (c_1+c_2x)e^{2x}$ (2)

Here P.I. can be found out using the above case twice which is laborious. We will find P.I. in another way.

Now $\quad P.I. = \frac{1}{f(D)}Q(x) = \frac{1}{D^2-4D+4}[8x^2e^{2x}sin\,2x]$

$= 8e^{2x}\frac{1}{(D+2)^2-4(D+2)+4}[x^2 sin\,2x]$,

since $\frac{1}{f(D)}[e^{ax}V(x)] = e^{ax}\frac{1}{f(D+a)}V(x)$

$= 8e^{2x}\frac{1}{D^2}[x^2 sin\,2x]$ = Imaginary Part of $8e^{2x}\frac{1}{D^2}\left[x^2e^{i2x}\right]$

$= \text{I. P. of } 8e^{2x}e^{i2x}\frac{1}{(D+i2)^2}x^2$

$= \text{I. P. of } 8e^{2x}e^{i2x}\frac{1}{4i^2\left(1+\frac{D}{2i}\right)^2}x^2$

$= \text{I. P. of}(-2e^{2x})e^{i2x}\left(1+\frac{D}{2i}\right)^{-2}x^2$

$= \text{I. P. of}(-2e^{2x})e^{i2x}\left(1-2\frac{D}{2i}+3\frac{D^2}{4i^2}+\cdots\right)x^2$

$= \text{I. P. of}(-2e^{2x})e^{i2x}\left(x^2-\frac{2x}{i}+\frac{3}{2i^2}\right)$

$= \text{I. P. of}(-2e^{2x})e^{i2x}\left(x^2+i2x-\frac{3}{2}\right)$

$= \text{I. P. of}(-2e^{2x})(cos\,2x+i\,sin\,2x)\left[\left(x^2-\frac{3}{2}\right)+i2x\right]$

$= (-2e^{2x})\left[2x\,cos\,2x+\left(x^2-\frac{3}{2}\right)\,sin\,2x\right]$ (3)

Therefore the general solution of (1) is

$$y = C.F.+P.I. = (c_1+c_2x)e^{2x}-2e^{2x}\left[2x\,cos\,2x+\left(x^2-\frac{3}{2}\right)\,sin\,2x\right]$$

3. Solve $\boldsymbol{(D^2+9)y = x\,sin\,2x.}$

Solution: Given differential equation is

$(D^2+9)y = x\,sin\,2x$ (1)

$i.e.,\quad [f(D)]y = x\,sin\,2x$

where $f(D) = D^2+9$

Now the auxiliary equation is $f(m) = 0, i.e.,\ m^2+9$=0

$i.e.,\quad m = \pm i3$

The roots are real and different.

$\therefore\quad C.F. = c_1cos\,3x+c_2sin\,3x$ (2)

Now $P.I. = \frac{1}{f(D)}[x\, sin\, 2x] = \frac{1}{D^2+9}[x\, sin\, 2x]$

$$= \left[x - \frac{2D}{D^2+9}\right]\frac{1}{D^2+9} sin\, 2x\,, \qquad \text{since} \frac{1}{f(D)}[xV(x)] = \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}V(x)$$

$$= \left[x - \frac{2D}{D^2+9}\right]\frac{1}{(-4+9)} \sin 2x\;, \qquad \text{since} \frac{1}{f(D^2)} sin\, ax = \frac{sin\, ax}{f(-a^2)}$$

$$= \frac{x\, sin\, 2x}{5} - \frac{2D}{5(D^2+9)} sin\, 2x$$

$$= \frac{x\, sin\, 2x}{5} - \frac{2D}{5(-4+9)} sin\, 2x, \qquad \text{since} \frac{1}{f(D^2)} sin\, ax = \frac{sin\, ax}{f(-a^2)}$$

$$= \frac{x\, sin\, 2x}{5} - \frac{2D}{25} sin\, 2x$$

$$= \frac{x\, sin\, 2x}{5} - \frac{4}{25} cos\, 2x \qquad (3)$$

Therefore the general solution of (1) is

$$y = C.F. + P.I. = c_1 cos\, 3x + c_2 sin\, 3x + \frac{x\, sin\, 2x}{5} - \frac{4}{25} cos\, 2x$$

<u>Method of Variation of Parameters:</u>

<u>Wronskian:</u>Wronskian of two functions $u(x)$and $v(x)$ is denoted by $W(u,v)$ and is defined by

$$W(u,v) = \begin{vmatrix} u & v \\ \frac{du}{dx} & \frac{dv}{dx} \end{vmatrix} \text{ or } \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = u\frac{dv}{dx} - v\frac{du}{dx}$$

<u>Working Rule:</u>To solve $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ by the method of variation of parameters, follow these steps

1. Reduce the given equation to the standard form , if necessary.
2. Find the solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ and let the solution be

   $C.F. = y_c = c_1 u(x) + c_2 v(x)$
3. Take $P.I. = y_p = A\,u(x) + B\,v(x)$, where A and B are functions of x.
4. Find $W(u,v) = u\frac{dv}{dx} - v\frac{du}{dx}$.
5. Find A and B using

   $A = -\int \frac{vR}{W(u,v)}dx\,, \quad B = \int \frac{uR}{W(u,v)}dx$
6. Write the general solution of the given equation as

   $y = y_c + y_p$

Examples

1. Solve $(D^2 + a^2)y = tan\, ax$ by method of variation of parameters.

Solution: Given differential equation is

$(D^2 + a^2)y = tan\, ax \qquad (1)$

$i.e., \qquad [f(D)]y = R$

where$f(D) = D^2 + 3D + 2$ and $R = tan\ ax$

Now the auxiliary equation is $f(m) = 0, i.e.,\ m^2 + a^2$=0

$$i.e., \quad m = \pm ia$$

The roots are complex.

$\therefore \quad C.F. = c_1 cos\ ax + c_2 sin\ ax$ (2)

Consider $P.I. = A\ cos\ ax + B\ sin\ ax$ (3)

Here $u = cos\ ax, v = sin\ ax$

Then $W(u,v) = u\dfrac{dv}{dx} - v\dfrac{du}{dx} = cos\ ax\ (a\ cos\ ax) - sin\ ax\ (-a\ sin\ ax) = a$

Where A and B are given by

$$A = -\int \frac{vR}{W(u,v)}dx = -\int \frac{sin\ ax\ tan\ ax}{a}dx = -\frac{1}{a}\int \frac{1 - cos^2 ax}{cos\ ax}dx$$

$$= -\frac{1}{a}\int (sec\ ax - cos\ ax)\,dx = -\frac{1}{a}\left[\frac{\log(sec\ ax + tan\ ax)}{a} - \frac{sin\ ax}{a}\right]$$

$$= \frac{1}{a^2}[sin\ ax - \log(sec\ ax + tan\ ax)]$$

$$B = \int \frac{uR}{W(u,v)}dx = \int \frac{cos\ ax\ tan\ ax}{a}dx = \frac{1}{a}\int sin\ ax\,dx = -\frac{1}{a^2}cos\ ax$$

$\therefore$ (3) $\Rightarrow$ $P.I. = \dfrac{1}{a^2}[sin\ ax - \log(sec\ ax + tan\ ax)]\ cos\ ax - \dfrac{1}{a^2}cos\ ax\ sin\ ax$

$$= -\frac{1}{a^2}cos\ ax\,.\log(sec\ ax + tan\ ax)$$

Hence the general solution of (1) is

$$y = C.F. + P.I.$$

$$y = c_1 cos\ ax + c_2 sin\ ax - \frac{1}{a^2}cos\ ax\,.\log(sec\ ax + tan\ ax)$$

2. Solve $(D^2 - 2D)y = e^x sin\ x$ by the method of variation of parameters.

Solution: Given differential equation is

$(D^2 - 2D)y = e^x sin\ x$ (1)

$$i.e., \quad [f(D)]y = R$$

where$f(D) = D^2 - 2D$ and $R = e^x sin\ x$

Now the auxiliary equation is $f(m) = 0, i.e.,\ m^2 - 2m = 0$

$$i.e., \quad m(m-2) = 0, i.e., \quad m = 0, 2$$

The roots are real and different.

$\therefore \quad C.F. = c_1 + c_2 e^{2x}$ (2)

By the method of variation of parameters

Consider $P.I. = A\ + B\ e^{2x}$ (3)

Here $u = 1, v = e^{2x}$

Then $W(u,v) = u\dfrac{dv}{dx} - v\dfrac{du}{dx} = 1\ (2e^{2x}) - e^{2x}(0) = 2e^{2x}$

Where A and B are given by

$$A = -\int \frac{vR}{W(u,v)}dx = -\int \frac{e^{2x}e^x sin\ x}{2e^{2x}}dx = -\frac{1}{2}\int e^x sin\ x\,dx$$

$$= -\frac{1}{4}[e^x sin\ x - e^x cos\ x]$$

$$B = \int \frac{uR}{W(u,v)} dx = \int \frac{1.e^x \sin x}{2e^{2x}} dx = \frac{1}{2}\int e^{-x} \sin x \, dx$$

$$= \frac{1}{4}[-e^{-x} \sin x - e^{-x} \cos x]$$

$$\therefore (3) \Rightarrow \quad P.I. = -\frac{1}{4}[e^x \sin x - e^x \cos x] + \frac{1}{4}[-e^{-x} \sin x - e^{-x} \cos x].e^{2x}$$

$$= -\frac{e^x \sin x}{2}$$

Hence the general solution of (1) is

$$y = C.F. + P.I.$$

$$\therefore \quad y = c_1 + c_2 e^{2x} - \frac{e^x \sin x}{2}$$

3. Solve the equation using method of variation of parameters: $\boldsymbol{(D^2 + 3D + 2)y = e^x + x^2}$**.**

Solution: Given differential equation is

$(D^2 + 3D + 2)y = e^x + x^2$ (1)

$$i.e., \quad [f(D)]y = e^x + x^2$$

where$f(D) = D^2 + 3D + 2$

Now the auxiliary equation is $f(m) = 0, i.e.,\ m^2 + 3m + 2$=0

$$i.e., \quad m = -1, -2$$

The roots are real and different.

$\therefore \quad C.F. = c_1 e^{-x} + c_2 e^{-2x}$ (2)

Consider $\quad P.I. = A\, e^{-x} + B\, e^{-2x}$ (3)

Here $u = e^{-x}, v = e^{-2x}$

Then $\quad W(u,v) = u\frac{dv}{dx} - v\frac{du}{dx} = -2e^{-x}e^{-2x} + e^{-2x}e^{-x} = -e^{-3x}$

Where A and B are given by

$$A = -\int \frac{vR}{W(u,v)} dx = -\int \frac{e^{-2x}(e^x + x^2)}{-e^{-3x}} dx = \int (e^{2x} + e^x x^2)\, dx$$

$$= \frac{e^{2x}}{2} + (x^2 - 2x + 2)e^x$$

$$B = \int \frac{uR}{W(u,v)} dx = \int \frac{e^{-x}(e^x + x^2)}{-e^{-3x}} dx = -\int (e^{3x} + e^{2x}x^2)\, dx$$

$$= -\left[\frac{e^{3x}}{3} + \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right)e^{2x}\right]$$

$$\therefore (3) \Rightarrow \quad P.I. = \left[\frac{e^{2x}}{2} + (x^2 - 2x + 2)e^x\right]e^{-x} - \left[\frac{e^{3x}}{3} + \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right)e^{2x}\right]e^{-2x}$$

$$= \frac{e^x}{2} + (x^2 - 2x + 2) - \frac{e^x}{3} - \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right)$$

$$= \frac{e^x}{6} + \frac{1}{4}(2x^2 - 6x + 7)$$

Hence the general solution of (1) is

$$y = C.F. + P.I.$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{e^x}{6} + \frac{1}{4}(2x^2 - 6x + 7)$$

4. Solve $(D^2+1)y = cosec\, x$ by method of variation of parameters.

Solution: Given differential equation is

$(D^2+1)y = cosec\, x$ (1)

$i.e.,\quad [f(D)]y = R$

where $f(D) = D^2+1$ and $R = cosec\, x$

Now the auxiliary equation is $f(m) = 0,\ i.e.,\ m^2+1=0$

$i.e.,\quad m = \pm i$

The roots are complex.

$\therefore\quad C.F. = c_1 cos\, x + c_2 sin\, x$ (2)

Consider $P.I. = A\, cos\, x + B\, sin\, x$ (3)

Here $u = cos\, x, v = sin\, x$

Then $W(u,v) = u\frac{dv}{dx} - v\frac{du}{dx} = cos\, x\,(cos\, x) - sin\, x\,(-sin\, x) = 1$

Where A and B are given by

$$A = -\int \frac{vR}{W(u,v)}dx = -\int \frac{sin\, x\, cosec\, x}{1}dx = -\int dx = -x$$

$$B = \int \frac{uR}{W(u,v)}dx = \int \frac{cos\, x\, cosec\, x}{1}dx = \int cot\, x\, dx = \log(sin\, x)$$

$\therefore (3) \Rightarrow\ P.I. = -x\, cos\, x + sin\, x.\log(sin\, x)$ (3)

Hence the general solution of (1) is

$$y = C.F. + P.I.$$

$$y = c_1 cos\, x + c_2 sin\, x - x\, cos\, x + sin\, x.\log(sin\, x)$$

ELECTRICAL CIRCUIT PROBLEMS

L**-**C**-**R **Circuit:** Cinsider the discharge of a condenser C through an induction L and the resistance R. Since the voltage drop across L, C and R respectively $L\frac{d^2q}{dt^2}, \frac{q}{c}$ and $R\frac{dq}{dt}$.

$\therefore$ By Kirchoff's law, $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = 0$

1. A condenser of capacity C discharged through an inductance L and resistance R in series and the charge q at time t satisfies the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = 0$. Given that $L = 0.25$ henries, $R = 250$ ohms, $C = 2\times10^{-6}$ farads, and that when $t = 0$, charge $q = 0.002$ coulombs and the current $\frac{dq}{dt} = 0$, obtain the value of q in terms of t.

Solution: Given differential equation is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = 0 \text{ or } \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{Lc} = 0$$

Substituting the given values in (1), we get

$$\frac{d^2q}{dt^2} + \frac{250}{0.25}\frac{dq}{dt} + \frac{q}{0.25\times2\times10^{-6}} = 0$$

$$\text{or } \frac{d^2q}{dt^2} + 1000\frac{dq}{dt} + 2\times10^{6}\, q = 0$$

or $\left(D^2+1000D+2\times10^6\right)q=0$ (1)

Its auxiliary equation is $m^2+1000m+2\times10^6=0$

$$\Rightarrow m=\frac{-1000\pm\sqrt{10^6-8\times10^6}}{2}=-500\pm i\,500\sqrt{7}=-500\pm i\,1323$$

Thus the solution of (1) is

$$q=e^{-500t}\left(c_1\cos1323t+c_2\sin1323t\right) \qquad (2)$$

when $t=0,\ q=0.002\ \therefore c_1=0.002$

$$\frac{dq}{dt}=-500\,e^{-500t}\left(c_1\cos1323t+c_2\sin1323t\right)$$

$$+e^{-500t}\times1323\left(-c_1\sin1323t+c_2\cos1323t\right) \qquad (3)$$

when $t=0,\ \frac{dq}{dt}=0\Rightarrow c_2=0.0008$

Hence the required solution is $q=e^{-500t}\left(0.002\cos1323t+0.0008\sin1323t\right)$

2. The charge $q(t)$ on the capacitor is given by the D.E., $10\frac{d^2q}{dt^2}+120\frac{dq}{dt}+1000\,q=17\sin2t$ **. at time zero the current is zero and the charge on the capacitor is $\frac{1}{2000}$ coulomb. Find the charge on the capacitor for $t>0$.**

Solution: Given differential equation is $10\frac{d^2q}{dt^2}+120\frac{dq}{dt}+1000\,q=17\sin2t$

$$\Rightarrow \frac{d^2q}{dt^2}+12\frac{dq}{dt}+100\,q=\frac{17}{10}\sin2t$$

$$\Rightarrow \left(D^2+12D+100\right)q=\frac{17}{10}\sin2t \qquad (1)$$

Its auxiliary equation is $m^2+12m+100=0$

$$\Rightarrow m=\frac{-12\pm\sqrt{144-400}}{2}=-6\pm i\,8$$

$$\therefore C.F.=e^{-6t}\left(c_1\cos8t+c_2\sin8t\right) \qquad (2)$$

Now $P.I.=\frac{1}{D^2+12D+100}\frac{17}{10}\sin2t$

$$=\frac{17}{10}\left[\frac{1}{-4+12D+100}\sin2t\right], \text{ Put } D^2=-4$$

$$=\frac{17}{10}\left[\frac{1}{12(D+8)}\sin2t\right]$$

$$=\frac{17}{120}\left[\frac{D-8}{D^2-64}\sin2t\right]$$

$$=\frac{17}{120}\left[\frac{D-8}{-4-64}\sin2t\right], \text{ Put } D^2=-4$$

$$= \frac{17}{120}\left[\frac{1}{-68}(2\cos 2t - 8\sin 2t)\right]$$

$$= \frac{1}{240}(4\sin 2t - \cos 2t) \qquad (3)$$

Thus the solution of (1) is $q = C.F. + P.I.$

$$q = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) + \frac{1}{240}(4\sin 2t - \cos 2t) \qquad (4)$$

when $t = 0$, $q = \frac{1}{2000} \Rightarrow \frac{1}{2000} = c_1 - \frac{1}{240} \Rightarrow c_1 = \frac{7}{1500}$

$$\frac{dq}{dt} = -6e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) + e^{-6t}(-8c_1 \sin 8t + 8c_2 \cos 8t)$$

$$+ \frac{1}{240}(8\cos 2t + 2\sin 2t) \qquad (3)$$

when $t = 0$, $\frac{dq}{dt} = 0 \Rightarrow -6c_1 + 8c_2 + \frac{1}{30} = 0$

$$\Rightarrow 8c_2 = 6c_1 - \frac{1}{30} \Rightarrow 8c_2 = \frac{7}{250} - \frac{1}{30} = \frac{-4}{750} \Rightarrow c_2 = \frac{-1}{1500}$$

Hence the required solution is

$$q = \frac{e^{-6t}}{1500}(7\cos 8t - \sin 8t) + \frac{1}{240}(4\sin 2t - \cos 2t)$$

and $i(t) = \frac{dq}{dt} = -\frac{e^{-6t}}{30}(\cos 8t + \sin 8t) + \frac{1}{120}(4\cos 2t + \sin 2t)$

here the current is a sum of two parts, namely transient part and steady state part.

Transient part $= -\frac{e^{-6t}}{30}(\cos 8t + \sin 8t)$

It is named so, because it decreses as 't' increases.

Steady state part $= \frac{1}{120}(4\cos 2t + \sin 2t)$

3. An uncharged condenser of capacity C is charged by applying an e.m.f. $E\sin\left(\frac{t}{\sqrt{LC}}\right)$, through leads of self-inductance L and negligible resistance. Prove that at any time t, the charge on one of the plates is $\frac{EC}{2}\left[\sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{t}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)\right]$.

Solution: Let q be the charge on the condenser, the differential equation of the circuit is

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E\sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{q}{LC} = \frac{E}{L}\sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$\Rightarrow \left(D^2 + \frac{1}{LC}\right)q = \frac{E}{L}\sin\left(\frac{t}{\sqrt{LC}}\right) \qquad (1)$$

Its auxiliary equation is $m^2 + \frac{1}{LC} = 0 \Rightarrow m = \pm i \frac{1}{\sqrt{LC}}$

$$\therefore C.F. = c_1 \cos\left(\frac{t}{\sqrt{LC}}\right) + c_2 \sin\left(\frac{t}{\sqrt{LC}}\right) \qquad (2)$$

Now $P.I. = \frac{1}{D^2 + \frac{1}{LC}} \frac{E}{L} \sin\left(\frac{t}{\sqrt{LC}}\right)$

$= \frac{E}{L} \frac{1}{D^2 + \frac{1}{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right)$, Put $D^2 = -\frac{1}{LC}$, we get denominator as zero

$$= \frac{E}{L}\left[\frac{-t}{2\sqrt{\frac{1}{LC}}} \cos\left(\frac{t}{\sqrt{LC}}\right)\right], \qquad \left[\because \frac{1}{D^2 + a^2} \sin at = -\frac{t}{2a} \cos at\right]$$

$$= -\frac{Et}{2}\sqrt{\frac{C}{L}} \cos\left(\frac{t}{\sqrt{LC}}\right) \qquad (3)$$

Thus the solution of (1) is $q = C.F. + P.I.$

$$q = c_1 \cos\left(\frac{t}{\sqrt{LC}}\right) + c_2 \sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{Et}{2}\sqrt{\frac{C}{L}} \cos\left(\frac{t}{\sqrt{LC}}\right) \qquad (4)$$

when $t = 0$, $q = 0 \Rightarrow c_1 = 0$

$$\therefore \quad q = c_2 \sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{Et}{2}\sqrt{\frac{C}{L}} \cos\left(\frac{t}{\sqrt{LC}}\right) \qquad (5)$$

Differentiating with respect to t, we get

$$\frac{dq}{dt} = \frac{c_2}{\sqrt{LC}} \cos\left(\frac{t}{\sqrt{LC}}\right) - \frac{E}{2}\sqrt{\frac{C}{L}}\left[\cos\left(\frac{t}{\sqrt{LC}}\right) - \frac{t}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right)\right] \qquad (6)$$

when $t = 0$, $\frac{dq}{dt} = 0 \Rightarrow \frac{c_2}{\sqrt{LC}} - \frac{E}{2}\sqrt{\frac{C}{L}} = 0 \Rightarrow c_2 = \frac{EC}{2}$

Substituting c_2 in (5), we get the required solution is

$$q = \frac{EC}{2}\left[\sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{t}{\sqrt{LC}} \cos\left(\frac{t}{\sqrt{LC}}\right)\right]$$

Unit-III

PARTIAL DIFFERENTIAL DIFFERENTIAL EQUATIONS

Introduction:

Partial differential equations arise in geometry, physics and in engineering branches when the number of independent variables in the given problem under discussion is two or more. In such cases any dependent variable is likely to be a function of more than one variables, so that it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several variables. For example, in the study of thermal effects in a solid body the temperature u may vary from point to point in the solid as well as from time to time, and, as a consequence, the derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial t}$, will, in general, be non zero. In general it may happen that higher derivatives of the types $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^3 u}{\partial x^3}$, etc. may be of physical significance.

When the laws of physics are applied to a problem of this kind, we may sometimes obtain a relation between the derivatives of the kind

$$\phi\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \ldots \ldots, \frac{\partial^2 u}{\partial x \partial y}\right) = 0$$

Such an equation relating partial derivatives is called a "**Partial Differential Equation**".

Simply, a partial differential equation is an equation involving a function of two or more variables and some of its partial derivatives. Therefore a partial differential equation contains one dependent variable and more than one independent variable. Hence the main difference between partial and ordinary differential equations if the number of independent variables involved in the equations.

Examples:

1. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$ where u- dependent variable; x, y-independent variables.

2. $\left(\frac{\partial u}{\partial x}\right)^3 + \frac{\partial u}{\partial y} = 0$ where u- dependent variable; x, y-independent variables.

3. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = 0$ where u- dependent variable; x, y-independent variables.

The **order** of a partial differential equation is the order of the highest partial derivative occurring in the equation.

In the above, example 1 is a second order equation in two variables, example 2 is a first order equation in two variables and example 3 is first order equation in three variables.

Now the students are able to understand what a partial differential equation is and how to identify whether a given differential equation is a partial differential or ordinary differential equation.

Now we are going to see how a partial differential equation is formed by using a given equation. Actually there are two methods to form a partial differential equation as given below.

Formation of Partial Differential Equations:

In practice, there are two methods to form a partial differential equation.

(i) By elimination of arbitrary constants

(ii) By elimination of arbitrary functions

Formation of Partial Differential Equations by Elimination of Arbitrary Constants:

Let $f(x, y, z, a, b) = 0$ (1)

be an equation which contains two arbitrary constants 'a' and 'b'. We know that, to eliminate two constants we need atleast three equations. Therefore partially differentiating equation (1) with respect to x and y we get two more equations. From these three equations we can eliminate the two constants 'a' and 'b'. Similarly, for eliminating three constants we need four equations and so on.

Note 1: If the number of arbitrary constants to be eliminated is equal to the number of independent variables, elimination of constants gives a first order partial differential equation. If the number of arbitrary constants to be eliminated is greater than the number of independent variables, then the elimination of constants gives a second or higher order partial differential equations.

Note 2: In this chapter we use the following notations.

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y} \text{ and } t = \frac{\partial^2 z}{\partial y^2}$$

EXAMPLES

1. Form the partial differential equation by eliminating the arbitrary constants from

$$z = ax + by + a^2 + b^2.$$

Solution: Given $z = ax + by + a^2 + b^2$ (1)

Partially differentiating (1) with respect to 'x' and 'y', we get

$$p = \frac{\partial z}{\partial x} = a \qquad (2)$$

$$q = \frac{\partial z}{\partial y} = b \qquad (3)$$

From equations (2) and (3), we get

$$a = p \text{ and } b = q$$

Substituting these values of a and b in (1), we get

$$z = px + qy + p^2 + q^2$$

This is the required partial differential equation.

2. Form the partial differential equation by eliminating the arbitrary constants from

$$z = (x - a)^2 + (y - b)^2 + 1.$$

Solution: Given $z = (x - a)^2 + (y - b)^2 + 1$ (1)

Partially differentiating (1) with respect to 'x' and 'y', we get

$$p = \frac{\partial z}{\partial x} = 2(x - a) \qquad (2)$$

$$q = \frac{\partial z}{\partial y} = 2(y - b) \qquad (3)$$

From equations (2) and (3), we get

$$a = x - \frac{p}{2} \text{ and } b = y - \frac{q}{2}$$

Substituting these values of a and b in (1), we get

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 + 1$$

$$4z = p^2 + q^2 + 4$$

This is the required partial differential equation.

3. Form the partial differential equation by eliminating the arbitrary constants from

$$z = (x^2 + a)(y^2 + b).$$

Solution: Given $z = (x^2 + a)(y^2 + b)$ (1)

Partially differentiating (1) with respect to 'x' and 'y', we get

$$p = \frac{\partial z}{\partial x} = 2x(y^2 + b) \quad (2)$$

$$q = \frac{\partial z}{\partial y} = 2y(x^2 + a) \quad (3)$$

From equations (2) and (3), we get

$$y^2 + b = \frac{p}{2x} \quad (4)$$

$$\text{and} x^2 + a = \frac{q}{2y} \quad (5)$$

Substituting (4) and (5) in (1), we get

$$z = \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right) \quad \text{or} \quad pq = 4xyz$$

This is the required partial differential equation.

4. Find the differential equation of all spheres of radius 5 having their centre's in the xy-plane.

Solution: The equation of the given spheres is

$(x - a)^2 + (y - b)^2 + z^2 = 25$ (1)

Partially differentiating (1) with respect to 'x' and 'y', we get

$2(x - a) + 2zp = 0 \Rightarrow x - a = -zp$ (2)

$2(y - b) + 2zq = 0 \Rightarrow y - b = -zq$ (3)

Substituting (2) and (3) in (1), we get

$$z^2p^2 + z^2q^2 + z^2 = 25$$

$$z^2(p^2 + q^2 + 1) = 25$$

This is the required partial differential equation.

5. Form the partial differential equation by eliminating the constants from

$$\boldsymbol{z = axe^y + \frac{1}{2}a^2e^{2y} + b.}$$

Solution: Given $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ (1)

Partially differentiating (1) with respect to 'x' and 'y', we get

$$\frac{\partial z}{\partial x} = p = ae^y \Rightarrow a = \frac{p}{e^y} \quad (2)$$

$$\frac{\partial z}{\partial y} = q = axe^y + \frac{1}{2}a^2e^{2y}(2)$$

$$i.e., \frac{\partial z}{\partial y} = \frac{p}{e^y}xe^y + \left(\frac{p}{e^y}\right)^2 e^{2y}, \text{using (1)}$$

$$i.e., \quad q = px + p^2$$

This is the required partial differential equation.

6. Form the partial differential equation by eliminating the constants 'a' and 'b' from

$$\boldsymbol{z = a\log\left[\frac{b(y-1)}{1-x}\right].}$$

Solution: Given $z = a\log\left[\frac{b(y-1)}{1-x}\right]$ (1)

Partially differentiating (1) with respect to 'x' and 'y', we get

$$\frac{\partial z}{\partial x} = p = a\left[\frac{1-x}{b(y-1)}\right].b(y-1)\left[\frac{-1}{(1-x)^2}\right](-1)$$

$$i.e., \quad p = \frac{a}{1-x} \Rightarrow a = p(1-x) \quad (2)$$

$$\frac{\partial z}{\partial y} = q = a\left[\frac{1-x}{b(y-1)}\right]\frac{b}{1-x} = \frac{a}{y-1}$$

$$i.e., \quad a = q(y-1) \qquad (3)$$

From (2) and (3), we get

$$p(1-x) = q(y-1) \text{or} \qquad px + qy = p + q$$

This is the required partial differential equation.

7. Form the partial differential equation by eliminating the constants 'a' and 'b' from

$$\boldsymbol{2z = (x+a)^{1/2} + (y-a)^{1/2} + b.}$$

Solution: Given $\quad 2z = (x+a)^{1/2} + (y-a)^{1/2} + b \qquad (1)$

Partially differentiating (1) with respect to 'x' and 'y', we get

$$2\,\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x+a}} \;\Rightarrow\; 2p = \frac{1}{2\sqrt{x+a}}$$

$$i.e., \quad \sqrt{x+a} = \frac{1}{4p} \qquad (2)$$

$$2\,\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{y-a}} \;\Rightarrow\; 2q = \frac{1}{2\sqrt{y-a}}$$

$$i.e., \quad \sqrt{y-a} = \frac{1}{4q} \qquad (3)$$

From(2), $\quad x + a = \frac{1}{16p^2} \;\Rightarrow\; a = \frac{1}{16p^2} - x \qquad (4)$

From(3), $\quad y - a = \frac{1}{16q^2} \;\Rightarrow\; a = y - \frac{1}{16q^2} \qquad (5)$

From (4) and (5), we get

$$x + y = \frac{1}{16}\left(\frac{1}{p^2} + \frac{1}{q^2}\right) \text{or} \frac{1}{p^2} + \frac{1}{q^2} = 16(x+y)$$

This is the required partial differential equation.

8. Form the partial differential equation by eliminating the constants 'a' and 'b' from

$$\boldsymbol{(x-a)^2 + (y-b)^2 = z^2 cot^2 \alpha.}$$

Solution: Given $(x + a)^2 + (y - b)^2 = z^2 cot^2\alpha$ (1)

Partially differentiating (1) with respect to 'x' and 'y', we get

$$2(x - a) = 2z \frac{\partial z}{\partial x} cot^2\alpha \;\Rightarrow\; x - a = z\,p cot^2\alpha \quad (2)$$

$$2(y - b) = 2z \frac{\partial z}{\partial y} cot^2\alpha \;\Rightarrow\; y - b = z\,q cot^2\alpha \quad (3)$$

Using (2) and (3) in (1), we get

$$z^2p^2cot^4\alpha + z^2q^2cot^4\alpha = z^2cot^2\alpha \qquad \Rightarrow\; p^2 + q^2 = tan^2\alpha$$

This is the required partial differential equation.

9. Find the partial differential equation of all planes having equal intercepts on the x and y axis.

Solution: The equation of such plane is

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{b} = 1 \qquad (1)$$

Partially differentiating (1) with respect to 'x' and 'y', we get

$$\frac{1}{a} + \frac{p}{b} = 0 \;\Rightarrow\; p = -\frac{b}{a} \qquad (2)$$

$$\frac{1}{a} + \frac{q}{b} = 0 \;\Rightarrow\; q = -\frac{b}{a} \qquad (3)$$

From (2) and (3), we get $p = q$

This is the required partial differential equation.

10. Form the partial differential equation by eliminating the constants 'a' and 'b' from

$$\boldsymbol{z = ax^n + by^n}.$$

Solution: Given $z = ax^n + by^n$ (1)

Partially differentiating (1) with respect to 'x' and 'y', we get

$$p = \frac{\partial z}{\partial x} = a\,nx^{n-1} \;\Rightarrow\; a = \frac{p}{nx^{n-1}} \quad (2)$$

$$q = \frac{\partial z}{\partial y} = b\, ny^{n-1} \;\;\Rightarrow\;\; b = \frac{q}{ny^{n-1}} \quad (3)$$

Substituting (2) and (3) in (1), we get

$$z = \frac{p}{nx^{n-1}} x^n + \frac{q}{ny^{n-1}} y^n$$

$$z = \frac{1}{n}(px + qy)$$

This is the required partial differential equation.

11. Form the partial differential equation by eliminating the constants 'a' and 'b' from

$$(\boldsymbol{x} - \boldsymbol{a})^2 + (\boldsymbol{y} - \boldsymbol{b})^2 + \boldsymbol{z}^2 = \mathbf{1}.$$

Solution: Given $\quad (x + a)^2 + (y - b)^2 + z^2 = 1 \qquad\qquad (1)$

Partially differentiating (1) with respect to 'x' and 'y', we get

$$2(x - a) + 2z\,\frac{\partial z}{\partial x} = 0 \;\;\Rightarrow\;\; x - a = -z\,p \quad (2)$$

$$2(y - b) + 2z\,\frac{\partial z}{\partial y} = 0 \;\;\Rightarrow\;\; y - b = -z\,q \quad (3)$$

Substituting (2) and (3) in (1), we get

$z^2p^2 + z^2q^2 + z^2 = 1$

$p^2 + q^2 + 1 = \dfrac{1}{z^2}$

This is the required partial differential equation.

12. Derive a partial differential equation by eliminating the constants from the equation

$$\mathbf{2}\boldsymbol{z} = \frac{\boldsymbol{x}^2}{\boldsymbol{a}^2} + \frac{\boldsymbol{y}^2}{\boldsymbol{b}^2}.$$

Solution: Given $\quad 2z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \qquad\qquad (1)$

Partially differentiating (1) with respect to 'x' and 'y', we get

$$2\,\frac{\partial z}{\partial x} = \frac{2x}{a^2} \;\Rightarrow\; \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x} \qquad (2)$$

$$2\,\frac{\partial z}{\partial y} = \frac{2y}{b^2} \;\Rightarrow\; \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y} \qquad (3)$$

Substituting (2) and (3) in (1), we get

$$2z = xp + yq$$

This is the required partial differential equation.

13. Find the differential equation of all spheres of the same radius 'c' having their centres on the yz-plane.

Solution: The equation of spheres whose radius is 'c' and the centres $(0, a, b)$ lies on yz-plane is

$$x^2 + (y-a)^2 + (z-b)^2 = c^2 \qquad (1)$$

Differentiating (1) partially with respect to 'x' and 'y', we get

$$2x + 2(z-b)\frac{\partial z}{\partial x} = 0 \;\Rightarrow\; z - b = -\frac{x}{p} \qquad (2)$$

$$2(y-a) + 2(z-b)\frac{\partial z}{\partial y} = 0 \;\Rightarrow\; y - a = \frac{qx}{p} \, (3) \quad (\text{using}(2))$$

Substituting (2) and (3) in (1), we get

$$x^2 + \left(\frac{qx}{p}\right)^2 + \left(-\frac{x}{p}\right)^2 = c^2 \;\; i.e., \quad x^2(1 + p^2 + q^2) = c^2p^2$$

This is the required partial differential equation.

14. Find the differential equation of all spheres whose centres lie on the z-axis.

Solution: The equation of such spheres is

$$x^2 + y^2 + (z-c)^2 = r^2 \qquad (1)$$

wherer, c are constats

Differentiating (1) partially with respect to 'x' and 'y', we get

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0 \;\Rightarrow\; z - c = -\frac{x}{p} \qquad (2)$$

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0 \;\Rightarrow\; z - c = -\frac{y}{q} (3)$$

From (2) and (3), we get

$\frac{x}{p} = \frac{y}{q}$, i.e., $qx = py$

This is the required partial differential equation.

15. Derive a partial differential equation by eliminating the constants a and b from

$$\boldsymbol{log(az - 1) = x + ay + b}\,.$$

Solution: Given $log(az-1) = x + ay + b$ (1)

Partially differentiating (1) with respect to 'x' and 'y', we get

$\frac{a}{az-1}\,p = 1$ (2)

and $\frac{a}{az-1}\,q = a$ (3)

From (2) and (3), we get

$$a = \frac{1}{z-p} (4) \text{and } az - 1 = q \qquad (5)$$

Substituting (4) in (5), we get

$$q = \frac{z}{z-p} - 1 \text{ or } q(z-p) = p \text{ or } p(q+1) = zq$$

This is the required partial differential equation.

16. Form the partial differential equation by eliminating the constants from the equation

$$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}.$$

Solution: Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (1)

Partially differentiating (1) with respect to 'x' and 'y', we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\, p = 0 \qquad (2)$$

$$\frac{2y}{b^2} + \frac{2z}{c^2}\, q = 0 \qquad (3)$$

Partially differentiating equation (2) with respect to x, we get

$$\frac{2}{a^2} + \frac{2}{c^2}(zr + p^2) = 0$$

$$\frac{c^2}{a^2} + (zr + p^2) = 0 \qquad (4)$$

From equation (2), we get

$$\frac{c^2}{a^2} = -\frac{pz}{x} \qquad (5)$$

Substituting (5) in (4), we get

$$-\frac{pz}{x} + (zr + p^2) = 0 i.e., \qquad zxr + xp^2 - zp = 0$$

This is the required partial differential equation.

Formation of partial differential equations by elimination of arbitrary functions:

Formation of partial differential equations by elimination of arbitrary functions from the given relation is explained in the following examples.

Note: The elimination of one arbitrary function from a given relation gives a partial differential equation of first order while elimination of two arbitrary functions from a given relation gives a second or higher order partial differential equations.

EXAMPLES

17. Form the partial differential equation by eliminating the arbitrary function 'f' from

$$\boldsymbol{z = e^{ax+by} f(ax - by)}.$$

Solution: Given $z = e^{ax+by} f(ax - by)$ (1)

Differentiating (1) partially with respect to 'x' and 'y', we get

$$p = \frac{\partial z}{\partial x} = e^{ax+by} f'(ax - by)\,.a + ae^{ax+by} f(ax - by)$$

$$p = ae^{ax+by} f'(ax - by) + az$$

$$\Rightarrow \; f'(ax - by) = \frac{p - az}{ae^{ax+by}} \qquad (2)$$

and $$q = \frac{\partial z}{\partial y} = e^{ax+by} f'(ax - by)\,.(-b) + be^{ax+by} f(ax - by)$$

$$q = -b\left(\frac{p - az}{a}\right) + bz \quad \text{using (2)}$$

$$aq = -pb + abz + abz$$

$$pb + aq = 2abz$$

This is the required partial differential equation.

18. Form the partial differential equation by eliminating the arbitrary function from

$$\boldsymbol{z = (x + y)\,\phi(x^2 - y^2).}$$

Solution: Given $z = (x + y)\,\phi(x^2 - y^2)$ (1)

Differentiating (1) partially with respect to 'x' and 'y', we get

$$p = \frac{\partial z}{\partial x} = (x + y)\phi'(x^2 - y^2)\,.2x + \phi(x^2 - y^2)$$

$$\Rightarrow p = 2x\,(x + y)\phi'(x^2 - y^2) + \frac{z}{x + y}$$

$$\Rightarrow p - \frac{z}{x + y} = 2x\,(x + y)\phi'(x^2 - y^2) \qquad (2)$$

and $$q = \frac{\partial z}{\partial y} = (x + y)\phi'(x^2 - y^2)\,.-2y + \phi(x^2 - y^2)$$

$$\Rightarrow q = -2y\,(x + y)\phi'(x^2 - y^2) + \frac{z}{x + y}$$

$$\Rightarrow q - \frac{z}{x+y} = -2y\,(x+y)\phi'(x^2 - y^2) \qquad (3)$$

Division gives $\dfrac{p - \frac{z}{x+y}}{q - \frac{z}{x+y}} = -\dfrac{x}{y}$

$$\Rightarrow\ [p(x+y) - z]y + [q(x+y) - z]x = 0$$

$$\Rightarrow\ (x+y)(py+qx) - z(x+y) = 0$$

$$\Rightarrow\ py + qx = z$$

This is the required partial differential equation.

19. Form the partial differential equation by eliminating the arbitrary functions from

$$\boldsymbol{z = f(x+at) + g(x-at)}.$$

Solution: Given $\quad z = f(x+at) + g(x-at) \qquad (1)$

Differentiating (1) partially with respect to 'x' and 't', we get

$$\frac{\partial z}{\partial x} = f'(x+at) + g'(x-at),\ \ \frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \qquad (2)$$

$$\frac{\partial z}{\partial t} = a\,f'(x+at) - a\,g'(x-at)$$

and $\dfrac{\partial^2 z}{\partial t^2} = a^2 f''(x+at) + a^2 g''(x-at) = a^2 \dfrac{\partial^2 z}{\partial x^2}$ From(2)

thus the required partial differential equation is

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Which is an equation of the second order and (1) is its solution.

20. Form the partial differential equation by eliminating the arbitrary function 'f' from

$$\boldsymbol{z = f(x^2 - y^2)}.$$

Solution: Given $\quad z = f(x^2 - y^2) \qquad (1)$

Differentiating (1) partially with respect to 'x' and 'y', we get

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)\,.2x$$

$$\Rightarrow \frac{p}{2x} = f'(x^2 - y^2) \qquad (2)$$

and $$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)\,.-2y$$

$$\Rightarrow -\frac{q}{2y} = f'(x^2 - y^2) \qquad (3)$$

From (2) and (3), we get

$$\frac{p}{2x} = -\frac{q}{2y} \Rightarrow py + qx = 0$$

This is the required partial differential equation.

21. Form the partial differential equation by eliminating the arbitrary functions from

$$\boldsymbol{z = f(x) + e^y g(x).}$$

Solution: Given $z = f(x) + e^y g(x)$ (1)

Differentiating (1) partially with respect to 'x' and 'y', we get

$$p = \frac{\partial z}{\partial x} = f'(x) + e^y g'(x) \qquad (2)$$

$$q = \frac{\partial z}{\partial y} = e^y g(x) \qquad (3)$$

From (3), we get

$$\frac{\partial^2 z}{\partial y^2} = e^y g(x) = \frac{\partial z}{\partial y} \text{From}(3)$$

Therefore the required partial differential equation is $\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y}$

22. Eliminate $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$ from $\boldsymbol{z = f_1(x) f_2(y)}$.

Solution: Given $z = f_1(x) f_2(y)$ (1)

Differentiating (1) partially with respect to 'x' and 'y', we get

$$p = \frac{\partial z}{\partial x} = f_1{}'(x) f_2(y) \qquad (2)$$

$$q = \frac{\partial z}{\partial y} = f_1(x) f_2{}'(y) \qquad (3)$$

Differentiating (3) with respect to 'x' we get

$$s = \frac{\partial^2 z}{\partial x\, \partial y} = f_1{}'(x) f_2{}'(y) \qquad (4)$$

$(2) \times (3) \;\Rightarrow\; pq = f_1{}'(x) f_2(y) . f_1(x) f_2{}'(y)$

$\Rightarrow\; pq = s\, z$, Using (1) and (4)

This is the required partial differential equation.

23. Form the partial differential equation by eliminating f and ϕ from $z = f(y) + \phi(x + y + z)$.

Solution: Given $z = f(y) + \phi(x + y + z)$(1)

Differentiating (1) partially with respect to 'x' and 'y', we get

$$p = \frac{\partial z}{\partial x} = \phi'(x + y + z).(1 + p) \qquad (2)$$

$$q = \frac{\partial z}{\partial y} = f'(y) + \phi'(x + y + z).(1 + q) \qquad (3)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \phi'(x + y + z).r + \phi''(x + y + z).(1 + p)^2 \qquad (4)$$

$$s = \frac{\partial^2 z}{\partial x\, \partial y} = \phi'(x + y + z).s + \phi''(x + y + z).(1 + p)(1 + q) \qquad (5)$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''(y) + \phi'(x + y + z).t + \phi''(x + y + z).(1 + q)^2 \qquad (6)$$

From (4), $r[1 - \phi'(x + y + z)] = (1 + p)^2 \phi''(x + y + z)$ (7)

From (5), $s[1 - \phi'(x + y + z)] = (1 + p)(1 + q)\phi''(x + y + z)$ (8)

Now $\frac{(7)}{(8)} \;\Rightarrow\; \frac{r}{s} = \frac{1 + p}{1 + q}$

This is the required partial differential equation.

24. Eliminate the arbitrary function f from $z = f\left(\frac{xy}{z}\right)$ and form the partial differential equation.

Solution: Given $z = f\left(\frac{xy}{z}\right)$ (1)

Differentiating (1) partially with respect to 'x' and 'y', we get

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right).\frac{zy - xy.p}{z^2} \qquad (2)$$

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right).\frac{zx - xy.q}{z^2} \qquad (3)$$

Now $\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{zy - xy.p}{zx - xy.q} \Rightarrow px = qy$

This is the required partial differential equation.

25. Form the partial differential equation by eliminating the arbitrary functions f and $g$from $z = f(2x + y) + g(3x - y)$.

Solution: Given $z = f(2x + y) + g(3x - y)$ (1)

Differentiating (1) partially with respect to 'x' and 'y', we get

$$p = \frac{\partial z}{\partial x} = 2\,f'(2x + y) + 3\,g'(3x - y)$$

(Or) $p = 2\,f' + 3\,g'$ (2)

Where f' means $f'(2x + y)$ and g' means $g'(3x - y)$

and $q = \frac{\partial z}{\partial y} = f'(2x + y) - g'(3x - y)$

(Or) $q = f' - g'$ (3)

From(2), $r = \frac{\partial^2 z}{\partial x^2} = 4\,f'' + 9\,g''$ (4)

Where f'' means $f''(2x + y)$ and g'' means $g''(3x - y)$

From(2), $s = \frac{\partial^2 z}{\partial x\, \partial y} = 2\, f'' - 3\, g''$ (5)

From(3), $t = \frac{\partial^2 z}{\partial y^2} = f'' + g''$ (6)

Eliminating f'' and g'' from (4), (5) and (6), we get

$$\begin{vmatrix} 4 & 9 & r \\ 2 & -3 & s \\ 1 & 1 & t \end{vmatrix} = 0 \quad \text{[Using determinant]}$$

$i.e.,$ $4(-3t - s) - 9(2t - s) + r(2 + 3) = 0$

$i.e.,$ $-12t - 4s - 18t + 9s + 5r = 0$

$i.e.,$ $5r + 5s - 30t = 0$

$i.e.,$ $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\, \partial y} - 6\, \frac{\partial^2 z}{\partial y^2} = 0$

This is the required partial differential equation.

26. Form the partial differential equation by eliminating the arbitrary function g from the relation $z = y^2 + 2\, g\left(\frac{1}{x} + \log y\right)$.

Solution: Given $z = y^2 + 2\, g\left(\frac{1}{x} + \log y\right)$ (1)

Here we have to eliminate the only arbitrary function g.

For, differentiating partially (1) with respect to 'x' and 'y' we get

$$p = \frac{\partial z}{\partial x} = 2\, g'\left(\frac{1}{x} + \log y\right).\left(\frac{-1}{x^2}\right)$$

$i.e.,$ $p = \frac{-2}{x^2} g'\left(\frac{1}{x} + \log y\right)$

$i.e.,$ $2\, g'\left(\frac{1}{x} + \log y\right) = -p\, x^2$ (2)

and $q = \frac{\partial z}{\partial y} = 2y + 2\, g'\left(\frac{1}{x} + \log y\right).\left(\frac{1}{y}\right)$

$i.e.,$ $q = 2y + \frac{2}{y} g'\left(\frac{1}{x} + \log y\right)$ (3)

$i.e.,\qquad q = 2y - \frac{p\,x^2}{y}$ Using (2)

$i.e.,\qquad p\,x^2 + qy = 2y^2$

This is the required partial differential equation.

27. Form the partial differential equation by eliminating the arbitrary function $\boldsymbol{\phi}$ from

$$\boldsymbol{xyz = \phi(x^2 + y^2 - z^2).}$$

Solution: Given $xyz = \phi(x^2 + y^2 - z^2)$ (1)

This equation contains only one arbitrary function ϕ and we have to eliminate it.

For, differentiating (1) partially with respect to 'x' and 'y' we get

$$yz + xyp = \phi'(x^2 + y^2 - z^2).(2x - 2zp)$$

and$xz + xyq = \phi'(x^2 + y^2 - z^2).(2y - 2zq)$

$$\Rightarrow \quad \phi'(x^2 + y^2 - z^2) = \frac{yz + xyp}{2x - 2zp} \qquad (2)$$

and$\phi'(x^2 + y^2 - z^2) = \frac{xz + xyq}{2y - 2zq}$ (3)

From (2) and (3), we get

$$\frac{yz + xyp}{2x - 2zp} = \frac{xz + xyq}{2y - 2zq}$$

$i.e.,\qquad (yz + xyp)(2y - 2zq) = (xz + xyq)(2x - 2zp)$

$i.e.,\qquad y\,(z + xp)(y - zq) = x\,(z + yq)(x - zp)$

$i.e.,\qquad px\,(y^2 + z^2) - qy\,(x^2 + z^2) = z\,(x^2 - y^2)$

This is the required partial differential equation.

28. Form the partial differential equation by eliminating the arbitrary functions from

$$\boldsymbol{z = x\,f_1(x + t) + f_2(x + t).}$$

Solution: Given $z = x\,f_1(x + t) + f_2(x + t)$ (1)

Differentiating (1) partially with respect to 'x' and 'y' we get

$$\frac{\partial z}{\partial x} = x\, f_1'(x+t) + f_1(x+t) + f_2'(x+t) \qquad (2)$$

$$\frac{\partial z}{\partial t} = x\, f_1'(x+t) + f_2'(x+t) \qquad (3)$$

$$\frac{\partial^2 z}{\partial x^2} = x\, f_1''(x+t) + 2f_1'(x+t) + f_2''(x+t) \qquad (4)$$

$$\frac{\partial^2 z}{\partial t^2} = x\, f_1''(x+t) + f_2''(x+t) \qquad (5)$$

$$\frac{\partial^2 z}{\partial x\, \partial t} = x\, f_1''(x+t) + f_1'(x+t) + f_2''(x+t)$$

$$i.e., \quad \frac{\partial^2 z}{\partial x\, \partial t} = \frac{\partial^2 z}{\partial t^2} + f_1'(x+t) [\text{Using } (5)] \qquad (6)$$

Substituting (5) in (4), we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2} + 2\, f_1'(x+t)$$

$$i.e., \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2} + 2\left[\frac{\partial^2 z}{\partial x\, \partial t} - \frac{\partial^2 z}{\partial t^2}\right] \qquad [\text{Using}(6)]$$

$$i.e., \quad \frac{\partial^2 z}{\partial x^2} = 2\frac{\partial^2 z}{\partial x\, \partial t} - \frac{\partial^2 z}{\partial t^2}$$

This is the required partial differential equation.

29. Form the partial differential equation by eliminating arbitrary function from

$$\boldsymbol{z = e^{my} f(x - y)}.$$

Solution: Given $z = e^{my} f(x-y) \qquad (1)$

Differentiating (1) partially with respect to 'x' and 'y' we get

$$p = \frac{\partial z}{\partial x} = e^{my} f'(x-y) \qquad (2)$$

$$q = \frac{\partial z}{\partial y} = -e^{my} f'(x-y) + me^{my} f(x-y)$$

$$i.e., \quad q = -p + me^{my} f(x-y) \qquad [\text{Using } (2)]$$

$i.e.,\quad p + q = mz$ [Using (1)]

This is the required partial differential equation.

30. Form the partial differential equation by eliminating the arbitrary functions 'f' and 'g' from $z = x^2 f(y) + y^2 g(x)$.

Solution: Given $z = x^2 f(y) + y^2 g(x)$ (1)

Differentiating (1) partially with respect to 'x' and 'y' we get

$$p = \frac{\partial z}{\partial x} = 2x\, f + y^2 g' \quad (2)$$

$$q = \frac{\partial z}{\partial y} = x^2 f' + 2y\, g \quad (3)$$

$$r = \frac{\partial^2 z}{\partial x^2} = 2f + y^2 g'' \quad (4)$$

$$s = \frac{\partial^2 z}{\partial x\, \partial y} = 2x f' + 2y\, g' \quad (5)$$

$$t = \frac{\partial^2 z}{\partial y^2} = x^2 f'' + 2\, g \quad (6)$$

$$(3) \Rightarrow f' = \frac{q - 2\, g\, y}{x^2} \quad (7)$$

$$(2) \Rightarrow g' = \frac{p - 2\, f\, x}{y^2} \quad (8)$$

Substituting (7) and (8) in (5), we get

$$s = 2x \left[\frac{q - 2\, g\, y}{x^2}\right] + 2y \left[\frac{p - 2\, f\, x}{y^2}\right]$$

$$i.e.,\quad s = 2\left[\frac{y(q - 2\, g\, y) + x(p - 2\, f\, x)}{xy}\right]$$

$$i.e.,\quad xys = 2[qy + px - 2(x^2 f + y^2 g)]$$

$$i.e.,\quad xys = 2[px + qy - 2z] \quad \text{From (1)}$$

This is the required partial differential equation.

31. Obtain the partial differential equation by eliminating 'f' from $xy + yz + zx = f\left(\frac{z}{x+y}\right)$.

Solution: Given $xy + yz + zx = f\left(\frac{z}{x+y}\right)$ (1)

Here we have to eliminate the only one arbitrary function 'f'. For differentiating (1) partially with respect to 'x' we get

$$y + yp + z + xp = f'\left(\frac{z}{x+y}\right)\left\{\frac{(x+y)p - z}{(x+y)^2}\right\}$$

$$i.e., \quad \frac{(x+y)^2}{(x+y)p - z}[p(x+y) + (y+z)] = f'\left(\frac{z}{x+y}\right) \quad (2)$$

Differentiating (1) partially with respect to 'y' we get

$$x + yq + z + xq = f'\left(\frac{z}{x+y}\right)\left\{\frac{(x+y)q - z}{(x+y)^2}\right\}$$

$$i.e., \quad \frac{(x+y)^2}{(x+y)q - z}[q(x+y) + (x+z)] = f'\left(\frac{z}{x+y}\right) \quad (3)$$

From (2) and (3), we get

$$[p(x+y) + (y+z)][(x+y)q - z] = [q(x+y) + (x+z)][(x+y)p - z]$$

$$i.e., \quad p(x+y)(x+2z) - q(x+y)(y+2z) = z(x-y)$$

This is the required partial differential equation.

32. Obtain the partial differential equation by eliminating the arbitrary functions f and ϕ from $z = x\,f\left(\frac{y}{x}\right) + y\,\phi(x)$.

Solution: Given $z = x\,f\left(\frac{y}{x}\right) + y\,\phi(x)$ (1)

Here we have to eliminate two arbitrary functions f and ϕ.

Differentiating (1) partially with respect to 'x' and 'y' we get

$$\frac{\partial z}{\partial x} = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right) + y\phi'(x) \quad (2)$$

and $\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) + \phi(x)$ (3)

Partially differentiating (3) with respect to 'y' and 'x', we get

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{x} f''\left(\frac{y}{x}\right) \quad (4)$$

and $\frac{\partial^2 z}{\partial x \partial y} = -\frac{y}{x^2} f''\left(\frac{y}{x}\right) + \phi'(x)$ (5)

Still we are unable to eliminate the two arbitrary functions. Hence we find one more partial derivatives i.e., third derivatives.

Differentiating (4) partially with respect to 'y' and 'x' we get

$$\frac{\partial^3 z}{\partial y^3} = \frac{1}{x^2} f'''\left(\frac{y}{x}\right) \quad (6)$$

and $\frac{\partial^3 z}{\partial x \partial y^2} = -\frac{1}{x^2} f''\left(\frac{y}{x}\right) - \frac{y}{x^3} f'''\left(\frac{y}{x}\right)$ (7)

Substituting (4) and (6) in (7), we get

$$\frac{\partial^3 z}{\partial x \partial y^2} = -\frac{1}{x^2}\left(x \frac{\partial^2 z}{\partial y^2}\right) - \frac{y}{x^3}\left(x^2 \frac{\partial^3 z}{\partial y^3}\right)$$

$$\frac{\partial^3 z}{\partial x \partial y^2} = -\frac{1}{x}\left[\frac{\partial^2 z}{\partial y^2} + y \frac{\partial^3 z}{\partial y^3}\right] \quad \text{or} \quad x \frac{\partial^3 z}{\partial x \partial y^2} + \frac{\partial^2 z}{\partial y^2} + y \frac{\partial^3 z}{\partial y^3} = 0$$

This is the required partial differential equation.

Formation of partial differential equations by elimination of arbitrary function f from $f(u, v) = 0$ where u and v are functions of x, y and z.

Let $f(u, v) = 0$ (1)

be a given function of u and v, where u and v are functions of x, y and z.

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=0 \qquad (2)$$

$$\text{and}\ \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=0 \qquad (3)$$

To eliminate f it is enough we eliminate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) and (3). Elimination of $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) and (3) gives

$$\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{vmatrix}=0 \qquad (4)$$

Where $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are to be determined from $u$and v, where u and v are functions of x, y and z.

EXAMPLES

33. Form the partial differential equation by eliminating the function f from the relation $f(x^2+y^2+z^2,\ xyz)=0$.

Solution: Given $f(x^2+y^2+z^2,\ xyz)=0$ (1)

Let $u=x^2+y^2+z^2$ (2)

$v=xyz$ (3)

Equation (1) becomes $f(u,\ v)=0$ (4)

This is of the above type. We know that elimination of f from (4) gives

$$\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{vmatrix}=0 \qquad (5)$$

From (2), we get $\frac{\partial u}{\partial x}=2x+2zp$ (6)

$$\frac{\partial u}{\partial y} = 2y + 2zq \qquad (7)$$

From (3), we get $\frac{\partial v}{\partial x} = xyp + yz \qquad (8)$

$$\frac{\partial v}{\partial y} = xyq + xz \qquad (9)$$

Substituting (6), (7), (8) and (9) in (5), we get

$$\begin{vmatrix} 2x + 2zp & 2y + 2zq \\ xyp + yz & xyq + xz \end{vmatrix} = 0$$

$$i.e., \qquad (2x + 2zp)(xyq + xz) - (2y + 2zq)(xyp + yz) = 0$$

$$i.e., \qquad px(z^2 - y^2) + qy(x^2 - z^2) = z(y^2 - x^2)$$

This is the required partial differential equation.

34. Form the partial differential equation by eliminating the function f from the relation

$$\boldsymbol{f\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0}$$

Solution: Given $f\left(\frac{y}{x},\ x^2 + y^2 + z^2\right) = 0 \qquad (1)$

Let $\quad u = y/x \qquad (2)$

$v = x^2 + y^2 + z^2 \qquad (3)$

Equation (1) becomes $\quad f(u,\ v) = 0 \qquad (4)$

This is of the above type. We know that elimination of f from (4) gives

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \qquad (5)$$

From (2), we get $\frac{\partial u}{\partial x} = -\frac{y}{x^2},\ \frac{\partial u}{\partial y} = \frac{1}{x} \qquad (6)$

From (3), we get $\frac{\partial v}{\partial x} = 2x + 2zp,\ \frac{\partial v}{\partial y} = 2y + 2zq \qquad (7)$

Substituting (6) and (7) in (5), we get

$$\begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 2x + 2zp & 2y + 2zq \end{vmatrix} = 0$$

$$i.e., \quad -\frac{y}{x^2}(2y + 2zq) - \frac{1}{x}(2x + 2zp) = 0$$

$$i.e., \quad xzp + yzq + x^2 + y^2 = 0$$

This is the required partial differential equation.

VECTOR CALCULUS

Unit-IV

Vector Differentiation

INTRODUCTION

The main objective is to introduce vector calculus to the engineering student which consists of differentiation and integration of vector functions. This, naturally, leads to the study of new concepts like gradient, divergence and curl of scalar and vector respectively, which in turn will facilitate the study of solenoidal, conservative and irrotational fields. These are important to engineering branches like electrical and electronics engineering and mechanical engineering. Finally, vector integration with useful theorems like Green;s Stokes; and Gauss' divergence theorems are introduced.

We have studies the differential and integral calculus of functions of a single variable and several variables. We are also familiar with the study of vectors. All these topics together form a branch of engineering mathematics known as vector calculus.

Vector calculus is used to model a vast range of engineering problems. For example, it is used in electrostatic charges, electromagnetic fields, air flow around air craft, cars and other solid objects, fluid flow around ships and heat flow in nuclear reactors. One can appreciate the actual use of vector calculus while dealing with different topics in it.

VECTOR FUNCTIONS

If to each value of a scalar variable $t,$ there corresponds a value of vector $\bar{r},$ then $\bar{r}$is called a vector function of a scalar variable t and we write $\bar{r} = \bar{r}\,(t)$ or $\bar{r} = \bar{f}(t)$.

For example the position vector $\bar{r}$ of a particle moving along a curved path is a vector function of time $t,$ a scalar.

Since every vector can be uniquely expressed as a linear combination of three fixed non co-planar vectors, therefore, we may write

$$\bar{f}(t) = f_1(t)\bar{\imath} + f_2(t)\bar{\jmath} + f_3(t)\bar{k}$$

where $\bar{\imath}, \bar{\jmath}, \bar{k}$ denote unit vectors along the axes of x, y, zrespectively, $f_1(t), f_2(t) and f_3(t)$ are called the components of the vector $\bar{f}(t)$ along the coordinate axes.

SCALAR AND VECTOR FIELDS

Consider a region$R$ of space such that every point P in this region is connected with some physical property. Let the physical property be expressed by a quantity which has a definite value at every such point, P. The region in which the physical property is specified is called a field.

Now, fields are of two kinds Scalar and Vector, according to the quantity expressing the physical property being the scalar or a vector.

Thus a scalar field is one where the physical property in question is given by a scalar quantity. This scalar quantity will have different values at the different points of the region. In the other words, its value at a point P in R will depend on the coordinates of P.Hence this variable quantity is a function of position. It is known as the scalar point function.

For example, in the study of temperature distribution is a heated body, the region occupied by that body will be a scalar field and the temperature at any point within it is a scalar point function. Other examples of scalar fields are distribution of density, electric potential or of any other non- directed and the pressure in the atmosphere.

$\emptyset(x, y, z) = x^2 + y^2 - z^2 - 3xyz$ define a scalar field.

If the physical property of a region is represented by a vector quantity, it is said to constitute a vectorfield.

A typical example of a vector field is the distribution of velocity at all points of a moving fluid.

The velocity at every point will be represented by a continuous vector function. At a particular point, the function is specified by a vector of certain magnitude and direction, both of which change continuously from point to point throughout the field. Such a function which represents the physical property by a vector quantity is known as vector point function.

Examples of vector fields are the velocity at any point in moving field, gravitational force on a particle in space and the earth's magnetic field.

$$\bar{F}(x,y,z) = (y-z)\bar{\imath} + (z-x)\bar{\jmath} + (x-y)\bar{k}$$

defines a vector field, where $\bar{\imath}, \bar{\jmath}, \bar{k}$ are unit vectors along x, y, z.

THE VECTOR DIFFERENTIAL OPERATOR DEL

The vector operator ∇ (read del) is defined as $\nabla = \bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

The vector operator possesses properties analogous to those of ordinary vectors. It is useful in defining their quantities which arise in practical applications and are known as the gradient, the divergence and the curl.

By its definition, ∇ is a symbolic vector consisting of three symbolic components along the axes $\bar{\imath}, \bar{\jmath}, \bar{k}$ the symbolic magnitudes of them being $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

So ∇ is a vector operator. It is also a differential operator, just as $\frac{d}{dx}$ is an operator in the differential calculus.

$$\text{Thus } \nabla\emptyset = \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\emptyset = \left(\bar{\imath}\frac{\partial\emptyset}{\partial x} + \bar{\jmath}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}\right)$$

So ∇ acts both as a differential operator and as a vector.

Note: The symbol $\nabla(del)$ was originally called "nabla" an also "atled" which is "delta" (Δ) reversed. It is called 'del'

GRADIENT OF A SCALAR FUNCTION

Let $\emptyset(x,y,z)$ be a scalar function of position throughout some region of space. Then the vector function $\bar{\imath}\frac{\partial\emptyset}{\partial x} + \bar{\jmath}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$ is known as the gradient of ∅ and is denoted by ∅ . In forming this new vector, it is assumed that the partial derivatives $\frac{\partial\emptyset}{\partial x}, \frac{\partial\emptyset}{\partial y}, \frac{\partial\emptyset}{\partial z}$ are exists. Such a vector exists corresponding each point of the region in which $\emptyset(x,y,z)$ is continuous and differentiable.

Hence $grad\emptyset = \bar{\imath}\frac{\partial\emptyset}{\partial x} + \bar{\jmath}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z} = \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\emptyset = \nabla\emptyset$

It is to be noted that $\nabla\emptyset$ defines a vector field.

Note : If $\emptyset$ is constant, then $\frac{\partial\emptyset}{\partial x} = \frac{\partial\emptyset}{\partial y} = \frac{\partial\emptyset}{\partial z} = 0$, so that $grad\emptyset = \bar{0}$.

IMPORTANT DEDUCTIONS

1. Gradient of the sum of the functions:

Let u and v be two scalar point functions

$$\nabla(u+v) = \bar{\imath}\frac{\partial(u+v)}{\partial x} + \bar{\jmath}\frac{\partial(u+v)}{\partial y} + \bar{k}\frac{\partial(u+v)}{\partial z}$$

$$= \bar{\imath}\frac{\partial u}{\partial x} + \bar{\jmath}\frac{\partial u}{\partial y} + \bar{k}\frac{\partial u}{\partial z} + \bar{\imath}\frac{\partial v}{\partial x} + \bar{\jmath}\frac{\partial v}{\partial y} + \bar{k}\frac{\partial v}{\partial z} = \nabla u + \nabla v$$

2. Gradient of the product of the functions $\nabla(uv) = u\nabla \mathrm{v} + v\nabla \mathrm{u}$

3. Gradient of a function:

Let $= f(u)$, $\nabla v = \nabla f(u) = \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) f(u)$

$$= f'(u)\left(\bar{\imath}\frac{\partial u}{\partial x} + \bar{\jmath}\frac{\partial u}{\partial y} + \bar{k}\frac{\partial u}{\partial z}\right) = f'(u)\nabla u.$$

Thus, as a differential operators, the operator ∇, follows the ordinary rules of calculus

EXAMPLES

1. If $\bar{r}$ is the positive vector joining the origin 0 of a coordinate system and any point (x,y,z). Prove that $\nabla(r^n) = nr^{n-2}\bar{r}$ where $\overline{op} = \bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ and $r^2 = x^2 + y^2 + z^2$.

Solution: Hence $2r\frac{\partial r}{\partial x} = 2x$, i.e., $\frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$. Also $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$

$$\nabla(r^n) = \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) r^n$$

$$= nr^{n-1}\left(\bar{\imath}\frac{\partial r}{\partial x} + \bar{\jmath}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial r}{\partial z}\right)$$

$$= nr^{n-1}\left(\frac{x}{r}\bar{\imath} + \frac{y}{r}\bar{\jmath} + \frac{z}{r}\bar{k}\right)$$

$$\nabla(r^n) = nr^{n-2}\bar{r}$$

2. If $\nabla\emptyset = yz\bar{\imath} + zx\bar{\jmath} + xy\bar{k}$, find $\emptyset$.

Solution: Let $\nabla\emptyset = \left(\bar{\imath}\frac{\partial\emptyset}{\partial x} + \bar{\jmath}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}\right) = yz\bar{\imath} + zx\bar{\jmath} + xy\bar{k}$

Equating the corresponding coefficients of the unit vectors, we get

$$\frac{\partial\emptyset}{\partial x} = yz \qquad (I)$$

$$\frac{\partial\emptyset}{\partial y} = zx \qquad (II)$$

$$\frac{\partial\emptyset}{\partial z} = xy \qquad (III)$$

Partially integrating $(I), (II)\ and\ (III)$ with respect to x, y, zrespectively, we get

$$\emptyset = \text{xyz} + \text{a constant independent of x}$$

$$\emptyset = \text{xyz} + \text{a constant independent of y}$$

$$\emptyset = \text{xyz} + \text{a constant independent of z}$$

Hence a possible form of $\emptyset$ is $\emptyset = xyz +$ a constant.

OPERATIONS INVOLVING∇:

The vector character of the operator ∇ suggests that ∇ can operate scalarly or vectorially on a vector point function, say $\bar{F}$. The dot product $\nabla \cdot \bar{F}$ and the cross product $\nabla \times \bar{F}$are known respectively as the divergence and curl of the vector function $\bar{F}$ and they are great importance in vector analysis.

THE DIVERGENCE OF A VECTOR

Let $\bar{F}(x, y, z)$be definedand differential at each point (x, y, z) in some region of space. i.e., $\bar{F}$ defines a differentiable vector field. Then the scalar product of the vector operator $\nabla and \bar{F}$ gives a scalar which is called the divergence of $\bar{F}$. Thus the divergence of $\bar{F}$ written $div\bar{F}$ or $\nabla\cdot\bar{F}$is defined as

$$\nabla\cdot\bar{F} = \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\bar{F} = \left(\bar{\imath}\frac{\partial F}{\partial x} + \bar{\jmath}\frac{\partial F}{\partial y} + \bar{k}\frac{\partial F}{\partial z}\right)$$

We can find the value of $\nabla\cdot\bar{F}$ in terms of the components of $\bar{F}$.

Let $\bar{F} = F_1\bar{\imath} + F_2\bar{\jmath} + F_3\bar{k}$, where F_1, F_2, F_3 are functions of x, y, z.

Then $\nabla\cdot\bar{F} = \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot(F_1\bar{\imath} + F_2\bar{\jmath} + F_3\bar{k})$

$= \bar{\imath}\frac{\partial F_1}{\partial x} + \bar{\jmath}\frac{\partial F_2}{\partial y} + \bar{k}\frac{\partial F_3}{\partial z}$ $(\because \bar{\imath}\cdot\bar{\imath} = \bar{\jmath}\cdot\bar{\jmath} = \bar{k}\cdot\bar{k} = 1)$

This formula enables us to compute the divergence of $\bar{F}$ when it is given in the form

$F_1\bar{\imath} + F_2\bar{\jmath} + F_3\bar{k}$. Clearly, the divergence of $\bar{F}$. i.e., $\nabla\cdot\bar{F}$ is a Scalar.

THE CURL OF A VECTOR

Let $\bar{v}(x, y, z)$be defined and differentiable at each point (x, y, z) in some region of space. i.e., $\bar{v}$ defines a differentiable vector field. Then the vector product of the vector operator $\nabla and \bar{v}$gives a vector which is called the $curl of \bar{v}$written $curl\bar{v}$ or $rot\bar{v} or \nabla\times\bar{v}$ is defined as

$$curl\bar{v} = \nabla\times\bar{v} = \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\times\bar{v}$$

$$= \bar{\imath}\times\frac{\partial\bar{v}}{\partial x} + \bar{\jmath}\times\frac{\partial\bar{v}}{\partial y} + \bar{k}\times\frac{\partial\bar{v}}{\partial z}$$

We can find the value of the $curl\bar{v}$ in terms of its components. Let $\bar{v} = v_1\bar{\imath} + v_2\bar{\jmath} + v_3\bar{k}$. where v_1, v_2, v_3 are function of x, y, z.

$$curl\bar{v} = \nabla\times\bar{v} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Note: $grad\emptyset = \nabla\emptyset = vector$

$$div\bar{v} = \nabla\cdot\bar{v} = scalar$$

$$curl\bar{v} = \nabla\times\bar{v} = vector$$

3. If $\bar{r} = x\bar{\imath} + \ y\bar{\jmath} + \ z\bar{k}$, then prove that

$$(\text{I})\nabla \cdot \bar{r} = 3 \ and (ii) \nabla \times \bar{r} = 0$$

Solution: $\nabla \cdot \bar{r} \ = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

$$\nabla \times \bar{r} = \left(\frac{\partial}{\partial x}\bar{\imath} + \bar{\jmath}\frac{\partial}{\partial y} + \ \bar{k}\frac{\partial}{\partial z}\right) \times \left(x\bar{\imath} + y\bar{\jmath} + z\bar{k}\right) = 0$$

SOME GEOMETRICAL CONSIDERATIONS

From the three dimensional analytic geometry, we recall that the equation of a plane is of the form

$$\emptyset(x, y, z) = \text{ constant}, c\text{say} \qquad (1)$$

where $\emptyset(x, y, z)$ is a linear function of x, y, z. Let S be the surface represented by (1), since $\emptyset(x, y, z) = c$, $\sigma constant on S$, we have $d\emptyset = 0 \ on S$. Thus $d\emptyset = \nabla\emptyset \cdot d\bar{r} = 0 \ on S$. (2)

Let $P(x, y, z)$ be a point on the surface S and $Q(x + dx, y + dy, Z + dz)$ be a neighbouring point on S. then $\overline{PQ} = \ \overline{OQ} - \overline{OP} = dx\bar{\imath} \ + \ dy\bar{\jmath} + \ dz\bar{k} = d\bar{r}$

Expression (2) implies that at a point P on a surface S, the vector $\nabla\emptyset$ is perpendicular to every directed line segment $\overline{PQ}$ that is tangential to S. This means that $\nabla\emptyset$ is along the normal to the surface S at the point P.

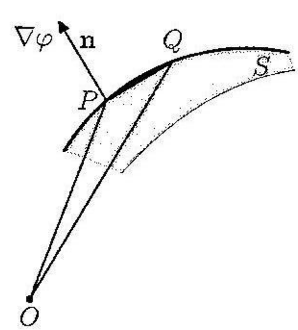

Figure.10.1

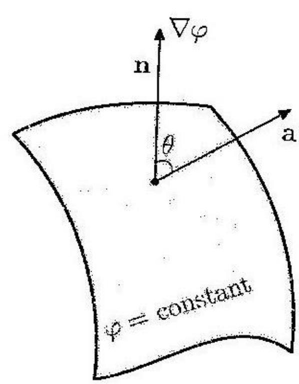

Figure.10.2

UNIT NORMAL: we denote the unit vector directed along $\nabla\emptyset$ by $\bar{n}$. Thus,

$$\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} \qquad (3)$$

The vector $\hat{n}$ is referred to as the unit vector to the surface S at the point $P(x, y, z)$.

Directional Derivative: consider a vector $\bar{a}$ inclined at an angle θ to the direction of $\nabla\emptyset$. thus the components of $\nabla\emptyset$ along $\bar{a}$, namely $\nabla\emptyset \cdot \bar{a}(where \hat{a} is unit vector \bar{a})$, is called the directional derivative of $\emptyset$ along $\bar{a}$. This is denoted by

$$\frac{d\emptyset}{d\bar{a}} = \nabla\emptyset \cdot \hat{a} = |\nabla\emptyset||\hat{a}|cos\theta = |\nabla\emptyset|cos\theta \qquad (4)$$

In particular, the directional derivative of $\emptyset$ along $\hat{n}$

$$\frac{d\emptyset}{d\bar{n}} = \nabla\emptyset \cdot \hat{n} = |\nabla\emptyset| \cdot \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{|\nabla\emptyset|^2}{|\nabla\emptyset|} = |\nabla\emptyset| \qquad (5)$$

This is called the normal derivative of $\emptyset$.

Since $cos\theta$ assumes the maximum value, when $\theta = 0,$ it follows from (4) and (5) that

$$\max\frac{\partial\emptyset}{\partial\bar{a}} = |\nabla\emptyset|\cos(0) = |\nabla\emptyset| = \frac{\partial\emptyset}{\partial\hat{n}} \qquad (6)$$

Thus, the directional derivative $\frac{\partial\emptyset}{\partial\bar{a}}$ is maximum when $\bar{a}$ is directed along $\hat{n}$, and the maximum is equal to the normal derivative. This means that $\emptyset$ varies most rapidly along $\nabla\emptyset$ and $|\nabla\emptyset|$ gives the maximum rate of variation.

4. Find the unit normal to the surface $yz + zx + xy = c$at the point $P(-1, 2, 3)$.

Solution: The equation of the given surface is $\emptyset(x, y, z) = c$, where $\emptyset = yz + zx + xy$.

This gives $\frac{\partial\emptyset}{\partial x} = z + y;\ \frac{\partial\emptyset}{\partial y} = z + x;\ \frac{\partial\emptyset}{\partial z} = x + y;$

$$\because \nabla\emptyset = (z + y)\bar{\imath} + (z + x)\bar{\jmath} + (x + y)\bar{k}$$

At the point $P(-1, 2, 3)$, this gives $\nabla\emptyset = 5\bar{\imath} + 2\bar{\jmath} + \bar{k}$

and $|\nabla\emptyset| = \sqrt{5^2 + 2^2 + 1^2} = \sqrt{30}$

Accordingly, the unit normal to the given surface at the given point P is

$$\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{1}{\sqrt{30}}\left(5\bar{\imath} + 2\bar{\jmath} + \bar{k}\right).$$

5. Find the angle between the directions of the normals to the surface $x^2yz = 1$ at the points $P(-1,1,1)$ and $Q(1,-1,1)$.

Solution: The given surface is $Q(x, y, z) = x^2yz = 1$.

At any point (x, y, z) of this surface, the normal is along the vector

$$\nabla\emptyset = 2xyz\bar{\imath} + x^2z\bar{\jmath} + x^2y\bar{k}$$

At the point $P(-1, 1, 1)$ the normal is along the vector $\bar{a} = [\nabla\emptyset]_p = 2\bar{\imath} - \bar{\jmath} + \bar{k}$

If θ is the angle between the directions of these normals, we have

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|a||b|} = \frac{-6}{\sqrt{6}\sqrt{6}} = -1$$

This gives $\theta = \pi$ as the required angle. Thus, at the given points the normals to the given surface are in opposite direction.

6. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

Solution: The angle between two surfaces at a common point P is defined to be equal to the angle between the normal to the surfaces at the point P.

Here, the given surfaces are S_1, whose equation is

$\emptyset(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$ (1)

and S_2, whose equation is

$\psi(x, y, z) = x^2 + y^2 + -3 - 3 = 0$ (2)

These gives $\frac{\partial\emptyset}{\partial x} = 2x,\ \frac{\partial\emptyset}{\partial y} = 2y, \frac{\partial\emptyset}{\partial z} = 2z$;

$$\frac{\partial\psi}{\partial x} = 2x,\ \frac{\partial\psi}{\partial y} = 2y, \frac{\partial\psi}{\partial z} = -1$$

$\nabla\emptyset = 2x\bar{\imath} + 2y\bar{\jmath} + 2z\bar{k}$ and $\nabla\psi = 2x\bar{\imath} + 2y\bar{\jmath} - \bar{k}$ (3)

At the given point $P(2, -1, 2)$ these become

$\nabla\emptyset = 4\bar{\imath} - 2\bar{\jmath} + 4\bar{k}$ and $\nabla\psi = 4\bar{\imath} - 2\bar{\jmath} - \bar{k}$ (4)

So that at P, $|\nabla\emptyset| = 6$ and $|\nabla\psi| = \sqrt{21}$ (5)

We note that $\nabla\emptyset$is along normal to surface S_1 and $\nabla\psi$is along normal to surface S_2.Therefore , if θ is angle between $\nabla\emptyset$ and $\nabla\psi$at point P. As such we have at P,

$\nabla\phi \cdot \nabla\psi = |\nabla\phi||\nabla\psi| \cos\theta$

using equations (4) and (5), this gives

$$\cos\theta = \frac{\nabla\phi \cdot \nabla\psi}{|\nabla\phi||\nabla\psi|} = \frac{(4\bar{i} - 2\bar{j} + 4\bar{k}) \cdot (4\bar{i} - 2\bar{j} - \bar{k})}{(6)(\sqrt{21})} = \frac{8}{3\sqrt{21}}$$

so that $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$. This is the required angle.

7. Find the equation of the tangent plane to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$.

Solution: The equation of the given surface S is $\phi(x, y, z) = 3$. Where $\phi(x, y, z) = x^3 + y^3 + 3xyz$.

This gives $\frac{\partial\phi}{\partial x} = 3x^2 + 3yz$, $\frac{\partial\phi}{\partial y} = 3y^2 + 3zx$, $\frac{\partial\phi}{\partial z} = 3xy$

$\nabla\phi = 3\{(x^2 + yz)\bar{i} + (y^2 + zx)\bar{j} + (xy)\bar{k}\}$

At point $P(1, 2, -1)$, this becomes

$\nabla\phi = 3(-\bar{i} + 3\bar{j} + 2\bar{k})$

This vector is directed along the normal to the given surface S at the given point P. The direction ratio's of this vector are $(-1, 3, 2)$

The tangent plane to the given surface S at the given point $P = (1, 2, -1)$ is the plane through P which is perpendicular to the normal to S at P, whose direction ratio's are $(-1, 3, 2)$.

Hence the equation of this tangent plane is

$(-1)(x - 1) + 3(y - 2) + 2(z + 1) = 0$ which implies to $x - 3y - 2z + 3 = 0$.

8. Find the constants a and b so that the surfaces $x^2 + ayz = 3x$ and $bx^2y + z^3 = (b - 8)y$ are orthogonal at the point $P = (1, 1, -2)$.

Solution: The given surfaces are S_1, whose equation is

$\phi(x, y, z) = x^2 + ayz - 3x = 0$ (1)

and S_2, whose equation is $\psi(x, y, z) = bx^2y + z^3(b - 8)y = 0$ (2)

Then $\frac{\partial\phi}{\partial x} = 2x - 3$, $\frac{\partial\phi}{\partial y} = az$, $\frac{\partial\phi}{\partial z} = ay$;

$\frac{\partial\psi}{\partial x} = 2by$, $\frac{\partial\psi}{\partial y} = bx^2 - b + 8$, $\frac{\partial\psi}{\partial z} = 3z^2$

$\nabla\phi = -\bar{i} - 2a\bar{j} + a\bar{k}$ and $\nabla\psi = 2b\bar{i} + 8\bar{j} + 12\bar{k}$ (3)

The Surfaces $S_1 and\ S_2$ are orthogonal at the point P if $\nabla\emptyset$ and $\nabla\psi$ given by (3) are orthogonal. That is if $\nabla\emptyset\cdot\nabla\psi = 0$ this yields

$-2b - 16a + 12a = 0\ (or)\ 4a + 2b = 0$(4)

Further, the point P must be common to the surfaces $S_1 and\ S_2$. That is the coordinates $(1,1,-2)$ of P must satisfy equations (1) and (2). This yields $a = -1$ consequently, (4) yields $b = 2$.

Thus, when $a = -1$ and $b = 2$, the given surfaces cut orthogonally at the point $(1, 1, -2)$.

9. Find the directional derivative of $\emptyset = x^2yz + 4z^2$ at the point $P(1, -2, -1)$ along the vector $\bar{a} = 2\bar{\imath} - \bar{\jmath} - 2\bar{k}$.

Solution: For the given $\emptyset$, we have

$$\frac{\partial\emptyset}{\partial x} = 2xyz,\ \ \frac{\partial\emptyset}{\partial y} = x^2z, \frac{\partial\emptyset}{\partial z} = x^2y + 8z$$

$$\nabla\emptyset = 4\bar{\imath} - \bar{\jmath} + 10\bar{k} \qquad (1)$$

Next, we find that for the given vector $\bar{a}$, $|\bar{a}| = 3$

The unit vector along $\hat{a}$ is $\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{3}\left(2\bar{\imath} - \bar{\jmath} - 2\bar{k}\right)$ (2)

From (1) and (2), we get $\nabla\emptyset\cdot\hat{a} = \frac{1}{3}(8 + 1 + 20) = \frac{29}{3}$

This is the directional derivative of the given function $\emptyset$ along the given vector $\bar{a}$at the given point P.

10. Find the directional derivative of $\emptyset = xyz$ along the tangent vector to the curve $x = t, y = t^2, z = t^3$at the point $P(-1, 1, -1)$.

Solution: For the given $\emptyset$, we find that

$$\nabla\emptyset = yz\bar{\imath} + zx\bar{\jmath} + xy\bar{k} \qquad (1)$$

The vector equation of the given is $\bar{r} = t\bar{\imath} + t^2\bar{\jmath} + t^3\bar{k}$ (2)

This gives $\frac{d\bar{r}}{dt} = \bar{\imath} + 2t\bar{\jmath} + 3t^2\bar{k} = \bar{a}$

And $|\bar{a}| = (1 + 4t^2 + 9t^4)^{1/2}$

Therefore the unit tangent vector to the curve is $\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{\bar{\imath} + 2t\bar{\jmath} + 3t^2\bar{k}}{(1 + 4t^2 + 9t^4)^{1/2}}$

(3)

From (2), we verify that the given point $P(-1, 1, -1)$ corresponds to

$t = -1$. Hence at P, we get from (1) and (3),

$$\nabla\emptyset = -\bar{\imath} + \bar{\jmath} - \bar{k}\,, \hat{a} = \frac{1}{\sqrt{14}}\left(-\bar{\imath} - 2\bar{\jmath} + 3\bar{k}\right)$$

Hence the required directional derivative of $\emptyset = x^3 - y^2 + z$ at the point $(-1, 1, -1)$

We first recall that $|\nabla\emptyset|$ is the maximal directional derivative of $\emptyset$.

For the given $\emptyset$, we find that $\nabla\emptyset = 3x^2\bar{\imath} - 2y\bar{\jmath} + \bar{k}$

At the point $(-1, 1, -1)$, this yields $\nabla\emptyset = 3\bar{\imath} - 2\bar{\jmath} + \bar{k}$ and $|\nabla\emptyset| = \sqrt{14}$

Thus $\sqrt{14}$is the required maximum directional derivative.

11. Find the directional derivative of $\emptyset = x^2yz + 4xz^2$ at the point $P(1, -2, -1)$ along the vector $\bar{a} = 2\bar{\imath} - \bar{\jmath} - 2\bar{k}$.

Solution: For the given $\emptyset$ we find that vector $\nabla\emptyset = (2xyz + 4z^2)\bar{\imath} + x^2z\,\bar{\jmath} + (x^2y + 8zx)\bar{k}$

At the point $P(1, -2, -1)$, this becomes

$$\nabla\emptyset = 8\bar{\imath} - \bar{\jmath} - 10\,\bar{k}$$

Next, we find that for the given vector $\bar{a}$, we have $|\bar{a}| = 3$. Therefore, the unit vector along $\bar{a}$ is $\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{3}(2\bar{\imath} - \bar{\jmath} - 2\bar{k})$

$$\nabla\emptyset \cdot \bar{a} = \frac{1}{3}(16 + 1 + 20) = \frac{37}{3}$$

This is the directional derivative of the given function $\emptyset$ along the given vector $\bar{a}$ at the given point P.

12. Find the direction from the point $P(3, 1, -2)$ along which the directional derivative of $\emptyset = x^2y^2z^4$ is maximum. Find also the magnitude of this maximum.

Solution: For the given $\emptyset$ we find that $\nabla\emptyset = 2xy^2z^4\bar{\imath} + 2x^2yz^4\bar{\jmath} + 4x^2y^2z^3\bar{k}$

At the point $P(3, 1, -2)$, this becomes

$$\nabla\emptyset = 96\bar{\imath} + 288\bar{\jmath} - 288\,\bar{k} = 96(\bar{\imath} + 3\bar{\jmath} - 3\,\bar{k}$$

Since the directional derivative of $\emptyset$ along the vector $\bar{c}$ is maximum when $\bar{c}$ is along $\nabla\emptyset$, it follows that the directional derivative of the given function $\emptyset$ at the given point P is maximum along the direction of the vector $96(\bar{\imath}+ 3\bar{\jmath}-3\,\bar{k}$. Also the magnitude of this maximum directional derivative is $\left|96(\bar{\imath}+ 3\bar{\jmath}-3\,\bar{k}\right| = 96\sqrt{19}$.

13. If thetemperature at any point in space is given by $T = xy + yz + zx$ find the direction in which temperature changes most rapidly with distance from the point $(1, 1, 1)$ and determine the maximum rate of change.The greatest of increase of T at any point is given in magnitude and direction by ∇T.

Solution: Here $\nabla T = \left(\bar{\imath}\frac{\partial}{\partial x}+ \bar{\jmath}\frac{\partial}{\partial y}+ \bar{k}\frac{\partial}{\partial z}\right)(xy + yz + zx)$

$$= (y+z)\bar{\imath} + (z+x)\bar{\jmath} + (x+y)\bar{k}$$

$= 2\bar{\imath} + 2\bar{\jmath} + 2\bar{k}$ at the point (1,1,1).

Magnitude of this vector $= \nabla T = \sqrt{12} = 2\sqrt{3}$.

Hence at the point (1,1,1), the temperature changes most rapidly in the direction given by the vector $2\bar{\imath} + 2\bar{\jmath} + 2\bar{k}$ and the greatest rate of increase $2\sqrt{3}$.

14. Prove that thedirectional derivative of $\emptyset = x^3y^2z$ at (1 ,2 ,3) is maximum along the direction $9\bar{\imath} + 3\bar{\jmath} + \bar{k}$. Also find the maximum directional derivative.

Solution: Let$\emptyset = x^3y^2z$,

$$\nabla\emptyset = \left(\bar{\imath}\frac{\partial}{\partial x}+ \bar{\jmath}\frac{\partial}{\partial y}+ \bar{k}\frac{\partial}{\partial z}\right)(x^3y^2z)$$

$= 36\bar{\imath} + 12\bar{\jmath} + 4\bar{k}$ at the point (1 ,2 ,3)

We know that the directional derivative of $\emptyset$ is maximum along the direction $\nabla\emptyset$

Hence, it is maximum along the direction of $4(9\bar{\imath} + 3\bar{\jmath} + \overline{k})$

The magnitude of this vector is $4\sqrt{91}$ and this is the maximum directional derivative.

PHYSICAL INTERPRETATION OF DIVERGENCE

Let us consider the case of a fluid flow. Consider a small rectangular parallelepiped of dimensions dx, dy, dz parallel to x, y and z axes respectively.

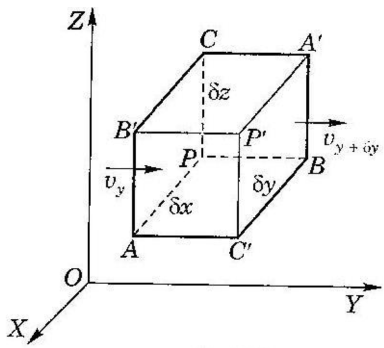

Figure 10.3

Let $\bar{V} = V_x\bar{\imath} + V_y\bar{\jmath} + V_z\bar{k}$ be the velocity of the fluid at $p(x, y, z)$

Mass of fluid flowing in through the face ABCper unit time $=$ $velocity \times area\ of\ the\ face = V_x(dy\,dz)$

Mass of fluid flowing out across the face $PQRS$ per unit time$= V_{x+dx}(dy\,dz) = \left(V_x + \frac{\partial V_x}{\partial x}\,dx\right) dy\,dz$

Net decrease in the mass of fluid in the parallelepiped corresponding to the flow along the x axis per unit time

$$V_x\,dy\,dz - \left(V_x + \frac{\partial V_x}{\partial x}\right) dy\,dz$$

$$= \frac{\partial V_x}{\partial x}\,dx\,dy\,dz \qquad (-ve\ sign\ shows\ decresing\ )$$

Similarly the decrease in mass of fluid to the flow along the y axis $= \frac{\partial V_y}{\partial y}\,dx\,dy\,dz$

Decrease in mass of fluid to the flow along the z axis $= \frac{\partial V_z}{\partial z}\,dx\,dy\,dz$

Total decrease of the amount of the fluid per unit time

$$= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dx\,dy\,dz$$

Thus the rate of loss of fluid per unit volume

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=\left(\bar{\imath}\frac{\partial}{\partial x}+\bar{\jmath}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot\left(\bar{\imath}V_x+\bar{\jmath}V_y+\bar{k}V_z\right)$$

$$=\nabla\cdot\bar{V}=div\,\bar{V}$$

If the fluid is incompressible, there can be no gain or no lose in the volume element. Hence $div\,\bar{V}=0\quad(1) and\ \bar{V}$ is called a Solenoidal vector function. Equation (1) is also called the equation of continuity.

PHYSICAL INTERPRETATION OF A CURL

We know that $\bar{V}=\omega\times\bar{r}$, where ω is the angular velocity, $\bar{V}$ is the linear velocity and $\bar{r}$ is the position vector of a point on the rotating body.

Result:If$\bar{V}=\bar{\omega}\times\bar{r}$, prove that

$\omega=\frac{1}{2}\,curl\,\bar{V},\ where\ \bar{\omega}\ is\ constsant\ vector\ and\ \bar{r}$ is the position vector .

- Let $\bar{\omega}=\omega_1\bar{\imath}+\omega_2\bar{\jmath}+\omega_3\bar{k}$,since $\bar{\omega}$is constant vector and $\omega_1,\omega_2,\omega_3$are constants

$$\bar{r}=x\bar{\imath}+y\bar{\jmath}+z\bar{k}\bar{\omega}\times\bar{r}=\begin{vmatrix}\bar{\mathrm{i}} & \bar{\mathrm{j}} & \bar{\mathrm{k}}\\ \omega_1 & \omega_2 & \omega_3\\ \mathrm{x} & \mathrm{y} & \mathrm{z}\end{vmatrix}$$

$$\bar{\omega}\times\bar{r}=(\omega_2z-\omega_3y)\mathrm{i}+(\omega_3x-\omega_1z)\mathrm{j}+(\omega_1y-\omega_2x)\mathrm{k}$$

$$curl\,(\bar{\omega}\times\bar{r})=\begin{vmatrix}\bar{\mathrm{i}} & \bar{\mathrm{j}} & \bar{\mathrm{k}}\\ \partial/\partial\mathrm{x} & \partial/\partial\mathrm{y} & \partial/\partial\mathrm{z}\\ \omega_2z-\omega_3y & \omega_3x-\omega_1z & \omega_1y-\omega_2x\end{vmatrix}$$

$$=\bar{\mathrm{i}}\left\{\frac{\partial}{\partial\mathrm{y}}(\omega_1y-\omega_2x)-\frac{\partial}{\partial\mathrm{z}}(\omega_3x-\omega_1z)\right\}$$

$$+\bar{\mathrm{j}}\left\{\frac{\partial}{\partial\mathrm{z}}(\omega_2z-\omega_3y)-\frac{\partial}{\partial\mathrm{x}}(\omega_1y-\omega_2x)\right\}$$

$$+\bar{\mathrm{k}}\left\{\frac{\partial}{\partial\mathrm{x}}(\omega_3x-\omega_1z)-\frac{\partial}{\partial\mathrm{y}}(\omega_2z-\omega_3y)\right\}$$

$$=2\omega_1\bar{\imath}+2\omega_2\bar{\jmath}+2\omega_3\bar{k}$$

$$=2\bar{\omega}$$

Hence $\omega=\frac{1}{2}\,curl\,\bar{V}$.

Thus, the angular velocity at any point is equal to half the curl of the linear velocity at that point of the body.

Definition: A vector is said to be solenoidal if its divergence is zero and irrotational if its curl is zero.

EXPANSION FORMULAE FOR OPERATORS INVOLVING ∇:

Let $\emptyset$ be a scalar point function and $\bar{u}, \bar{v}$ be vector point functions. We can form the following products between these point functions: $\emptyset\bar{u}$(Vector), $\bar{u} \cdot \bar{v}$(Scalar), $\bar{u} \times \bar{v}$(Vector). Operating with ∇ Scalarlly or vectorially we get the expressions: $\nabla \cdot \emptyset\bar{u}, \nabla \times \emptyset\bar{u}, \nabla(\bar{u} \cdot \bar{v}), \nabla \cdot (\bar{u} \times \bar{v}), \nabla \times (\bar{u} \times \bar{v})$.

1. To prove that $div\,(\emptyset\bar{u}) = \emptyset\, div\, \bar{u} + \bar{u} \cdot grad\, \emptyset$

- ❖ $\nabla \cdot \emptyset\bar{u} =: \emptyset\nabla \cdot \bar{u} + \bar{u} \cdot \nabla\emptyset$

  By definition,

$$div\,\bar{F} = \bar{\imath} \cdot \frac{\partial \bar{F}}{\partial x} + \bar{\jmath} \cdot \frac{\partial \bar{F}}{\partial y} + \bar{k} \cdot \frac{\partial \bar{F}}{\partial z}$$

Hence $div\,(\emptyset\bar{u}) = \bar{\imath} \cdot \frac{\partial(\emptyset\bar{u})}{\partial x} + \bar{\jmath} \cdot \frac{\partial(\emptyset\bar{u})}{\partial y} + \bar{k} \cdot \frac{\partial(\emptyset\bar{u})}{\partial z}$

$$= \bar{\imath} \cdot \left(\emptyset \frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial \emptyset}{\partial x}\right) + \bar{\jmath} \cdot \left(\emptyset \frac{\partial \bar{u}}{\partial y} + \bar{u}\frac{\partial \emptyset}{\partial y}\right) + \bar{k} \cdot \left(\emptyset \frac{\partial \bar{u}}{\partial z} + \bar{u}\frac{\partial \emptyset}{\partial z}\right)$$

$$= \emptyset\left\{\bar{\imath} \cdot \frac{\partial \bar{u}}{\partial x} + \bar{\jmath} \cdot \frac{\partial \bar{u}}{\partial y} + \bar{k} \cdot \frac{\partial \bar{u}}{\partial z}\right\} + \bar{u}\left\{\bar{\imath} \cdot \frac{\partial \emptyset}{\partial x} + \bar{\jmath} \cdot \frac{\partial \emptyset}{\partial y} + \bar{k} \cdot \frac{\partial \emptyset}{\partial z}\right\}$$

$$= \emptyset\, \nabla \cdot \bar{u} + \bar{u} \cdot \nabla\, \emptyset$$

i.e., $div\,(\emptyset\bar{u}) = \emptyset\, div\, \bar{u} + \bar{u} \cdot grad\, \emptyset$

2. To prove that $Curl\,(\emptyset\bar{u}) = \nabla\emptyset \times \bar{u} + \emptyset\, Curl\, \bar{u}$

- ❖ By definition

$$Curl\,\bar{F} = \bar{\imath} \times \frac{\partial \bar{F}}{\partial x} + \bar{\jmath} \times \frac{\partial \bar{F}}{\partial y} + \bar{k} \times \frac{\partial \bar{F}}{\partial z}$$

Hence $Curl\,(\emptyset\bar{u}) = \bar{\imath} \times \frac{\partial(\emptyset\bar{u})}{\partial x} + \bar{\jmath} \times \frac{\partial(\emptyset\bar{u})}{\partial y} + \bar{k} \times \frac{\partial(\emptyset\bar{u})}{\partial z}$

$$= \bar{\imath} \times \left(\emptyset \frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial \emptyset}{\partial x}\right) + \bar{\jmath} \times \left(\emptyset \frac{\partial \bar{u}}{\partial y} + \bar{u}\frac{\partial \emptyset}{\partial y}\right) + \bar{k} \times \left(\emptyset \frac{\partial \bar{u}}{\partial z} + \bar{u}\frac{\partial \emptyset}{\partial z}\right)$$

$$= \emptyset\left\{\bar{\imath}\times\frac{\partial\bar{u}}{\partial x}+\bar{\jmath}\times\frac{\partial\bar{u}}{\partial y}+\bar{k}\times\frac{\partial\bar{u}}{\partial z}\right\}+\bar{u}\left\{\bar{\imath}\times\frac{\partial\emptyset}{\partial x}+\bar{\jmath}\times\frac{\partial\emptyset}{\partial y}+\bar{k}\times\frac{\partial\emptyset}{\partial z}\right\}$$

$$= \emptyset\,\text{Curl}\,\bar{u}+(Grad\,\emptyset)\times\bar{u}$$

3. To prove that $div\,(\bar{u}\times\bar{v})=\bar{v}\cdot Curl\,\bar{u}-\bar{u}\cdot Curl\,\bar{v}$

❖ By definition,

$$div\,\bar{F}=\bar{\imath}\cdot\frac{\partial\bar{F}}{\partial x}+\bar{\jmath}\cdot\frac{\partial\bar{F}}{\partial y}+\bar{k}\cdot\frac{\partial\bar{F}}{\partial z}$$

Hence $div\,(\bar{u}\times\bar{v})=\bar{\imath}\cdot\frac{\partial(\bar{u}\times\bar{v})}{\partial x}+\bar{\jmath}\cdot\frac{\partial(\bar{u}\times\bar{v})}{\partial y}+\bar{k}\cdot\frac{\partial(\bar{u}\times\bar{v})}{\partial z}$

$$=\bar{\imath}\cdot\left(\bar{u}\times\frac{\partial\bar{v}}{\partial x}+\frac{\partial\bar{u}}{\partial x}\times\bar{v}\right)+\bar{\jmath}\cdot\left(\bar{u}\times\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{u}}{\partial y}\times\bar{v}\right)+\bar{k}\cdot\left(\bar{u}\times\frac{\partial\bar{v}}{\partial z}+\frac{\partial\bar{u}}{\partial z}\times\bar{v}\right)$$

$$=\bar{\imath}\cdot\bar{u}\times\frac{\partial\bar{v}}{\partial x}+\bar{\imath}\cdot\frac{\partial\bar{u}}{\partial x}\times\bar{v}+\bar{\jmath}\cdot\bar{u}\times\frac{\partial\bar{v}}{\partial y}+\bar{\jmath}\cdot\frac{\partial\bar{u}}{\partial y}\times\bar{v}+\bar{k}\cdot\bar{u}\times\frac{\partial\bar{v}}{\partial z}+\bar{k}\cdot\frac{\partial\bar{u}}{\partial z}\times\bar{v}$$

$$=-\bar{\imath}\cdot\frac{\partial\bar{v}}{\partial x}\times\bar{u}+\bar{\imath}\cdot\frac{\partial\bar{u}}{\partial x}\times\bar{v}-\bar{\jmath}\cdot\frac{\partial\bar{v}}{\partial y}\times\bar{u}+\bar{\jmath}\cdot\frac{\partial\bar{u}}{\partial y}\times\bar{v}-\bar{k}\cdot\frac{\partial\bar{v}}{\partial z}\times\bar{u}+\bar{k}\cdot\frac{\partial\bar{u}}{\partial z}\times\bar{v}$$

Now in each of the triple products in the right side, the dot and the cross can be interchanged.

Hence

$$div\,(\bar{u}\times\bar{v})=-\bar{\imath}\times\frac{\partial\bar{v}}{\partial x}\cdot\bar{u}+\bar{\imath}\times\frac{\partial\bar{u}}{\partial x}\cdot\bar{v}-\bar{\jmath}\times\frac{\partial\bar{v}}{\partial y}\cdot\bar{u}+\bar{\jmath}\times\frac{\partial\bar{u}}{\partial y}\cdot\bar{v}-\bar{k}\times\frac{\partial\bar{v}}{\partial z}\cdot\bar{u}+\bar{k}\times\frac{\partial\bar{u}}{\partial z}\cdot\bar{v}$$

$$=\left[\bar{\imath}\times\frac{\partial\bar{u}}{\partial x}+\bar{\jmath}\times\frac{\partial\bar{u}}{\partial y}+\bar{k}\times\frac{\partial\bar{u}}{\partial z}\right]\cdot\bar{v}-\left[\bar{\imath}\times\frac{\partial\bar{v}}{\partial x}+\bar{\jmath}\times\frac{\partial\bar{v}}{\partial y}+\bar{k}\times\frac{\partial\bar{v}}{\partial z}\right]\cdot\bar{u}$$

$$=(Curl\,\bar{u})\cdot\bar{v}-(Curl\,\bar{v})\cdot\bar{u}$$

$$div\,(\bar{u}\times\bar{v})=\bar{v}\cdot Curl\,\bar{u}-\bar{u}\cdot Curl\,\bar{v}$$

SECOND ORDER DIFFERENTIAL OPERATORS

1. To find the result of the operation $\nabla \cdot (\nabla \emptyset) = \text{div}(\text{grad } \emptyset)$

❖ Div grad $\emptyset = \nabla \cdot (\nabla \emptyset) = \nabla \cdot \left(\bar{\imath} \cdot \frac{\partial \emptyset}{\partial x} + \bar{\jmath} \cdot \frac{\partial \emptyset}{\partial y} + \bar{k} \cdot \frac{\partial \emptyset}{\partial z} \right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \emptyset}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \emptyset}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \emptyset}{\partial z}\right)$$

$$= \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} + \frac{\partial^2 \emptyset}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\emptyset$$

The operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator and it is denoted by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Hence we have $\nabla . \nabla \emptyset = \nabla^2 \emptyset$

The divergence of the gradient of a function is its Laplacian.

We notice that we can use the notation $\nabla \cdot \nabla = \nabla^2$, Similar to the notation $\bar{a} \cdot \bar{a} = \bar{a}^2$

2. **To prove the identity :**Curl (grad $\emptyset$) $= \bar{0}$, $\nabla \times (\nabla \emptyset)$

❖ $Grad\ \emptyset = \bar{\imath} \frac{\partial \emptyset}{\partial x} + \bar{\jmath} \frac{\partial \emptyset}{\partial y} + \bar{k} \frac{\partial \emptyset}{\partial z}$

Hence Curl (grad $\emptyset$) $= \nabla \times \nabla \emptyset = \nabla \times \left(\bar{\imath} \frac{\partial \emptyset}{\partial x} + \bar{\jmath} \frac{\partial \emptyset}{\partial y} + \bar{k} \frac{\partial \emptyset}{\partial z} \right)$

$$= \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \emptyset}{\partial x} & \frac{\partial \emptyset}{\partial y} & \frac{\partial \emptyset}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 \emptyset}{\partial y\, \partial z} - \frac{\partial^2 \emptyset}{\partial z\, \partial y}\right) + j\left(\frac{\partial^2 \emptyset}{\partial z\, \partial x} - \frac{\partial^2 \emptyset}{\partial x\, \partial z}\right) + k\left(\frac{\partial^2 \emptyset}{\partial x\, \partial y} - \frac{\partial^2 \emptyset}{\partial y\, \partial x}\right)$$

$$= \bar{0} + \bar{0} + \bar{0} = \bar{0}$$

The identity Curl (grad $\emptyset$) $= \bar{0}$ is true for all values of $\emptyset$ and is very important. If $Curl\, \bar{F} = \bar{0}$, then the vector $\bar{F}$is calledirrational. From the above, we have the result that $Curl\ (gradient) = \bar{0}$. Hence if $Curl\ \bar{F} = \bar{0}$, then the vector $\bar{F}$ can be expressed an the gradient of a scalar function.

3. To prove the identity : $div\ Curl\ \bar{F} = 0$,i.e., $\nabla \cdot (\nabla \times \bar{F}) = 0$

❖ Let $\bar{F} = F_1\bar{\imath} + F_2\bar{\jmath} + F_3\bar{k}$

$$Curl\ \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \bar{\imath}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \bar{\jmath}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \bar{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Hence $$div\ Curl\ \bar{F} = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z} + \frac{\partial^2 F_1}{\partial y\,\partial z} - \frac{\partial^2 F_3}{\partial y\,\partial x} + \frac{\partial^2 F_2}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial z\,\partial y} = 0$$

The identity $div\ Curl\ \bar{F} = 0$, i.e., $\nabla \cdot (\nabla \times \bar{F}) = 0$ is true for any vector $\bar{F}$.

4. To prove$Curl\ Curl\ \bar{F} = grad\ div\bar{F} - \nabla^2\bar{F}$

❖ Let $\bar{F} = F_1 i + F_2 j + F_3 k$, then $Curl\ \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$Curl\ \bar{F} = \bar{\imath}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \bar{\jmath}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \bar{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\text{Hence } Curl\ Curl\ \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) & \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) & \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \end{vmatrix}$$

$$= \bar{\imath}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\right]$$
$$+ \bar{\jmath}\left[\frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$
$$+ \bar{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\right]$$

$$= \bar{\imath}\left[\frac{\partial^2 F_2}{\partial y\,\partial x} + \frac{\partial^2 F_3}{\partial z\,\partial x}\right] + \bar{\jmath}\left[\frac{\partial^2 F_3}{\partial z\,\partial y} + \frac{\partial^2 F_1}{\partial x\,\partial y}\right] + \bar{k}\left[\frac{\partial^2 F_1}{\partial x\,\partial z} + \frac{\partial^2 F_3}{\partial y\,\partial z}\right]$$

$$- \bar{\imath}\left[\frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right] - \bar{\jmath}\left[\frac{\partial^2 F_2}{\partial z^2} + \frac{\partial^2 F_2}{\partial x^2}\right] - \bar{k}\left[\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2}\right]$$

$$= \bar{\imath}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y\,\partial x} + \frac{\partial^2 F_3}{\partial z\,\partial x}\right] + \bar{\jmath}\left[\frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial z\,\partial y} + \frac{\partial^2 F_1}{\partial x\,\partial y}\right]$$

$$+ \bar{k}\left[\frac{\partial^2 F_3}{\partial z^2} + \frac{\partial^2 F_1}{\partial x\,\partial z} + \frac{\partial^2 F_3}{\partial y\,\partial z}\right] - \bar{\imath}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right]$$

$$- \bar{\jmath}\left[\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2}\right] - \bar{k}\left[\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2}\right]$$

$$= \bar{\imath}\frac{\partial}{\partial x}\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] + \bar{\jmath}\frac{\partial}{\partial y}\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] + \bar{k}\frac{\partial}{\partial z}\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right]$$

$$- \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\left(F_1\bar{\imath} + F_2\bar{\jmath} + F_3\bar{k}\right)$$

$$= \left[\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right]\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] - \nabla^2\bar{\mathrm{F}}$$

$$= \text{grad div }\bar{\mathrm{F}} - \nabla^2\bar{\mathrm{F}}$$

Hence $Curl\ Curl\ \bar{\mathrm{F}} = \ grad\ div\bar{\mathrm{F}} - \nabla^2\bar{\mathrm{F}}$

MEANING OF THE OPERATION $(\bar{u}\ \cdot\nabla)\bar{v}$:

$\nabla$is a vector operator. Hence we first express the dot product $\bar{u}\ \cdot\nabla$ as a scalar operator.

$$\bar{u}\ \cdot\nabla = \left(u_1\bar{\imath} + u_2\bar{\jmath} + u_3\bar{k}\right).\left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)$$

$$= u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y} + u_3\frac{\partial}{\partial z}$$

Hence $(\bar{u}\ \cdot\nabla)\bar{v} = \left(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y} + u_3\frac{\partial}{\partial z}\right)\bar{v}$

$$= \left(u_1\frac{\partial\bar{v}}{\partial x} + u_2\frac{\partial\bar{v}}{\partial y} + u_3\frac{\partial\bar{v}}{\partial z}\right)$$

Now $(\bar{u}\ \cdot\nabla)\bar{v}$is written without the brackets as$\bar{u}\ \cdot\nabla\bar{v}$ but since $\cdot\nabla\bar{v}$ has no meaning, $\bar{u}\ \cdot\nabla\bar{v}$ means that $\bar{u}$associated with ∇ operates on $\bar{v}$.

Similarly we shall now prove that $(\bar{u}\ \cdot\nabla)\emptyset = \bar{u}\ \cdot\nabla\emptyset$

$\bar{u} \cdot \nabla$ as before gives the scalar operator $u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + u_3 \frac{\partial}{\partial z}$

Hence $(\bar{u} \cdot \nabla)\emptyset = \left(u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + u_3 \frac{\partial}{\partial z}\right) \emptyset = \left(u_1 \frac{\partial \emptyset}{\partial x} + u_2 \frac{\partial \emptyset}{\partial y} + u_3 \frac{\partial \emptyset}{\partial z}\right)$

Now $\bar{u} \cdot \nabla\emptyset = (u_1 x + u_2 y + u_3 z) \cdot \left(\frac{\bar{\imath}(\partial\emptyset)}{\partial x} + \frac{\bar{\jmath}(\partial\emptyset)}{\partial y} + \frac{\bar{k}(\partial\emptyset)}{\partial z}\right) = u_1 \frac{\partial \emptyset}{\partial x} + u_2 \frac{\partial \emptyset}{\partial y} + u_3 \frac{\partial \emptyset}{\partial z}$

Hence $(\bar{u} \cdot \nabla)\emptyset = \bar{u} \cdot \nabla\emptyset$

TWO MORE EXPANSION FORMULAS

1. To prove that $Curl(\bar{u} \times \bar{v}) = \bar{v} \cdot \nabla\bar{u} - \bar{u} \cdot \nabla\bar{v} + \bar{u} div\, \bar{v} - \bar{v}\, div\, \bar{u}$

- By the definition $Curl(\bar{F}) = \bar{\imath} \times \frac{\partial \bar{F}}{\partial x} + \bar{\jmath} \times \frac{\partial \bar{F}}{\partial y} + \bar{k} \times \frac{\partial \bar{F}}{\partial z}$

  Hence $Curl\,(\bar{u} \times \bar{v}) = \bar{\imath} \times \frac{\partial(\bar{u} \times \bar{v})}{\partial x} + \bar{\jmath} \times \frac{\partial(\bar{u} \times \bar{v})}{\partial y} + \bar{k} \times \frac{\partial(\bar{u} \times \bar{v})}{\partial z}$

$$= \bar{\imath} \times \left(\bar{u} \times \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial x} \times \bar{v}\right) + \bar{\jmath} \times \left(\bar{u} \times \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{u}}{\partial y} \times \bar{v}\right) + \bar{k} \times \left(\bar{u} \times \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{u}}{\partial z} \times \bar{v}\right)$$

$$= \bar{\imath} \times \left(\bar{u} \times \frac{\partial \bar{v}}{\partial x}\right) + \bar{\imath} \times \left(\frac{\partial \bar{u}}{\partial x} \times \bar{v}\right) + \bar{\jmath} \times \left(\bar{u} \times \frac{\partial \bar{v}}{\partial y}\right) + \bar{\jmath} \times \left(\frac{\partial \bar{u}}{\partial y} \times \bar{v}\right)$$

$$+ \bar{k} \times \left(\bar{u} \times \frac{\partial \bar{v}}{\partial z}\right) + \bar{k} \times \left(\frac{\partial \bar{u}}{\partial z} \times \bar{v}\right)$$

But we know that $\bar{a} \times \left(\bar{b} \times \bar{c}\right) = (\bar{a} \cdot \bar{c})\bar{b} - \left(\bar{a} \cdot \bar{b}\right)\bar{c}$

Hence $Curl(\bar{u} \times \bar{v}) = \left(\bar{\imath} \cdot \frac{\partial \bar{v}}{\partial x}\right)\bar{u} - (\bar{\imath} \cdot \bar{u})\frac{\partial \bar{v}}{\partial x} + (\bar{\imath} \cdot \bar{v})\frac{\partial \bar{u}}{\partial x} - \left(\bar{\imath} \cdot \frac{\partial \bar{u}}{\partial x}\right)\bar{v} + \left(\bar{\jmath} \cdot \frac{\partial \bar{v}}{\partial y}\right)\bar{u} - (\bar{\jmath} \cdot \bar{u})\frac{\partial \bar{v}}{\partial y} + (\bar{\jmath} \cdot \bar{v})\frac{\partial \bar{u}}{\partial y} - \left(\bar{\jmath} \cdot \frac{\partial \bar{u}}{\partial y}\right)\bar{v} + \left(\bar{k} \cdot \frac{\partial \bar{v}}{\partial z}\right)\bar{u} - \left(\bar{k} \cdot \bar{u}\right)\frac{\partial \bar{v}}{\partial z} + \left(\bar{k} \cdot \bar{v}\right)\frac{\partial \bar{u}}{\partial z} - \left(\bar{k} \cdot \frac{\partial \bar{u}}{\partial z}\right)\bar{v}$

$$= \left(\bar{\imath} \cdot \frac{\partial \bar{v}}{\partial x} + \bar{\jmath} \cdot \frac{\partial \bar{v}}{\partial y} + \bar{k} \cdot \frac{\partial \bar{v}}{\partial z}\right)\bar{u} - \bar{v}\left(\bar{\imath} \cdot \frac{\partial \bar{u}}{\partial x} + \bar{\jmath} \cdot \frac{\partial \bar{u}}{\partial y} + \bar{k} \cdot \frac{\partial \bar{u}}{\partial z}\right)$$

$$-\left((\bar{u} \cdot \bar{\imath})\frac{\partial \bar{v}}{\partial x} + (\bar{u} \cdot \bar{\jmath})\frac{\partial \bar{v}}{\partial y} + \left(\bar{u} \cdot \bar{k}\right)\frac{\partial \bar{v}}{\partial z}\right)$$

$$+\left((\bar{v} \cdot \bar{\imath})\frac{\partial \bar{u}}{\partial x} + (\bar{v} \cdot \bar{\jmath})\frac{\partial \bar{u}}{\partial y} + \left(\bar{v} \cdot \bar{k}\right)\frac{\partial \bar{u}}{\partial z}\right) \qquad (A)$$

Now $(\bar{u}\cdot\bar{\imath}) = (u_1\bar{\imath} + u_2\bar{\jmath} + u_3\bar{k})\cdot\bar{\imath} = u_1$

Similarly, $(\bar{u}\cdot\bar{\jmath}) = u_2,\ (\bar{u}\cdot\bar{k}) = u_3, (\bar{v}\cdot\bar{\imath}) = v_1, (\bar{v}\cdot\bar{\jmath}) = v_2, (\bar{v}\cdot\bar{k}) = v_3$

Also $\sum \bar{\imath}\cdot\frac{\partial\bar{v}}{\partial x} = div\,\bar{v}$ and $\sum \bar{\imath}\cdot\frac{\partial\bar{u}}{\partial x} = div\,\bar{u}$

Hence (A) becomes

$$\text{Curl }(\bar{u}\times\bar{v}) = (div\,\bar{v})\bar{u} - (div\,\bar{u})\bar{v} - \left(u_1\frac{\partial\bar{v}}{\partial x} + u_2\frac{\partial\bar{v}}{\partial y} + u_3\frac{\partial\bar{v}}{\partial z}\right) + \left(v_1\frac{\partial\bar{u}}{\partial x} + v_2\frac{\partial\bar{u}}{\partial y} + v_3\frac{\partial\bar{u}}{\partial z}\right)$$

$$= (div\,\bar{v})\bar{u} - (div\,\bar{u})\bar{v} - \left(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y} + u_3\frac{\partial}{\partial z}\right)\bar{v} + \left(v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + v_3\frac{\partial}{\partial z}\right)\bar{u}$$

$$= (div\,\bar{v})\bar{u} - (div\,\bar{u})\bar{v} - (\bar{u}\cdot\nabla)\bar{v} + (\bar{v}\cdot\nabla)\bar{u}$$

$$= (div\,\bar{v})\bar{u} - (div\,\bar{u})\bar{v} - \bar{u}\cdot\nabla\bar{v} + \bar{v}\cdot\nabla\bar{u}$$

$$= \bar{v}\cdot\nabla\bar{u} - \bar{u}\cdot\nabla\bar{v} + (div\,\bar{v})\bar{u} - (div\,\bar{u})\bar{v}$$

2. To Prove that grad **(**$\bar{u}\cdot\bar{v}) = \bar{v}\cdot\nabla\bar{u} + \bar{u}\cdot\nabla\bar{v} + \bar{v}\times curl\,\bar{u} + \bar{u}\times curl\,\bar{v}$

- By the definition grad **(**$\bar{u}\cdot\bar{v}) = \left(\bar{\imath}\cdot\frac{\partial}{\partial x} + \bar{\jmath}\cdot\frac{\partial}{\partial y} + \bar{k}\cdot\frac{\partial}{\partial z}\right)(\bar{u}\cdot\bar{v})$

$$= \bar{\imath}\cdot\left(\bar{u}\cdot\frac{\partial\bar{v}}{\partial x} + \frac{\partial\bar{u}}{\partial x}\cdot\bar{v}\right) + \bar{\jmath}\cdot\left(\bar{u}\cdot\frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{u}}{\partial y}\cdot\bar{v}\right) + \bar{k}\cdot\left(\bar{u}\cdot\frac{\partial\bar{v}}{\partial z} + \frac{\partial\bar{u}}{\partial z}\cdot\bar{v}\right) \quad (1)$$

Now, $\bar{v}\times curl\,\bar{u} = \bar{v}\times\left(\bar{\imath}\times\frac{\partial\bar{u}}{\partial x} + \bar{\jmath}\times\frac{\partial\bar{u}}{\partial y} + \bar{k}\times\frac{\partial\bar{u}}{\partial z}\right)$

$$= \bar{v}\times\left(\bar{\imath}\times\frac{\partial\bar{u}}{\partial x}\right) + \bar{v}\times\left(\bar{\jmath}\times\frac{\partial\bar{u}}{\partial y}\right) + \bar{v}\times\left(\bar{k}\times\frac{\partial\bar{u}}{\partial z}\right)$$

$$= \left(\bar{v}\cdot\frac{\partial\bar{u}}{\partial x}\right)\cdot\bar{\imath} - (\bar{v}\cdot\bar{\imath})\frac{\partial\bar{u}}{\partial x} + \left(\bar{v}\cdot\frac{\partial\bar{u}}{\partial y}\right)\cdot\bar{\jmath} - (\bar{v}\cdot\bar{\jmath})\frac{\partial\bar{u}}{\partial y} + \left(\bar{v}\cdot\frac{\partial\bar{u}}{\partial z}\right)\cdot\bar{k} - (\bar{v}\cdot\bar{k})\frac{\partial\bar{u}}{\partial z}$$

$$= \left(\bar{v}\cdot\frac{\partial\bar{u}}{\partial x}\right)\cdot\bar{\imath} - \overline{v_1}\cdot\frac{\partial\bar{u}}{\partial x} + \left(\bar{v}\cdot\frac{\partial\bar{u}}{\partial y}\right)\cdot\bar{\jmath} - \bar{v}_2\cdot\frac{\partial\bar{u}}{\partial y} + \left(\bar{v}\cdot\frac{\partial\bar{u}}{\partial z}\right)\cdot\bar{k} - \overline{v_3}\cdot\frac{\partial\bar{u}}{\partial x} (\because \bar{v}\cdot\bar{\imath} = \overline{v_1}\ etc).$$

$$= \left(\bar{v} \cdot \frac{\partial \bar{u}}{\partial x}\right) \cdot \bar{\imath} + \left(\bar{v} \cdot \frac{\partial \bar{u}}{\partial y}\right) \cdot \bar{\jmath} + \left(\bar{v} \cdot \frac{\partial \bar{u}}{\partial z}\right) \cdot \bar{k} - \left(v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + v_3 \frac{\partial}{\partial z}\right) \bar{u}$$

$$= \left(\bar{v} \cdot \frac{\partial \bar{u}}{\partial x}\right) \cdot \bar{\imath} + \left(\bar{v} \cdot \frac{\partial \bar{u}}{\partial y}\right) \cdot \bar{\jmath} + \left(\bar{v} \cdot \frac{\partial \bar{u}}{\partial z}\right) \cdot \bar{k} - (\bar{v} \cdot \nabla)\bar{u}$$

$$= \left(\bar{v} \cdot \frac{\partial \bar{u}}{\partial x}\right) \cdot \bar{\imath} + \left(\bar{v} \cdot \frac{\partial \bar{u}}{\partial y}\right) \cdot \bar{\jmath} + \left(\bar{v} \cdot \frac{\partial \bar{u}}{\partial z}\right) \cdot \bar{k} - \bar{v} \cdot \nabla \bar{u} \qquad (2)$$

Similarly, $\bar{u} \times curl\, \bar{v} = \left(\bar{u} \cdot \frac{\partial \bar{v}}{\partial x}\right) \cdot \bar{\imath} + \left(\bar{u} \cdot \frac{\partial \bar{v}}{\partial y}\right) \cdot \bar{\jmath} + \left(\bar{u} \cdot \frac{\partial \bar{v}}{\partial z}\right) \cdot \bar{k} - \bar{u} \cdot \nabla \bar{u}$ (3)

Adding (2) and (3), we get

$\bar{v} \times curl\, \bar{u} + \bar{u} \times curl\, \bar{v}$

$$= \left(\bar{u} \cdot \frac{\partial \bar{v}}{\partial x} + \bar{v} \cdot \frac{\partial \bar{u}}{\partial x}\right) \cdot \bar{\imath} + \left(\bar{u} \cdot \frac{\partial \bar{v}}{\partial y} + \bar{v} \cdot \frac{\partial \bar{u}}{\partial y}\right) \cdot \bar{\jmath} + \left(\bar{u} \cdot \frac{\partial \bar{v}}{\partial z} + \bar{v} \cdot \frac{\partial \bar{u}}{\partial z}\right) \cdot \bar{k} - \bar{v} \cdot \nabla \bar{u} - \bar{u} \cdot \nabla \bar{u}$$

$$= \text{grad}(\bar{u} \cdot \bar{v}) - \bar{v} \cdot \nabla \bar{u} - \bar{u} \cdot \nabla \bar{v} \qquad [\text{from}(1)]$$

grad $(\bar{u} \cdot \bar{v}) = \bar{v} \cdot \nabla \bar{u} + \bar{u} \cdot \nabla \bar{v} + \bar{v} \times curl\, \bar{u} + \bar{u} \times curl\, \bar{v}$

SOLENOIDAL AND IRROTATIONAL FIELDS

We shall mention here only two kinds of vector fields, having different associations of Curl and Divergence:

(i) If the divergence of a vector is zero, everywhere in a field, that field is termed Solenoidal.

Suppose $div\, \bar{v} = 0$ (1)

Then v determines a solenoidal field, we have the identity

$$div\, Curl\, \bar{F} = 0 \qquad (2)$$

Hence from (1) & (2) , we have $\bar{v} = Curl\, \bar{F}$

i.e., the solenoidal field $\bar{v}$ can be expressed as the curl of another vector $\bar{F}$. This is an important characteristic of a solenoidal field.

In the motion of an incompressible field, the divergence of the velocity vector is zero. Hence the velocity field is solenoidal.

(ii) If the curl of a vector is zero everywhere in a field, that field is termed irrotational or lamellar. Suppose $Curl\ \bar{v} = 0$ (3)

We have the identity that, if $\emptyset$ is a scalar function. Then $curl\ grad\ \emptyset = \bar{0}$

From (3) & (4), we get $\bar{v} = grad\ \emptyset$

i.e., the irrotational field $\bar{v}$ can be expressed as the gradient of a scalar function. This is an important characteristic of an irrotational field. Since $\bar{v} = grad\ \emptyset$, the vector field $\bar{v}$ can be derived from a scalar field $\emptyset$. $\bar{v}$ is called a Conservative vector field and $\emptyset$ is called the scalar potential.

15. Show that the vector $\bar{v} = (x+3y)\bar{\imath} + (y-3z)\bar{\jmath} + (x-2z)\bar{k}$ *is solenoidal.*

Solution: Let $\bar{v} = (x+3y)\bar{\imath} + (y-3z)\bar{\jmath} + (x-2z)\bar{k}$

$$div\ \bar{F} = \nabla\cdot\ \bar{F} = \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left\{(x+3y)\bar{\imath} + (y-3z)\bar{\jmath} + (x-2z)\bar{k}\right\}$$

$$= \left(\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z)\right)$$

$$= 1+1-2 = 0\bar{v}\ \ is\ solenoidal$$

16. Show that $\nabla f(r) = \frac{\bar{r}}{r}\left(\frac{\partial f}{\partial r}\right)$and $(\nabla f(\bar{r}) \times \bar{r} = 0$

Solution: Where $|\bar{r}| = |xi + yj + zk\,|, r^2 = x^2 + y^2 + z^2, \frac{\partial r}{\partial x} = \frac{x}{r}$

$$\nabla f(r) = \left(\bar{\imath}\frac{\partial f(r)}{\partial x} + \bar{\jmath}\frac{\partial f(r)}{\partial y} + \bar{k}\frac{\partial f(r)}{\partial z}\right)$$

$$= \bar{\imath}\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \bar{\jmath}\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial f}{\partial r}\frac{\partial r}{\partial z}$$

$$\{\nabla f(r)\} \times \bar{r} = \left(\frac{\bar{r}}{r}\frac{\partial f}{\partial r}\right) \times \bar{r} = \frac{1}{r}\frac{\partial f}{\partial r}(\bar{r} \times \bar{r}) = \bar{0}$$

17. Show that $div\ (r^n\bar{r}) = (n+3)r^n$ and $Curl\ (r^n\bar{r}) = \bar{0}$.

Solution: where $\bar{r} = xi + yj + zk,\ r^2 = x^2 + y^2 + z^2, \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$div\,(r^n\bar{r}) = div\,\left(r^n(xi+yj+zk)\right) = \nabla\cdot\left(r^n(xi+yj+zk)\right)$$
$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$
$$= 3r^n + nr^{n-1}\left\{\frac{\partial r}{\partial x}x + \frac{\partial r}{\partial y}y + \frac{\partial r}{\partial z}z\right\}$$
$$= 3r^n + nr^{n-1}\left(\frac{x^2+y^2+z^2}{r}\right) = 3r^n + nr^{n-1}r = 3r^n + nr^n$$
$$= (n+3)r^n$$

18. If $\bar{A}\ and\ \bar{B}$ are irrotational, show that $\bar{A} \times \bar{B}$ is **solenoidal**.

Solution: Let $\bar{A}\ and\ \bar{B}$ are irrotational

$$Curl\ \bar{A} = \bar{0}\ and\ Curl\ \bar{B} = \bar{0}\,, i.e., \nabla \times \bar{A} = \bar{0}\ and\ \nabla\times \bar{B} = \bar{0}(1)$$
$$div\,(\bar{A} \times \bar{B}) = \nabla\cdot\bar{A} \times \bar{B} = \bar{B}\cdot(\nabla\times\bar{A}) - \bar{A}\cdot(\nabla\times\bar{B})$$
$$= \bar{B}\cdot\bar{0} - \bar{A}\cdot\bar{0} = 0 \qquad using\ (1)$$

By the definition, a vector is solenoidal if its divergence is zero.
$\bar{A} \times \bar{B}$ is solenoidal if $\bar{A}\ and\ \bar{B}$ are irrotational.

19. Show that $r^n\bar{r}$ is an irrotational vector for any value of $n,$ but it is solenoidal only if $n = -3$.

Solution: $\nabla\times(r^n\bar{r}) = \bar{0}\,, r^n\bar{r}$ is an irrotational vector for any value of $n, \nabla\cdot(r^n\bar{r}) = (n+3)r^n$

But the vector is solenoidal, $\nabla\cdot(r^n\bar{r}) = 0$
$\Rightarrow n+3 = 0 \Rightarrow n = -3$

20. Determine the constant a so that the vector $\bar{F} = (x+3y)\bar{\imath} + (y-3z)\bar{\jmath} + (x-az)\bar{k}$ is solenoidal

Solution: $\nabla\cdot\bar{F} = 1 + 1 + a = a + 2$

For a solenoidal field, $\nabla\cdot\bar{F} = 0$
$i.e.,\qquad a+2 = 0\ or\ a = -2$

21. If $\emptyset\ and\ \Psi$ are differential scalar fields, prove that $(\nabla\emptyset\cdot\nabla\Psi)$ is solenoidal.

Solution: $div\,(\nabla\emptyset\cdot\nabla\Psi) = \nabla\Psi\cdot curl\ \emptyset - \nabla\emptyset\cdot \text{curl}\ \nabla\Psi \qquad (1)$

But $curl\ (gradient\ ) = \bar{0}$ identically

Hence $curl\ grad\ \emptyset = \bar{0} = curl\ grad\ \Psi$

So (1) becomes $div\ (\nabla\emptyset \cdot \nabla\Psi) = 0$, $\nabla\emptyset \times \nabla\Psi$ is solenoidal.

22. Show that $\nabla^2 r^n = n\ (n+1) r^{n-2}$

Solution: We have $\nabla r^n = n\ r^{n-2}\bar{r}$ (1)

$\nabla r^n = \nabla \cdot \nabla r^n$, since $\nabla \cdot \nabla\emptyset = \nabla^2\emptyset$

$$= \nabla \cdot n\ r^{n-2}\bar{r} = n r^{n-2}\nabla \cdot \bar{r} + \nabla(r^{n-2}) \cdot \bar{r}$$

But $\nabla \cdot \bar{r} = 3$

$$\nabla^2 r^n = 3nr^{n-2} + n\ \nabla(r^{n-2}) \cdot \bar{r} \qquad (2)$$

Changing n into $n-2$ in (1), we have

$$\nabla \cdot r^{n-2} = (n-2) r^{n-4}\bar{r}$$

Substituting in (2), we have

$$\nabla^2 r^n = 3nr^{n-2} + n\ (n-2)\ r^{n-4}\bar{r} \cdot \bar{r}$$

$$= 3nr^{n-2} + n\ (n-2)\ r^{n-2}$$

$$= n(3+n-2)\ r^{n-2} = n(n+1)\ r^{n-2}$$

23. Prove that $curl\ (\emptyset\ \nabla\emptyset) = \bar{0}$.

Solution: $\nabla \times (\emptyset\ \nabla\emptyset) = \emptyset(\nabla \times \nabla\emptyset) + \nabla\emptyset \times \nabla\emptyset = \bar{0}$

24. A Vector field is given by $\bar{F} = (x^2 - y^2 + x)\bar{\imath} - (2xy + y)\bar{\jmath}$ show that the field is irrotational and find its scalar potential.

Solution: Since $\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - y^2 + x & -2xy - y & 0 \end{vmatrix}$

$$= \bar{\imath}(0-0) + \bar{\jmath}(0-0) + \bar{k}(-2y - +2y) = \bar{0}$$

$\bar{F}$ is irrotational field and the vector $\bar{F}$ can be expressed as the gradient of a scalar potential. i.e., $\bar{F} = \nabla\emptyset$.

$$(x^2 - y^2 + x)\bar{\imath} - (2xy + y)\bar{\jmath} = \frac{\partial\emptyset}{\partial x}\bar{\imath} + \frac{\partial\emptyset}{\partial y}j$$

$$\frac{\partial\emptyset}{\partial x} = x^2 - y^2 + x \quad (1), \quad \frac{\partial\emptyset}{\partial y} = -2xy - y \qquad (2)$$

Integrating (1) $w.r.t.$ x, keeping y constant, we get

$$\emptyset = \frac{x^3}{3} - y^2x + \frac{x^2}{2} + f(y) \qquad (3)$$

Integrating $(2) w.r.t.\ y$, keeping x constant, we get

$$\emptyset = -xy^2 - \frac{y^2}{2} + g(x) \qquad (4)$$

Equating (3) & (4), we get $\frac{x^3}{3} - y^2x + \frac{x^2}{2} + f(y) = -xy^2 - \frac{y^2}{2} + g(x)$

$$f(y) = -\frac{y^2}{2} \qquad and\ \ g(x) = \frac{x^3}{3} + \frac{x^2}{2}$$

Hence $\emptyset = \frac{x^3}{3} - y^2x + \frac{x^2}{2} - \frac{y^2}{2}$

25. A fluid motion is given by $\bar{F} = (y+z)\bar{\imath} + (z+x)\bar{\jmath} + (x+y)\bar{k}$. Is this motion irrotational? If so, find the scalar potential.

Solution:

$$\text{curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & z+x & x+y \end{vmatrix}$$

$$= \bar{\imath}(1-1) + \bar{\jmath}(1-1) + \bar{k}(1-1) = \bar{0}$$

This motion is irrational and if $\emptyset$ is the scalar potential then $\bar{F} = \nabla\emptyset$

ie., $(y+z)\bar{\imath} + (z+x)\bar{\jmath} + (x+y)\bar{k} = \bar{\imath}\frac{\partial\emptyset}{\partial x} + \bar{\jmath}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$

$$\frac{\partial\emptyset}{\partial x} = (y+z)(1); \quad \frac{\partial\emptyset}{\partial y} = (z+x)(2); \frac{\partial\emptyset}{\partial z} = (x+y) \quad (3)$$

Integrating these, we get

$\emptyset = (y+z)x + f_1(y,z) \qquad (4)$

$\emptyset = (z+x)y + f_2(z,x) \qquad (5)$

$\emptyset = (x+y)z + f_3(x,y) \qquad (6)$

Equating (4), (5) & (6), we get $f_1(y,z) = yz, f_2(z,x) = zx, f_3(x,y) = xy$

Hence $\emptyset = yz + zx + xy$

Unit-V

Vector Integration

INTEGRATION OF A VECTOR FUNCTIONS

Let $\bar{f}(t)$ and$\bar{F}(t)$ be two vector functions of a scalar variable t such that$\frac{d\bar{F}(t)}{dt} = \bar{f}(t)$, then$\bar{F}(t)$ is called an integral of $\bar{f}(t)$ with respect to t and we write $\int \bar{f}(t)\, dt = \bar{F}(t)$.

If $\bar{c}$ is any arbitrary constant vector independent of t, then$\frac{d(\bar{F}(t)+\bar{c})}{dt} = \bar{f}(t)$

This is equivalent to $\int \bar{f}(t)\, dt = \bar{F}(t) + \bar{c}$.

$\bar{F}(t)$ is called the indefinite integral of $\bar{f}(t)$. The constant vector $\bar{c}$ is called the constant of integration and can be determined if some initial conditions are given.

The definite integral of $\bar{f}(t)$ between the limits $t = a \; and \; t = b$ is written as

$$\int_a^b \bar{F}(t)\, dt = [\bar{F}(t)]_a^b = \bar{F}(b) - \bar{F}(a)$$

Note 1: If $\bar{f}(t) = f_1(t)\bar{\imath} + f_2(t)\bar{\jmath} + f_3(t)\bar{k}$, then

$$\int \bar{f}(t)\, dt = \bar{\imath}\int f_1(t)dt + \bar{\jmath}\int f_2(t)\, dt + \bar{k}\int f_3(t)\, dt$$

Thus in order to integrate a vector function, integrate the components.

Note 2: we can obtain some standard results for integration of vector functions by considering the derivatives of suitable vector functions. For example,

(i) $\frac{d(\bar{r}\cdot\bar{s})}{dt} = \frac{d\bar{r}}{dt}\cdot\bar{s} + \bar{r}\cdot\frac{d\bar{s}}{dt} \Rightarrow \int\left(\frac{d\bar{r}}{dt}\cdot\bar{s} + \bar{r}\cdot\frac{d\bar{s}}{dt}\right)dt = \bar{r}\cdot\bar{s} + c$

Here c is a scalar quantity. Since the integrand is a scalar.

(ii) $\frac{d\bar{r}^2}{dt} = 2\bar{r}\cdot\frac{d\bar{r}}{dt} \Rightarrow \int\left(2\bar{r}\cdot\frac{d\bar{r}}{dt}\right)dt = \bar{r}^2 + c$

Here c is a scalar quantity. Since the integrand is a scalar.

(iii) $\frac{d}{dt}\left(\bar{r}\times\frac{d\bar{r}}{dt}\right) = \frac{d\bar{r}}{dt}\times\frac{d\bar{r}}{dt} + \bar{r}\times\frac{d^2\bar{r}}{dt^2} = \bar{r}\times\frac{d^2\bar{r}}{dt^2}$

$$\Rightarrow \int\left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)dt = \bar{r}\times\frac{d\bar{r}}{dt} + \bar{c}$$

Here $\bar{c}$ is a vector quantity. Since the integrand is a vector.

(iv) If $\bar{a}$ is a constant vector, then

$$\frac{d}{dt}(\bar{a}\times\bar{r}) = \frac{d\bar{a}}{dt}\times\bar{r} + \bar{a}\times\frac{d\bar{r}}{dt} = \bar{a}\times\frac{d\bar{r}}{dt}$$

$$\Rightarrow \int\left(\bar{a}\times\frac{d\bar{r}}{dt}\right)dt = \bar{a}\times\bar{r}+\bar{c}$$

Here $\bar{c}$ is a vector quantity. Since the integrand is a vector.

1. The acceleration of a particle at time t is given by $\bar{a} = 18\cos 3t\,\bar{\imath} - 8\sin 2t\,\bar{\jmath} + 6t\,\bar{k}$. If the velocity $\bar{v}$and displacement $\bar{r}$ are zero at $t=0$, find $\bar{v}$and $\bar{r}$ at any point t .

Solution: Given $\bar{a} = \frac{d^2\bar{r}}{dt^2} = 18\cos 3t\,\bar{\imath} - 8\sin 2t\,\bar{\jmath} + 6t\,\bar{k}$

Integrating, we get

$$\bar{v} = \frac{d\bar{r}}{dt} = \int\left(18\cos 3t\,\bar{\imath} - 8\sin 2t\,\bar{\jmath} + 6t\,\bar{k}\right)dt$$

$$= \bar{\imath}\int 18\cos 3t\;dt - \bar{\jmath}\int 8\sin 2t\;dt + \bar{k}\int 6t\;dt$$

$$= 6\sin 3t\,\bar{\imath} + 4\cos 2t\,\bar{\jmath} + 3t^2\bar{k} + \bar{c}$$

At $t=0, \bar{v}=\bar{0} \Rightarrow \bar{0} = 4\bar{\jmath}+\bar{c}\;\; or\;\; \bar{c} = -4\bar{\jmath}$

$$\therefore \bar{v} = 6\sin 3t\,\bar{\imath} + 4\,(\cos 2t - 1)\bar{\jmath} + 3t^2\bar{k}$$

Integrating again, we get

$$\bar{r} = \int\left(6\sin 3t\,\bar{\imath} + 4\,(\cos 2t-1)\bar{\jmath} + 3t^2\bar{k}\right)dt$$

$$= \bar{\imath}\int 6\sin 3t\;dt - \bar{\jmath}\int 4\,(\cos 2t - 1)\;dt + \bar{k}\int 3t^2\;dt$$

$$= -2\cos 3t\,\bar{\imath} + (2\sin 2t - 4t)\bar{\jmath} + t^3\bar{k} + \bar{d}$$

At $t=0, \bar{r}=\bar{0} \Rightarrow \bar{0} = -2\,\bar{\imath}+\bar{d}\;\; or\;\; \bar{d} = 2\bar{\imath}$

$$\therefore \bar{r} = 2(1-\cos 3t)\bar{\imath} + (2\sin 2t - 4t)\bar{\jmath} + t^3\bar{k}$$

2. If $\bar{f}(t) = (3t^2-2t)\bar{\imath} + (6t-4)\bar{\jmath} + 4t\,\bar{k}$, evaluate $\int_2^3 \bar{f}(t)\,dt$.

Solution: $\int_2^3 \bar{f}(t)\,dt = \int_2^3 (3t^2-2t)\bar{\imath} + (6t-4)\bar{\jmath} + 4t\,\bar{k}\,dt$

$$= \left[(t^3-t^2)\bar{\imath} + (3t^2-4t)\bar{\jmath} + 2t^2\bar{k}\right]_2^3$$

$$= 14\bar{\imath} + 11\bar{\jmath} + 10\,\bar{k}$$

LINE INTEGRAL OF A VECTOR FUNCTION

Any integral which is to be evaluated along a curve is called a line integral.

Consider a vector $\bar{f}$ defined over a region R in three dimensional space. Let C be curve in this region and let its vector equation be $\bar{r} = x(t)\bar{\imath} + y(t)\bar{\jmath} + z(t)\bar{k}(1)$

Where t is a real parameter. Let A and B be end points of C, which corresponds to $t = a\ and\ t = b$ respectively, where $a < b$. Then, as t increases $a\ to\ b$, a variable point $P\ (x, y, z)$ describes the curve C from the initial point A to the terminal point B. If C is a closed curve, then the point B becomes coincident with the point A.

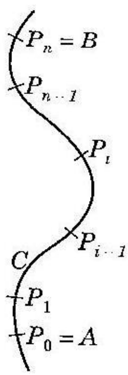

Figure 10.4

We note that $\frac{d\bar{r}}{dt}$is along the tangent vector to the curve C at the point $P\ (x, y, z)$and that the unit vector to C, at P is given by

$$\hat{t} = \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} \qquad (2)$$

Now, consider the scalar function $\bar{f} \cdot \frac{d\bar{r}}{dt}$, since $\bar{f}$ is a function of x, y, z and x, y, z are functions of the parameter t on C, and $\frac{d\bar{r}}{dt}$ is a function of t, it follows that $\bar{f} \cdot \frac{d\bar{r}}{dt}$ is a function of t on C. Suppose we integrate this function with respect to t from $t = a\ to\ t = b$. The resulting integral is called the scalar line integral of $\bar{f}$ along the curve C and is denoted by $\int_C \bar{\boldsymbol{f}}\,.dr$.

Thus, we have by definition

$$\int_C \bar{\boldsymbol{f}} \,.\, dr = \int_a^b \left(\bar{f} \cdot \frac{d\bar{r}}{dt}\right) dt \qquad (3)$$

If $C$is a closed curve, then the integral sign $\int_C$ is replaced by $\oint_C$.

Circulation: In fluid dynamics, if $\bar{v}$ represent s the velocity of a fluid particle and C is a closed curve then the integral $\int_C \bar{v} \cdot d\bar{r}$ is called the circulation of $\bar{v}$ around the curve C.

If the circulation of $\bar{v}$around every closed curve in a region R vanishes then $\bar{v}$ is said to be irrotational in R.

Remarks:

1. While defining $\int_C \bar{\boldsymbol{f}} \,.\, dr$ through the relation (3) it is customary to take the curve C as positively oriented. A space curve C is said to be positively oriented if its projection on the $xy -$ plane is described in the anti-clockwise sense. The sign of the integral $\int_C \bar{\boldsymbol{f}} \,.\, dr$ changes, when the sense of description of C is reversed.

2. In Cartesians, expression (3) becomes

$$\int_C \bar{\boldsymbol{f}} \,.\, dr = \int_C (f_1 dx + f_2 dy + f_3 dz)$$

$$= \int_a^b \left(f_1 \frac{dx}{dt} + f_2 \frac{dy}{dt} + f_3 \frac{dz}{dt}\right) dt \qquad (4)$$

3. In the special case where R is a region in the $xy -$ plane so that C is a plane curve (in this region), expression (4) becomes

$$\int_C f_1 dx + f_2 dy = \int_a^b \left(f_1 \frac{dx}{dt} + f_2 \frac{dy}{dt}\right) dt \qquad (5)$$

4. If $\bar{f}$ represents a force under which a particle moves from one end of curve C to the other end (along the curve), then $\int_C \bar{f} \,.\, dr$ represents the corresponding **total work** done by $\bar{f}$.

3. If $\bar{\boldsymbol{f}} = (5xy - 6x^2)\bar{\imath} + (2y - 4x)\bar{\jmath}$, evaluate $\int_C \bar{\boldsymbol{f}}\,.dr$,where C is the curve $y = x^3$, from the point$(1,1)$ to the point$(2,8)$.

Solution: The given curve C is a curve in the xy – plane. Therefore, $z \equiv 0$ at every point of the curve. Setting $x = t$ in the equation of C, we get $y = t^3$. Thus, the parametric equations of the given curve C may be taken as $x = t, y = t^3$, z = 0.Since C is from the point $(1,1)$to the point $(2,8)$, the x- coordinate of a point on the curve from 1 to 2. Thus, since we have set $x = t$ on C, we have $1 \leq t \leq 2$.

We find that , on C,

$$r = x\bar{\imath} + y\bar{\jmath} + z\bar{k} = t\bar{\imath} + t^3\bar{\jmath} + 0\bar{k}, \qquad \frac{dr}{dt} = \bar{\imath} + 3t^2\bar{\jmath}$$

and $\qquad f = (5t^4 - 6t^2)\bar{\imath} + (2t^3 - 4t)\bar{\jmath},$

so that $\quad f.\frac{dr}{dt} = (5t^4 - 6t^2) + 3t^2(2t^3 - 4t) = 6t^5 + 5t^4 - 12t^3 - 6t^2$.

Therefore,

$$\int_c f\,.dr = \int_1^2 (f\,.\frac{dr}{dt})\,dt = \int_1^2 (6t^5 + 5t^4 - 12t^3 - 6t^2)dt = 35$$

4. Evaluate$\int_c \bar{\boldsymbol{f}}\,.dr$ along the circle$x^2 + y^2 = a^2$, where$\bar{\boldsymbol{f}} = 3xyi - yj + 2zk$.

Solution: The parametric equations of the given curve C can be taken as $x = acost,\ y = asint$, z = 0, $0 \leq t \leq 2\pi$. Hence, on C,

$$r = x\bar{\imath} + y\bar{\jmath} + z\bar{k} = (acost)\bar{\imath} + (asint)\bar{\jmath} + 0k, \qquad \frac{dr}{dt}$$

$$= a(-sint\bar{\imath} + cost\bar{\jmath}),$$

and $f = 3(acost)(asint)\bar{\imath} - (asint)\bar{\jmath} + 2 \cdot 0\bar{k}$

so that $\qquad f.\frac{dr}{dt} = -3a^3sin^2t\,cost - a^2sint\,cost = -a^2(3a\,sin^2t + sint\,)\,cost$

Therefore,

$$\int_c f\,.dr = \int_0^{2\pi} (f\,.\frac{dr}{dt})\,dt = \int_0^{2\pi} \{-a^2(3a\,sin^2t + \ \ sint\,)\,cost\}\,dt$$

$$= -a^2[asin^3t + \tfrac{1}{2}sin^2t]_0^{2\pi} = 0$$

5. If $\bar{\boldsymbol{f}} = (2y+3)\bar{\imath} + xz\bar{\jmath} + (yz-x)\bar{k}$, evaluate the integral $\int_c f\,.dr$, where C is the curve $x = 2t^2,\ y = t, z = t^3$ from the point $(0,0,0)$ to the point $(2,1,1)$.

Solution: The vector equation of the given curve C is

$r = x\bar{\imath} + y\bar{\jmath} + z\bar{k} = 2t^2\bar{\imath} + t\bar{\jmath} + t^3\bar{k}$,

so that $\frac{dr}{dt} = 4t\bar{\imath} + \bar{\jmath} + 3t^2\bar{k}$.

Also on C, we have

$$f = (2t+3)\bar{\imath} + 2t^5\bar{\jmath} + (t^4 - 2t^2)\bar{k}.$$

Therefore, on C,

$$f.\frac{dr}{dt} = 4t(2t+3) + 2t^5 + 3t^2(t^4 - 2t^2)$$

$$= 3t^6 + 2t^5 - 6t^4 + 8t^2 + 12t.$$

Since, the curve C is from the point $(0,0,0)$ to the point $(2,1,1)$ and $y = t$ on C, we note that, on the curve $C, y = t$ varies from 0 to 1. Hence,

$$\int_c f\,.dr = \int_0^1 (f\,.\frac{dr}{dt})\,dt = \int_0^1 (3t^6 + 2t^5 - 6t^4 + 8t^2 + 12t)\,dt = \frac{288}{35}$$

6. If $\bar{\boldsymbol{f}} = (3x^2 + 6y)\bar{\boldsymbol{\imath}} - 14yz\bar{\boldsymbol{\jmath}} + 20xz^2\bar{\boldsymbol{k}}$, evaluate $\int_c f\,.dr$, from $(0,0,0)$ to the point $(1,1,1)$ along the curve C given by $x = t,\ y = t^2, z = t^3$.

Solution: The vector equation of the given curve C is $r = t\bar{\boldsymbol{\imath}} + t^2\bar{\boldsymbol{\jmath}} + t^3\bar{\boldsymbol{k}}$, so that

$$\frac{dr}{dt} = \bar{\boldsymbol{\imath}} + 2t\bar{\boldsymbol{\jmath}} + 3t^2\bar{\boldsymbol{k}}.$$

Also on C, we have

$$f = 9t^2\bar{\imath} - 14t^5\bar{\jmath} + 20t^7\bar{\boldsymbol{k}}.$$

Therefore, on C,

$$f.\frac{dr}{dt} = 9t^2 - 28t^6 + 60t^9$$

We note that along the curve C is from the point (0,0,0) to the point $(1, 1, 1)$ the parameter t increases from 0 to 1. therefore,

$$\int_c \bar{\boldsymbol{f}}\,.dr = \int_0^1 (\bar{\boldsymbol{f}}\,.\frac{dr}{dt})\,dt = \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = 5.$$

7. Find the total work done by a force $\boldsymbol{f} = 2xy\bar{\imath} - 4z\bar{\jmath} + 5x\bar{\boldsymbol{k}}$ along the curve by $x = t^2,\ y = 2t + 1,\ z = t^3$ from the point $t = 1$ to the point $t = 2$.

Solution: On the given curve C, we have

$$r = t^2\bar{\imath} + (2t + 1)\,\bar{\jmath} + t^3\bar{\boldsymbol{k}},$$

$\frac{dr}{dt} = 2t\bar{\imath} + 2\bar{\jmath} + 3t^2\bar{\boldsymbol{k}}.$ and $\bar{\boldsymbol{f}} = 2t^2(2t + 1)\bar{\imath} - 4t^3\bar{\jmath} + 5t^2\bar{\boldsymbol{k}},$

So that

$$\bar{\boldsymbol{f}}.\frac{dr}{dt} = 4t^3(2t + 1) - 8t^3 + 15t^4 = 23t^4 - 4t^3, \qquad \mathbf{1 \le t \le 2}$$

Hence, the required work is

$$\int_c \bar{\boldsymbol{f}}\,.dr = \int_1^2 (\bar{\boldsymbol{f}}\,.\frac{dr}{dt})\,dt = \int_1^2 (23t^4 - 4t^3)\,dt = \frac{638}{5}.$$

8. *Find the total work done by a force* $\bar{\boldsymbol{f}} = (2y - x^2)\bar{\imath} + \boldsymbol{6}yz\bar{\jmath} - 8xz^2\bar{\boldsymbol{k}}$ *from the point* (0,0,0) *to the point* $(1, 1,1)$ *along the straight line joining these points.*

Solution: Here the path (curve) C along which the work is done is the straight line from the origin (0,0,0) to the point (1,1,1).The Cartesian equations of this straight line are $\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t\ (say)$.These equations yield $x = t,\ y = t, z = t, 0 \le t \le 1$ as the parametric equations of the path C. Thus, here,

$$r = t\,(\bar{\imath} + \bar{\jmath} + \bar{\boldsymbol{k}}), \qquad \frac{dr}{dt} = \bar{\imath} + \bar{\jmath} + \bar{\boldsymbol{k}}.$$

Hence on C, $\boldsymbol{f} = (2t - t^2)\bar{\boldsymbol{\imath}} + 6t^2\bar{\boldsymbol{\jmath}} - 8t^3\bar{\boldsymbol{k}}$,

and $\bar{\boldsymbol{f}}.\frac{dr}{dt} = (2t - t^2) + 6t^2 - 8t^3 = -8t^3 + 5t^2 + 2t$.

Therefore, the required work is q

$$\int_c \bar{\boldsymbol{f}} \,.\, dr = \int_0^1 (\bar{\boldsymbol{f}} \,.\frac{dr}{dt})\, dt = \int_0^1 (-8t^3 + 5t^2 + 2t)\, dt = \frac{2}{3}.$$

SURFACE INTEGRAL OF A VECTOR FUNCTION

Any integral which is to be evaluated over a surface is called a surface integral.

Consider a surface S in a three dimensional region. Suppose we setup an orthogonal coordinate system $(u, v) on\, S$. Let $P(x, y, z)$ be any point on S. Then x, y, z are functions are $u\ and\ v$,so that on S,

$$\bar{r} = x(u, v)\bar{\imath} + y(u, v)\bar{\jmath} + z(u, v)\bar{k} = \bar{r}(u, v), say \qquad (1)$$

This expression for $\bar{r}$ holds for any $P(x, y, z)$; therefore, this is (taken as) the vector equation of S, with u and v as parameters. As (x, y, z)vary over S; the parameter pair $(u\ , v)$ varies over region $\bar{S}$ in the $uv -$ plane.

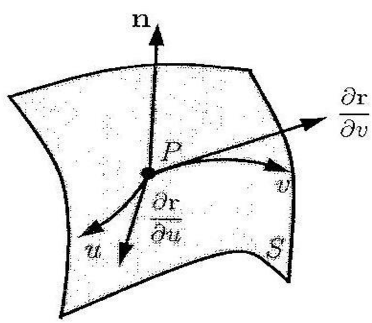

Figure 10.5

Now consider a vector function $\bar{f}$ defined over the region R. Then on S, $\bar{f}$ is a u and v. Let $\hat{n}$be the unit normal vector to S. Then it can be proved that $\hat{n}$ is along the vector $\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}$.

Suppose we the double integral of the scalar function $\bar{f} \cdot \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right)$ over the plane region $\bar{S}$. This double is called the scalar surface integral of $\bar{f}$ over S and is denoted by $\int_S \bar{f} \,.\, \hat{n}$ ds.

Thus we have by definition

$$\int_S \bar{f}\,.\hat{n}\,\mathrm{ds} = \iint_{\bar{S}} \bar{\boldsymbol{f}}\,.\left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right) du\, dv \qquad (2)$$

The term $\left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right) dudv$ present in the integral on the R.H.S of the above expression is referred to as the vectorial area element on S and is denoted by $\hat{n}$ dS. Thus by definition

$$\hat{n}\,\mathrm{ds} = \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right) dudv \qquad (3)$$

Remark: Integrals of the form $\int_S \bar{f}\,.\hat{n}\,\mathrm{ds}$ arise in many physical situations. In fluid flow problems the integral $\int_S \bar{f}\,.\hat{n}\,\mathrm{d}S$ gives the **flux** across S(= mass of fluid crossing S per unit time) when $\bar{f} = \rho\bar{v}$, where ρ is the density of the fluid and $\bar{v}$ is the velocity vector of the flow for this reason, the integral $\int_S \bar{f}\,.\hat{n}\,\mathrm{dS}$ is often referred to as the **flux integral** of the vector $\bar{f}$(across S).

CARTESIAN EXPRESSION

Suppose the Cartesian equation of S is of the form $z = f(x,y)$. Then $x\ and\ y$ serve as parameters defining$S$. Consequently, the region $\bar{S}$ on which these parameters vary is the projection of S on the xy −plane, and expression (3) yields

$$\hat{n}\,\mathrm{ds} = \left(\frac{\partial \bar{r}}{\partial x} \times \frac{\partial \bar{r}}{\partial y}\right) dxdy = \left(\bar{\imath} + \frac{\partial z}{\partial x}\bar{k}\right) \times \left(\bar{\jmath} + \frac{\partial z}{\partial y}\bar{k}\right) dxdy$$

$$= \left\{\bar{k} - \left(\frac{\partial z}{\partial x}\bar{\imath} + \frac{\partial z}{\partial y}\bar{\jmath}\right)\right\} dx \qquad (4)$$

Thus, if the equation of the surface S is of the form $z = z(x,y)$, we have

$$\int_S \bar{f}\,.\hat{n}\,\mathrm{ds} = \iint_{\bar{S}} \bar{\boldsymbol{f}}\,.\left\{\bar{k} - \left(\frac{\partial z}{\partial x}\bar{\imath} + \frac{\partial z}{\partial y}\bar{\jmath}\right)\right\} dx\, dy \qquad (5)$$

From (4), we find$\bar{k} \cdot (\hat{n}\,\mathrm{ds}) = dxdy$ (6)

We note that$\bar{k} \cdot (\hat{n}\,\mathrm{ds})$ is the projection of the vectorial area elements $(\hat{n}\,\mathrm{ds})$ on the xy −plane, and (6) shows that this projection is equal to $dxdy$, which is the area element in the xy −plane.

Similarly, we can show that the projections of $(\hat{n}\ \mathrm{ds})$ on $yz -$ and $zx -$ planes are $dydz$ and $dzdx$ respectively. That is,

$$\bar{\imath} \cdot (\hat{n}\ \mathrm{ds}) = dydz, \bar{\jmath} \cdot (\hat{n}\ \mathrm{ds}) = dzdx \qquad (7)$$

In view of (6) and (7), we obtain the following Cartesian expression for the vectorial area element.

$$(\hat{n}\ \mathrm{ds}) = (dydz)\bar{\imath} + (dzdx)\bar{\jmath} + (dxdy)\bar{k} \qquad (8)$$

9. If S denotes that the part of the plane $2x + y + 2z = 6$ which lies in the positive octant, and $\bar{\boldsymbol{f}} = 4x\bar{\mathbf{\imath}} + \mathrm{y}\bar{\mathbf{\jmath}} + \mathrm{z}\bar{\mathbf{k}}$,evalute$\int_S \bar{f} . \hat{n}\ \mathrm{ds}$.

Solution: The intercepts of the given plane on the positive *x*-, *y*- and *z*-axes are 3, 6 and 3 respectively. Therefore, in the first octant, we have $0 \le x \le 3, 0 \le y \le 6, 0 \le z \le 3$.

With (x, y) as parameters, the parametric equations of S are $x = x,\ y = y, 3 - x - \frac{1}{2}y$.

Therefore, at a point of S,

$$\bar{r} = x\bar{\imath} + y\bar{\jmath} + (3 - x - \frac{1}{2}y)\bar{k}$$

and $\frac{\partial \boldsymbol{r}}{\partial x} = \bar{\boldsymbol{\imath}} - \bar{\boldsymbol{k}}$, $\quad \frac{\partial \boldsymbol{r}}{\partial y} = \bar{\boldsymbol{\jmath}} - \frac{\mathbf{1}}{\mathbf{2}}\bar{\boldsymbol{k}}$,

so that $\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} = \bar{\boldsymbol{\imath}} + \frac{\mathbf{1}}{\mathbf{2}}\bar{\boldsymbol{\jmath}} + \bar{\boldsymbol{k}}$.

Therefore

$$\bar{\boldsymbol{f}} . \left(\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right) = 4x + \frac{1}{2}y + z = 4x + \frac{1}{2}y + \left(3 - x - \frac{1}{2}y\right) = 3(x+1).$$

According to the virtue of expression (2), we have

$$\int_S \bar{f} . \hat{n}\ \mathrm{ds.} = \iint_{\bar{S}} \bar{\boldsymbol{f}} . \left(\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right) dx\, dy = \iint_{S} 3(x+1) dx\, dy$$

Hence $\bar{S}$ is the projection of S on the xy – plane; this projection is the triangular area having vertices $O = (0,0)$, $A = (3,0)$, $B = (0,6)$. See figure . Thus

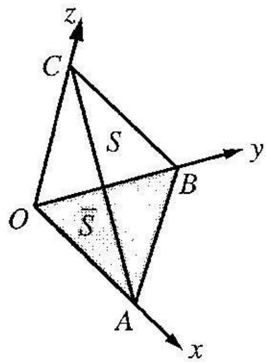

Figure 10.6

$$\int_S \bar{f}\,.\hat{n}\,\mathrm{ds} = \int_{x=0}^{3}\int_{y=0}^{6-2x} 3(x+1)\,dy\,dx = 3\int_0^3 (x+1)(6-2x)\,dx = 54.$$

10. Evaluate $\int_S \bar{f}\,.\hat{n}\,\mathrm{ds}$, where S is the part of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant, and $\bar{\boldsymbol{f}} = yz\,\bar{\boldsymbol{\imath}} + zx\bar{\boldsymbol{\jmath}} + xy\bar{\boldsymbol{k}}$.

Solution: The given surface S has the Cartesian equation

$$z^2 = a^2 - x^2 - y^2 \qquad x > 0,\ y > 0, z > 0$$

(i)

From this, we find $2z\frac{\partial z}{\partial x} = -2x$, so that $\frac{\partial z}{\partial x} = -\frac{x}{z} = -\frac{x}{\sqrt{a^2-x^2-y^2}}$

Similarly, $\frac{\partial z}{\partial y} = -\frac{x}{z} = -\frac{y}{\sqrt{a^2-x^2-y^2}}$

Therefore, if $\bar{S}$ is the projection of S on the xy – plane, we have

$$\int_S \bar{f}\,.\hat{n}\,\mathrm{ds} = \iint_{\bar{S}} \bar{\boldsymbol{f}}\,.\left\{\bar{\boldsymbol{k}} - \left(\frac{\partial \boldsymbol{z}}{\partial x}\bar{\boldsymbol{\imath}} + \frac{\partial z}{\partial y}\bar{\boldsymbol{\jmath}}\right)\right\} dx\,dy$$

$$= \iint_{\bar{S}} \bar{\boldsymbol{f}}\,.\left\{\bar{\boldsymbol{k}} + \left(\frac{\boldsymbol{x}\bar{\boldsymbol{\imath}} + \boldsymbol{y}\bar{\boldsymbol{\jmath}}}{\sqrt{a^2 - x^2 - y^2}}\right)\right\} dx\,dy$$

On S, the given vector is

$$\boldsymbol{f} = yz\bar{\boldsymbol{\imath}} + zx\bar{\boldsymbol{\jmath}} + xy\bar{\boldsymbol{k}} = \iint_{\bar{S}} \left\{\left(\sqrt{a^2 - x^2 - y^2}\right)\left(\frac{yx+xy}{\sqrt{a^2-x^2-y^2}}\right)\right\} dx\,dy$$

$$= \iint_{\bar{S}} (3xy)\,dx\,dy$$

Since S is the part of spherical surface $x^2 + y^2 + z^2 = a^2$ in the first octant, its projection $\bar{S}$ on the xy -plane is the area bounded by the circle $x^2 + y^2 = a^2$ in the first quadrant. As such, changing to polar coordinates, expression (iii) becomes

$$\int_S \bar{f}\,.\hat{n}\,\mathrm{ds} = \int_{r=0}^{a}\int_{\theta=0}^{\pi/2} 3\big((r\cos\theta)(r\sin\theta)(r dr d\theta)\big)$$

$$= 3\int_0^a r^3 dr \times \int_0^{\pi/2} sin\theta cos\theta d\theta = 3\cdot\frac{a^4}{4}\cdot\frac{1}{2} = \frac{3}{8}a^4.$$

Thus, the given surface integral is evaluated.

VOLUME INTEGRAL OF A VECTOR FUNCTION

Any integral which is to be evaluated over a volume is called a volume integral.

Consider a vector field $\bar{f}$ defined over region of volume V in three dimensional space. If $\bar{f} = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$, then f_1, f_2, f_3 are scalar functions of x, y, z over region. The vector whose $x-$, $y-$, $z-$ components of the volume integrals of f_1, f_2, f_3 respectively over V is called the vector volume integral of $\bar{f}$ over V; it is denoted by $\int_V \bar{f}dV$.

Thus, we have by definition

$$\int_V \bar{f}dV = \int_V \left(f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}\right) dV = \bar{\imath}\int_V f_1 dV + \bar{\jmath}\int_V f_2 dV + \bar{k}\int_V f_3 dV$$

11. If $\bar{f} = 2xz\bar{\imath} - x\bar{\jmath} + y^2\bar{k}$, evaluate $\int_V \bar{f}dV$, where V is the volume of the region bounded by the surfaces $x = 0, x = 2, y = 0, y = 6, z = x^2, z = 4$.

Solution: Here $f_1 = 2xz, f_2 = -x, f_3 = y^2$. Therefore,

$$\int_V \bar{f}\,dV = \left\{\int_V f_1 dV\right\}i + \left\{\int_V f_2 dV\right\}j + \left\{\int_V f_3 dV\right\}k$$

$$= \left\{2\int_V xzdV\right\}\bar{\imath} - \left\{\int_V xdV\right\}\bar{\jmath} + \left\{\int_V y^2 dV\right\}\bar{k} \qquad (I)$$

Now $\int_V xz\,dV = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} xz\,dz\,dy\,dx = \int_0^2\int_0^6 x\left(\frac{16}{2} - \frac{x^4}{2}\right)dy\,dx$

$$= 3\int_0^2 x(16 - x^4)\,dx = 64 \qquad (II)$$

$$\int_V x\,dV = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dz\,dy\,dx = \int_0^1\int_0^6 x(4-x^2)\,dy\,dx$$

$$= 6\int_0^2 x(4-x^2)\ dx = 24 \qquad (III)$$

$$\int_V y^2\,dV = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\,dz\,dy\,dx = \int_0^2\int_0^6 y^2(4-x^2)\,dy\,dx$$

$$= 72\int_0^2 (4-x^2)\ dx = 384. \qquad (IV)$$

Putting (II), (III), and (IV) into (I), we get

$$\int_V \bar{f}\,dV = 128\bar{\imath} - 24\bar{\jmath} + 384\,k.$$

12. *If* $\bar{\boldsymbol{f}} = (2x^2 - 3z)\bar{\boldsymbol{\imath}} - 2xy\bar{\boldsymbol{\jmath}} - 4\bar{\boldsymbol{k}}$,*evaluate* $\int_V div\bar{\boldsymbol{f}}dV$ *and* $\int_V curl\bar{\boldsymbol{f}}dV$ *where V is the volume of the region bounded by the surfaces x=0, y=0, z=0 and 2x+2y+z=4.*

Solution: For the given $\bar{\boldsymbol{f}}$,we find that $div\,\bar{\boldsymbol{f}} = 2$x. Therefore,

$$\int_V div\bar{\boldsymbol{f}}dV = \int_V 2x\ dV$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} x\,dz\,dy\,dx = \int_0^2\int_0^{2-x} x(4-2x-2y)\,dy\,dx$$

$$= 2\int_0^2 \{x(4-2x)(2-x) - x(2-x)^2\}dx$$

$$= 2\int_0^2 (x^3 - 4x^2 + 4x)\ dx = \frac{8}{3} \qquad (i)$$

Next, we find that, for the given **f**, we have $curl\,\bar{\boldsymbol{f}} = \bar{\boldsymbol{\jmath}} - 2y\bar{\boldsymbol{k}}$.Therefore,

$$\int_V curl\,\bar{\boldsymbol{f}}\,dV = \left\{\int_V 1\,dV\right\}\bar{\boldsymbol{\jmath}} - \mathbf{2}\left\{\int_V y\,dV\right\}\bar{\boldsymbol{k}} \qquad (\boldsymbol{ii})$$

Now $\int_V dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} dz\, dy\, dx = \int_0^2 \int_0^{2-x} (4-2x-2y)\, dy\, dx$

$$= \int_0^2 \{(4-2x)(2-x) - (2-x)^2\} dx$$

$$= \int_0^2 (x^2 - 4x + 4)\, dx = \frac{8}{3} \qquad (iii)$$

$$\int_V y\, dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y\ dz\, dy\, dx = \int_0^2 \int_0^{2-x} y(4-2x-2y)\, dy\, dx$$

$$= \frac{1}{3} \int_0^2 (2-x)^3 dx = \frac{4}{3} \qquad (iv)$$

Putting (iii) and (iv) into (i), we get

$$\int_V curl\, \bar{\boldsymbol{f}}\, dV = \frac{8}{3}(\boldsymbol{j} - \boldsymbol{k}).$$

GREEN'S THEOREM IN THE PLANE

Let $P(x,\ y)$ and $Q(x,\ y)$ be two functions defined in a region R in the xy-plane with a simple closed curve C as its boundary.

Then $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\ dy$

EXAMPLES

1. (a) If C is a simple closed curve in the xy-plane, prove by using Green's theorem that the integral $\int_C \frac{1}{2}(x\,dy - y\,dx)$ represents the area A enclosed by C.

(b) Hence find the areas enclosed by the following curves:

(i) The ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ii) The asteroid : $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Solution:

(a) According to Green's theorem

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\ dy$$

Take P = - y and Q = x in this result, we get

$$\oint_C (-y)dx + (x)dy = \iint_R 2\ dx\ dy = 2\iint_A dx\,dy = 2A$$

or $A = \int_C \frac{1}{2}(x\,dy - y\,dx)$ This proves the required result.

(b) (i) The parametric equations of the given ellipse are
$x = a\cos\theta,\ \ y = b\sin\theta, 0 \le \theta \le 2\pi$

The area bounded by this curve is

$$A = \int_C \frac{1}{2}(x\,dy - y\,dx) = \frac{1}{2}ab\int_0^{2\pi} d\theta = \pi\,ab$$

(ii) The parametric equations of the given asteroid are

$$x = a\cos^3\theta,\ \ y = a\sin^3\theta, 0 \le \theta \le 2\pi$$

Hence, the area enclosed by this curve is

$$A = \int_C \frac{1}{2}(x\,dy - y\,dx) = \frac{3}{2}a^2 \int_0^{2\pi} \left(\cos^4\theta \sin^2\theta + \sin^4\theta\cos^2\theta\right)d\theta$$

$$= \frac{3}{2}a^2 \int_0^{2\pi} \left(\cos^2\theta + \sin^2\theta\right)\cos^2\theta \sin^2\theta\ d\theta = 6a^2 \int_0^{\frac{\pi}{1}} \cos^2\theta \sin^2\theta\ d\theta$$

$$= 6a^2\,\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{8}\pi a^2$$

2. By using Green's theorem, evaluate $\int_C [(3x + 4y)dx + (2x - 3y)dy]$ where C is the circle. $x^2 + y^2 = 4$.

Solution: If A is the area enclosed by the given circle, we have, by Green's theorem

$$\int_C [(3x + 4y)dx + (2x - 3y)dy] = \iint_R \left[\frac{\partial}{\partial x}(2x - 3y) - \frac{\partial}{\partial y}(3x + 4y)\right]dx\,dy$$

$$= -2\iint_R dx\ dy = -2A = -2(4\pi) = -8\pi$$

3. By using Green's theorem, evaluate $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$ where C is the squire formed by the lines $x = \pm 1\ \ y = \pm 1$.

Solution: Here the region bounded by C is the square region in which both x and y increase from -1 to +1, therefore by taking $P = x^2 + xy\ \ and Q = x^2 + y^2$ in the Green's theorem, we get

$$\int_C \left[\left(x^2+xy\right)dx+\left(x^2+y^2\right)dy\right]$$

$$=\iint_R \left[\frac{\partial}{\partial x}\left(x^2+y^2\right)-\frac{\partial}{\partial y}\left(x^2+xy\right)\right]dx\,dy$$

$$=\iint_R x\ dx\ dy=\int_{-1}^{1}\int_{-1}^{1} x\ dx\ dy=0$$

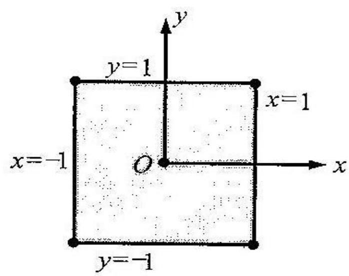

Figure 10.7

4. By using Green's theorem, evaluate $\int_C \left[(y-\sin x)dx+(\cos x)dy\right]$ where C is the triangle in the xy-plane bounded by the lines $y=0\ x=\frac{\pi}{2}\ and\ y=\frac{2x}{\pi}$.

Solution: By using Green's theorem, we get

$$\int_C \left[(y-\sin x)dx+(\cos x)dy\right]=\iint_R \left[\frac{\partial}{\partial x}(\cos x)-\frac{\partial}{\partial y}(y-\sin x)\right]dx\ dy$$

$$=\iint_R (1+\sin x)\ dx\ dy=-\int_{x=0}^{\frac{\pi}{2}}\int_{y=0}^{\frac{2x}{\pi}}(1+\sin x)dy\ dx=-\left(\frac{\pi}{4}+\frac{2}{\pi}\right)$$

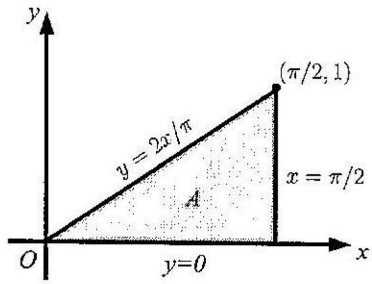

Figure 10.8

5. By using Green's theorem evaluate $\int_C \left[(2x^2 - y^2)dx + (x^2 + y^2)dy\right]$ where C is the boundary of the region in the xy-plane enclosed by the x axis and the upper half of the circle $x^2 + y^2 = a^2$.

Solution: By using Green's theorem, we get

$$\int_C \left[(2x^2 - y^2)dx + (x^2 + y^2)dy\right] == \iint_R \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(2x^2 - y^2)\right] dx\, dy$$

$$= 2\iint_R (x + y) dx\, dy \qquad (1)$$

Where R is the region shown in figure above, in this region r various from 0 to a and θvarious from 0 to π, where (r, θ) are the plane polar coordinates. Also $dx\, dy = r\, dr\, d\theta$.

$$\therefore \iint_R (x + y) dx\, dy = \int_{\theta=0}^{\pi} \int_{r=0}^{a} r(\cos\theta + \sin\theta) r\, dr\, d\theta = \frac{2}{3} a^3 .$$

Putting this into equation (1), we get

$$\int_C \left[(2x^2 - y^2)dx + (x^2 + y^2)dy\right] = \frac{4}{3} a^3$$

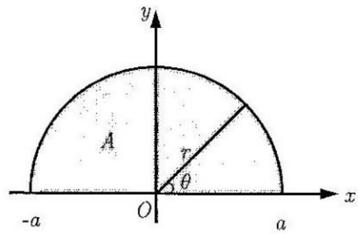

Figure 10.9

6. Using the Green's theorem, find the area enclosed between the parabolas $y^2 = 4ax \text{ and } x^2 = 4ay$

Solution: The region between the given the parabolas is shown in figure below.

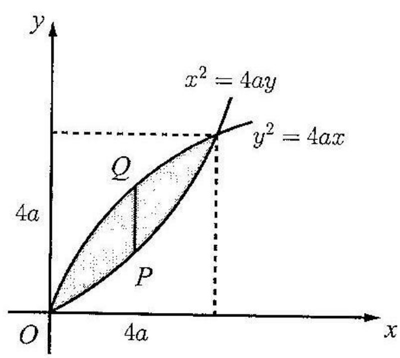

Figure 10.10

Let us denote the parabola $x^2 = 4ay \text{ by } C_1$ and the parabola $y^2 = 4ax \text{ by } C_2$. Then the boundary C of the region is made up of $C_1 \text{ and } C_2$.

On C_1 x increases from 0 to 4a, and $y = \frac{x^2}{4a}\left(\text{so that } dy = \frac{x}{2a}dx\right)$

On C_2 y increases from 4a to 0, and $x = \frac{y^2}{4a}\left(\text{so that } dx = \frac{y}{2a}dy\right)$

Now, by virtue of the green's theorem, we find that the required area is given by

$$A=\int_C \frac{1}{2}(x\,dy-y\,dx)=\frac{1}{2}\int_{C_1}(x\ dy-y\ dx)+\int_{C_2}(x\ dy-y\ dx)$$

$$=\frac{1}{2}\int_0^{4a}\left(x\frac{x}{2a}dx-\frac{x^2}{4a}dx\right)+\int_{4a}^{0}\left(\frac{y^2}{4a}dy-y\frac{y}{2a}\ dy\right)$$

$$=\frac{1}{8a}\int_0^{4a} x^2dx+\int_0^{4a}y^2\ dy=\frac{1}{8a}\left[\left[\frac{x3}{3}\right]_0^{4a}+\left[\frac{y^3}{3}\right]_0^{4a}\right]$$

$$=\frac{1}{8a}\left[\frac{(4a)^3}{3}+\frac{(4a)^3}{3}\right]=\frac{16}{3}a^3$$

7. Using Green theorem, evaluate $\int_C (xy^2dy-x^2y\,dx)$ where C is the cardioids, $r=a(1-\cos\theta)$.

Solution:

By using Green's theorem we find

$$\int_C (xy^2dy-x^2y\,dx)=\int_C(-x^2y\ dx+xy^2\ dy)=\iint_A\left[\frac{\partial}{\partial x}(xy^2)-\frac{\partial}{\partial y}(-x^2y)\right]dx\ dy$$

$=\iint_A \left[y^2+x^2\right]dx\ dy$ where A is the area bounded by the given cardioids

$=\iint_A r^2[r\ dr\ d\theta]$ on changing over to polar coordinates.

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{a(1-\cos\theta)} r^3\ dr\ d\theta=\int_{\theta=0}^{2\pi}\left[\left[\frac{r^4}{4}\right]_0^{a(1-\cos\theta)}\right]d\theta$$

$$=\frac{a^4}{4}\int_{\theta=0}^{2\pi}[1-\cos\theta]^4\ d\theta=\frac{a^4}{4}\int_{\theta=0}^{2\pi}\left[2\sin^2\left(\frac{\theta}{2}\right)\right]^4 d\theta=4a^4\int_{\theta=0}^{2\pi}\sin^8\left(\frac{\theta}{2}\right)d\theta$$

$$=16a^4\int_{\theta=0}^{\frac{\pi}{2}}\sin^8 t\ dt\quad where\ \ t=\frac{\theta}{2}$$

$$= 16a^4 \frac{7.5.3.1}{8.6.4.2} \cdot \frac{\pi}{2} = \frac{35}{16} \pi \ a^4$$

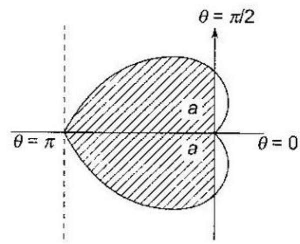

Figure 10.11

8. Verify Green's theorem for $\int_C \left[\left(3x^2 - 8y^2\right)dx + \left(4y - 6xy\right)dy\right]$

where C is the boundary of the region enclosed by the line x = 0, y =0, and x+y = 1

Solution: The given region is shown in figure below

We note that the boundary curve C is made up of three parts:

(i) The line OP on which y = 0 and x increases from 0 to 1

(ii) The line PQ on which y = 1-x, and x varies from 1 to 0 and

(iii) The line QO on which x = 0 and y varies from 1 to 0

Therefore taking $P = 3x^2 - 8y^2$ *and* $Q = 4y - 6xy,$ we find the given integral is

$$\int_C (Pdx + Qdy) = \int_{OP} (Pdx + Qdy) + \int_{PQ} (Pdx + Qdy) + \int_{QO} (Pdx + Qdy)$$

$$= \int_0^1 3x^2 dx + \int_1^0 \left[\left(3x^2 - 8(1-x)^2\right)dx + \left[4(1-x) - 6x(1-x)\right](-dx)\right] + \int_1^0 4y \ dy = \frac{5}{3}$$

(1)

on evaluating the integrals.

On the other hand, we find that

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \iint_A 10y\, dx\, dy = 10 \int_{x=0}^{1} \int_{y=0}^{1-x} y \ dy\, dx = 10 \int_{x=0}^{1} \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= 5\int_{x=0}^{1} (1-x)^2\, dx = \frac{5}{3}$$

(2)

Expression (1) and (2) show that

$$\int_C (Pdx + Qdy) = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

Thus, the Green's theorem is verified for the given integral.

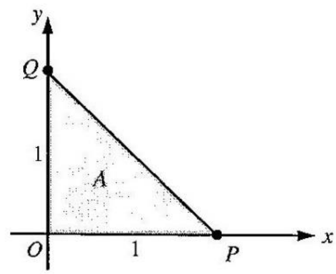

Figure 10.12

9. Verify Green's theorem for $\int_C \left[(xy + y^2)dx + x^2 dy\right]$ where C is the closed curve made up of the line y = x and the parabola, $y = x^2$.

Solution: The two parts C_1 and C_2 of the given curve C and the region bounded by the C are shown in figure below.

We note that along C_1: $y = x^2$ and x varies from 0 to 1 and

along C_2: y = x and x varies from 1 to 0.

Therefore taking $P = xy + y^2$ *and* $Q = x^2$, we find that given integral is

$$\int_C (Pdx + Qdy) = \int_{C_1} (Pdx + Qdy) + \int_{C_2} (Pdx + Qdy)$$

$$= \int_{C_1} (Pdx + Qdy) = \int_0^1 \left[\left(x(x^2) + (x^2)^2\right)dx + x^2\left(d(x^2)\right)\right]$$, because y = x^2 on C_1

$$= \int_0^1 \left[\left(x^3 + x^4\right)dx + x^2(2x\,dx)\right] = \int_0^1 \left[\left(3x^3 + x^4\right)dx\right] = \frac{19}{20} \qquad (1)$$

And

$$= \int_{C_2} (Pdx + Qdy) = \int_1^0 \left[\left(\left(x^2 + x^2\right)dx + \left(x^2\right)dx\right)\right]$$, because y = x on C_2

$$= \int_1^0 \left[3x^2 dx\right] = -1 \qquad (2)$$

Adding (1) and (2) we get the integral as

$$\int_C (Pdx + Qdy) = \frac{19}{20} - \frac{1}{20} = \frac{-1}{20} \qquad (3)$$

We find that

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_A (x - 2y)\,dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[-x^2 - x^3\right] dx = \frac{-1}{20} \qquad (4)$$

From (3) and (4), we note that

$$\int_C (Pdx + Qdy) = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Thus, the Green's theorem is verified for the given integral.

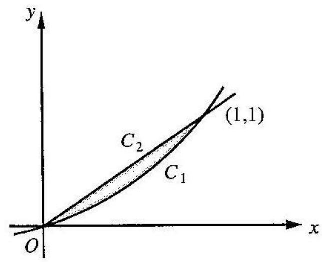

Figure 10.13

10. Verify Green's theorem for $\int_C [(e^{-x}\sin y)dx + (e^{-x}\cos y)dy]$ where C is the rectangle whose vertices are (0, 0), (π, 0), (π, π/2), (0, π/2) .

Solution: Here the given rectangular boundary C is made up of the four lines OA, AB, BC, CO shown in figure below.

Therefore taking $P = e^{-x}\sin y \;\; and \;\; Q = e^{-x}\cos y,$ we find that given integral is $\int_C (Pdx + Qdy) = \int_{OA} (Pdx + Qdy) + \int_{AB} (Pdx + Qdy) + \int_{BC} (Pdx + Qdy) + \int_{CO} (Pdx + Qdy)$

Along OA: y =0, dy = 0, x varies from 0 to π

Along AB: x =π, dx = 0, y varies from 0 to π/2

Along BC: y =π/2, dy = 0, x varies from π to 0

Along CO: x =0, dx = 0, y varies from π/2 to 0

$$\int_{OA} (Pdx + Qdy) = 0 \qquad (1)$$

$$\int_{AB} (Pdx + Qdy) = \int_0^{\frac{\pi}{2}} e^{-\pi}\cos y \; dy = e^{-\pi} \qquad (2)$$

$$\int_{BC} (Pdx + Qdy) = \int_{\pi}^{0} e^{-x}dx = e^{-\pi} - 1 \qquad (3)$$

$$\int_{CO} (Pdx + Qdy) = \int_{\frac{\pi}{2}}^{0} \cos y \, dy = -1 \qquad (4)$$

Adding equations (1), (2), (3) and (4)we get the given integral

$$\int_C (Pdx + Qdy) = 0 + e^{-\pi} + (e^{-\pi} - 1) + (-1) = 2(e^{-\pi} - 1) \quad (5)$$

We find that

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R \left(-2e^{-x}\cos y\right) dx\,dy = -2\int_{x=0}^{\pi}\int_{y=0}^{\frac{\pi}{2}} \left(e^{-x}\cos y\right) dy\,dx$$

$$= 2(e^{-\pi} - 1) \quad (6)$$

From (5) and (6), we note that

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Thus, the Green's theorem is verified for the given integral.

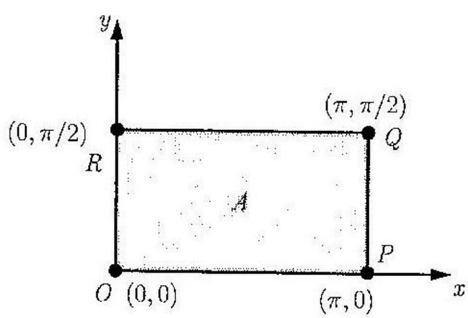

Figure 10.14

STOKE'S THEOREM

If 'S' be an open surface bounded by a closed curve C and $\overline{F} = F_1\overline{i} + F_2\overline{j} + F_3\overline{k}$ be vector point function having continuous first order partial derivatives, then $\int_C F \cdot d\overline{r} = \iint_S curl\ \overline{F} \cdot \overline{n}\ ds$ where $\overline{n}$ is a unit normal vector at any point of S drawn in the sense in which a right in the sense of description of C.

EXAMPLES

1. Using Stoke's theorem, evaluate $\iint_S curl\ \overline{f} \cdot \overline{n}\ ds$ for $\overline{f} = (y - z + 2)\overline{i} + (yz + 4)\overline{j} - xz\overline{k}$ where S is the cubical surface formed by the planes x=0, y=0, x =2, y = 2 and z= 2.

Solution: The rim C of the given surface is the square OPQR in the xy-plane, where O(0, 0), P(2, 0), Q(2, 2), R(0, 2) we note that z=0 on the whole of C, x = constant on PQ and RO, and y=constant on OP and QR.

$\therefore$ By using Stoke's theorem, we get

$$\iint_S curl\ \overline{f} \cdot \overline{n}\ ds = \int_C \overline{f} \cdot d\,\overline{r}$$

$$= \int_{OP} f_1 dx + \int_{PQ} f_2 dy + \int_{QR} f_1 dx + \int_{RO} f_2 dy$$

$$= \int_{OP} (y - z + 2)dx + \int_{PQ} (yz + 4)dy + \int_{QR} (y - z + 2)dx + \int_{RO} (yz + 4)dy$$

$$= \int_0^2 2\,dx + \int_0^2 4\,dy + \int_2^0 4\,dx + \int_2^0 4\,dy = -4$$

2. Evaluate $\int_C \overline{f} \cdot \overline{dr}$ by Stoke's theorem, where $\overline{f} = y^2 \overline{i} + x^2 \overline{j} - (x+z)\overline{k}$ and C is the boundary of the triangle with vertices at (0,0, 0), (1, 0, 0) and (1, 1, 0).

Solution: Since z- coordinates of each vertex of the triangle is zero, therefore, the triangle lies in the xy-plane and $\overline{n} \cdot \overline{k}$

$$curl\ \overline{f} = \overline{j} + 2(x-y)\overline{k}$$

$$curl\ \overline{f} \cdot \overline{n} = \left(\overline{j} + 2(x-y)\overline{k}\right) \cdot \overline{k} = 2(x-y)$$

The equation of the line OB is y=x

$\therefore$ By using Stoke's theorem, we get

$$\int_C \overline{f} \cdot d\,\overline{r} = \iint_S curl\ \overline{f} \cdot \overline{n}\ ds$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} 2(x-y)dy\,dx = 2\int_{x=0}^{1} \left[xy - \frac{y^2}{2}\right]_0^x dx$$

$$= 2\int_{x=0}^{1} \left[x^2 - \frac{x^2}{2}\right] dx = \int_{x=0}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

3. Using Stoke's theorem, evaluate $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$ where C is the boundary of the triangle with vertices at P(1, 0, 0), Q(0, 2, 0) and R(0, 0, 3).

Solution: We have

$$\int_C [(x+y)dx + (2x-z)dy + (y+z)dz] = \int_C \left[(x+y)\overline{i} + (2x-z)\overline{j} + (y+z)\overline{k}\right] \cdot \mathrm{d}\overline{\mathrm{r}}$$

$$= \int_C \overline{\mathrm{f}} \cdot \mathrm{d}\overline{\mathrm{r}} = \iint_S curl\ \overline{f} \cdot \overline{n}\,ds$$ by Stoke's theorem.

Where $\overline{f} = (x+y)\overline{i} + (2x-z)\overline{j} + (y+z)\overline{k}$ and S is any surface having C as its rim.

We may take the plane bounded by the given triangle PQR itself as S. The equation of this plane is $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1$, so that the direction ratios of its normal are (1, ½, 1/3).

Therefore $\bar{n}=\dfrac{\bar{i}+\left(\frac{1}{2}\right)\bar{j}+\left(\frac{1}{3}\right)\bar{k}}{\sqrt{1+\frac{1}{4}+\frac{1}{9}}}=\frac{1}{7}\left(6\bar{i}+3\bar{j}+2\bar{k}\right)$

Hence $Curl\,\bar{f}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+y & 2x-z & y+z\end{vmatrix}=2\bar{i}+\bar{k}$

Therefore $curl\,\bar{f}\cdot\bar{n}=\left(2\bar{i}+\bar{k}\right)\cdot\frac{1}{7}\left(6\bar{i}+3\bar{j}+2\bar{k}\right)=\frac{1}{7}(12+2)=2$

Hence $\iint_S curl\,\bar{f}\cdot\bar{n}\,ds=\iint_S 2ds=2A$ where A is the area of the triangle PQR.

We note that the area of the triangle PQR, $A=\frac{1}{2}\left|\overline{PQ}\times\overline{PR}\right|=\frac{7}{2}$

Therefore $\int_C\left[(x+y)dx+(2x-z)dy+(y+z)dz\right]=2\left(\frac{7}{2}\right)=7$

4. Verify Stoke's theorem for $\bar{f}=y\bar{i}+z\bar{j}+x\bar{k}$ for the upper part of the sphere $x^2+y^2+z^2=a^2$

Solution: The rim C of the given surface is the circle $x^2+y^2=a^2$ in the xy-plane. Therefore the parametric equation C are $x=a\cos t,\ \ y=a\sin t,\ \ z=0;\ 0\le t\le 2\pi$

Hence, $\int_C \bar{\mathrm{f}}\cdot d\bar{\mathrm{r}}=\iint_C y\,dx$ because z = 0 on C.

$$\int_0^{2\pi}(a\sin t - a\cos t)\,dt = -4a^2\int_0^{\frac{\pi}{2}}\sin^2 t\ dt = -\pi a^2 \qquad (1)$$

The given surface, S for which C is the rim is the upper part of the sphere $x^2 + y^2 + z^2 = a^2$

Therefore on S, $z^2 = a^2 - x^2 - y^2 \ \ z > 0$ (2)

From this we find, $2\frac{\partial z}{\partial x} = -x$, so that $\frac{\partial z}{\partial x} = \frac{-x}{z} = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}$

Similarly, $\frac{\partial z}{\partial y} = \frac{-y}{z} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$,

$$\iint_S curl\,\bar{f}\cdot\bar{n}\,ds = \iint_{\bar{S}} curl\,\bar{f}\cdot\left[\bar{k} + \left(\frac{x\bar{i} + y\bar{j}}{\sqrt{a^2 - x^2 - y^2}}\right)dx\,dy\right] \qquad (3)$$

Here $\bar{S}$ is the projection of s on the xy-plane which is the area bounded by the circle $x^2 + y^2 = a^2$.

For the given *f*, we find that

$$curl\ f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(i + j + k)$$

Using this in the r.h.s of (3), we get

$$\int_{\bar{S}} (curl\ f)\cdot n\,ds = -\iint_{\bar{S}}\left[\left(1 + \frac{x + y}{\sqrt{a^2 - x^2 - y^2}}\right)dx\,dy\right] \qquad (4)$$

Changing to polar coordinates (r, θ) and noting that, in the circular area $\bar{S}$, *r* varies from 0 to *a* and θ varies from 0 to 2π, expression (4) reads

$$\int_S (curl\ f)\cdot n\,ds = -\int_{r=0}^{a}\int_{\theta=0}^{2\pi}\left(1 + \frac{r(\cos\theta + \sin\theta)}{\sqrt{a^2 - r^2}}\right) r\ dr\ d\theta$$

$$= -\left[\int_{r=0}^{a} \int_{\theta=0}^{2\pi} r\, dr\, d\theta + \left(\int_0^a \frac{r^2}{\sqrt{a^2 - r^2}} dr \right) \times \left(\int_0^{2\pi} (\cos\theta + \sin\theta) d\theta \right) \right]$$

$$= -\left[2\pi \frac{a^2}{2} + 0 \right] = -\pi a^2 \qquad (5)$$

From (1) and (5), we note that $\int_C \bar{f} \cdot d\bar{r} = \int_S curl\ \bar{f} \cdot \bar{n}\, ds$

Thus, stokes' theorem is verified in the given case.

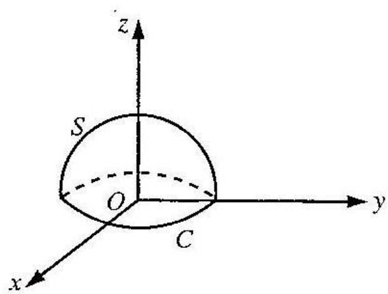

Figure 10.15

5. Verify Stokes theorem for the vector field $\bar{f} = (2x - y)\bar{i} - (yz^2)\bar{j} - (y^2 z)\bar{k}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy-plane.

Solution: Let S be the upper half surface of $x^2 + y^2 + z^2 = 1$. The boundary C of S is a circle in the xy-plane of radius unity and centre O (or origin).

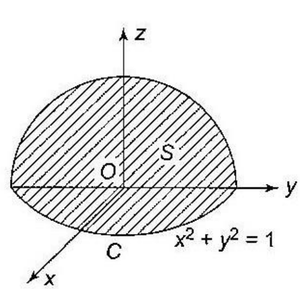

Figure 10.16

The equation of C are $x^2 + y^2 = 1, \ z = 0$.

Therefore the parametric equations of C are $x = \cos t, \ y = a\sin t, \ z = 0; \ 0 \le t \le 2\pi$

Hence, $\int_C \overline{\mathrm{f}} \cdot d\overline{\mathrm{r}} = \iint_C \left[(2x - y)dx - yz^2 dy - y^2 z\, dz\right]$ because z = 0 on C.

$$\int_0^{2\pi} (2\cos t - \sin t)\frac{dx}{dt}\, dt = \int_0^{2\pi} (2\cos t - \sin t)(\sin t)dt = \int_0^{2\pi} \left(-2\sin t \cos t + \sin^2 t\right)dt$$

$$= \int_0^{2\pi} \left(-\sin 2t + \frac{1}{2}(1 - \cos 2t)\right)dt = \left(\frac{1}{2}\cos 2t + \frac{1}{2}\left(t - \frac{1}{2}\sin 2t\right)\right)_0^{2\pi} = \pi \qquad (1)$$

Also $\quad Curl\ \overline{f} = \overline{k}$

$$Curl\ \overline{f} \cdot \overline{n} = \overline{k} \cdot \overline{n}$$

$$\iint_S curl\ \overline{f} \cdot \overline{n}\, ds = \iint_S \overline{n} \cdot \overline{k}\ ds = \iint_R \left(\overline{n} \cdot \overline{k}\right)\frac{\mathrm{dx\, dy}}{\left|\overline{n} \cdot \overline{k}\right|}$$

where R is the projection of x on xy-plane.

$$= \iint_R dx\, dy = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx = 2\int_{-1}^{1} \sqrt{1 - x^2}\, dx = 4\int_0^1 \sqrt{1 - x^2}\, dx$$

$$= 4\left[\frac{x}{2}\sqrt{1 - x^2}\, dx + \frac{1}{2}\sin^{-1} x\right]_0^1 = 4\left[\frac{1}{2}\ \frac{\pi}{2}\right] = \pi$$

(2)

From (1) and (2) we get

$$\int_C \overline{\mathrm{f}} \cdot d\overline{\mathrm{r}} = \iint_S curl\ \overline{f} \cdot \overline{n}\, ds$$

Therefore Stokes theorem verified.

6. If C is the circle of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane x+z=a, prove that $\int_C y\,dx + 2dy + x dz = \frac{-\pi}{\sqrt{2}} a^2$

Solution: We note that

$$\int_C y\,dx + 2dy + xdz = \int_C \left(y\bar{i} + z\bar{j} + x\bar{k}\right)\cdot d\bar{r} \;=\; \iint_S curl\,\bar{f}\cdot\bar{n}\,ds$$

where $\bar{f} = y\bar{i} + z\bar{j} + x\bar{k}$ (1)

by stokes theorem. Here S is any, surface for which C is the rim. We can take the portion of the plane x+z=a bounded by C itself as S. for this plane, the direction ratios of the normal are (1, 0, 1). Therefore,

$$\bar{n} = \frac{1}{\sqrt{2}}\left(\bar{i} + \bar{k}\right) \quad (2)$$

Also, $curl\,\bar{f} = -\left(\bar{i} + \bar{j} + \bar{k}\right)$ (3)

Putting (2) and (3) into (1), we get

$$\int_C \bar{f}\cdot d\bar{r} = -\iint_S \left(\bar{i} + \bar{j} + \bar{k}\right)\cdot\frac{1}{\sqrt{2}}\left(\bar{i} + \bar{k}\right)ds = -\sqrt{2}\iint_S ds = -\sqrt{2}A \quad (4)$$

Where A is the area of the plane x+z=a bounded by C.

Since C is the circle of inter section of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane x+z=a the radius of C is $R = \sqrt{a^2 - p^2}$, where p is the length of the perpendicular from the centre of the sphere onto the plane. Since the origin is the centre of the sphere, we note that $p = \frac{a}{\sqrt{2}}$.

(The length of the perpendicular from the origin onto the plane $ax+by+cz+d = 0$ is $\frac{|d|}{\sqrt{\Sigma a^2}}$)

$\therefore\ R = \left(a^2 - \frac{a^2}{2}\right)^{\frac{1}{2}} = \frac{a}{\sqrt{2}}$ consequently, the area bounded by C is

$A = \pi R^2 = \pi\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{\pi a^2}{2}$. Putting this into (4), we get

$$\int_C \overline{f} \cdot d\,\overline{r} = -\sqrt{2}\,\frac{\pi a^2}{2} = \frac{-\pi a^2}{\sqrt{2}} .$$

DIVERGENCE THEOREM

Let 'S' be the enclosed boundary surface of a region of volume V. Then, for a vector field $\overline{f}$ defined in V and on S, $\int_S \overline{f} \cdot \overline{n}\ ds = \int_V div\ \overline{f}\ dV$. (1)

Here $\overline{n}$ is the unit outward normal to S.

Note: If we take $\overline{f} = f_1\overline{i} + f_2\overline{j} + f_3\overline{k}$ and use expression

$\overline{n}\ ds = (dy\ dz)\overline{i} + (dz\ dx)\overline{j} + (dx\ dy)\overline{k}$, we get

$\overline{f} \cdot \overline{n}\ ds = f_1 dy\ dz + f_2 dz\ dx + f_3 dy\ dx$ then equation (1) stated above as follows:

$$\iint_S \left(f_1 dy\,dz + f_2 dz\,dx + f_3 dx\,dy\right) = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\ dy\ dz \qquad (2)$$

This is the Cartesian form of the divergence theorem.

Remark: Whereas Stoke's theorem converts a surface integral taken on an open surface into the line integral over its boundary curve (rim), the divergence theorem converts a surface integral on a closed surface into the volume integral over the region enclosed by the surface.

EXAMPLES

1. For any closed surface S, prove that $\iint_S curl\ \overline{f} \cdot \overline{n}\ ds = 0$.

Solution: By the divergence theorem, we have

$$\iint_S curl\ \bar{f}\cdot\bar{n}\ ds = \iiint_V \left(div\ curl\ \bar{f}\right)dv = 0 \qquad \left(\sin ce\ \ div\ curl\ \bar{f}\right) = 0$$

Where V is the volume enclosed by S.

2. Evaluate $\iint_S curl\ \bar{r}\cdot\bar{n}\ ds$ where S is a closed surface.

Solution: By the divergence theorem, we have

$\iint_S curl\ \bar{r}\cdot\bar{n}\ ds = \iiint_V \left(div\ \ \bar{r}\right)dv$, where V is the volume enclosed by S.

$$= \iiint_V 3\ \ dv = 3V \qquad \left(div\ \bar{r} = 3\right)$$

3. Use divergence theorem to show that $\int_S \nabla r^2 \cdot d\bar{s} = 6v$, where S is any closed surface enclosing a volume V.

Solution: By the divergence theorem, we have

$$\iint_S \nabla\left(r^2\right) d\bar{s} = \iiint_V \left(div\ \ \bar{r}\right)dv$$

$$= \iiint_V \left(\frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y} + \frac{\partial^2}{\partial z}\right) r^2\ dV = \iiint_V 6\ dV = 6V$$

where V is the volume enclosed by S.

4. Evaluate $\iint_S \bar{f}\cdot\bar{n}\ ds$, where $\bar{f} = (2x+3z)\bar{i} - (xz+y)\bar{j} + \left(y^2+2z\right)\bar{k}$ where S is the surface of the sphere having centre at (3, -1, 2) and radius 3.

Solution: Let $\bar{f} = (2x+3z)\bar{i} - (xz+y)\bar{j} + \left(y^2+2z\right)\bar{k}$, we have $div\ \bar{f} = 3$

Let V be the volume enclosed by the surface S. Then by the divergence theorem, we have

$$\iint_S \overline{f}\cdot\overline{n}\,ds = \iiint_V \left(div\,\overline{f}\right)dv = 3\iiint_V dV = 3V$$

But V is the volume of a sphere of a radius 3.

$$\therefore \frac{4}{3}\pi(3)^3 = 36\pi$$

Hence $\iint_S \overline{f}\cdot\overline{n}\,ds = 3\times 36\pi = 108\pi$

5. Evaluate $\iint_S \overline{f}\cdot\overline{n}\,ds$, where $\overline{f} = x\overline{i} + y\overline{j} + z\overline{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution: Let $\overline{f} = x\overline{i} + y\overline{j} + z\overline{k}$ we have $div\,\overline{f} = 3$

Let V be the volume enclosed by the surface S. Then by the divergence theorem, we have $\iint_S \overline{f}\cdot\overline{n}\,ds = \iiint_V \left(div\,\overline{f}\right)dv = 3\iiint_V dV = 3V$

But V is the volume of a given sphere $= \frac{4}{3}\pi(a)^3 = \frac{4}{3}\pi a^3$

Hence $\iint_S \overline{f}\cdot\overline{n}\,ds = 3\times\frac{4}{3}\pi a^3 = 4\pi a^3$

6. If S is a closed circuit enclosing a volume V and $\overline{f} = ax\overline{i} + by\overline{j} + cz\overline{k}$, where a, b, c are constants. Prove that $\iint_S \overline{f}\cdot\overline{n}\,ds = (a+b+c)V$.

Solution: Let $\overline{f} = ax\overline{i} + by\overline{j} + cz\overline{k}$ we have $div\,\overline{f} = a+b+c$

Let V be the volume enclosed by the surface S. Then by the divergence theorem, we have

$$\iint_S \overline{f}\cdot\overline{n}\,ds = \iiint_V \left(div\,\overline{f}\right)dv = (a+b+c)\iiint_V dV = (a+b+c)V$$

7. Using the divergence theorem, evaluate $\iint_S \overline{f}\cdot\overline{n}\,ds$, where $\overline{f} = 4xz\overline{i} - y^2\overline{j} + yz\overline{k}$ and S is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

Solution: Let $\overline{f} = 4xz\overline{i} - y^2\overline{j} + yz\overline{k}$ we have $div\,\overline{f} = 4z - y$.

Now, the divergence theorem yields

$$\iint_S \overline{f}\cdot\overline{n}\,ds = \iiint_V \left(div\,\overline{f}\right)dv = \iiint_V (4z-y)dV$$

Where V is the volume of a given cube

Hence $\iint_S \overline{f}\cdot\overline{n}\,ds = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z-y)dz\,dy\,dx$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{1}\left[2z^2 - yz\right]_0^1 dy\right]dx == \int_{x=0}^{1}\left[\int_{y=0}^{1}\left[2-y\right]_0^1 dy\right]dx$$

$$= \int_{x=0}^{1}\left[\left[2y - \frac{y^2}{2}\right]_0^1\right]dx = \int_{x=0}^{1}\left[\left[2-\frac{1}{2}\right]\right]dx = \int_{x=0}^{1}\left[\frac{3}{2}\right]dx = \frac{3}{2}$$

8. If S is the sphere $x^2 + y^2 + z^2 = a^2$, prove that

$$\iint_S \left(x^3 dydz + y^3 dz\,dx + z^3 dx\,dy\right) = \frac{12}{5}\pi a^5$$

Solution: We recall that $\overline{f} = f_1\overline{i} + f_2\overline{j} + f_3\overline{k}$

$$\iint_S \overline{f}\cdot\overline{n}\,ds = \iint_S \left(f_1 dy\ dz + f_2 dz\ dx + f_3 dx\ dy\right) \qquad (1)$$

According we can write

$$\iint_S \overline{f}\cdot\overline{n}\,ds = \iint_S \left(x^3 dy\ dz + y^3 dz\ dx + z^3 dx\ dy\right) \qquad (2)$$

By taking $f_1 = x^3, \ f_2 = y^3, \ f_3 = z^3$ or equivalently
$\overline{f} = x^3\overline{i} + y^3\overline{j} + z^3\overline{k}$ (3)

Now if V is the volume of the given sphere, the divergence theorem yields

$$\iint_S \overline{f}\cdot\overline{n}\ ds = \iiint_V div\ \overline{f}\ dv = 3\iiint_V \left(x^2+y^2+z^2\right)dV \qquad (4)$$

Over the given sphere we have $0 \le r \le a = 1$,
$0 \le \theta \le \pi \ \ and \ \ 0 \le \phi \le 2\pi$

Where (r, θ , ϕ) are spherical polar coordinates. Also $dV = r^2 \sin\theta\, dr\ d\theta\ d\phi$

$\therefore$ Equation (4) becomes

$\iint_S \overline{f}\cdot\overline{n}\ ds = 3\int_{r=0}^{s}\int_{\theta=0}^{\pi}\int_{\pi=0}^{2\pi} r^2\left(r^2 \sin\theta\, dr\, d\theta\ d\phi\right) = \frac{12}{5}\pi a^5$ using this in expression (2) we get the required results.

9. If S is the sphere $x^2 + y^2 + z^2 = k^2$, prove that

$$\iint_S \left(ax^2 + by^2 + cz^2\right)ds = \frac{4\pi}{3}(a+b+c)k^4$$

Solution: Here the given surface S is $\phi(x,\ y,\ z) = \left(x^2+y^2+z^2\right)k^2 = 0$, so that the unit outward normal to this surface is

$$\overline{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\left(x\overline{i} + y\overline{j} + z\overline{k}\right)}{2\sqrt{x^2+y^2+z^2}} = \frac{x\overline{i} + y\overline{j} + z\overline{k}}{k} \qquad (1)$$

Here we consider vector $\overline{f} = f_1\overline{i} + f_2\overline{j} + f_3\overline{k}$ then we have

$$\overline{f}\cdot\overline{n} = \frac{1}{k}\left(x\ f_1 + y\ f_2 + z\ f_3\right)$$

According we can write

$$\iint_S \overline{f}\cdot\overline{n}\ ds = \iint_S \left(ax^2 + by^2 + cz^2\right)ds \qquad (2)$$

By taking $f_1 = kax, \quad f_2 = kby, \quad f_3 = kcz$ or equivalently

$$\overline{f} = k\left(ax\overline{i} + by\overline{j} + cz\overline{k}\right) \qquad (3)$$

Thus, with $\overline{f}$ given by equation (3), we find that, $div\,\overline{f} = k(a+b+c)$

Now the divergence theorem yields

$$\iint_S \overline{f}\cdot\overline{n}\,ds = \iiint_V div\,\overline{f}\,dv = k\iiint_V (a+b+c)dV = k(a+b+c)V$$

$$= k(a+b+c)\frac{4}{3}\pi k^3 = \frac{4\pi}{3}(a+b+c)k^4$$

(Since volume of the sphere, $V = \frac{4}{3}\pi k^3$)

10. Evaluate $\int_S \left(yz\overline{i} + zx\overline{j} + xy\overline{k}\right)\cdot ds$ where S is the surface of the sphere, prove that $x^2 + y^2 + z^2 = a^2$ in the first octant.

Solution: The surface of the region OABC is piece wise smooth and is comprised of four surfaces

(i) S_1 – circular quadrant OBC in the yz-plane
(ii) S_2 – circular quadrant OCA in the zx-plane
(iii) S_3 – circular quadrant OAB in the xy-plane
(iv) S – surface ABC of the sphere in the first octant.

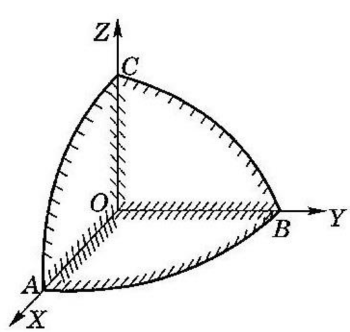

Figure 10.17

Also in the yz-plane $\overline{f} = yz\overline{i} + zx\overline{j} + xy\overline{k}$ by divergence theorem

$$\int_V div\,\overline{f}\,dv = \int_{S_1} \overline{f}\cdot d\overline{s} + \int_{S_2} \overline{f}\cdot d\overline{s} + \int_{S_3} \overline{f}\cdot d\overline{s} + \int_{S} \overline{f}\cdot d\overline{s} \qquad (1)$$

Now $div\ \overline{f} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0$ for the surface S_1, x = 0

$$\therefore \int_{S_1} \overline{f}\cdot d\overline{s} = \int_0^a \int_0^{\sqrt{a^2-y^2}} (yz\,i)\cdot\left(-dy\,dz\,\overline{i}\right) = -\int_0^a \int_0^{\sqrt{a^2-y^2}} (yz)\,dy\,dz = \frac{-a^4}{8}$$

Similarly $\int_{S_2} \overline{f}\cdot d\overline{s} = \frac{-a^4}{8}$ $\quad \int_{S_3} \overline{f}\cdot d\overline{s} = \frac{-a^4}{8}$

Thus equation (1) becomes $0 = \frac{-3a^4}{8} + \int_s \overline{f}\cdot d\overline{s}$

Hence $\int_s \overline{f}\cdot d\overline{s} = \frac{3a^4}{8}$

11. Verify the divergence theorem for $\overline{f} = (x^2 - yz)\overline{i} + (y^2 - zx)\overline{j} + (z^2 - xy)\overline{k}$ over the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Solution: For the given $\overline{f} = (x^2 - yz)\overline{i} + (y^2 - zx)\overline{j} + (z^2 - xy)\overline{k}$, we have

$$div\ \overline{f} = 2(x + y + z).$$

$\therefore$ If V is the volume of the given parallelepiped, we have

$$\int_V div\ \overline{f}\,dv = \int_V 2(x+y+z)\,dv = 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} (x+y+z)\,dz\,dy\,dx \qquad (1)$$

$$= abc(a+b+c)$$

Next, we note that the boundary surface S of the given parallelepiped is made up of the following six faces:

S_1 : OABC, S_2: OPQA, S_3: OCSP, S_4: PQRS , S_5: CSRB, S_6: ABRQ

The unit outward normals to these faces are: $-\overline{k},\ -\overline{j}, -\overline{i},\ \overline{k},\ \overline{j}$ and $\overline{i}$ respectively.

$$\iint_{S_1} \overline{f} \cdot \overline{n}\, ds = \iint_{OABC} \overline{f} \cdot \left(-\overline{k}\right) ds = -\iint_{OABC} xy\, dx\, dy = \int_{x=0}^{a} \int_{y=0}^{b} xy\, dy\, dx = \frac{a^2 b^2}{4} \quad (2)$$

Similarly we find that

$$\iint_{S_2} \overline{f} \cdot \overline{n}\, ds = \frac{c^2 a^2}{4} \quad (3)$$

$$\iint_{S_3} \overline{f} \cdot \overline{n}\, ds = \frac{b^2 c^2}{4} \quad (4)$$

Next,

$$\iint_{S_4} \overline{f} \cdot \overline{n}\, ds = \iint_{PQRS} \overline{f} \cdot \left(-\overline{k}\right) ds = -\iint_{PQRS} \left(c^2 - xy\right) dx\, dy = \int_{x=0}^{a} \int_{y=0}^{b} \left(c^2 - xy\right) dy\, dx$$

$$= abc^2 - \frac{a^2 b^2}{4} \quad (5)$$

Similarly find that

$$\iint_{S_5} \overline{f} \cdot \overline{n}\, ds = ab^2 c - \frac{c^2 a^2}{4} \quad (6)$$

$$\iint_{S_6} \overline{f} \cdot \overline{n}\, ds = a^2 bc - \frac{b^2 c^2}{4} \quad (7)$$

Adding expression (2) to (7), we get

$$\iint_{S} \overline{f} \cdot \overline{n}\, ds = abc^2 + ab^2 c + a^2 bc = abc\left(a + b + c\right) \quad (8)$$

From equations (1) and (8) we obtain

$$\int_V div\ \overline{f}\, dv = \int_S \overline{f} \cdot \overline{n}\, ds$$

Thus for the given $\overline{f}$ and for the given region, the divergence theorem is verified.

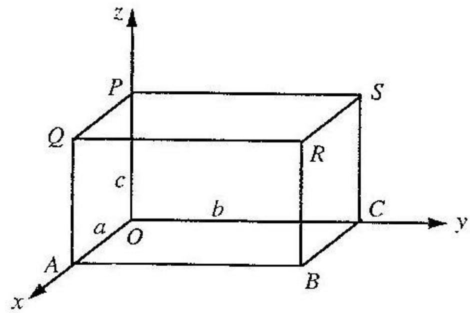

Figure 10.18

12. Verify divergence theorem for $\overline{f} = 4x\overline{i} - 2y^2\overline{j} + z^2\overline{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 4, \quad z = 0, \quad z = 3$..

Solution: For the given $div\ \overline{f} = 4 - 4y + 2z$.

$$\iiint_V div\,\overline{f}\,dv = \iiint_V \int_V (4 - 4y + 2z)\,dx\,dy\,dz = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{3} (4 - 4y + 2z)\,dz\,dy\,dx$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\,dy\,dx = \int_{x=-2}^{2} 42\sqrt{4 - x^2}\,dx = 84\left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left[\sin^{-1} 1\right] = 84\left[2 \times \frac{\pi}{2}\right] = 84\pi \qquad (1)$$

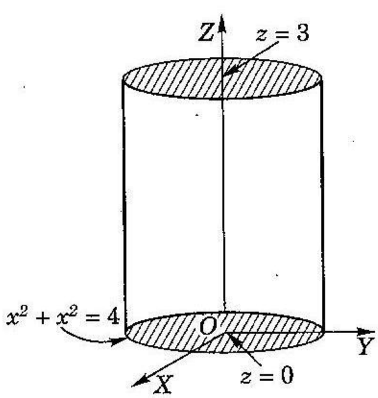

Figure 10.19

To evaluate the surface integral, divide the closed surface S of the cylinder into 3 parts.

S_1: The circular base in the plane z=0

S_2: The circular top in the plane z=3

S_3: The curved surface of the cylinder given by the equation $x^2 + y^2 = 4$

Also

$$\iint_S \bar{f}\cdot\bar{n}\, ds = \iint_{S_1} \bar{f}\cdot\bar{n} ds + \iint_{S_2} \bar{f}\cdot\bar{n} ds + \iint_{S_3} \bar{f}\cdot\bar{n} ds$$

On S_1 (z=0), we have $\bar{n} = -\bar{k}$, $\bar{f} = 4x\bar{i} - 2y^2\bar{j}$

So that $\bar{f}\cdot\bar{n} = \left(4x\bar{i} - 2y^2\bar{j}\right)\cdot\left(-\bar{k}\right) = 0$

$$\iint_{S_1} \bar{f}\cdot\bar{n}\, ds = 0 \qquad (2)$$

On S_2 (z=3), we have $\bar{n} = \bar{k}$, $\bar{f} = 4x\bar{i} - 2y^2\bar{j} + 9\bar{k}$

So that $\bar{f}\cdot\bar{n} = \left(4x\bar{i} - 2y^2\bar{j} + 9\bar{k}\right)\cdot\left(\bar{k}\right) = 9$

$$\iint_{S_2} \bar{f}\cdot\bar{n}\, ds = \iint_{S_2} 9\, dx\; dy = 9\iint_{S_2} dx\, dy$$

$$= 9 \times \text{area of surface } S_2 = 9\left(\pi\cdot 2^2\right) = 36\pi \qquad (3)$$

On S_3, $x^2 + y^2 = 4$

A vector normal to the surface S_3 is given by

$$\nabla\left(x^2 + y^2\right) = 2x\bar{i} + 2y\bar{j}$$

$\therefore \bar{n}$ = a unit vector normal to the surface S_3

$$= \frac{2x\bar{i} + 2y\bar{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2x\bar{i} + 2y\bar{j}}{\frac{1}{2}\left(x\bar{i} + y\bar{j}\right)} \qquad \left(\because x^2 + y^2 = 4\right)$$

$$\overline{f}\cdot\overline{n} = \left(4x\overline{i} - 2y^2\overline{j} + z^2\overline{k}\right)\cdot\left(\frac{x\overline{i} + y\overline{j}}{2}\right) = 2x^2 - y^3$$

Also, on S_3 i.e., $x^2 + y^2 = 4$, x = 2cosθ, y = 2sinθ and ds = dx dy = 2dθ dz

To cover the whole surface S_3, z varies from 0 to 3 and θ varies from 0 to 2π

$$\therefore \iint_{S_3} \overline{f}\cdot\overline{n}\ ds = \int_{\theta=0}^{2\pi}\int_{z=0}^{3}\left[2(2\cos\theta)^2 - (2\sin\theta)^3\right]2\,dz\,d\theta$$

$$= 16\int_{\theta=0}^{2\pi}\int_{z=0}^{3}\left[\cos^2\theta - \sin^3\theta\right]dz\,d\theta = 16\int_{\theta=0}^{2\pi}\left[\cos^2\theta - \sin^3\theta\right]\left[z\right]_{z=0}^{3}\,d\theta$$

$$= 48\int_{\theta=0}^{2\pi}\left[\cos^2\theta - \sin^3\theta\right]d\theta = 48\left[\int_{\theta=0}^{2\pi}\cos^2\theta\,d\theta - \int_{\theta=0}^{2\pi}\sin^3\theta\ d\theta\right]$$

$$= 48\left[4\int_{\theta=0}^{\pi/2}\cos^2 d\theta - 0\right] = 48\times 4\times\frac{1}{2}\times\frac{\pi}{2} = 48\pi \qquad (4)$$

Adding expression (2), (3) and (4), we get

$$\iint_S \overline{f}\cdot\overline{n}\ ds = 0 + 36\pi + 48\pi = 84\pi \qquad (5)$$

From equations (1) and (5) we obtain

$$\int_V div\ \overline{f}\,dv = \int_S \overline{f}\cdot\overline{n}\ ds$$

Thus for the given $\overline{f}$ and for the given region, the divergence theorem is verified.