# EMTL

## Electro Magnetic Theory & Transmission Lines

**Charge:-** * charge is defined as the product of current and time. It is represented with letter "Q".

$$\boxed{Q = I \times T}$$

* charge is responsible for delivering both electric field and Magnetic field.

**EM Waves:-** Electromagnetic waves or EM waves are waves that are created as a result of vibrations between an Electric field and Magnetic field.

Ex:- Light.

**Field:-** Field is a function that specifies a quantity everywhere in a region or a space.

**Magnetic Field:-** Field is produced due to Magnetic effects. is called Magnetic field.

**Electric field:-** An electric charge produces a field around it, which is called as an Electric field.

**Electromagnetic field:-** Moving charges produce a current and current carrying conductor produces a magnetic field. In such case electric & Magnetic fields are related to each other. Such field is called Electro magnetic Field.

The study of electric, magnetic & combined fields is nothing but the Engineering Electromagnetics. Such fields may be time varying or time independent.

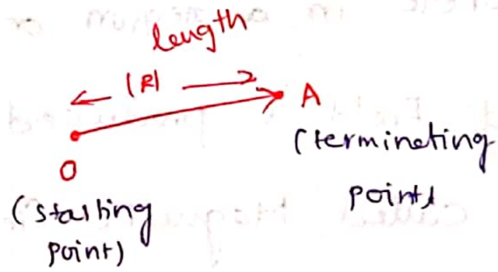
## Scalar & Vector:- Various Quantities are involved in Study of electromagnetics. Mainly we have 2 types.
1. Scalar.
2. Vector.

**Scalar:-** Scalar is a quantity which has only magnitude. there is no direction. Scalar is represented by a letter. A, B, V, S

Ex:- Temperature, Mass, Volume, Density, Speed, Electric charge etc.

**Vector:-** A vector is a quantity which has magnitude and direction. vector is represented with $\vec{A}, \vec{V}, \vec{B}, \vec{F}$.

Ex:- Force, velocity, Displacement, electric Field intensity, Magnetic field intensity, Acceleration etc.

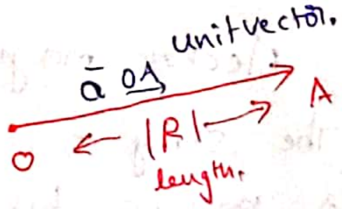



$|\overline{OA}| = R$

**Unit Vector:-** A unit vector has a function to indicate the direction. It's magnitude is always unity, irrespective of the direction.

When a simple vector is divided by it's own magnitude. a new vector is created known as the unit vector.

$$\boxed{\text{Unit vector:-}\ \bar{a}_{OA} = \frac{\overline{OA}}{|\overline{OA}|}}$$



**Position Vector/Radius Vector ($\overline{OP}$) :-** A position vector define the position of ~~th~~ a point (P) in Space which is relative to the origin (O). Hence ~~vector~~ position vector is another way to denote a point in Space.

$$\boxed{\overline{OP} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z}$$

**Displacement vector:-** Displacement vector is the displacement or the shortest distance from one point to another. distance between 2 points = $|\sqrt{\text{distance vector}}|$

**Vector Multiplication :-** When two vectors are multiplied the result is either a scalar or a vector depending on how they are multiplied. vector multiplication is performed by 2 types. 1. Dot product / scalar product

2. Cross product / vector product.

1. **Dot product :-** (A·B)

Dot product of 2 vectors A & B defined as

$$\bar{A}\cdot\bar{B} = |\bar{A}|.|\bar{B}|\cos\theta_{AB}$$

$\theta_{AB}$ = angle b/w A & B

range. 0 to $\pi$.

$$0 \le \theta_{AB} \le \pi$$

result AB is a scalar.

② **Cross product:-** (A×B)

Cross product of two vectors A & B in given as

$$\bar{A}\times\bar{B} = |\bar{A}|\times|\bar{B}|\sin\theta_{AB}\,\bar{a}_N$$

L unit vector.

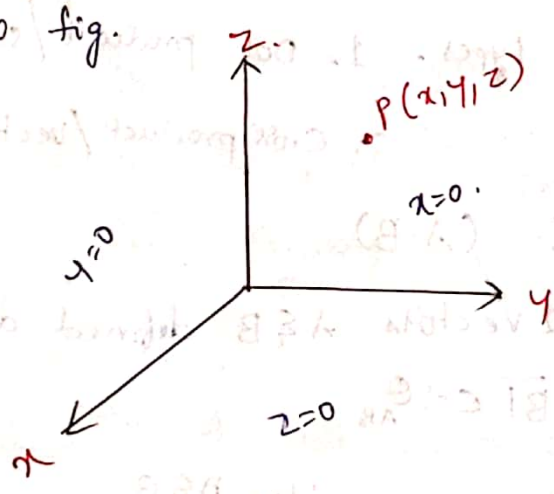
# Co-ordinate system's and it's transformation

**Co-ordinate system:-** Co-ordinate System is a System which is used to represent a point in Space. Basically there are 3 types of Co-ordinate Systems. They are.

1. Cartesian / Rectangular.
2. Circular / Cylindrical
3. Spherical

## ① Cartesian (or) Rectangular co-ordinate system.

A point in Cartesian co-ordinate System is represented as $p(x, y, z)$ as shown in the below fig.




Where x, y, z are Space Variables (or) Co-ordinates of Cartesian Co-ordinate System measured in meter's and are mutually perpendicular to each other.

* the Space variables / Co-ordinates ranges are as follows.

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty$$

* A vector in Cartesian Co-ordinate System is represented as

$$\vec{A} = A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z}.$$

$$\vec{A} = (A_x, A_y, A_z).$$

where $A_x, A_y$ & $A_z$ are Components of $\vec{A}$ along x, y, z directions respectively and $\vec{a_x}, \vec{a_y}, \vec{a_z}$ are unit vectors of $\vec{A}$ along x, y, z directions respectively.

## properties of unit vectors:-

1) $\vec{a_x}\cdot\vec{a_x} = \vec{a_y}\cdot\vec{a_y} = \vec{a_z}\cdot\vec{a_z} = 1$ $(\cos(0) = 1)$
2) $\vec{a_x}\cdot\vec{a_y} = \vec{a_y}\cdot\vec{a_z} = \vec{a_z}\cdot\vec{a_x} = 0$ $[\cos(90) = 0)$
3) $\vec{a_x}\times\vec{a_y} = \vec{a_z}$
4) $\vec{a_y}\times\vec{a_z} = \vec{a_x}$
5) $\vec{a_z}\times\vec{a_x} = \vec{a_y}$
6) $\vec{a_x}\times\vec{a_y} = -\vec{a_y}$
7) $\vec{a_z}\times\vec{a_y} = -\vec{a_x}$
8) $\vec{a_y}\times\vec{a_x} = -\vec{a_z}$

## Circular / Cylindrical Co-ordinate System. $P(\rho, \phi, z)$

A point in Circular Co-ordinate System is represented as $P(\rho, \phi, z)$ as shown in the below fig.

where $\rho$ = radius of cylinder, unit of $\rho$ is metre

$\phi$ = Azimuth angle, unit of $\phi$ is degree.

$z$ = Same as in Cartesian co-ordinate system, unit of z is metre.

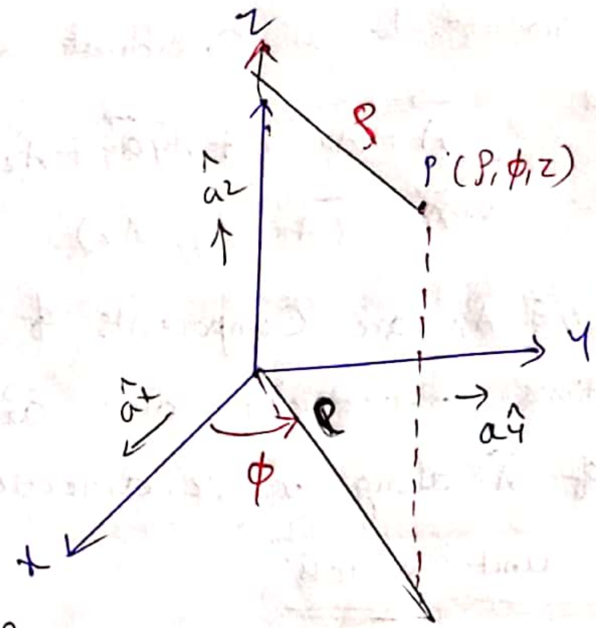



fig:- point in Circular Co-ordinate system.

The ranges of $\rho$ from $0 \leq \rho < \infty$

$\phi$ from $0 \leq \phi \leq 2\pi$

$z$ from $-\infty < z < \infty$

A vector in Circular Co-ordinate system is represented as $\vec{A} = A\rho \vec{a\rho} + A\phi \vec{a\phi} + Az \vec{az}$

$\vec{A} = A\rho + A\phi + Az$.

where $A\rho, A\phi, Az$ are the Components of $\vec{A}$ along the $\rho, \phi, z$ directions. and $\vec{a\rho}, \vec{a\phi}, \vec{az}$ are unit vectors along $\rho, \phi, z$ directions respectively.

The relation between $x, y, z$ and $\rho, \phi, z$:

$x = \rho \cos\phi$ and $\rho = \sqrt{x^2+y^2}$

$y = \rho \sin\phi$ $\quad \phi = \tan^{-1}(y/x)$

$z = z$. $\quad z = z$

* Dot product of $ax, ay,$ and $az$ with $a\rho, a\phi$ and $az$ are given by.

$a_x \cdot a_\rho = \cos\phi$
$a_x \cdot a_\phi = -\sin\phi$
$a_y \cdot a_\rho = \sin\phi$
$a_y \cdot a_\phi = \cos\phi$
$a_z \cdot a_\rho = 0$
$a_z \cdot a_\phi = 0$

| | $a_\rho$ | $a_\phi$ | $a_z$ |
|---|---|---|---|
| $a_x$ | $\cos\phi$ | $-\sin\phi$ | 0 |
| $a_y$ | $\sin\phi$ | $\cos\phi$ | 0 |
| $a_z$ | 0 | 0 | 1 |

$a_x \cdot a_z = 0$
$a_y \cdot a_z = 0$
$a_z \cdot a_z = 1$

## properties of unit vector in cylindrical co-ordinates

1. $\vec{a_\rho} \cdot \vec{a_\rho} = \vec{a_\phi} \cdot \vec{a_\phi} = \vec{a_z} \cdot \vec{a_z} = 1$
2. $\vec{a_\rho} \cdot \vec{a_\phi} = \vec{a_\phi} \cdot \vec{a_z} = \vec{a_z} \cdot \vec{a_\rho} = 0$
3. $\vec{a_\rho} \times \vec{a_\phi} = \vec{a_z}$
4. $\vec{a_\phi} \times \vec{a_z} = \vec{a_\rho}$
5. $\vec{a_z} \times \vec{a_\rho} = \vec{a_\phi}$
6. $\vec{a_\rho} \times \vec{a_z} = -\vec{a_\phi}$
7. $\vec{a_z} \times \vec{a_\phi} = -\vec{a_\rho}$
8. $\vec{a_\phi} \times \vec{a_\rho} = -\vec{a_z}$

## 3. Spherical Co-ordinate System. $P(r, \theta, \phi)$

A point in Spherical co-ordinate System is represented as $P(r, \theta, \phi)$ as shown in fig below.

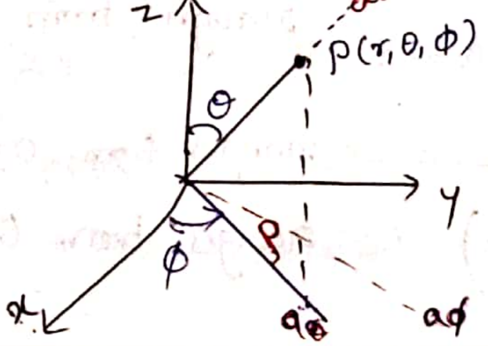



fig:- point in Spherical co-ordinate System.

Where $r$ is the radius of the Spherical Co-ordinate System measured in metres and

$\theta$ is the angle of elevation measured from z-axis. ~~and~~ measured in degrees. and

$\phi$ is the azimuthal angle measured from x-axis. measured in degree's

"$r$" ranges from $0 \leq r < \infty$

"$\theta$" ranges from $0 \leq \theta \leq \pi$

"$\phi$" ranges from $0 \leq \phi < 2\pi$

* A vector in Spherical co-ordinate system is represented as.

$$\vec{A} = A_r.\vec{a_r} + A_\theta.\vec{a_\theta} + A_\phi.\vec{a_\phi}$$

$$\vec{A} = A_r, A_\theta, A_\phi$$

Where $A_r$, $A_\theta$, and $A_\phi$ are the Components of $\vec{A}$ along $r, \theta, \phi$ directions. $\vec{a_r}, \vec{a_\theta}, \vec{a_\phi}$ are the unit vectors along $r, \theta, \phi$ directions respectively.

The variables / co-ordinates of Cartesian and Spherical Co-ordinates are related by

$x = r \sin\theta \cos\phi$ ranges from $-\infty < x < \infty$

$y = r \sin\theta \sin\phi$ ranges from $-\infty < y < \infty$

$z = r \cos\theta$ ranges from $-\infty < z < \infty$

and

$r = \sqrt{x^2+y^2+z^2}$ ranges from $0 \leq r \leq \infty$

$\theta = \cos^{-1}\left(\frac{z}{r}\right)$ ranges from $0 \leq \theta \leq \pi$

$\phi = \tan^{-1}(y/x)$ ranges from $0 \leq \phi \leq 2\pi$

The relations between the variables of cylindrical and Spherical Co-ordinates.

$\rho = r\sin\theta$

$\phi = \phi$

$z = r\cos\theta$.

$r = \sqrt{\rho^2+z^2}$

$\theta = \tan^{-1}(z/r)$

$\phi = \phi$

* The dot products of $a_x$, $a_y$, and $a_z$ with $a_r$, $a_\theta$ and $a_\phi$ are given by

$a_x \cdot a_r = \sin\theta \cdot \cos\phi$

$a_x \cdot a_\theta = \cos\theta \cdot \cos\phi$

$a_x \cdot a_\phi = -\sin\phi$

$a_y \cdot a_r = \sin\theta \cdot \sin\phi$

$a_y \cdot a_\theta = \cos\theta \cdot \sin\phi$

$a_y \cdot a_\phi = \cos\phi$

$a_z \cdot a_r = \cos\theta$

$a_z \cdot a_\theta = -\sin\theta$

$a_z \cdot a_\phi = \phi$.

| | $a_r$ | $a_\theta$ | $a_\phi$ |
|---|---|---|---|
| $a_x$ | $\sin\theta.\cos\phi$ | $\cos\theta\cos\phi$ | $-\sin\phi$ |
| $a_y$ | $\sin\theta\sin\phi$ | $\cos\theta.\sin\phi$ | $\cos\phi$ |
| $a_z$ | $\cos\theta$ | $-\sin\theta$ | $0$ |

Here $A_r = A_x\sin\theta\cos\phi + A_y\sin\theta.\sin\phi + A_z\cos\theta$.

$A_\theta = A_x\cos\theta.\cos\phi + A_y\cos\theta\cdot\sin\phi - A_z\sin\theta$.

$A_\phi = -A_x\sin\phi + A_y\cos\phi$

The point $A(x,y,z) = A(\rho,\phi,z) = A(r,\theta,\phi)$ in Cartesian, Cylindrical and Spherical Co-ordinate

Systems is shown in a fig. below.

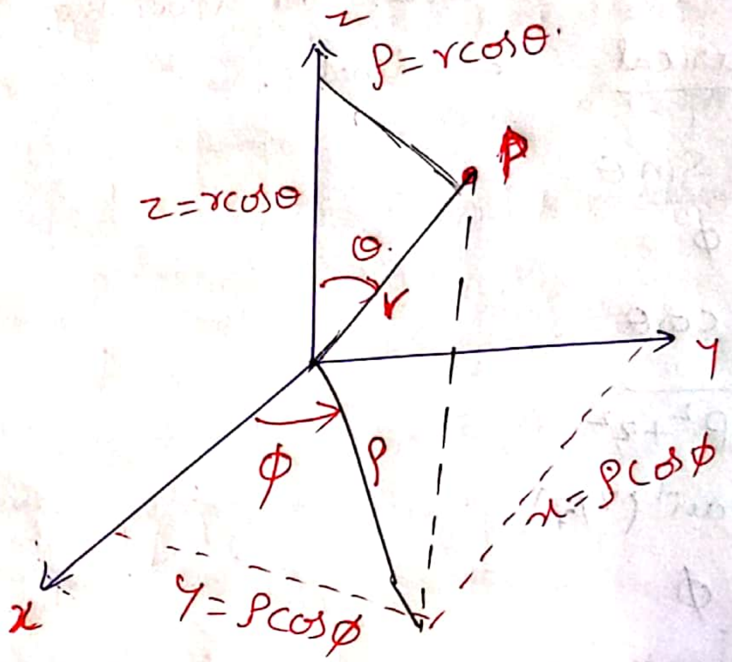



fig:- Co-ordinates of a point in all the three Systems.

## Differential Length, Area and Volume:

1. Cartesian/Rectangular:-

Let us take incremental length to find length.

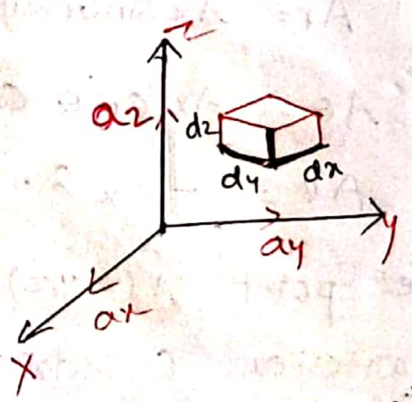
length : $\vec{dL} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$

Area = $\vec{ds}$ = in x-plane = $dydz$

in y-plane = $dzdx$

in z-plane = $dx.dy$

Volume = $dx.dy.dz$



## Representation of vector form

Consider two 2 different vectors.

1. $x_1, y_1, z_1$
2. $x_2\ y_2\ z_2$

$$\vec{A} = (x_2 - x_1)\hat{a_x} + (y_2 - y_1)\hat{a_y} + (z_2 - z_1)\hat{a_z}.$$

NOTE:- A unit vector is calculated by direction of unit vector and divided by it's magnitude.

$$\vec{A} = \frac{\vec{A}}{|A|} = \frac{(x_2 - x_1)\hat{a_x} + (y_2 - y_1)\hat{a_y} + (z_2 - z_1)\hat{a_z}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

Ex:- Specify the unit vector extending from the origin towards the point $G(-2, -2, 1)$

Sol:- Vector is from $O(0,0,0)$ to $G(-2,-2,1)$

$$\vec{G} = (-2-0)\hat{a_x} + (-2-0)\hat{a_y} + (1-0)\hat{a_z}$$

$$= -2\hat{a_x} - 2\hat{a_y} + \hat{a_z}$$

$$|\vec{G}| = \sqrt{2^2 + 2^2 + 1} = 3$$

unit vector in $\vec{G} = \dfrac{-2\hat{a_x} - 2\hat{a_y} + \hat{a_z}}{3}$

$$\boxed{\hat{a}_G = -0.667\hat{a_x} - 0.667\hat{a_y} + 0.33\hat{a_z}}$$

2. Cylindrical / Circular :-

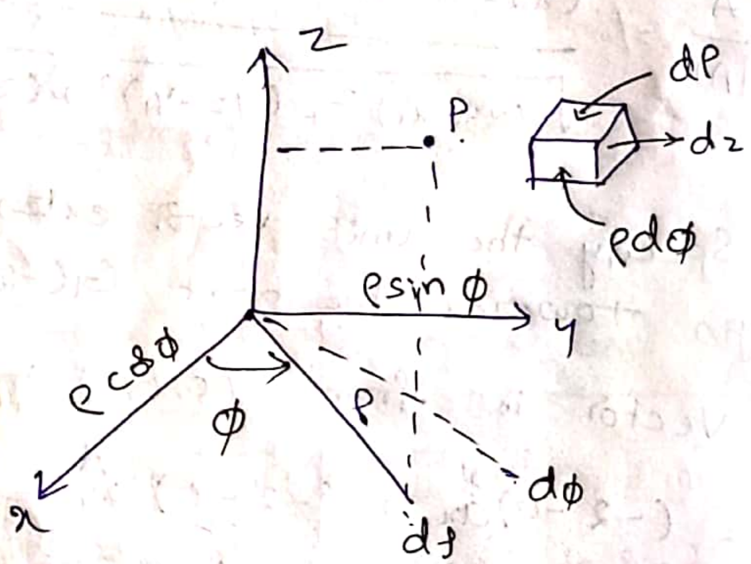
length $d\vec{L} = d\rho\,\hat{a_\rho} + \rho d\phi . \hat{a_\phi} + dz\,\hat{a_z}$

Area in z plane $= \rho d\rho d\phi$ (Top plane of cylinder)

$\rho$ - plane $= \rho d\phi dz$ (curve plane)

$\phi$ - plane $= d\rho dz$ (centre plane)

Volume :- $\rho d\rho d\phi dz$ (or) $d\rho dz \rho d\phi$




Formulae of arc. = $\boxed{\theta = \frac{l}{r}}$

$\rho = d\rho$

$\phi = \rho d\phi$

$z = dz$

$d\phi = \frac{l}{\rho}$ – arc length.

$\boxed{l = \rho d\phi}$

3 Spherical Co-ordinate System :-

length : $dL = dr\,\hat{a_r} + r d\theta\,\hat{a_\theta} + r\sin\theta\, d\phi\,\hat{a_\phi}$

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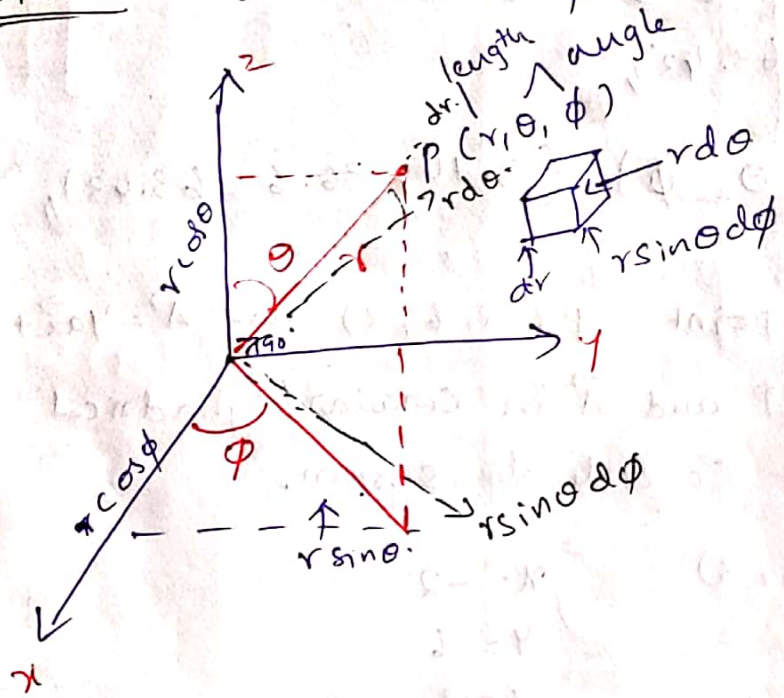
**Area:-** in r plane $= rd\theta . r\sin\theta . d\phi$

$\theta$ plane $= dr . r\sin\theta . d\phi$

$\phi$ plane $= dr . rd\theta$.

**volume:-** $r^2 \sin\theta . dr d\theta d\phi$




where $r =$ distance from orgin to the point P.

$\theta =$ angle made on z - axis.

$\phi =$ projection of point from p to angle on x - y plane.

**problems:-**

① Convert the given point P(1,2,3) to cylindrical and Spherical Co-ordinate system.

point P (1,2,3) $\quad x = 1, \; y = 2, \; z = 3$.

For cylindrical $=$

$\rho = \sqrt{x^2 + y^2} = \sqrt{5} = 2.23 m$

$\phi = \tan^{-1}(y/x) = \tan^{-1}(2) = 63.4^\circ$

$z = z = 3$

$P(\rho = 2.23m, \; \phi = 63.4^\circ, \; z = 3m)$

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For Spherical

$r = \sqrt{x^2+y^2+z^2} = \sqrt{1^2+2^2+3^2} = \sqrt{14} = 3.74\,m$

$\theta = \cos^{-1}\left[\frac{z}{r}\right] = \cos^{-1}\left(3/\sqrt{14}\right) = 36.6$

$\phi = 63.43°$ //

$P(r, \theta, \phi) = (3.74, 36.6, 63.43)$ //

(2) Given point $P(-2, 6, 3)$ and $\vec{A} = y\vec{a_x} + (x+y)\vec{a_y}$ express $P$ and $\vec{A}$ in Circular/cylindrical and Spherical Co-ordinate system.

$P(-2, 6, 3)$ $\quad x = -2$

$y = 6$

$z = 3$

$P(\rho, \phi, z)$

$\rho = \sqrt{x^2+y^2} = \sqrt{(-2)^2+6^2} = \sqrt{4+36} = \sqrt{40} = 6.32.$

$\phi = 180 - \tan^{-1}\left(\cancel{6}^{3}/\cancel{-2}\right)$

$= 180 - \tan^{-1}(-3)$

$= 108.43°$

$z = 3$

For Spherical:- $\quad P(r, \theta, \phi)$

$r = \sqrt{x^2+y^2+z^2} = \sqrt{4+36+9} = \sqrt{49} = 7$ //

$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\frac{3}{\sqrt{49}}\right) = 64.62°$

$\phi = \tan^{-1}(y/x) = 180 - \tan^{-1}(-3) = 108.43°$

$z = 3$

## Vector transformation :-

$x = -2$
$y = 6$
$z = 3$
$\theta = 64.62^\circ$
$\phi = 108.43^\circ$

$\vec{A} = y a_x + (x+y) a_y$

**Cylindrical** . $\vec{A} = (A_\rho, A_\phi, A_z)$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Given $\vec{A} = y a_x + (x+y) a_y$

$\vec{A} = (y, (x+y), 0)$

$A_x = y, \quad A_y = x+y \quad A_z = 0$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x+y \\ 0 \end{bmatrix}$$

$A_\rho = y \cos\phi + (x+y) \sin\phi$

$= 6 \cos(108.43^\circ) + (4) \sin(108.43^\circ)$

$\boxed{A_\rho = -0.94}$

$A_\phi = y(-\sin\phi) + (x+y)\cos\phi$

$= -6 \sin(108.43^\circ) + (4) \cos(108.43^\circ)$

$\boxed{A_\phi = -6}$

$A_z = 0$

$\therefore \vec{A} = (A_\rho, A_\phi, A_z)$.

$\vec{A} = (-0.94, -6, 0)$

(B) Spherical Co-ordinate System

$\vec{A}(A_r, A_\theta, A_\phi)$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \begin{bmatrix} \sin\theta.\cos\phi & \sin\theta.\sin\phi & \cos\theta \\ \cos\theta.\cos\phi & \cos\theta.\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y \\ x+y \\ 0 \end{bmatrix}$$

$A_r = y \sin\theta.\cos\phi + (x+y)\sin\theta.\sin\phi$

$= 6\sin(64.62)\cos(108.43) + (4)\sin(64.62)$.

$\cos(108.43^\circ)$

$$\boxed{A_r = -0.85}$$

$A_\theta = y\cos\theta.\cos\phi + (x+y)\cos\theta.\sin\phi$

$= 6\cos(64.62)\cos(108.43) + (4)\cos(64.62)$.

$\sin(108.43^\circ)$

$$\boxed{A_\theta = -0.4}$$

$$\boxed{A_\phi = -6}$$

$\vec{A}(A_r, A_\theta, A_\phi) = (-0.85, -0.4, -6)$

# CO-ordinate System and It's transformation.

| Co-ordinate System transformation | Point transformation | Vector transformation. |
|---|---|---|
| Cartesian Co-ordinate to Cylindrical Co-ordinate | $P(x,y,z) = P(\rho, \phi, z)$<br>$\rho = \sqrt{x^2+y^2}$ $\quad \phi = \tan^{-1}(y/x)$<br>$z = z.$ | $\vec{A} = A(A_x, A_y, A_z) \rightarrow \vec{A}(A_\rho, A_\phi, A_z)$<br>$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$ |
| Cylindrical Co-ordinate to Cartesian Co-ordinate | $P(\rho,\phi,z) \rightarrow P(x,y,z)$<br>$x = \rho\cos\phi$<br>$y = \rho\sin\phi$<br>$z = z$ | $\vec{A}(A_\rho, A_\phi, A_z) = \vec{A}(A_x, A_y, A_z)$<br>$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$ |
| Cartesian to Spherical | $P(x,y,z) \rightarrow P(r,\theta,\phi)$<br>$r = \sqrt{x^2+y^2+z^2}$<br>$\phi = \tan^{-1}(y/x)$<br>$\theta = \cos^{-1}(z/r)$ | $\vec{A}(A_x, A_y, A_z) = \vec{A}(A_r, A_\theta, A_\phi)$<br>$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cdot\cos\phi & \sin\theta\sin\phi & \cos\phi \\ \cos\theta\cos\phi & \cos\theta\cdot\sin\phi & -\sin\phi \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$ |
| Spherical to Cartesian | $P(r,\theta,\phi) = P(x,y,z)$<br>$x = r\sin\theta\cdot\cos\phi$<br>$y = r\sin\theta\cdot\sin\phi$<br>$z = r\cos\theta$ | $\vec{A} = (A_r, A_\theta, A_\phi) = \vec{A}(A_x, A_y, A_z)$<br>$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cdot\cos\phi & \cos\theta\cdot\cos\phi & -\sin\phi \\ \sin\theta\cdot\sin\phi & \cos\theta\cdot\sin\phi & \cos\phi \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$ |

# VECTOR CALCULUS

The relations for differential length, area and volume.

| Co-ordinate System. | $dl$ | $ds$ | $dv$ |
|---|---|---|---|
| Cartesian | $dx\,a_x + dy\,a_y + dz\,a_z$ | $dx\,dy\,a_z$ or $dy\,dz\,a_z$ or $a_z\,d_x\,a_y$ | $dx\,dy\,dz$ |
| Cylindrical | $d\rho\,a_\rho + \rho\,d\phi\,a_\phi + dz\,a_z$ | $\rho\,d\rho\,d\phi\,a_z$ or $\rho\,d\phi\,dz\,a_\rho$ or $d\rho\,dz\,a_\phi$ | $\rho\,d\rho\,d\phi\,dz$ |
| Spherical | $dr\,a_r + r\,d\theta\,a_\theta +$ $r\sin\theta\,d\phi\,a_\phi$ | $r\,dr\,d\theta\,a_\phi$ (or) $r\sin\theta\,d_r\,d\phi\,a_\theta$ (or) $r^2\sin\theta\,d\theta\,d\phi\,a_r$ | $r^2\sin\theta\,dr\,d\theta\,d\phi$ |

# Line, Surface and volume Integrals:-

① **Line Integral:-** * The open line integral of vector field A is given as

$$\boxed{\int_L \vec{A} \cdot \vec{dl}}$$

* The closed line integral of vector field A is given as $\boxed{\oint_L \vec{A} \cdot \vec{dl}}$

**Note:-** Closed line integral is also called as Circulation vector field "$\vec{A}$" around a path "L"

**Surface Integral:-** The open Surface integral of vector field "$\vec{A}$" is given as

$$\boxed{\int_S \vec{A} \cdot \vec{ds}}$$

The total outward flux (or) closed Surface integral of vector field $\vec{A}$ is given as

$$\boxed{\oint_S \vec{A} \cdot \vec{ds}}$$

**Volume Integral:-** The volume integral vector field $\vec{A}$ is given as

$$\boxed{\int_V \vec{A} \cdot dv}$$

**Integral theorem's:-**

Basically there are two integral theorem's.

① Stoke's theorem

② Divergence theorem.

## Stoke's Theorem:-

Stoke's theorem is given as

$$\boxed{\oint_L \vec{A} \cdot \vec{dl} = \int_S (\vec{\nabla} \times \vec{A}) \, \vec{ds}}$$

Stoke's theorem states that the circulation of vector field $\vec{A}$ around a closed path "L" is equal to open surface integral of $(\nabla \times A)$ Curl of vector field $\vec{A}$ bounded by the path. (or) "L".

## Divergence theorem:-

Divergence theorem is given as

$$\boxed{\oint_S \vec{A} \cdot \vec{ds} = \int_V (\vec{\nabla} \cdot \vec{A}) \, dv}$$

The total outward flux of vector field $\vec{A}$ through a closed surface integral is equal to volume integral of divergence of vector field $\vec{A}$.

## Del($\nabla$)operator:-

**Definition:** The del or nabla is known as differential vector operator and is defined as

① In cartesian:- $\boxed{\nabla = a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}}$

② In Cylindrical :

$$\boxed{\vec{\nabla} = \frac{\partial}{\partial \rho}\vec{a_\rho} + \frac{1}{\rho}\frac{\partial}{\partial \phi}\vec{a_\phi} + \frac{\partial}{\partial z}\vec{a_z}}$$

③ In Spherical :-

$$\boxed{\vec{\nabla} = \frac{\partial}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\vec{a_\phi}}$$

Del has units of 1/metre (1/m)

## Gradient of a scalar V ($= \nabla V$)

Gradient of a Scalar is a vector and is defined as

① In Cartesian :-

$$\boxed{\vec{\nabla} V = \frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z.}$$

② In Cylindrical :-

$$\boxed{\vec{\nabla} V = \frac{\partial V}{\partial \rho}\vec{a_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a_\phi} + \frac{\partial V}{\partial z}\vec{a_z}}$$

③ In Spherical :-

$$\boxed{\vec{\nabla} V = \frac{\partial V}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a_\phi}}$$

* Gradient of vector is not possible.

* Examples are gradient of temperature, gradient of electrical potential

# Divergence of a vector, A $(= \nabla \cdot A)$

Divergence of a vector is a scalar and is defined as

In Cartesian:

$$\boxed{\nabla \cdot A = \operatorname{div} A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

In Cylindrical:-

$$\boxed{\nabla \cdot A = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}A_\phi + \frac{\partial}{\partial z}A_z}$$

In Spherical:-

$$\boxed{\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \cdot \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}}$$

Divergence means spreading or diverging of a quantity from a point. It is applicable to vectors only. The divergence of a vector indicates the net flow of quantities like gas, fluid, vapour, electric and magnetic flux lines.

(or)

In other words, It is a measure of the difference between outflow and inflow.

* The divergence of a vector is positive if the net flow is outward.

* The divergence of a vector is negative if the netflow is inward.

* The fluid is said to be incompressible if the divergence is zero, that is, $\nabla \cdot A = 0$ is the condition of incompressibility.

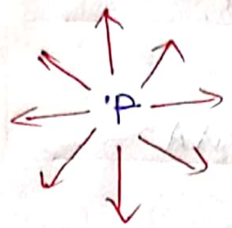


fig: positive divergence

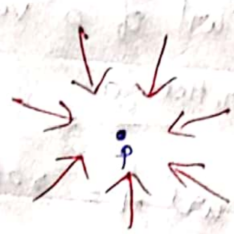


fig: -ve divergence (or) covergence

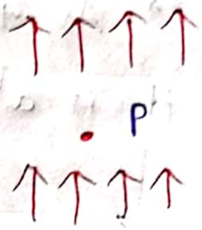


fig:- Divergence less (or) Solenoidal

## ③ Curl of vector $\vec{A}$ (or) Rotational of vector $\vec{A}$

Curl of a vector is a vector and is defined as

**In Cartesian:-**

$$\text{Curl } A = \vec{\nabla} \times \vec{A} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= a_x \left[ \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y \right] + a_y \left[ \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z \right] + a_z \left[ \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \right]$$

**In Cyclidrical:-**

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \vec{a_\rho} & \rho \vec{a_\phi} & \vec{a_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & A_\phi & A_z \end{vmatrix}$$

**In Spherical:-**

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{a_r} & r\vec{a_\theta} & r\sin\theta \, \vec{a_\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta . A_\phi \end{vmatrix}$$

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Note :- If $\vec{\nabla} \times \vec{A} = 0$ then it is irrotational field

* As the Curl of a vector represents rotation, it is also written as

$$\text{Curl } A = \text{rot } A = \nabla \times A$$

It may be noted that Curl (gradient of a Scalar) = $\nabla \times (\nabla V)$ is zero. This means that the gradient of fields is irrotational. Also div (curl) = 0.

## Laplacian Operator ($\nabla^2$)

It is defined as $\nabla \cdot \nabla$. It's unit is $\frac{1}{m^2}$. It is a Scalar differential operator. It is operated on a Scalar as well as a vector.

~~In Cartesian~~ :-

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

~~In Cartesian~~ : * Laplacian of a Scalar electric
~~In Cylindrical~~ :- potential ($\nabla^2 V$), is a Scalar.

x - * Laplacian of an electric field ($\nabla^2 E$) is a vector.

problems :-

(1) A scalar function, V is given by $V = xyz^2$.
Find the gradient of V.

Sol:- Grad $V = \nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$

$$= \frac{\partial}{\partial x}(xyz^2)a_x + \frac{\partial}{\partial y}(xyz^2)a_y + \frac{\partial}{\partial y}(xyz^2)a_z$$

$$\boxed{\nabla V = yz^2 a_x + xz^2 a_y + 2xyz a_z}$$

(2) If a vector, $B = 4xy^2 a_x + 2y^3 a_y + xyz a_z$, find the divergence of B.

Sol:- Div $B = \nabla . B$

$$= \frac{\partial}{\partial x} B_x + \frac{\partial}{\partial y} B_y + \frac{\partial}{\partial z} B_z$$

$$= \frac{\partial}{\partial x}(4xy^2) + \frac{\partial}{\partial y}(2y^3) + \frac{\partial}{\partial z}(xyz)$$

$$= 4y^2 + 6y^2 + xy = 10y^2 + xy.$$

$$\boxed{\nabla . B = 10y^2 + xy}$$

(3) Given a vector. $A = 3x a_x + y a_y + 5z a_z$, find the curl of A.

Sol:- Curl A $= \nabla \times A = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

$$= a_x\left[\frac{\partial}{\partial y}(5z) - \frac{\partial}{\partial z}(y)\right] + a_y\left[\frac{\partial}{\partial z}(3x) - \frac{\partial}{\partial x}(5z)\right] + a_z\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(3x)\right]$$

$$\boxed{\nabla \times A = 0}$$

(4) If the Scalar potential is given by $V = x^2 - y^2 - z^2$ volts. find the Laplacian of v.

Sol:- $V = x^2 - y^2 - z^2$ volts.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= 2 - 2 - 2$$

$$= 2 - 4$$

$$\boxed{\nabla^2 V = -2}$$

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## Electrostatic fields :-

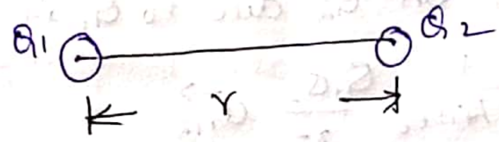
Electrostatic fields are also called Static electric fields or Steady electric fields. These fields are not variant with time. They are produced by Static Charge or charge distributions. These fields are estimated by either Coulomb's law (or) Gauss law.

## Applications of Electrostatic fields :-

* To produce potential
* LCD's
* Touch pads
* Computer peripherals.
* X-ray Machines
* In electrostatic generators.
* To spin cotton
* In electric power transmission.

## Coulomb's Law :-

Consider two point charges $Q_1, Q_2$ seperated by a distance $r$ as shown in the figure below.



**Statement :-** Coulomb's law states that the force exists between $Q_1$ & $Q_2$ will be

(i) directly proportional to the product of $Q_1$ & $Q_2$.

(ii) Inversely proportional to the square of the distance between them.

i.e., $F \propto \frac{Q_1 Q_2}{r^2}$

$$F = K . \frac{Q_1 Q_2}{r^2}$$

Where $K$ = proportionality constant $= 1/4\pi\epsilon$

$\epsilon$ = permitivity or dielectric constant of the medium

$\epsilon = \epsilon_0 + \epsilon_r$.

Where $\epsilon_0$ = permitivity of the free space = $8.854 \times 10^{-12}$ F/m

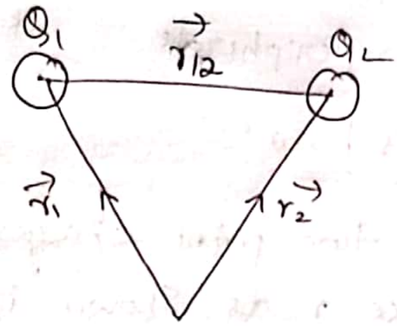
$\epsilon_r$ = relative permittivity of the medium.

∴ For free space $\epsilon_r = 1$, $\epsilon = \epsilon_0$.

then. $\boxed{F = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r^2}}$ Scalar form of Coulomb's Law.

## Vector form of Coulomb's law:-

Let the point charges $Q_1$, $Q_2$ are located at points with position vectors $\vec{r_1}$ and $\vec{r_2}$ respectively as shown in fig.




Then the force on $Q_2$ due to $Q_1$ is

$$\vec{F_{21}} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r^2} \vec{a_{r_{12}}}$$

where $\vec{a_{r_{12}}}$ is the unit vector along the direction vector $\vec{r_{12}}$.

$$\vec{r_{12}} = \vec{r_2} - \vec{r_1} \quad \text{and}$$

The magnitude of $\vec{r_{12}}$ is $|\vec{r_{12}}|$

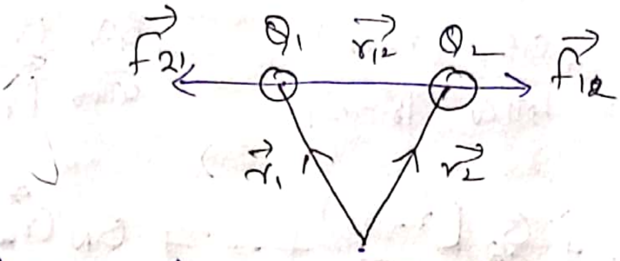
$$r\ |\vec{r_{12}}| = |\vec{r_2} - \vec{r_1}|$$

then $$\vec{F_{12}} = \frac{Q_1 Q_2}{4\pi\epsilon_0} \frac{(\vec{r_2} - \vec{r_1})}{|\vec{r_2} - \vec{r_1}|^3} \longrightarrow (1)$$

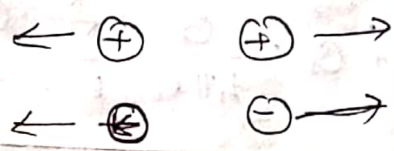
This is the vector form of Coulomb's law. the force on $Q_1$ and $Q_2$ is

$$\vec{F_{21}} = \frac{Q_1 Q_2}{4\pi\epsilon_0} \frac{(\vec{r_1} - \vec{r_2})}{|\vec{r_1} - \vec{r_2}|^3}$$

From $Q_1$ & $Q_2$ $\boxed{\vec{F}_{12} = -\vec{F}_{21}}$




* Like charges are always repels to each other. Therefore force between the two like charges is called "Repulsive force".



* unlike charges are always attract each other. Therefore, force between the two unlike charges is called "Attractive Force".

In attractive force charge migration will appear

$$\boxed{\frac{Q_1 - Q_2}{2}}$$

## Coulomb's law using Superposition principle:-

If there are more than 2 point charges then the force on a particular charge can be determined by "Superposition principle".

For example:- If there are N number of charges $Q_1, Q_2, Q_3, Q_4 \cdots Q_N$. Located at points with position Vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3 \cdots \vec{r}_N$, respectively. Then according to Superposition principle the resultant force on Charge Q located at a point with position vector $\vec{r}$ due to N number of charges is equal to the Vector Sum of forces Caused by each individual Charge that Can be written as

$$\vec{F}_Q = \vec{F}_{1Q} + \vec{F}_{2Q} + \vec{F}_{3Q} + \cdots + \vec{F}_{NQ} \cdot N$$

$$= \frac{QQ_1}{4\pi\epsilon_0}\frac{(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3} + \frac{QQ_2}{4\pi\epsilon_0}\frac{(\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3} + \cdots + \frac{QQ_N}{4\pi\epsilon_0}\frac{(\vec{r}-\vec{r}_N)}{|\vec{r}-\vec{r}_N|^3}$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{Q_1(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3} + \frac{Q_2(\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3} + \cdots + \frac{Q_N(\vec{r}-\vec{r}_N)}{|\vec{r}-\vec{r}_N|^3}\right]$$

$$\boxed{\vec{F}_Q = \frac{Q}{4\pi\epsilon_0}\sum_{k=1}^{N}\frac{Q_k(\vec{r}-\vec{r}_k)}{|\vec{r}-\vec{r}_k|^3}} \ N.$$

## Applications of coulomb's law:-

Coulomb's law is used to.

* To find the force between a pair of charges.
* find the potential at a point due to a fixed charge.
* find the electric field at a point due to a fixed charge
* find the displacement flux density indirectly.
* Find the charge if the force and the electric field are known.

## Limitation of Coulomb's law:-

It is difficult to apply the law when charges are of arbitary shape.

problems :-

① Find the force of interaction between two charges spaced 20cm. apart in a vaccum the charges are $4\mu C$ and $5\mu C$ respectively.

② If the same charges are seperated by same distance in kirosene ($\epsilon_r = 2$). what is the corresponding force of interaction.

(A) Given data. $r = 20cm = 20\times10^{-2}m$

$Q_1 = 4\mu C = 4\times10^{-6}C$

$Q_2 = 5\mu C = 5\times10^{-6}C$

Formulae :-

① when the charges are in free space.

$$F = \frac{1}{4\pi\epsilon_0}\frac{Q_1Q_2}{r^2}$$

$\pi = 3.14$

$\epsilon_0 = 8.854\times10^{-12}\ F/m$

$$\frac{1}{4\pi\epsilon_0} = \frac{1}{4\times3.14\times8.854\times10^{-12}}$$

$$= 9\times10^9.$$

$$= \frac{9\times10^9 \times (4\times10^{-6})(5\times10^{-6})}{(20\times10^{-2})^2}$$

$$\boxed{F = 4.5\,N}$$

② when the charges are in kirosene.

$\epsilon_r = 2.$

$$F = \frac{1}{4\pi\epsilon_0\epsilon_r}\frac{Q_1Q_2}{r^2}$$

we know that $\frac{1}{4\pi\epsilon_0}\cdot\frac{Q_1Q_2}{r^2} = 4.5N.$

$$F = \frac{4.5}{2} = 2.25N.$$

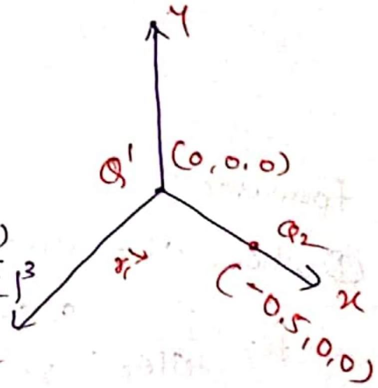
$$\boxed{F = 2.25N.}$$

② A charge of $Q_1 = -1\mu c$ is placed at the origin of a rectangular co-ordinate system and a second charge $Q_2 = -10mc$ is placed on the x-axis at a distance of 50cm from the origin. Find the force on $Q_1$ due to $Q_2$ if they are in free space.

Ⓐ Given data $Q_1 = -1\mu c = -1\times10^{-6}c$

$Q_2 = -10mc = -10\times10^{-3}c$

Formulae $= \frac{1}{4\pi\epsilon_0} Q_1 Q_2 \frac{|\vec{r_1} - \vec{r_2}|}{|\vec{r_1} - \vec{r_2}|^3}$




$= (9\times10^9)(-1\times10^{-6})(-10\times10^{-3}) \frac{(-0.5,0,0)}{|\sqrt{(-0.5)^2+0^2+0^2}|^3}$

$$\boxed{\vec{F_{21}} = -360\vec{a_x}\,N}$$

③ Two identical spheres are $2nC$ & $-0.5nC$ are placed with a distance of 4cm.

(i) What is the force between them

(2) If they are brought into contact and then seperated 4cm. then what is the force between them.

Given data: $Q_1 = 2\times10^{-9}c$

$Q_2 = -0.5\times10^{-9}c$

$r = 4cm = 4\times10^{-2}m$

(i) $F = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r^2}$

$$F = \frac{(9\times10^9)(2\times10^{-9})(-0.5\times10^{-9})}{(4\times10^{-2})^2} = -5.625\mu N$$

(ii) Charge migration $\frac{(Q_1 - Q_2)}{2}$

$$= \frac{(2-(-0.5))}{2} = \frac{2+0.5}{2} = \frac{\overset{1.25}{\cancel{2.5}}}{\cancel{2}} = 1.25$$

$Q_1 = 1.25$

$Q_2 = 1.25$

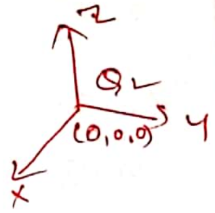
$$\vec{F} = \frac{(9\times10^{9})(1.25\times10^{-9})(1.25\times10^{-9})}{(4\times10^{-2})^{2}}$$

$$\vec{F} =$$

(4) A point charge $Q_1 = 2\mu C$ is at $(2,3,6)$ and another charge $Q_2 = 5\mu C$ at $(0,0,0)$ in free space. Find the force on $Q_1$ due to $Q_2$.

Given data. $Q_1 = 2\mu C = 2\times10^{-6} C$ at $(2,3,6)$

$Q_2 = 5\mu C = 5\times10^{-6} C$ at $(0,0,0)$ at origin




Formulae:- $\vec{F}_{21} = \frac{1}{4\pi\epsilon_0} Q_1 Q_2 \frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}$

$$\vec{F}_{21} = (9\times10^{9})(2\times10^{-6})(5\times10^{-6})\frac{(2,3,6)}{\left(\sqrt{2^2+3^2+6^2}\right)^3}$$

$$= 0.09 \frac{(2,3,6)}{\sqrt{2^2+3^2+6^2})^3}$$

$$= 2.623\times10^{-4}(2,3,6)$$

$$= (0.262(2,3,6))10^{-3}$$

$$= (0.524, 0.786, 1.57)\ MN$$

$$\vec{F}_{21} = (0.524\,\vec{a_x} + 0.786\,\vec{a_y} + 1.57\,\vec{a_z})\ MN.$$

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## Electric field (or) Electric field intensity (or) Electric field Strength.

**Definition:-** The Electric field due to a charge is defined as the Coulomb's force per unit charge.

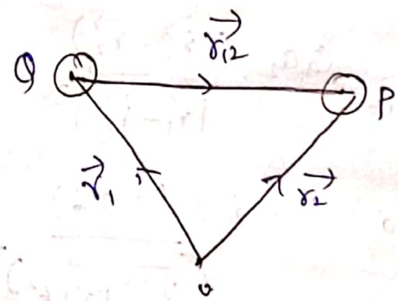
Electric field, $E = \frac{F}{Q}$, N/C or V/m.

where F = Coulomb's Force

Q = charge.

* The direction of Electric field $\vec{E}$ is as same as Force $\vec{F}$.

Let a point charge Q located at a point with position vector $\vec{r_1}$ as shown in the fig below. The Electric field point P with a position vector $\vec{r_2}$ due to charge Q can be written as




$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\vec{a_{rQ}}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q(\vec{r_2}-\vec{r_1})}{|\vec{r_2}-\vec{r_1}|^3} \quad V/m$$

Let's consider N number of charges $Q_1, Q_2, \ldots Q_n$ located at points with position vectors $\vec{r_1}, \vec{r_2}, \ldots \vec{r_N}$ then the electric field at a point P with a position vector $\vec{r}$ can be written as

$$\vec{E} = \vec{E_1} + \vec{E_2} + \ldots \ldots + \vec{E_N}$$

$$\vec{E} = \frac{Q_1}{4\pi\epsilon_0}\frac{(\vec{r}-\vec{r_1})}{|\vec{r}-\vec{r_1}|^3} + \frac{Q_2}{4\pi\epsilon_0}\frac{(\vec{r}-\vec{r_2})}{|\vec{r}-\vec{r_2}|^3} + \ldots \ldots + \frac{Q_n}{4\pi\epsilon_0}\frac{(\vec{r}-\vec{r_n})}{|\vec{r}-\vec{r_n}|^3}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{K=1}^{N} Q_N \frac{(\vec{r}-\vec{r}_k)}{|\vec{r}-\vec{r}_k|^3}$$

**Features of Electric field Intensity:-**

* It is a vector
* It's direction is the same as that of Coulomb's force
* It depends on the location of the charges
* It depends on the permitivity of the medium
* It has both magnitude and direction.

**problems :-**

1) If Coulomb's force, $F = 2a_x + a_y + a_z$ N, is acting on a charge of 10C, find the electric field intensity, its magnitude and direction.

Sol:- Force, $F = 2a_x + a_y + a_z$, N.

$Q = 10C$

$E = F/Q$

$$= \frac{2a_x + a_y + a_z}{10}$$

$E = 0.2a_x + 0.1a_y + 0.1a_z$, N/C

The magnitude of E is

$$\boxed{a_E = \frac{\vec{E}}{|E|}}$$

$E = |E| = \sqrt{(0.2)^2 + (0.1)^2 + (0.1)^2}$

$= \sqrt{0.04 + 0.01 + 0.01} = \sqrt{0.06}$

$= 0.2449$ V/m

The direction of E is

$$a_E = \frac{E}{E} = \frac{0.2a_x + 0.1a_y + 0.1a_z}{0.2449}$$

$a_E = 0.816a_x + 0.408a_y + 0.408a_z$

② There are three charges which are given by $Q_1 = 1\mu C$, $Q_2 = 2\mu C$, $Q_3 = 3\mu C$. The field due to each charge at a point, P in free space is $a_x + 2a_y - a_z$, $a_y + 3a_z$ and $2a_x - a_y$ N/c. Find the total field at a the point, P due to all the three charges.

Sol: The field, $E_1$ at P due to $Q_1$ ($\mu c$)

$$= a_x + 2a_y - a_z, \ N/c$$

The field, $E_2$ due to $Q_2$ ($2\mu c$)

$$= a_y + 3a_z, \ N/c$$

The field $E_3$ due to $Q_3$ ($3\mu c$)

$$= 2a_x - a_y, \ N/c$$

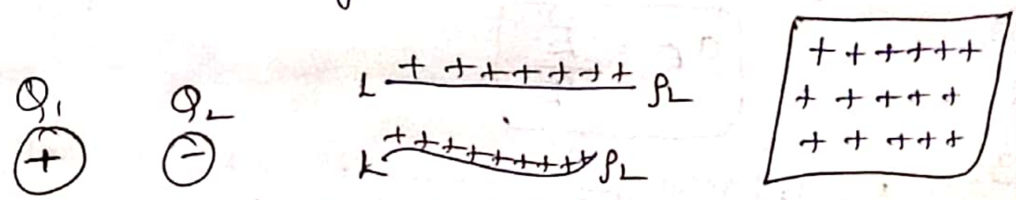
The total field at P, $E = E_1 + E_2 + E_3$

$$E = a_x + 2a_y - a_z + a_y + 3a_z + 2a_x - a_y$$

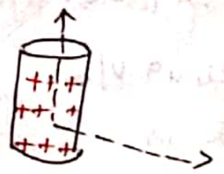
$$E = 3a_x + 2a_y + 2a_z, \ N/c$$

## Different types of charge distributions:-

In addition to a point charge, there is a possibility to have a continuous/uniformly distribution charge along a line, over a surface or in a volume. as in the below figure.




(a) point charge (b) line charge (c) Surface charge



(d) volume charge

① **point charge:-** The point charge has a position but not the dimensions. point charge can be positive or negative. $\oplus^{Q_1}$ $\ominus^{Q_2}$

(a) point charge

② **Line charge density:-**

when the charge is continuously distributed along a line then the term line charge density ($\rho_L$) is used. and it is defined as charge per length.

i.e., $$\rho_L = \frac{dQ}{dl} \quad (C/m)$$

$$dQ = \rho_L dl$$

where $dQ$ = differential charge in Coulomb's
$dl$ = differential length in metre
& $\rho_l$ = line charge density in c/metre

The total charge $Q = \int_L \rho_L dL$

∴ Electric field $\vec{E}$ due to point charge is

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2} \vec{a_r}$$

Electric field due to line charge is

$$\vec{E} = \int_L \frac{\rho_L dl}{4\pi\epsilon r^2} \vec{a_r}$$

③ **Surface charge density ($\rho_s$):-**

when a ~~Surface~~ charge is continuously distributed over a surface then the term charge density is used, and it is defined as charge per surface. i.e. $\rho_s = \frac{dQ}{ds} \; (C/m^2)$

$$dQ = \rho_s \, ds$$

Where $dQ$ = differential charge in Coulomb's

$\rho_s$ = Surface charge density in Coulomb's/$m^2$

$ds$ = differential Surface in $(\text{metre})^2$

Total Charge $Q = \int_S \rho_s \, ds$

∴ Electric field $\vec{E}$ due to point charge is

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2} \vec{a_r}$$

∴ Electric field due to Surface charge is

$$\vec{E} = \int_S \frac{\rho_s \, ds}{4\pi\epsilon r^2} \vec{a_r}$$

(4) Volume Charge density ($\rho_v$) :-

when a charge is continuously distributed over a volume then the term volume charge density ($\rho_v$) is used. and is defined as charge per volume.

i.e., $\rho_v = \frac{dQ}{dv}$ $(C/m^3)$

$$dQ = \rho_v \, dv.$$

Where $dQ$ = differential charge in Coulomb's

$dv$ = differential charge ~~density~~ in $(m^3)$

$\rho_v$ = Volume charge density in $C/m^3$.

Total charge $Q = \int \rho_v \, dv.$

∴ Electric field $\vec{E}$ due to point charge is

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2} \vec{a_r}$$

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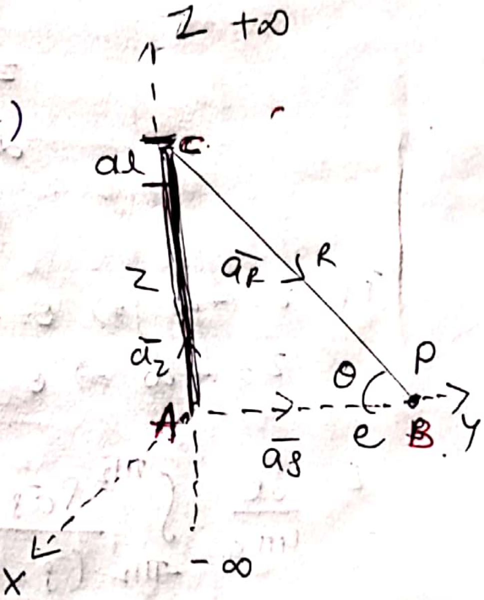
electric field $\vec{E}$ due to volume charge is

$$\vec{E} = \int_V \frac{\rho_v dv}{4\pi \epsilon r^2} \vec{a_r}$$

## Electric Field Intensity due to Infinite line charge:-

let's consider

$P(0, \rho, 0)$ & $dl(0,0,z)$




$\bar{R} = (0, \rho, 0)$ & $dl(0,0,z)$

$\bar{R} = (0, \rho, 0) - (0,0,z)$

$= 0 + \rho \bar{a_y} - z\bar{a_z}$

$\bar{R} = \rho \bar{a_y} - z\bar{a_z}$

$$\bar{a_R} = \frac{\bar{R}}{|R|} = \frac{\rho \bar{a_\rho} - z\bar{a_z}}{\sqrt{\rho^2 + z^2}}$$

$$\Rightarrow E = \frac{1}{4\pi\epsilon_0 R^2} \int_l \rho_l dl \bar{a_R} \rightarrow ①$$

$$dl = \underbrace{dx\bar{a_x} + dy\bar{a_y}}_{0} + dz\bar{a_z}$$

$dl = dz\bar{a_z}$

$dl = dz$ [$\because$ unit vector $= 1$]

$$E = \frac{1}{4\pi\epsilon_0} \frac{1}{(\sqrt{\rho^2+z^2})^2} \int_{-\infty}^{\infty} \rho_l dz \left( \frac{\rho \bar{a_\rho} - z\bar{a_z}}{\sqrt{\rho^2+z^2}} \right)$$

$$E = \frac{\rho_l}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\rho \bar{a_\rho} - z\bar{a_z}}{(\rho^2+z^2)^{3/2}} dz \longrightarrow ②$$

Note:- For every charge on +ve z-axis there will be equal & opposite charge on -ve z-axis

$\boxed{\therefore z\bar{a_z} = 0}$

$$E = \frac{\rho_l}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\rho \bar{a}_\rho}{(\rho^2+z^2)^{3/2}} dz$$

from $\Delta ABC$, $\sin\theta = z/R$

$\cos\theta = \frac{\rho}{R}$

$\tan\theta = z/\rho$

$z = \rho \tan\theta$

$dz = \rho \sec^2\theta \, d\theta$.

$\theta = \tan^{-1}(z/\rho)$

If $z = -\infty \Rightarrow \theta = -\pi/2$

$z = +\infty \Rightarrow \theta = \pi/2$

$$E = \frac{\rho_l}{4\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} \frac{\rho \bar{a}_\rho}{(\rho^2+\rho^2\tan^2\theta)^{3/2}} \rho \sec^2\theta \, d\theta$$

$$E = \frac{\rho_l}{4\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} \frac{\rho^2 \sec^2\theta}{(\rho^2 \sec^2\theta)^{3/2}} d\theta . \bar{a}_\rho$$

$$E = \frac{\rho_l}{4\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} \frac{1 . \sec^2\theta}{\rho . \sec^3\theta} d\theta . \bar{a}_\rho$$

$$E = \frac{\rho_l}{4\pi\epsilon_0 \rho} \int_{-\pi/2}^{\pi/2} \frac{1}{\sec\theta} d\theta . \bar{a}_\rho$$

$$E = \frac{\rho_l}{4\pi\epsilon_0 \rho} \int_{-\pi/2}^{\pi/2} \cos\theta . d\theta . \bar{a}_\rho \quad \left(\because \frac{1}{\sec\theta} = \cos\theta\right)$$

$$E = \frac{\rho_l}{4\pi\epsilon_0 \rho} [\sin\theta]_{-\pi/2}^{\pi/2} \bar{a}_\rho$$

$$E = \frac{\rho_l}{4\pi\epsilon_0 \rho} \left(\sin\frac{\pi}{2} + \sin\pi/2\right) \bar{a}_\rho$$

$$E = \frac{\rho_l}{\cancel{2}4\pi\epsilon_0 \rho} (\cancel{2}) . \bar{a}_\rho$$

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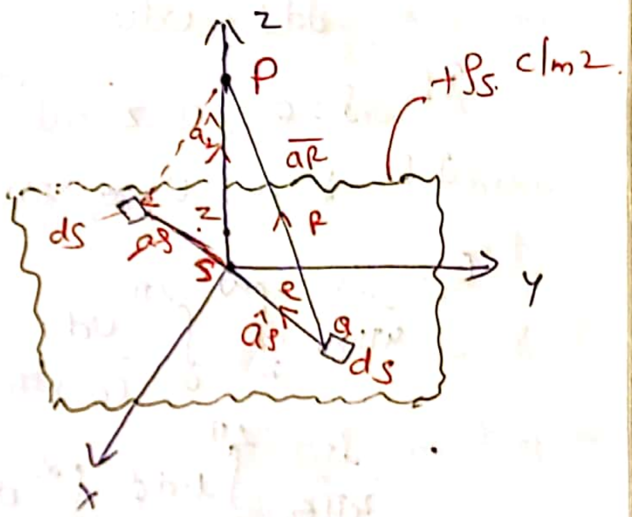
$E = \frac{\cancel{2}\rho_l}{\cancel{4}\pi\epsilon_0\rho}\bar{a_\rho}$ . $E = \frac{\rho_l}{2\pi\epsilon_0\rho}\bar{a_\rho}$ N/m or V/m

## Electric Field Intensity due to infinite Surface charge

The differential surface in cylindrical co-ordinate System along z-direction is ds.

$ds = \rho\, d\rho\, d\phi$




from $\Delta PSQ$

$\Delta PSQ$: $\rho\bar{a_\rho} + R\bar{a_R} = z\bar{a_z}$

$R\bar{a_R} = z\bar{a_z} - \rho\bar{a_\rho}$

$\bar{R} = z\bar{a_z} - \rho\bar{a_\rho}$

$|\bar{R}| = \sqrt{z^2+\rho^2}$

Unit vector. $\bar{a_R} = \frac{\bar{R}}{|\bar{R}|} = \frac{z\bar{a_z} - \rho\bar{a_\rho}}{\sqrt{z^2+\rho^2}}$.

Electric Field Intensity. $E = \frac{1}{4\pi\epsilon_0 R^2}\int_S \rho_s\, ds.\bar{a_R}$

$$E = \frac{\rho_s}{4\pi\epsilon_0} \cdot \frac{1}{(z^2+\rho^2)}\int_S \frac{\rho\, d\rho\, d\phi\,(z\bar{a_z} - \rho\bar{a_\rho})}{\sqrt{z^2+\rho^2}}$$

$$= \frac{\rho_s}{4\pi\epsilon_0} \cdot \frac{1}{(z^2+\rho^2)}\int_S \frac{\rho\, d\rho\, d\phi \cdot z\bar{a_z}}{\sqrt{z^2+\rho^2}}$$

Note:- For every charge on positive axis, there exists a equal charge in opposite direction.

$\Rightarrow$ $\rho\bar{a_\rho}$ & $-\rho\bar{a_\rho'}$ are cancel each other at point P. ($\rho\bar{a_\rho} = 0$)

$$E = \frac{\rho_s}{4\pi\epsilon_0} \int_0^{\infty}\int_0^{2\pi} \frac{1}{(\rho^2+z^2)^{3/2}} \rho\, d\rho\, d\phi \cdot z\bar{a_z}$$

[$\because$ $0-\infty$ for $\rho$ & $0-2\pi$ for $\phi$ limits]

let $\rho^2+z^2 = u^2$

$2\rho\, d\rho = 2u\, du$

$\rho\, d\rho = u\, du$

If $\rho = 0$ ; $u = z$

If $\rho = \infty$ ; $u = \infty$

$$\therefore E = \frac{\rho_s}{4\pi\epsilon_0} \int_z^{\infty}\int_0^{2\pi} \frac{u\, du}{(u^2)^{3/2}} z\cdot\bar{a_z}\, d\phi$$

$$= \frac{\rho_s}{4\pi\epsilon_0} \int_0^{2\pi} 1\cdot d\phi \int_z^{\infty} \frac{u}{u^{3}} du\cdot z\bar{a_z}$$

$$= \frac{\rho_s}{4\pi\epsilon_0} \int_0^{2\pi} 1\cdot d\phi \int_z^{\infty} \frac{1}{u^2} du\cdot z\bar{a_z}$$

$$= \frac{\rho_s}{\cancel{4}\pi\epsilon_0} (\cancel{2}\pi)\cdot\left[-\frac{1}{u}\right]_z^{\infty} z\bar{a_z}$$

$$= \frac{\rho_s}{2\cancel{\pi}\epsilon_0} = \frac{\rho_s}{2\epsilon_0}\left[-\frac{1}{\infty}+\frac{1}{z}\right] z\bar{a_z}$$

$$E = \frac{\rho_s}{2\epsilon_0}\cdot\frac{1}{\cancel{z}}\cdot\cancel{z}a_z$$

$$\boxed{E = \frac{\rho_s}{2\epsilon_0}\bar{a_z}}$$ N/C (or) V/m

## Electric flux density [D] :

(a) **Electric flux ($\psi$)** :- The total no. of flux lines in an electric field is known as "electric flux" (or) "Displacement flux" or (stream lines). It is denoted by letter "$\psi$". and units of electric flux is Coulomb's. It is a scalar quantity. A electric charge produces an electric flux.

Electric flux is directly proportional to electric field area and angle between them.

i.e., $\psi = EA\cos\theta$

$\psi = |\vec{E}||\vec{A}|\cos\theta.$

$\boxed{\psi = \vec{E}\cdot\vec{A}}$ C. This is the symmetric surface of electric flux.

* The asymmetric surface, the differential flux is given as $d\psi = E\cdot ds\cos\theta.$

$$\psi = \int_S E ds \cos\theta.$$

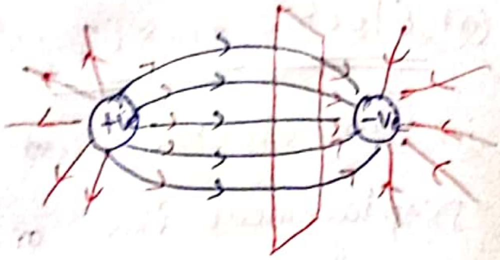
$$\psi = \int_S |\vec{E}||\vec{ds}|\cos\theta.$$

$$\boxed{\psi = \int_S \vec{E}\cdot\vec{ds}}\ C$$

For closed surface $\boxed{\psi = \oint_S \vec{E}\cdot d\vec{s}}$ C . This is the total electric flux for ~~Asymmetric~~ closed surface. 1 coulomb of charge will generates 1 coulomb of flux lines. i.e, $1Q = 1\psi$.

**Electric flux density (D)** :- It is defined as electric flux per unit surface and is expressed in C/m². i.e, $\vec{D} = \frac{\psi}{S}\vec{a_r}$ C/m². → Symmetric surface

$$\vec{D} = \frac{d\psi}{ds}\,\vec{a_r}$$




$$d\psi = \vec{D}\cdot ds\,\vec{a_r}$$

$$d\psi = \vec{D}\cdot\vec{ds}$$

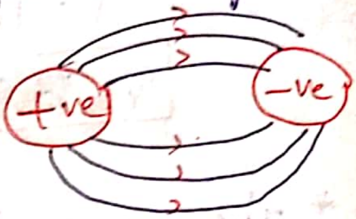
$$\boxed{\psi = \int_s \vec{D}\cdot\vec{ds}}$$

$$\boxed{\psi = \int_s |\vec{D}|\,|\vec{ds}|\cos\theta} \quad C \quad \ldots \text{Asymmetric Surface}$$

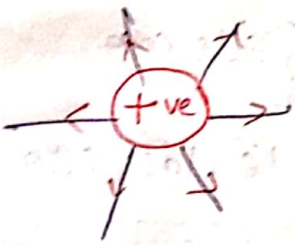
For a closed Surface $\boxed{\psi = \oint_s \vec{D}\cdot\vec{ds}}$ C

Properties of Flux lines :-

* All the flux lines starts at positive Charge and ends at negative charge.




* If Negative charge is absent then the flux lines starts at positive charge and ends at infinity.




* If positive charge is absent then the flux lines starts at infinity and ends at Negative charge.

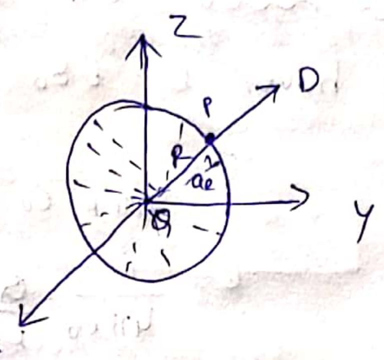

* All the flux lines are parallel to each other and never cross each other.

* If the Charge at a particular point is $\pm Q$ then the total no. of flux lines originating or

termination from the charge is equal to the electric flux i.e, $\boxed{\psi = Q}$

## Electric Flux density due to different charge distribution.

① point charge:-




Let us consider an Imaginary Sphere where point charge 'Q' is located at the centre of an imaginary sphere with radius of R.

* From the definition of electric flux density.

$$D = \frac{\psi}{S}$$

$$1Q = 1\psi.$$

$$D = \frac{Q}{S}, \quad \text{where}$$

$S \rightarrow$ area of imaginary sphere $\Rightarrow S = 4\pi R^2$.

$$\boxed{\therefore D = \frac{Q}{4\pi R^2}\bar{a_R}}$$

## Relation between D & $\bar{E}$:

We know that, $D = \frac{Q}{4\pi R^2}\bar{a_R}$ &

$$E = \frac{Q}{4\pi\epsilon_0 R^2}\bar{a_R}$$

Now, $\frac{D}{E} = \frac{\cancel{Q}}{4\pi R^2} \times \frac{4\pi R^2 \cdot \epsilon_0}{\cancel{Q}} \cdot \bar{a_R} \cdot \bar{a_R}$

$[\because \bar{a_R} \cdot \bar{a_R} = 1]$

$\frac{D}{E} = \epsilon_0.$ $\boxed{D = \epsilon_0 E}$

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* D is always directly proportional to 'ε' with a constant of $\epsilon_0$ (permitivity of free space).

## EFD due to line charge

We have, $D = E\epsilon_0$

$$E = \frac{1}{4\pi R^2 \epsilon_0} \cdot Q \cdot \bar{a}_R$$

$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{R^2} \bar{a}_R$$

$$E = \frac{1}{4\pi\epsilon_0 R^2} \int_l \rho_l dl \cdot \bar{a}_R$$

$$D = \cancel{\epsilon_0} \frac{Q}{4\pi R^2 \cancel{\epsilon_0}} \int_l \rho_l dl \, \bar{a}_R$$

$$D = \frac{1}{4\pi R^2} \int_l \rho_l dl \, \bar{a}_R$$

## D due to Infinite line:-

WKT. $D = E\epsilon_0$

$$E = \frac{\rho_l}{2\pi\epsilon_0 \rho}$$

Then $D = \cancel{\epsilon_0} \cdot \frac{\rho_l}{2\pi\cancel{\epsilon_0} \cdot \rho}$

$$D = \frac{\rho_l}{2\pi\rho}$$

## EFD due to Surface charge:-

We have, $D = E\epsilon_0$

$$E = \frac{1}{4\pi\epsilon_0 R^2} \int_S \rho_s ds \, \bar{a}_R$$

Now, $D = \cancel{\epsilon_0} \frac{1}{4\pi \cancel{\epsilon_0} R^2} \int_S \rho_s \, ds \, \bar{a_R}$

$$\boxed{D = \frac{1}{4\pi R^2} \int_S \rho_s \, ds \, \bar{a_R}}$$

* **EFD due to Infinite sheet or surface:-**

WKT, $D = \epsilon_0 E$

$E = \frac{\rho_s}{2\epsilon_0} \bar{a_z}$

Then $D = \cancel{\epsilon_0} \cdot \frac{\rho_s}{2\cancel{\epsilon_0}} \bar{a_z}$

$$\boxed{D = \frac{\rho_s}{2} \bar{a_z}}$$

**EFD due to volume charge:-**

We have, $D = E\epsilon_0$.

$E = \frac{1}{4\pi \epsilon_0 R^2} \int_V \rho_v \, dv \, \bar{a_R}$

Now $D = \cancel{\epsilon_0} \cdot \frac{1}{4\pi \cancel{\epsilon_0} R^2} \int_V \rho_v \, dv \, \bar{a_R}$

$$\boxed{D = \frac{1}{4\pi R^2} \int_V \rho_v \, dv \, \bar{a_R}}$$

# Gauss Law:-

Gauss Law states that the total electric flux passing through a closed surface is equal to total charge enclosed by the same surface mathematically.

$$\boxed{\therefore \psi = Q\,(\text{enclosed})}$$

**proof:-** Let us consider an imaginary sphere where the point charge Q is located at the centre of an image sphere with radius of R.

From the definition of EFD

$$D = \frac{\psi}{s} \rightarrow (1)$$

$$\psi = Q$$

$$D = \frac{Q}{s} \longrightarrow S = 4\pi R^2$$

$$D = \frac{Q}{4\pi R^2} \bar{a}_R \longrightarrow (2)$$

We have, $\psi = \int_s D.ds \longrightarrow (3)$

$$D.ds = \frac{Q}{4\pi R^2} \bar{a}_R .ds$$

$$D.ds = \frac{Q}{4\pi R^2} ds \quad (\because \bar{a}_R = 1)$$

$$\int_s D.ds = \int_s \frac{Q}{4\pi R^2} ds$$

$$\int_s D.ds = \frac{Q}{4\pi R^2} \int_s 1.ds$$

$$\int_s D.ds = \frac{Q}{4\pi R^2} . S \Rightarrow \frac{Q}{\cancel{4\pi R^2}} . \cancel{4\pi R^2}$$

$$\oint_S D \cdot ds = Q$$

From eq$^n$ (3), we have

$$\oint_S D \cdot ds = \psi$$

$$\boxed{\therefore \psi = Q}$$

## I - MAXWELL Equation for Electrostatic fields:

Maxwell I eqn was derived with the help of Gauss Law. i.e, $\boxed{\nabla D = \rho_v}$

= divergence of EFD = volume charge density

W.K.T

$$D = \frac{\psi}{s}$$

$$D = \frac{d\psi}{s}$$

$$d\psi = D \cdot ds$$

$$\psi = \oint_S D \cdot ds \longrightarrow (1) \qquad \boxed{\psi = Q}.$$

$$\oint_S D \cdot ds = Q \longrightarrow (2)$$

while calculating the charge we will consider its total charge with its volume charge density as $\rho_v$

i.e, $Q = \int_V \rho_v dv$

then eqn (2) can be given as

$$\oint_S D \cdot ds = \int_V \rho_v \cdot dv \longrightarrow (3)$$

Eqn (3) is called as Integral form of Maxwell I equation.

→ Apply divergence theorem to L.H.S. side of eq,n (3)

$$\int_S D\cdot ds = \int_V \nabla\cdot D \, dv \rightarrow (4)$$

$$\because \int_V \nabla\cdot\bar{A}\, ds = \int_S A\cdot ds$$

Then $\int_V \nabla\cdot D\, dv = \int_V \rho_v\, dv$

$\boxed{\nabla D = \rho_v}$ → Differential form of Maxwell I - equation.

## Limitations of Gauss Law:-

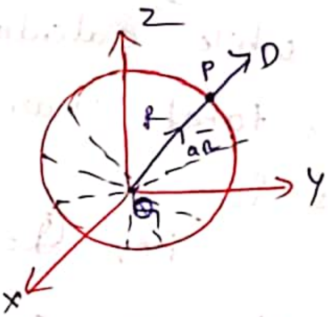
* It is applicable only if the surface is a Gaussian surface (or) closed surface.

## Applications:-

* point charge using Gauss law
* Infinite line charge using Gauss law.
* Infinite surface charge using Gauss law.

## 1. point charge using Gauss Law:-

We have $D = \frac{\psi}{s}$




$\boxed{\psi = Q}$

$D = \frac{Q}{s}$

Then. $D = \frac{Q}{4\pi R^2}\bar{a}_R \quad C/m^2$

W.K.T, $D = E\epsilon_0$

$E = \frac{D}{\epsilon_0}$

$E = \frac{Q}{4\pi\epsilon_0 R^2}\cdot\bar{a}_R \quad V/m$

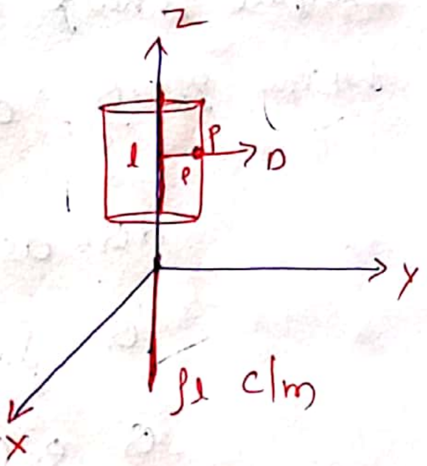
# Infinite line charge using Gauss Law

Consider a uniform line charge density of infinite line as $\rho_l$, which lies on z-axis

* From the limitations of Gauss law. let us Consider a closed Surface in the form of a cylinder.

where $l \rightarrow$ represents length of cylinder
$\rho \rightarrow$ distance from z-axis to point P.

W.K.T. $\rho_l = \frac{Q}{l} = \frac{dQ}{dl}$




$dQ = \rho_l dl$

$Q = \int_l \rho_l dl$

$Q = \rho_l \cdot \int_l dl$

$\boxed{Q = \rho_l \cdot l} \longrightarrow (1)$

From the Concept of Gauss Law.

$\psi = Q.$

$\int_s D \cdot ds = \rho_l \cdot l.$

$D \cdot \int_s 1 \cdot ds = \rho_l \cdot l$

$D(s) = \rho_l \cdot l$

$D = \frac{\rho_l \cdot l}{s}$

where s represents area of the cylinder.

$\therefore \quad S = 2\pi \rho l$

Now, $D = \frac{\rho_l \cdot \cancel{l}}{2\pi \rho \cancel{l}}$

$\boxed{D = \frac{\rho_l}{2\pi \rho} \bar{a}_\rho} \quad c/m^2$

We have, $D = E\epsilon_0$

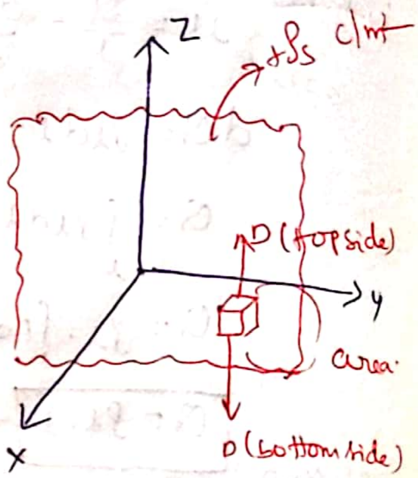
$$E = \frac{D}{\epsilon_0} = \frac{\rho_l}{2\pi\epsilon_0 \rho}\,\bar{a_\rho}$$

$$\boxed{E = \frac{\rho_l}{2\pi\epsilon_0 \rho}\,\bar{a_\rho}} \quad V/m$$

③ Infinite Surface / sheet charge using Gauss Law.

Consider an infinite sheet with uniform Surface charge density $\rho_s$.

WKT. $\rho_s = \frac{Q}{S} = \frac{dQ}{ds}$




$$dQ = ds.\rho_s$$

$$Q = \int_S \rho_s . ds$$

$$Q = \rho_s \int_S 1.ds$$

$$Q = \rho_s . S$$

From the Concept of Gauss law.

$\psi = Q \quad \Rightarrow \quad Q = \rho_s . S$

$$\int_S D.ds = \rho_s . S$$

$$D \int_S 1.ds = \rho_s . S$$

$$D\left[\int_{top} 1.ds + \int_{Bottom} 1.ds\right] = \rho_s S$$

$$D[2S] = \rho_s . S$$

$$D = \frac{\rho_s}{2}\, a_z \quad c/m^2$$

We have, $D = E\epsilon_0$

$$E = \frac{D}{\epsilon_0} = \frac{\rho_s}{2\epsilon_0} \bar{a}_z$$

$$\boxed{E = \frac{\rho_s}{2\epsilon_0} \bar{a}_z} \quad V/m$$

problem :-

If electric flux density $D = (2y^2+3)\bar{a}_x + 4xy\bar{a}_y + x\bar{a}_z$ $C/m^2$

① Determine volume charge density.

② Determine the total flux passing through a Cube is defined as $0 \le x \le 1 : 0 \le y \le 1, 0 \le z \le 1$.

③ Calculate the total charge enclosed by the Same cube.

Sol :- Given $D = (2y^2+3)a\bar{x} + 4xy\bar{a}_y + \bar{x}a_z$

① $\rho_v = \nabla \cdot D$

$$\rho_v = \left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right) \cdot D$$

$$= \left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right)\left((2y^2+3)\bar{a}_x + 4xy\bar{a}_y + x\bar{a}_z\right)$$

$$= \frac{\partial}{\partial x}(2y^2+3) + \frac{\partial}{\partial y}4xy + \frac{\partial}{\partial z}\cdot x \quad (\because a_x \cdot a_x = 1)$$

$$= 0 + 4x + 0$$

$$\boxed{\rho_v = 4x}$$

② $\psi = Q$

$$Q = \int_v \rho_v dv = \int_0^1\int_0^1\int_0^1 4x\,dx\,dy\,dz$$

$$= \int_0^1 4x\,dx \int_0^1 1\,dy \int_0^1 1\,dz$$

$$= 4\left[\frac{x^2}{2}\right]_0^1 \times 1 \times 1 \quad = \frac{4^2}{2} \times 1 \times 1 \quad \boxed{\psi = 2\ C}$$

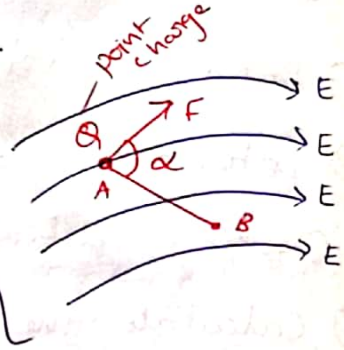
(3) Total Charge

From $\psi = Q$

$Q = 2C$

## Electric potential (or) potential Difference (V).

Let us Consider an Electric field lines, in that Q is a point charge.




* From the diagram a force is exists between Electric charge & point Charge. which is moving along the direction of E.

* We would like to move a point charge from A to B with a differential length. Then we can say that a small work is done from A→B

→ Mathematically,

$$dw = F dl \cos\alpha.$$

$$E = \frac{F}{Q}$$

$$F = Q.E$$

$$dw = QE.dl.\cos\alpha$$

If $\alpha = 90°$, $dw = 0$

If $\alpha = 180°$, $dw = -QE\,dl\,\cos\alpha$

$$w = -\int_l QE.dl$$

$$\boxed{W = -Q\int_A^B E.dl} \cdot \text{Joules} \longrightarrow (1)$$

The Electric potential or potential difference is defined as total work done per unit charge.

i.e, $\boxed{V = \frac{W}{Q}\ J/C}$

$$V_{AB} = \frac{-\cancel{Q}\int_A^B E \cdot dl}{\cancel{Q}}$$

$$V_{AB} = -\int_A^B E \cdot dl \longrightarrow ②$$

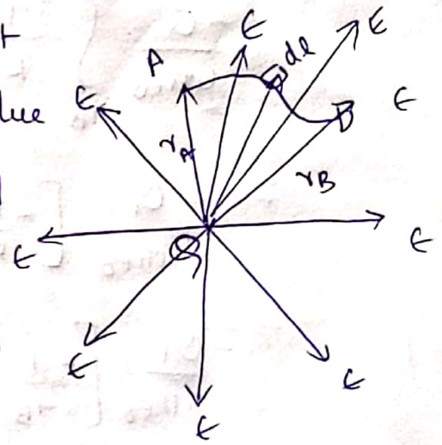
$$\boxed{V = -\int_l E\, dl} \longrightarrow ③$$

Note:- ① If $V_{AB}$ = −ve value, ie, there is a loss of potential energy while moving from $A \to B$

② If $V_{AB}$ = +ve value, there is a gain of potential energy while moving from $A \to B$.

## Electric potential due to point charge:-

Let us Consider a point charge Q at origin, due to charge Electric field lines are distributed in all directions as shown above.




⇒ $r_A$ & $r_B$ → position vectors from origin → A & origin → B

⇒ When a point charge is moving from $A \to B$ with a differential length dl, then we can say that a small work is done

Mathematically From the definition of Electrical potential,

$$V_{AB} = -\int_l E\,dl$$

$$V_{AB} = -\int_A^B E \cdot dl$$

$$E = \frac{Q}{4\pi\epsilon_0 R^2}\bar{a_R}$$

$$V_{AB} = -\int_A^B \frac{Q}{4\pi\epsilon_0 R^2}\bar{a_R}\,dl.$$

Assume. Spherical Coordinate System.

$$dl = dR\bar{a_R} + Rd\theta\bar{a_\theta} + R\sin\theta\, d\phi\, \bar{a_\phi}$$

$$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 R^2}\bar{a_R}\left[dR\bar{a_R} + Rd\theta\bar{a_\theta} + R\sin\theta d\phi \bar{a_\phi}\right]$$

$$= -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 R^2}\bar{a_R}\cdot dR\cdot\bar{a_R} \qquad \left[\because \begin{array}{l}\bar{a_R}\cdot\bar{a_R} = 1\\ \bar{a_R}\cdot\bar{a_\theta} = 0\end{array}\right]$$

$$= \frac{-Q}{4\pi\epsilon_0}\int_{r_A}^{r_B} \frac{1}{R^2}\,dR$$

$$= \frac{-Q}{4\pi\epsilon_0}\left[-\frac{1}{R}\right]_{r_A}^{r_B}$$

$$= \frac{-Q}{4\pi\epsilon_0}\left[-\frac{1}{r_B} + \frac{1}{r_A}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_0 r_B} - \frac{Q}{4\pi\epsilon_0 r_A} \longrightarrow ①$$

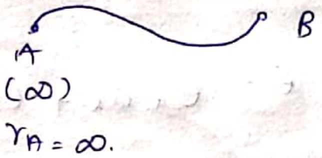
$$V_{AB} = V_B - V_A \longrightarrow ②$$

where $V_B = \dfrac{Q}{4\pi\epsilon_0 r_B}$ & $V_A = \dfrac{Q}{4\pi\epsilon_0 r_A}$.

Note:- If the point charge is moving from $\infty \rightarrow B$, the Electric potential becomes

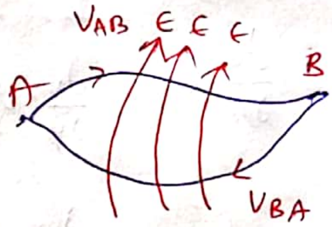
$$V_{AB} = \frac{Q}{4\pi\epsilon_0 r_B}$$

$$V = \frac{Q}{4\pi\epsilon_0 r}$$




From Eqn (1) $r_A = \infty$, then Electric potential becomes as only eqn $\Rightarrow r_B$. $\left[\because V = \dfrac{Q}{4\pi\epsilon_0}\left[\dfrac{1}{r_B} - \dfrac{1}{r_A}\right]\right]$

## Relation between E & V:-




$V_{AB} \rightarrow$ Electric potential from $A \rightarrow B$.

$V_{BA} \rightarrow$ Electric potential from $B \rightarrow A$

Then $V_{AB} = -V_{BA}$

$$V_{AB} + V_{BA} = 0$$

$$-\oint_l E\,dl = 0 \longrightarrow (1)$$ Called as "Integral form of Maxwell II eqn.

Apply stokes theorem

$$\int_S (\nabla \times A)\cdot ds = \oint_l A\,dl$$

$$\int_S (\nabla \times E)\,ds = \oint_l E\,dl \qquad \left(\because \oint_l E\,dl = 0\right)$$

$$\int_S (\nabla \times E)\,ds = 0$$

$\boxed{\nabla \times E = 0}$ — ② Called as "differential form of Max well II - eqn.

We know that

$$V = -\int_l E\,dl$$

$$dv = -E\,dl \longrightarrow (3)$$

let $E = E_x \bar{a_x} + E_y \bar{a_y} + E_z \bar{a_z}$

$$dl = dx\,\bar{a_x} + dy\,\bar{a_y} + dz\,\bar{a_z}$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz$$

Substitute above Values in eqn (3)

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = -\left[E_x dx + E_y dy + E_z dz\right]$$

$$E_x = -\frac{\partial v}{\partial x}$$

$$E_y = -\frac{\partial v}{\partial y}$$

$$E_z = -\frac{\partial v}{\partial z}$$

from $E = E_x \bar{a_x} + E_y \bar{a_y} + E_z \bar{a_z}$

$$E = -\frac{\partial v}{\partial x} a_x - \frac{\partial v}{\partial y} a_y - \frac{\partial v}{\partial z} a_z$$

$$E = -\left[\frac{\partial}{\partial x} \bar{a_x} + \frac{\partial}{\partial y} \bar{a_y} + \frac{\partial}{\partial z} a_z\right] \cdot V$$

$$\boxed{E = -\nabla \cdot V}$$

# Maxwell 2 eqn for Electrostatic Fields

## Maxwell - I eqn.

We know that $D = \frac{\psi}{S}$

$$D = \frac{d\psi}{ds}$$

$$d\psi = D \cdot ds$$

$$\psi = \int_S D \cdot ds$$

$$\Rightarrow \quad \psi = Q$$

$$\int_S D \cdot ds = Q$$

$$Q = \int_V \rho_v \, dv$$

$$\int_S D \cdot ds = \int_V \rho_v \, dv \longrightarrow ① \text{ Integral form}$$

Apply divergence theorem on L.H.S of eqn ①

$$\int_V (\nabla \cdot D) \, dv = \int_S D \cdot ds$$

$$\int_V (\nabla \cdot D) \, \cancel{dv} = \int_V \rho_v \, \cancel{dv} \quad [\because \text{from integral eqn}]$$

$$\boxed{\nabla \cdot D = \rho_v} \Rightarrow \text{differential form}$$

## Maxwell II - eqn.

We have $V_{AB} \Rightarrow$ Electric potential from $A \longrightarrow B$

$V_{BA} \longrightarrow$ Electric potential from $B \longrightarrow A$

Then $V_{AB} = -V_{BA}$

$$V_{AB} + V_{BA} = 0$$

$$-\oint_l E \, dl = 0 \longrightarrow \text{Integral form of II - eqn.}$$

⇒ Apply Stoke's theorem.

$$\boxed{\int_S (\nabla \times A) \cdot ds = \int_l A \cdot dl}$$ – Stoke's theorem formulae.

$$\int_S (\nabla \times E)\, ds = -\int_l E \cdot dl$$

$$\int_S (\nabla \times E)\, ds = 0 \quad \left[\because -\int_l E\, dl = 0\right]$$

$$\boxed{\nabla \times E = 0}$$ ⇒ differential form

## Convection & Conduction Currents.

① Electric Current:- It is defined as (flow of) rate of flow of electrons or charges through a medium. mathematically it is defined as

$$\boxed{I = -\frac{dQ}{dt}}$$ c/sec (or) Amp

* The "–"ve Sign indicates rate of decrement of charge (or) flow of current is opposite to flow of electrons.

* 1 Coulomb of charge which is passing through a closed surface in 1 second. then we can say that 1 ampere of current is flowing through the Surface.

② Conductivity ($\sigma$):-

It is the ratio of conduction current density to electric field intensity.

$$\boxed{\sigma = \frac{J}{E}}$$ ℧/m (or) Siemens/m

$$\boxed{J = \sigma E}$$

which is also called as "point form of ohm's law".

* Depending upon a Conductivity of a material, materials are classified as:-

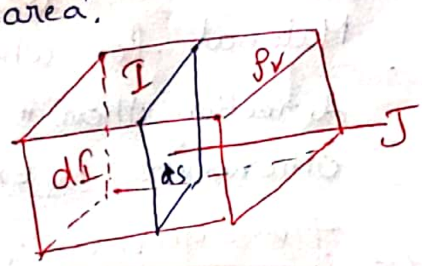
1. Conductors [$\sigma >> 1$]
2. Insulators [$\sigma << 1$]
3. Semiconductors [$\sigma$ lies between conductors & Insulators].

* A material having infinite conductivity which are called as "Super Conductors".

## Conduction Current Density [J]:-

It occurs due to drifting of electrons under the influence of applied electric field. Then conduction current density is defined "as flow of current for unit surface area".

→ Mathematically:




$$\boxed{J = \frac{I}{S}} \text{ Amp/m}^2$$

$$J = \frac{dI}{ds}$$

$$dI = J \cdot ds$$

$$\boxed{I = \int_S J \cdot ds} \quad \text{—— (1)}$$

## Convection Current:- (or) Displacement Current:-

It occurs due to flow of current in an insulating medium like solids or liquids.

→ It does not satisfies ohm's law.

Ex:- Current passing through tube light & capacitors.

Ohm's law: Current density is directly proportional to the electric field intensity.

## Linear, Isotropic, Homogeneous & Dielectric Constants:-

(1) **Linear**:- A medium is said to be linear, the Electric flux density is always directly proportional to electric field intensity with a proportional constant. $\boxed{\varepsilon_0 = 8.845 \times 10^{-12}}$

(2) **Isotropic**:- A medium is said to be isotropic $\varepsilon_0$ is a scalar quantity.

(Or)

Material for which $\vec{D}$ & $\vec{E}$ are in the same direction then it is isotropic and non-isotropic otherwise. Ex:- Mica.

(3) **Homogeneous**:- A medium is said to be homogeneous when the physical characteristics of the medium does not changes from point to point. Ex:- Human Body.

(4) **Dielectric Constant / Relative permitivity**:- ($\epsilon_r$)

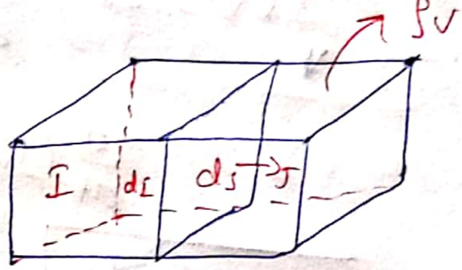
It is the ratio of permitivity to the permitivity of free-space.

$$\epsilon = \epsilon_0 \epsilon_r$$

$$\boxed{\epsilon_r = \frac{\epsilon}{\epsilon_0}}$$

## Continuity Equation:-

It states that the current density diverging from a small volume is equal to rate of decrease of charge per unit volume mathematically. $\boxed{\nabla \cdot J = -\frac{d\rho_v}{dt}}$




Let us consider a small volume with a volume charge density $\rightarrow \rho_v$.

$\Rightarrow$ Let $I$ be the current which is passing through the volume and $dI$ is a differential current which passes through a differential surface.

W.K.T., Electric current, $I = -\frac{dQ}{dt} \rightarrow (1)$

$\rho_v = \frac{Q}{V}$

J = Conduction current density.

$\rho_v = \frac{dQ}{dv}$

$dQ = \rho_v dv$

$\therefore Q = \int_v \rho_v dv$

$\Rightarrow I = -\frac{d}{dt}\left[\int_v \rho_v dv\right] \rightarrow (2)$

$J = \frac{I}{S} = \frac{dI}{ds}$

$dI = J \cdot ds \Rightarrow I = \int_S J \cdot ds \rightarrow (3)$

$$\oint_S J \cdot ds = -\frac{d}{dt}\left[\int_V \rho_v \cdot dv\right] \quad \text{--- (4)}$$

Apply divergence theorem to the L.H.S side.

$$\int_V (\nabla \cdot J)\, dv = \oint_S J \cdot ds$$

$$\int_V (\nabla \cdot J)\, dv = -\frac{d}{dt}\left[\int_V \rho_v\, dv\right]$$

$$\nabla \cdot J = -\frac{d}{dt}(\rho_v) \quad \text{--- (5)}$$

$$\boxed{\nabla \cdot J = -\frac{d\rho_v}{dt}}$$

* For Constant Surface the derivative becomes partial derivative.

## Relaxation time (or) Re-arrangement time $(\tau)$.

From Maxwell I - equation.

**Definition:** Relaxation time is defined as the time at which the volume charge density decays 36.7% of it's initial value. [$e^{-1}$ times the initial value].

$$\nabla \cdot D = \rho_v$$

$$D = \varepsilon_0 E$$

$$\nabla(E\varepsilon_0) = \rho_v$$

$$\boxed{\nabla E = \frac{\rho_v}{\varepsilon_0}}$$

Multiply with "$\sigma$" on both sides

$$\nabla(\sigma \cdot E) = \frac{\sigma}{\varepsilon_0} \cdot \rho_v$$

$$\boxed{J = \sigma \cdot E}$$

$$\nabla \cdot J = \frac{\sigma}{\varepsilon_0} \cdot \rho_v$$

Note:- $\tau_r$ is very small for good conductors

$\tau$ is very fast for good Insulators.

From the equation of Continuity.

$$-\frac{d\rho_v}{dt} = \frac{\sigma}{\varepsilon_0}\rho_v$$

$$\frac{d\rho_v}{dt} + \frac{\sigma}{\varepsilon_0}\cdot\rho_v = 0$$

The above equation is a first order differential equation and its sol$^n$ is. $\boxed{\rho_v = \rho_{v0}e^{-t/\tau}}$

$\boxed{\tau = \frac{\varepsilon_0}{\sigma}}$

$$\frac{d\rho_v}{dt} = -\frac{\sigma}{\varepsilon_0}\cdot\rho_v$$

$$\frac{d\rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon_0}\cdot dt$$

$$\int\frac{d\rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon_0}\int 1.dt$$

$$\int\frac{1}{\rho_v}d\rho_v = -\frac{\sigma}{\varepsilon_0}\cdot t$$

$$\log(\rho_v) + c = -\frac{\sigma}{\varepsilon_0}\cdot t \longrightarrow (1)$$

$\rho_v = \rho_{v0}$ at $t=0$

$$\log(\rho_{v0}) + c = 0$$

$$c = -\log(\rho_{v0})$$

$$\log(\rho_v) - \log(\rho_{v0}) = -\frac{\sigma}{\varepsilon_0}t.$$

$$\log(\rho_v/\rho_{v0}) = -\frac{\sigma}{\varepsilon_0}\cdot t$$

$$\frac{\rho_v}{\rho_{v0}} = e^{-\frac{\sigma}{\varepsilon_0}\cdot t}$$

$\boxed{\frac{1}{\tau} = \frac{\sigma}{\varepsilon_0}}$

$$\boxed{\rho_v = \rho_{v0}\cdot e^{-\frac{t}{\tau}}}$$

$$\tau = \frac{\varepsilon_0}{\sigma}$$

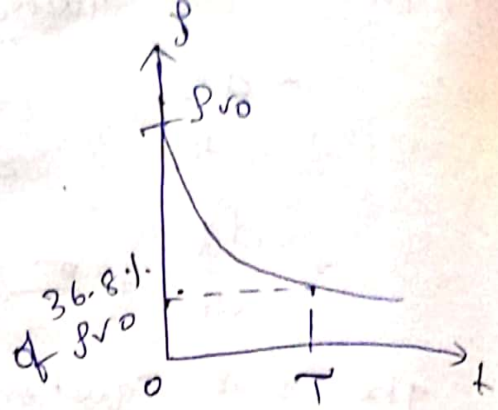
at $t = \tau$

$\rho_v = \rho_{v0} e^{-\tau/\tau}$

$\rho_v = \rho_{v0} e^{-1}$

$= \dfrac{\rho_{v0}}{\rho_v}$

$\boxed{\rho_v = 0.368\, \rho_{v0}}$




* At $t=0$, $\rho_v = \rho_{v0}$ and $t = \tau$ $\rho_v$ becomes $\rho_v = 36.8\%\, \rho_{v0}$. then the relaxation time is defined as the time takes by time density it is decreases from $36.8\, \rho_{v0}$.

## * Poisson's & Laplace equation:-

Consider maxwell -I equation.

$\nabla \cdot D = \rho_v$

The relation between $D$ & $\epsilon$ $= \boxed{D = E\epsilon_0}$

$\nabla \cdot E\epsilon_0 = \rho_v$ $\qquad E = -\nabla V$

$\nabla \cdot E = \dfrac{\rho_v}{\epsilon_0}$.

$\boxed{E = -\nabla . V}$

$\nabla(-\nabla V) = \dfrac{\rho_v}{\epsilon_0}$

$-\nabla^2 V = \dfrac{\rho_v}{\epsilon_0}$

poission eqn $\boxed{\nabla^2 V = -\dfrac{\rho_v}{\epsilon_0}}$

Laplace eqn. $\boxed{\nabla^2 V = 0}$

∴ If there is no charge inside volume, then volume charge density $\rho_v = 0$.

The poission's Equation becomes Laplace.

$$\nabla = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

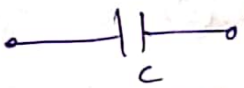
$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

## Capacitance:-

A Capacitance of a Capacitor is defined as it is the ratio of magnitude of the charge to the electric potential (or) potential difference.

i.e., $\boxed{C = \frac{Q}{V}}$ F

## Capacitor:-




Any two conducting bodies seperated by an insulating medium is known as a capacitor.

~~Mathematically~~ $\cancel{C} = Q$

There are three types of configurations.

① Spherical Capacitance Configuration.

② Co-axial Capacitance Configuration.

③ parallel plate Capacitance Configuration.

## Spherical Capacitance Configuration

① Let us Consider 2 conducting imaginary Spheres with radius a & b.

* Inner Conductor Carries a positive charge and outer Conductor Carries a negative charge.

* From the limitation of Gauss law we will

Consider a Gaussian Surface / Closed Surface.

We know that.

$$\psi = Q.$$

$$Q = \int_S D \cdot ds$$

As $D = \epsilon_0 E = \int_S \epsilon_0 \cdot E \, ds$

$$E = E_R \bar{a_R}$$

$$E = \int_S \epsilon_0 E_R \bar{a_R} \, ds$$

$$Q = \epsilon_0 E_R \int_S 1 \cdot ds$$

$$Q = \epsilon_0 E_R . (S)$$

Where, $S = 4\pi R^2$.

$$Q = \epsilon_0 . E_R 4\pi R^2$$

$$E_R = \frac{Q}{4\pi \epsilon_0 R^2}$$

$$E = E_R \bar{a_R}$$

$$E = \frac{Q}{4\pi \epsilon_0 R^2} \cdot \bar{a_R}$$

Now, $V = -\int_l E dl$

$$= -\int_l \frac{Q}{4\pi \epsilon_0 R^2} dl . \bar{a_R}$$

$$= -\int_l \frac{Q}{4\pi \epsilon_0 R^2} [dR \bar{a_R} + \underbrace{R d\theta \bar{a_\theta} + R \sin\theta d\phi \bar{a_\phi}}_{0}] \cdot \bar{a_R}$$

$$= -\int_l \frac{Q}{4\pi \epsilon_0 R^2} dR \qquad [\bar{a_R} . \bar{a_R} = 1]$$

$$V = -\int_a^b \frac{Q}{4\pi\epsilon_0 R^2}\, dR$$

$$V = \frac{-Q}{4\pi\epsilon_0} \int_a^b \frac{1}{R^2}\, dR$$

$$V = -\frac{Q}{4\pi\epsilon_0} \left[ -\frac{1}{R} \right]_a^b$$

$$V = \frac{-Q}{4\pi\epsilon_0} \left[ -\frac{1}{b} + \frac{1}{a} \right]$$

$$V = +\frac{Q}{4\pi\epsilon_0} \left[ -\left( \frac{1}{b} - \frac{1}{a} \right) \right]$$

$$V = \frac{Q}{4\pi\epsilon_0} \left[ \frac{1}{b} - \frac{1}{a} \right]$$

We have. $C = \frac{Q}{V}$

$$C = \frac{\cancel{Q}}{\frac{\cancel{Q}}{4\pi\epsilon_0}\left[\frac{1}{b} - \frac{1}{a}\right]}$$

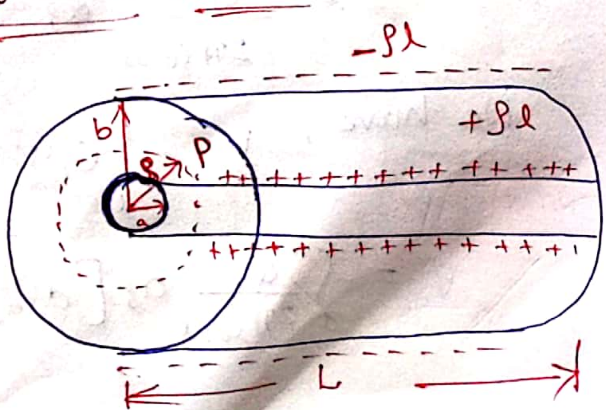
$$C = \frac{4\pi\epsilon_0}{\left[\frac{1}{b} - \frac{1}{a}\right]} \cdot F.$$

If $b = \infty$ then $\frac{1}{b} \to 0$. $C = 4\pi\epsilon a \;\; F$

This is the capacitance of isolated sphere.

## ③ Co-axial Capacitance Configuration (or) Capacitance of Co-axial Cable.

Let us consider Co-axial cable's with a radius of a & b



* Inner Co-axial Cable Carries a positive line charge density $\rho_l$.
* Outer Co-axial Cable Carries a negative line charge density $-\rho_l$.

From the limitation of Gauss Law. We will Consider a Gaussian Surface in the two Cables. Where $\rho$ is a radius from origin to P.

We know that.

$$\psi = Q$$

$$Q = \int_S D \cdot ds$$

$$\rightarrow \quad D = \epsilon_0 E$$

$$Q = \int_S \epsilon_0 E \cdot ds$$

$$E = E_\rho \bar{a}_\rho$$

$$\Rightarrow \quad Q = \int_S \epsilon_0 . E_\rho \bar{a}_\rho \, ds$$

$$= \epsilon_0 . E_\rho \int_S 1 . ds$$

$$Q = \epsilon_0 . E_\rho (S)$$

Where, $S = 2\pi \rho l$

$$Q = \epsilon_0 \epsilon_r 2\pi \rho l$$

$$E_\rho = \frac{Q}{2\pi \epsilon_0 \rho l}$$

$$E = \frac{Q}{2\pi \epsilon_0 \rho l} \bar{a}_\rho$$

We have, $V = -\int_l E \, dl$

$$V = -\int_a^b \frac{Q}{2\pi \epsilon_0 \rho l} \bar{a}_\rho \left[ d\rho \bar{a}_\rho + \underbrace{\rho d\phi \bar{a}_\phi + dz \bar{a}_z}_{0} \right]$$

$$V = -\int_a^b \frac{Q}{2\pi\epsilon_0 \rho l} \cdot d\rho \; \bar{a_\rho} \cdot \bar{a_\rho} \qquad [\because \bar{a_\rho} \cdot \bar{a_\rho} = 1]$$

$$V = -\int_a^b \frac{Q}{2\pi\epsilon_0 \rho l} \, d\rho$$

$$V = \frac{-Q}{2\pi\epsilon_0 L} \int_a^b \frac{1}{\rho} \, d\rho$$

$$V = -\frac{Q}{2\pi\epsilon_0 L} [\log b - \log a]$$

$$V = \frac{Q}{2\pi\epsilon_0 L} [\log a - \log b]$$

$$V = \frac{Q}{2\pi\epsilon_0 L} [\log [a/b]]$$

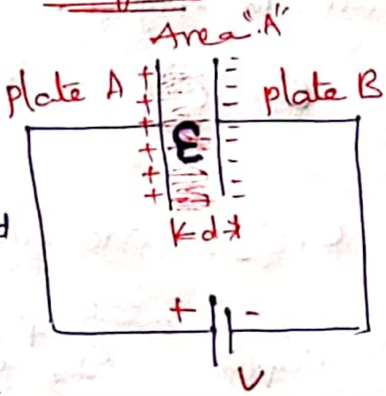
We know that $C = \frac{Q}{V}$

$$C = \frac{\cancel{Q}}{\frac{\cancel{Q}}{2\pi\epsilon_0 L} \times \log(a/b)}$$

$$\boxed{C = \frac{2\pi\epsilon_0 L}{\log(a/b)}} \quad F.$$

## Capacitance of a parallel plate :- (or)

## parallel plate Capacitance Configuration.




* Let us Consider a Capacitor which is having two parallel two Conducting plates named as plate A & B.

$$\boxed{E = \frac{F}{Q} = \frac{V}{d}}$$

* The two parallel Conducting plates filled with a dielectric medium whose permitivity is ε. with a distance Supportion of "d".

The two parallel Conducting plates are having equal & opposite charges due to this there exist a force which operates an electric field E.

$$E = \frac{V}{d}$$

$$D = \frac{\psi}{S}$$

$$\psi = Q$$

$$D = \frac{Q}{S}$$

From the figure Surface Area is A. $\boxed{S = A}$

$$D = \frac{Q}{A}$$

$$D = \epsilon_0 . E$$

$$\frac{Q}{A} = \epsilon_0 . E$$

$$Q = \epsilon_0 . A . \frac{V}{d}$$

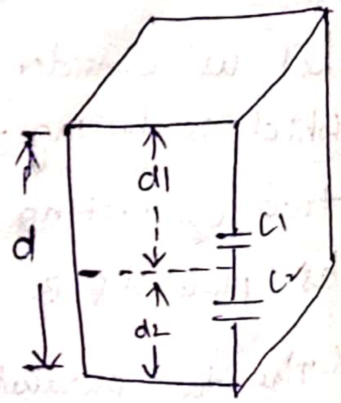
$$\frac{Q}{V} = \frac{\epsilon_0 . A}{d}$$

$$\boxed{C = \frac{\epsilon_0 . A}{d}} \ F.$$

## Parallel Plate Capacitors in Series:-

Consider two parallel plate Capacitor in Series as shown in figure.

$\because \ C_1 = \frac{\epsilon_1 A}{d_1} \qquad C_2 = \frac{\epsilon_2 A}{d_2}$




$\therefore$ $C_1$ & $C_2$ are in Series

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\frac{1}{C_{eq}} = \frac{1}{\frac{\epsilon_1 A}{d_1}} + \frac{1}{\frac{\epsilon_2 A}{d_2}}$$

$$\frac{1}{C_{eq}} = \frac{d_1}{\epsilon_1 A} + \frac{d_2}{\epsilon_2 A}.$$

$$\boxed{C_{eq} = \frac{1}{\frac{d_1}{\epsilon_1 A} + \frac{d_2}{\epsilon_2 A}}}$$

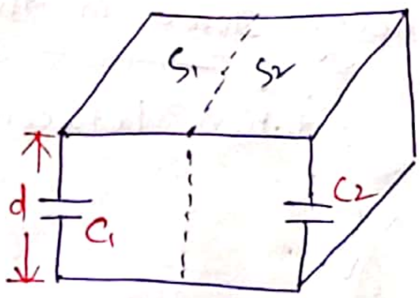
If there are 'n' no of Capacitors.

$$\boxed{C_{eq} = \frac{1}{\sum_{k=1}^{n} \frac{d_k}{\epsilon_k S}}} \quad F.$$

## Parallel plate Capacitors in parallel Combination:-

Consider two parallel plate Capacitors in parallel as shown in figure.

$\therefore \quad C_1 = \frac{\epsilon_1 A_1}{d} \qquad C_2 = \frac{\epsilon_2 A_2}{d}$




$C_1$ & $C_2$ are in parallel

$C_{eq} = C_1 + C_2$

$C_{eq} = \frac{\epsilon_1 A_1}{d} + \frac{\epsilon_2 A_2}{d}$

If n number of Capacitors Connected in parallel

$$\boxed{C_{eq} = \sum_{k=1}^{n} \frac{\epsilon_k A_k}{d}} \quad F$$

## Importance of poisson's and laplace's equations. (or) Application.

* It is used to determine charge density and electric field intensity in the region of interest irrespective of charge distributions.

**Energy density:-** It is defined as the energy stored per unit volume in a Static field. It is measured in Joules/$m^2$.

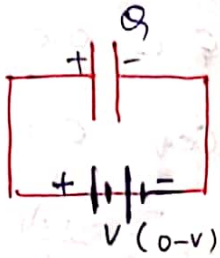
Consider. $$\boxed{W_E = \frac{1}{2}\int_V \vec{D}\cdot\vec{E}\cdot dv}$$ Joules.

differentiating on both Sides w.r.to v we get

$$\boxed{\frac{dw_E}{dv} = \frac{1}{2}\vec{D}\cdot\vec{E}}\ J/m^3$$

① **Energy Stored in the Capacitor:-**

The relation between Q, C & V is




$C = \frac{Q}{V}$

$Q = CV.$

From the above expression $Q \propto v$.

Work done in capacitor $dw = Qdv$

$$dw = Qdv$$

$$W = \int Qdv$$

$$= \int_0^V Cv\,dv$$

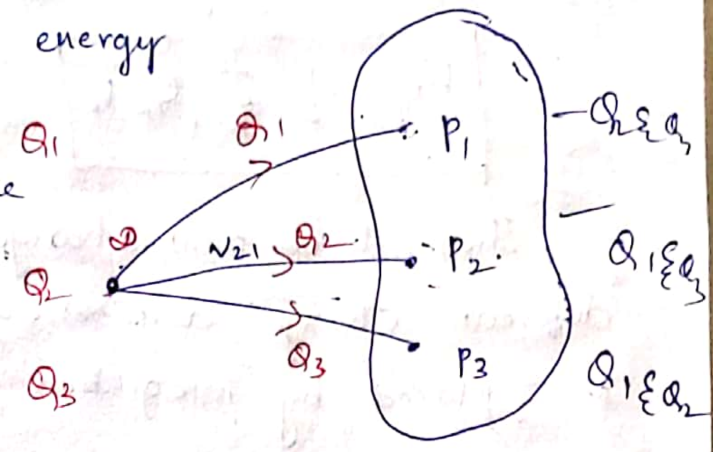
$$= C.\int_0^V v\,dv$$

$$= C\left[\frac{v^2}{2}\right]_0^V$$

$$\boxed{W = \frac{1}{2}Cv^2}$$ Joules

# Energy Stored in Electrostatics:

To determine the energy present we must first determine the amount of work required to assemble charges.




Consider $Q_1, Q_2$ & $Q_3$ are three charges. we wish to transfer or move from infinity to $P_1, P_2, P_3$ respectively. as shown in the figure.

Case I: No work is required to move $Q_1$ from infinity to $P_1$. because initially the space is charge free i.e, $W_1 = 0$.

Case-II:- The work done in moving $Q_2$ from infinity to $P_2$. is equal to the product of charge $Q_2$. and the potential at $P_2$ due to the charge $Q_1$

i.e., $W_2 = Q_2 V_{21}$

Case III :- The work done in moving charge $Q_3$ from infinity to $P_3$ is equal to the product of $Q_3$ and sum of potentials at $P_3$ due to the charges $Q_1$ and $Q_2$.

i.e., $W_3 = Q_3 (V_{31} + V_{32})$

Total work done $W = W_1 + W_2 + W_3$

$$W_E = 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32}) \longrightarrow ①$$

Let $V_1$ be the potential at point $P_1$ due to $Q_2$ & $Q_3$

$V_2$ be the potential at point $P_2$ due to $Q_1$ & $Q_3$

$V_3$ be the potential at point $P_3$ due to $Q_1$ & $Q_2$.

$$\therefore \; W_E = \frac{1}{2}\left[Q_1 V_1 + Q_2 V_2 + Q_3 V_3\right]$$

$$\boxed{W_E = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k}$$ Joules. point charge

Instead of point charge, if there are different charge distributions then summation is replaced by integration, and is given by

$$W_E = \frac{1}{2}\int_L \rho_L \, dl \cdot V \longrightarrow \text{Line charge}$$

$$W_E = \frac{1}{2}\int_S \rho_S \cdot ds \cdot V \longrightarrow \text{Surface charge}$$

$$W_E = \frac{1}{2}\int_V \rho_V \cdot dv \cdot V \longrightarrow \text{volume charge.}$$

Date
10/6/22

Unit - II

# Static Magnetic Fields & Time Varying Fields

* The Sources for Electro Static fields are electric Charge.
* The Sources for magnetostatic fields are
① Current Carrying Conductor.
② permanent Magnet.
* the Source for Electromagnetic waves are time varying electric & Magnetic fields.

## Static Magnetic Fields :- (Constant fields)

Steady currents produce steady magnetic fields. ~~or~~ which Steady magnetic fields are magnetic fields are Constant with time. These fields are also called "Static magnetic fields" or "Magnetostatic fields".

## Applications of Static Magnetic fields:

* Motor
* Transformer
* Microphone
* ElectroMagnetic pump.
* Compasses.

## Basic properties of the Magnetic fields

* The region around a magnet is called magnetic field. existence of magnetic field can be exiperienced with Compass.
* It has two poles, north (N) and South (S).

* Magnetic field is represented by an imaginary lines around the magnet which are called "Magnetic lines of force" (or) "Magnetic flux" as shown in below fig.

* Every Magnetic flux line starting from north pole must end at South pole and Complete the path from South to north internal to the magnet.

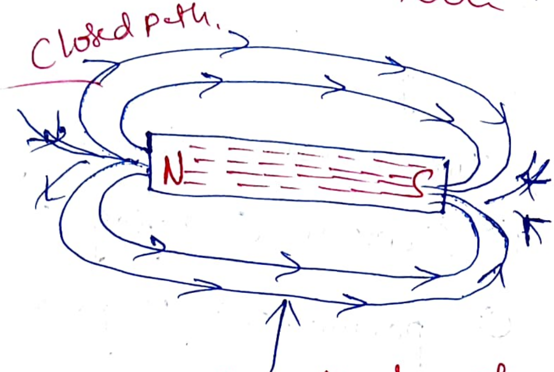



fig:- permanent Magnet and Magnetic lines of force.

* The Magnetic lines never diverge (or) Spreadi
* The Direction of Magnetic field is obtained from Flemming's Right hand thumb rule.

## Magnetic Field Intensity: [H]:

Magnetic field intensity (H) at any point in the magnetic field is defined as the force experienced by the unit north pole at that point.

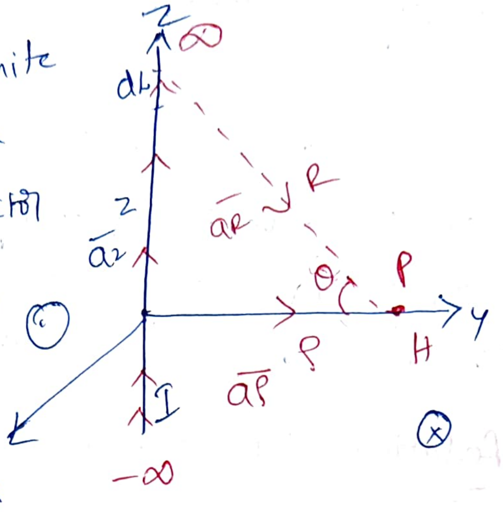
(or)

In Simple terms, it is a measure of how strong or weak any magnetic field is. It is measured in newtons/weber (N/wb) or amperes per meter (A/m).

It is a vector quantity.

## Magnetic Field Intensity due to infinite long Conductor:-

Initially consider a infinite long conductor with a current carrying conductor of current I flowing in it & the limits are from $-\infty$ to $+\infty$.




$$H = \int \frac{I\,dL \sin\theta}{4\pi R^2}\ A/m$$ — From Biot Savart Law

$$H = \int \frac{I\,d\bar{L} \times \bar{a}_R}{4\pi R^2}$$ — From vector form of Biot-Savart law.

Step 1:- $I d\bar{L} = I\,(\underbrace{dx\,\bar{a}_x + dy\,\bar{a}_y}_{0} + dz\,\bar{a}_z)$

$I d\bar{L} = I\,dz\,\bar{a}_z$

Step 2:- $\bar{a}_R = \frac{\bar{R}}{|\bar{R}|}$

$\bar{R} = \bar{OP} - \bar{OdL}$

$= \rho\bar{a}_\rho - z\bar{a}_z$

$|\bar{R}| = \sqrt{\rho^2 + z^2}$

$\sin\theta = \frac{z}{R}$, & $\cos\theta = \frac{\rho}{R}$, $\tan\theta = \frac{z}{\rho}$, $z = \rho\tan\theta$

$dz = \rho\sec^2\theta\,d\theta$

$\theta = \tan^{-1}(z/\rho)$

if $z = -\infty, \Rightarrow \theta = -\pi/2$, $z = \infty, \Rightarrow \theta = \pi/2$

$$H = \int_{-\pi/2}^{\pi/2} \frac{I\,dz\,\bar{a}_z \times \frac{(\rho\bar{a}_\rho - z\bar{a}_z)}{\sqrt{\rho^2+z^2}}}{4\pi(\sqrt{\rho^2+z^2})^2}$$

$$H = \frac{I}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{dz\,\bar{a}_z \times (\rho\bar{a}_\rho - z\bar{a}_z)}{(\rho^2+z^2)^{3/2}}$$

$$H = \frac{I}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{\rho\sec^2\theta\,d\theta\;\rho(\bar{a}_z\times\bar{a}_\rho) - (\bar{a}_z\times\bar{a}_z)z}{(\rho^2+\rho^2\tan^2\theta)^{3/2}}$$

$$H = \frac{I}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{\rho^2\sec^2\theta\,d\theta\,\bar{a}_\phi}{(\rho^2)^{3/2}[\sec^2\theta]^{3/2}}$$

$\left[\because \bar{a}_z\times\bar{a}_z = 0,\ \bar{a}_z\times\bar{a}_\rho = \bar{a}_\phi\right]$

$$= \frac{I}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{\rho\sec\theta}\,d\theta\,\bar{a}_\phi$$

$$= \frac{I}{4\pi\rho}\int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta\,\bar{a}_\phi$$

$$= I/4\pi\rho\,[\sin\theta]_{-\pi/2}^{\pi/2}\,\bar{a}_\phi$$

$$= \frac{I\cdot 2}{4\pi\rho}\,\bar{a}_\phi$$

$$\boxed{H = \frac{I}{2\pi\rho}\,\bar{a}_\phi}\ A/m$$

## Magnetic flux density:- [$\vec{B}$] The Magnetic flux

density (or) Magnetic induction is the number of lines of force passing through a unit area of material. It is denoted by $\vec{B}$.. $\phi = \int_S B.ds$

* It is a vector quantity.
* It is measured in weber per Square metre. ($Wb/m^2$) which is also Called as (Tesla) T.

## Relation ship between $\bar{B}$ and $\bar{H}$:

The relation ship between magnetic flux density and magnetic field intensity is given by.

$$\boxed{\vec{B} = \mu\vec{H} = \mu_0 \mu_r . \bar{H}}$$

where $\mu$ = permeability of the dielectric

$$\boxed{\mu = \mu_0 \mu_r}$$

$\mu_0 = 4\pi \times 10^{-7}$ H/m

$\mu_r$ = relative permeability of medium.

$\mu_0$ = permeability of free space

For free Space. $\boxed{\bar{B} = \mu_0 . \bar{H}}$

For all non-magnetic media $\mu_r = 1$. while for magnetic materials $\mu_r$ is greater than unity.

## Magnetic Field due to Current Carrying Conductor:-

After the experient performed by Oestered, a professor in 19th century, the relationship between electric Current and Magnetic field developed.

When a straight Conductor carries a dc Current, It produces a magnetic field around it, and the field lines of force are in the form of concentric circles at right angles to the conductor.

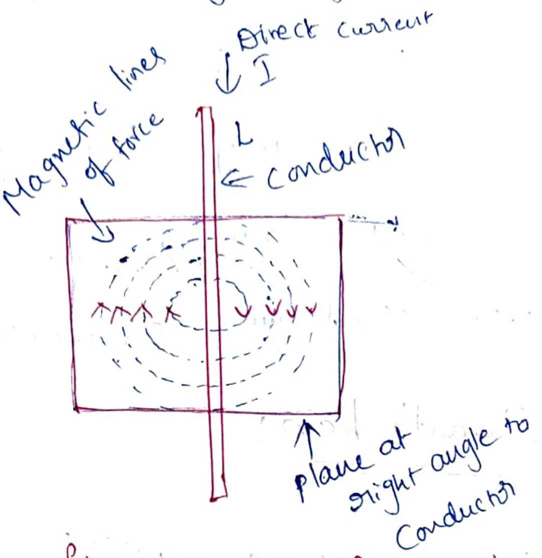
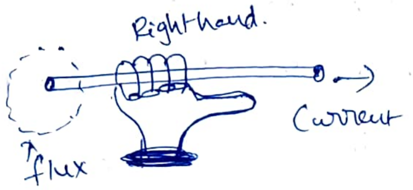


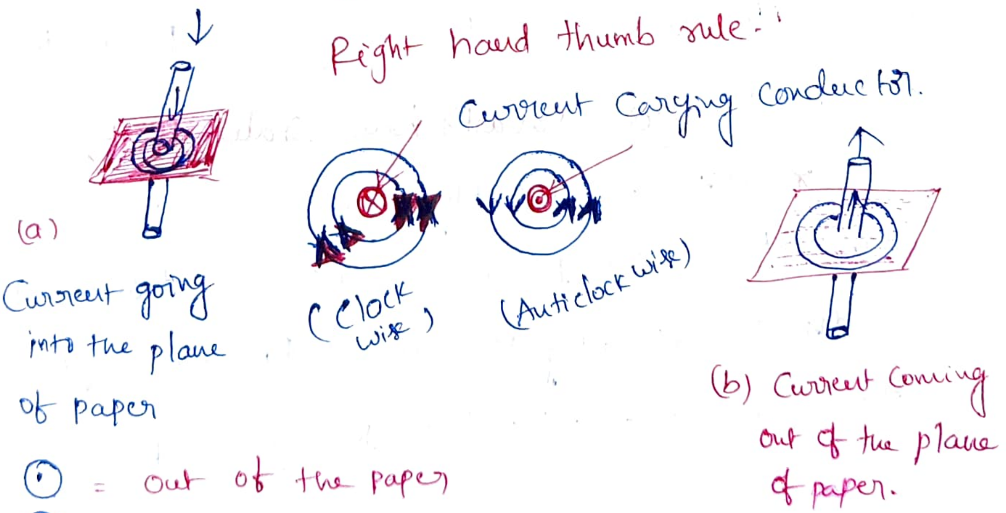

fig. Magnetic field due to conductor carrying direct current.

## Direction of Magnetic field Intensity:-

Flemming's right hand thumb rule is used to determine the direction of magnetic field around a conductor carrying a direct current. It states that, hold the current carrying conductor in the right hand such that the thumb pointing in the direction of current and parallel to the conductor, then curled fingers point in the direction of the magnetic lines of flux around it.




Right hand thumb rule:-




⊙ = out of the paper

⊗ = Into the paper.

Magnetostatic fields are produced by a Constant Current Source.

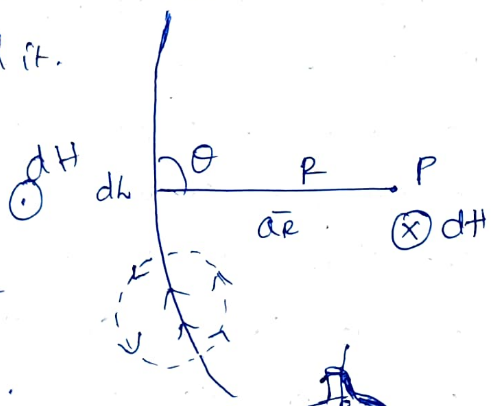
There are two major laws governing magnetostatic

1. Biot - Savart's law - Asymmetric current
2. Ampere's - Circuit law - Symmetrical current

## Biot-Savart's Law:

* Let us consider a current carrying conductor which carries a constant current I.
* Due to the current carrying conductor a magnetic field is developed around it.

This can be analyzed with the help of right hand thumb rule.




* The circle with cross mark indicates the direction of magnetic field is inward & the circle with dot ⊙ represents direction of magnetic field is outward.

Biot-Savart's Law states that,

* dH is directly proportional to product of current carrying in conductor & differential length.
* dH is directly proportional to sine angle between the dl and the line joining to P.

* dH is inversely proportional to square of the distance from dl to point P.

ie,
1) $dH \propto dLI$
2) $dH \propto \sin\theta$
3) $dH \propto \frac{1}{R^2}$

Now, $dH \propto \frac{I dL \sin\theta}{R^2}$

$$dH = K . \frac{I dL \sin\theta}{R^2}$$

$$dH = \frac{1}{4\pi} \frac{I dL \sin\theta}{R^2} \text{ A/m}$$

$$\boxed{H = \int \frac{1}{4\pi} \frac{I dL \sin\theta}{R^2}} \text{ A/m} \quad \text{—(1)}$$

* thumb indicates → direction of current of current carrying conductor.

* closed fingers → Magnetic field direction of current carrying conductor.

Vector Form of Biot-Savart's Law.

→ We know that $\bar{A}$ & $\bar{B}$ are two vectors.
Cross product of $\bar{A}$ & $\bar{B}$ is,

$\bar{A} \times \bar{B} = |\bar{A}||\bar{B}| \sin\theta.$

Consider $\bar{A} = I\bar{dL}$ & $\bar{B} = \bar{a}_R$

$I\bar{dL} \times \bar{a}_R = |I\bar{dL}||\bar{a}| . \sin\theta.$

$I\bar{dL} \times \bar{a}_R = I dL \sin\theta.$

Substitute the above in eqn (1)

$$\boxed{\therefore H = \int \frac{1}{4\pi} \frac{I\bar{dL} \times \bar{a}_R}{R^2}} \text{ A/m}$$

There are 3 different current distributions

1. Line current :- $H = \int_l \frac{1}{4\pi} \frac{I\,dL \times \bar{a}_R}{R^2}$

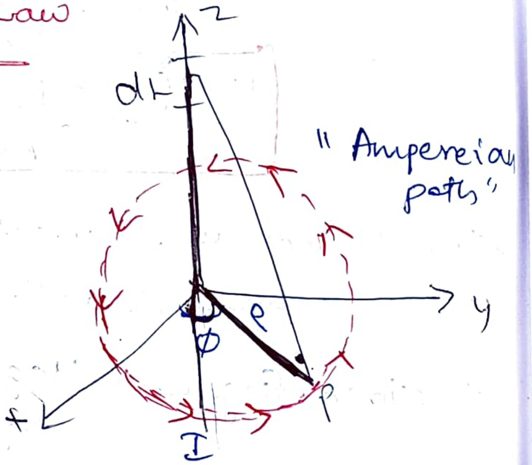
2. Surface current :- $H = \int_s \frac{1}{4\pi} \frac{K\,ds \times \bar{a}_R}{R^2}$

3. Volume current :- $H = \int_v \frac{1}{4\pi} \frac{J\,dv \times \bar{a}_R}{R^2}$.

~~Magnet~~

## Ampere's Circuital Law

→ Consider a current carrying conductor with current I.




Ampere's circuital Law states that "The line Integral of magnetic field Intensity around a closed path is equal to the total current enclosed by the same path".

→ Mathematically,

$$\boxed{\int_l H \cdot dl = I_{Cenclosed}}$$

Proof:- Consider an infinite long conductor which carries a constant current I on z-axis. Due to this a magnetic field is developed around the conductor. The closed path is called as Amperian path.

W.K.T, H due to infinite line conductor is

$H = \frac{I}{2\pi\rho} \bar{a}_\phi$ → from Magnetic field Intensity due to infinite long Conductor.

$H \cdot dl = \frac{I}{2\pi\rho} \bar{a}_\phi \, dl$

$H\,dl = \frac{I}{2\pi\rho} \bar{a}_\phi \left[ d\rho \bar{a}_\rho + \rho d\phi \bar{a}_\phi + dz \bar{a}_z \right]$.

$H\,dl = \frac{I}{2\pi\rho} \rho\, d\phi \quad [\because a_\rho \cdot \bar{a}_\phi = a_z \cdot \bar{a}_\phi = 0]$

$H\,dl = \frac{I}{2\pi} d\phi$

Apply line integral on both sides,

$\int_l H\,dl = \frac{I}{2\pi} \int_l 1 \cdot d\phi$

$\int_l H\,dl = \frac{I}{2\pi} \int_0^{2\pi} 1 \cdot d\phi$

$= \frac{I}{2\pi} [\phi]_0^{2\pi}$

$= \frac{I}{\cancel{2\pi}} \times \cancel{2\pi}$

$$\boxed{\int_l H\,dl = I}$$

## MAXWELL III - Equation In Magnetostatics:-

We know that

$\int_l H\,dl = I$ ——— ①

From the definition of Conduction Current density,

$J = \frac{I}{S}$

Let $dI \to ds \Rightarrow$ differential current passing through differential surface.

$J = \frac{dI}{ds}$

$dI = J \cdot ds$

$\boxed{I = \int_S J \cdot ds}$ ⟶ (2)

Comparing eqn (1) & (2), we get

$$\int_l H \cdot dl = \int_S J \cdot ds \rightarrow (3)$$ [Integral form of Maxwell - III eqn]

Apply Stoke's theorem for above eqn.

$$\int_S (\nabla \times H) ds = \int_l H \cdot dl$$

Now, $$\oint_S (\nabla \times H) ds = \oint_S J \cdot ds$$

$$\nabla \times H = J$$ → [differential form of Maxwell III eqn]

<u>Note</u>:- Divergence of electric field Intensity = 0 in a charge free region.

$$\nabla E = 0$$

$$\nabla \cdot V = -\frac{\rho_v}{\varepsilon_0}$$ → From poission's eqn.

Max-I — $\nabla \cdot D = \rho_v$ $\qquad D = \epsilon_0 E$

$$\nabla (\epsilon_0 \cdot E) = \rho_v$$

$$\nabla \cdot E = \frac{\rho_v}{\epsilon_0}$$

$$\nabla \cdot E = \frac{0}{\epsilon_0}$$

* If there is no charge then no volume charge density. $\rho_v = 0$ $\boxed{\nabla \cdot E = 0}$

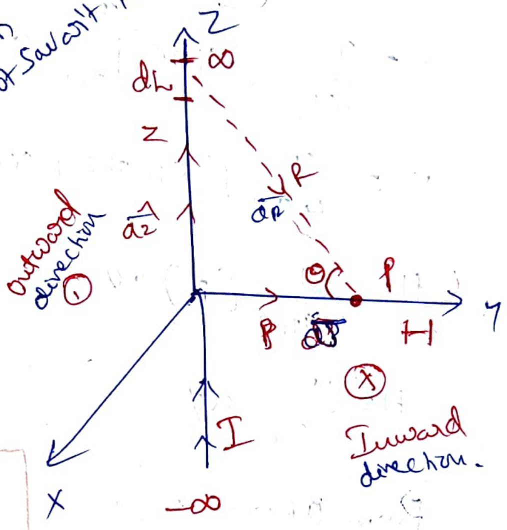
## Ampere's Circuit Law Applications.

1. Ampere's Circuital law using Infinite line Current.
2. Ampere's Circuital Law using finite line current.

## * Infinite line current of current carrying conductor

* Let us consider an infinite line current of current carrying conductor.

$H = \int \frac{I\,dL \sin\theta}{4\pi R^2}$ A/m — from Biot Savart's law.




$H = \int \frac{I\,\vec{dL} \times \vec{a_R}}{4\pi R^2}$

Step-1:- $I\,dL = I\,dz\,\vec{a_z}$

Step-2: $\vec{a_R} = \frac{R}{|R|}$

$\bar{R} = \bar{OP} - \bar{OdL}$

$= \rho\vec{a_\rho} - z\vec{a_z}$

$|R| = \sqrt{\rho^2 + z^2}$

$I\,dL = I(dx\,\vec{a_x} + dy\,\vec{a_y} + dz\,\vec{a_z})$

$I\,dL = I(dz\,\vec{a_z})$

$\sin\theta = \frac{z}{R}$ & $\cos\theta = \frac{\rho}{R}$

$\tan\theta = z/\rho$

$z = \rho\tan\theta$

$dz = \rho\sec^2\theta\,d\theta$ & $\theta = \tan^{-1}(z/\rho)$

if $z = -\infty \Rightarrow \theta = -\pi/2$

$z = +\infty \Rightarrow \theta = \pi/2$

$$H = \int_{-\pi/2}^{\pi/2} \frac{I\,dz\,\vec{a_z} \times \frac{(\rho\vec{a_\rho} - z\vec{a_z})}{\sqrt{\rho^2+z^2}}}{4\pi(\sqrt{\rho^2+z^2})^2}$$

$$H = \frac{I}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{dz\,\vec{a_z} \times (\rho\vec{a_\rho} - z\vec{a_z})}{(\rho^2+z^2)^{3/2}}$$

$$H = \frac{I}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{\rho\sec^2\theta\,d\theta \cdot \rho(\vec{a_z}\times\vec{a_\rho}) - (\vec{a_z}\times\vec{a_z})z}{(\rho^2+\rho^2\tan^2\theta)^{3/2}}$$

$$H = \frac{I}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{\rho^2\sec^2\theta\,d\theta\,\vec{a_\phi}}{(\rho^2)^{3/2}(\sec^2\theta)^{3/2}} \qquad \left[\because \begin{array}{l}\vec{a_z}\times\vec{a_z} = 0 \\ \vec{a_z}\times\vec{a_\rho} = \vec{a_\phi}\end{array}\right]$$

$$H = \frac{I}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{\rho \sec\theta} \, d\theta \, a_\phi$$

$$H = \frac{I}{4\pi\rho} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta \, \bar{a_\phi}$$

$$H = \frac{I}{4\pi\rho} \left[\sin\theta\right]_{-\pi/2}^{\pi/2} a_\phi$$

$$H = \frac{I}{4\pi\rho} \left[\sin\pi/2 + \sin\pi/2\right] a_\phi$$

$$H = \frac{I}{\cancel{4}\pi\rho} (\cancel{2}) a_\phi$$

$$H = \frac{I}{2\pi\rho} a_\phi$$

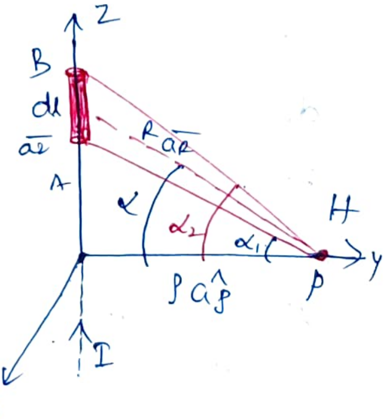
From above eqn: $\boxed{I = \frac{H 2\pi\rho}{a_\phi}}$ A

② Ampere's Circuit Law using finite Line Current:

From I - application of Ampere's Circuital law.

We have, eqn.

$$H = \frac{I}{4\pi\rho} \int_{\alpha_1}^{\alpha_2} \cos\theta \, d\theta \, \bar{a_\phi}$$




$$H = \frac{I}{4\pi\rho} \left[\sin\theta\right]_{\alpha_1}^{\alpha_2} \bar{a_\phi}$$

$$H = \frac{I}{4\pi\rho} \left[\sin\alpha_2 - \sin\alpha_1\right] \bar{a_\phi}$$

$$H = \frac{I\left[\sin\alpha_2 - \sin\alpha_1\right]\bar{a_\phi}}{4\pi\rho}$$

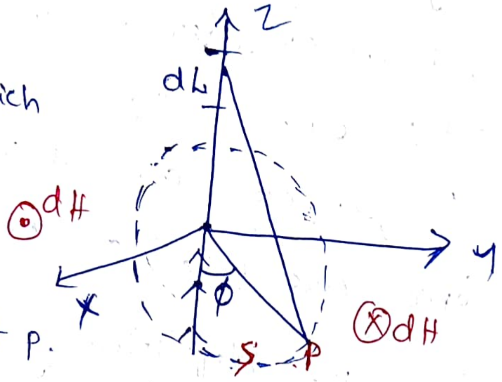
From above eqn,

$$\boxed{I = \frac{H \cdot 4\pi\rho}{\left[\sin\alpha_2 - \sin\alpha_1\right] a_\phi}}$$ A.

## Infinite line Current using Ampere's Circuit Law:-

Consider a Current Carrying Conductor which carries Current I.

p is the radius from origin to point p.




We know that, Ampere's Circuit Law.

$$\oint_l H \cdot dl = I$$

$$H = H_\phi \bar{a}_\phi$$

$$\int_l H_\phi \bar{a}_\phi [\underbrace{d\rho \bar{a}_\rho}_{0} + \rho d\phi \bar{a}_\phi + \underbrace{dz \bar{a}_z}_{0}] = I$$

$$\int_l H_\phi \rho d\phi = I$$

$$H_\phi \rho \int_l 1 \cdot d\phi = I$$

$$H_\phi \rho \int_0^{2\pi} 1 \cdot d\phi = I$$

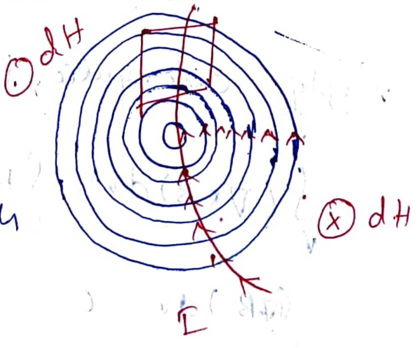
$$H_\phi \rho . 2\pi = I$$

$$\boxed{I = 2\pi \rho H_\phi}$$

## ~~Magnetic flux density (B):~~

## Magnetic Flux ($\phi$) density (B):

It is defined as the total magnetic flux which is passing through a closed surface or a closed loop ie,




$$\boxed{B = \frac{\phi}{S}}$$ Wb/m² (or) Tesla.

## Magnetic flux $\phi$

Let us Consider a Conductor which carries a Current I due to this Conductor. Magnetic field will be generated around it. These magnetic lines of forces are called as magnetic flux ($\phi$).

From the definition. $B = \frac{\phi}{S} = \frac{d\phi}{ds}$

$$d\phi = B\,ds$$

$$\boxed{\phi = \int B\,ds} \rightarrow (1)$$

The magnetic flux lines are continuous in nature which exists in the form of a closed loop.

→ The magnetic flux entering into the closed surface is equal to flux leaving it.

$\boxed{\text{i.e.}\ \ \phi = 0}$ [Total magnetic flux around a closed loop = 0]

Then $\boxed{\int_S B.ds = 0}$ from eqn (1)

"This eqn is called as "Integral form of Maxwell IV eqn".

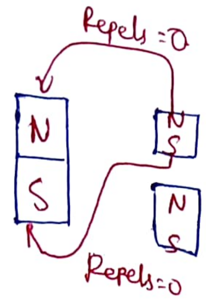
Apply divergence theorem to above eqn:

$$\int_V (\nabla \cdot B)\,dv = \int_S B.ds \qquad \boxed{\oint_S \vec{A}\,\vec{ds} = \int_V (\vec{\nabla}.\vec{A})\,dv}$$

$$\int_V (\nabla B.)\,dv = 0$$

$\boxed{\nabla \cdot B = 0}$ " Differential form of Maxwell-IV equation"

Note:- An isolated magnetic field does not exists.




## Maxwell's equations for Magnetostatic fields:-

we have from Ampere's Circuit law,

$$\boxed{\oint_l H \cdot dl = I} \rightarrow (1)$$

$$J = \frac{I}{S} = \frac{dI}{ds} \longrightarrow \text{from Conduction Current density}$$

$$dI = J\,ds$$

$$\boxed{I = \int_S J\,ds} \longrightarrow (2)$$

Compare 1 & 2 equations

$$\boxed{\oint_l H \cdot dl = \int_S J \cdot ds} \rightarrow (3)$$

$\longrightarrow$ Maxwell - III equation of In Integral form.

Apply Stoke's theorem

$$\int_S (\nabla \times H) ds = \oint_l H\,dl$$

$$\int_S (\nabla \times H)\,\cancel{ds} = \int_S J\,\cancel{ds} \quad [\because \text{from eqn } 3]$$

$$\boxed{\nabla \times H = J} \rightarrow \text{differential form of Maxwell eqn - III}$$

## Maxwell's - IV equation:-

From Ampere's Circuital Law

$$\oint_l H\,dl = I \longrightarrow (1)$$

We have total magnetic flux around a closed loop = 0.

ie. $\boxed{\phi = 0}$

We know that $B = \frac{\phi}{S}$

$$d\phi = ds$$

$$B = \frac{d\phi}{ds}$$

$$d\phi = B \cdot ds$$

$$\boxed{\phi = \int_S B \cdot ds} \rightarrow ②$$

from $\phi = 0$, $\phi = \int_S B \cdot ds$

$$\boxed{\int_S B\, ds = 0.} \rightarrow$$ Integral form of Maxwell IV equation.

Apply divergence theorem,

$$\int_V (\nabla \cdot B)\, dv = \int_S B \cdot ds$$

$$\int_V (\nabla \cdot B)\, dv = 0$$

$\boxed{\nabla \cdot B = 0}$ [Maxwell - IV equation in differential form]

Maxwell equations for static fields.

| S.No | Integral form | Differential form | Remarks. |
|---|---|---|---|
| 1 | $\int_S D \cdot ds = \int_V \rho_v dv$ | $\nabla \cdot D = \rho_v$ | Gauss law |
| 2 | $\int_l E \cdot dl = 0$ | $\nabla \times E = 0$ | law of Conservative |
| 3 | $\int_l H dl = \int_S J ds$ | $\nabla \times H = J$ | Ampere's Circuit law |
| 4 | $\int_S B ds = 0$ | $\nabla \cdot B = 0$ | Magnetic flux (or) Isolated Magnetic field. |

# Scalar Magnetic potential [$V_m$]

* The Scalar magnetic potential is defined in the region where $J=0$

* we know that, the electric potential in Electrostatistics.

$$V = -\int_l E\,dl$$

Similarly: In Magneostatics, the magnetic potential is

$$V_m = -\int_l H\,dl \quad \text{if } J=0 \text{ where}$$

$V_m$ = Scalar magnetic potential

$H$ = Magnetic field Intensity

W.K.T: $E = -\nabla V$ — from relation b/w E & V

Then $H \rightarrow V_m$ Similarly $\boxed{H = -\nabla V_m}$

Apply curl on both sides, $\boxed{\nabla \times H = J}$ - from diff form of Ampere's Circuit law

$$\nabla \times H = -\nabla \times (\nabla . V_m)$$

$\boxed{\nabla \times H = 0} \rightarrow (1)$ [∵ from Curl properties Curl of diverge = 0]

We know that, Maxwell-III eqn

$\boxed{\nabla \times H = J} \rightarrow (2)$

Compare (1) & (2) eqn's

$\boxed{J=0}$

The Scalar magnetic potential is exists in a Source free region

Note:- Scalar magnetic potential will satisfies Laplace eqn's

| | | |
|---|---|---|
| $\nabla^2 V = 0$ | $B = \mu_0 H$ | $\nabla(-\nabla . V_m) = 0$ |
| $\nabla^2 V_m = 0$ | $\nabla . (\mu_0 H) = 0$ | [from $H = -\nabla . V_m$] |
| $\nabla . B = 0$ | $\nabla . H = 0$ | $\boxed{\nabla^2 . V_m = 0}$ V |

# Vector Magnetic Potential

It is defined in a region where $J \neq 0$

From the properties of curl:-

$$\nabla \cdot (\nabla \times A) = 0 \quad \text{--- (1)}$$

From Maxwell - IV eqn. $\nabla \cdot B = 0$ --- (2)

Compare the above two eqn's

(1) ← $B = \nabla \times A$, Where a → vector magnetic potential

W.K.T, Maxwell - III eqn, $\nabla \times H = J$

$$B = \mu_0 H$$

$$H = \frac{B}{\mu_0}$$

$$\nabla \times \frac{B}{\mu_0} = J$$

$$\nabla \times B = \mu_0 J$$

$$\nabla \times (\nabla \times A) = \mu_0 J$$

$$\nabla \times \nabla \times A = \nabla \cdot (\nabla \cdot A) - \nabla^2 A$$

$$\nabla \cdot (\nabla \cdot A) - \nabla^2 A = \mu_0 J$$

if $\nabla \cdot A = 0$

$$0 - \nabla^2 A = \mu_0 J$$

$$\nabla^2 A = -\mu_0 J \quad \text{--- (2)}$$

$$\nabla^2 (A_x \bar{a_x} + A_y \bar{a_y} + A_z \bar{a_z}) = -\mu_0 [J_x \bar{a_x} + J_y \bar{a_y} + J_z \bar{a_z}]$$

$$\nabla^2 A_x = -\mu_0 J_x$$

$$\nabla^2 A_y = -\mu_0 J_y$$

$$\nabla^2 A_z = -\mu_0 J_z.$$

Note:- Vector magnitude potential due to line Current is

$$A = \frac{\mu_0}{4\pi} \int_l \frac{I\,dL}{R}$$

⇒ Due to Surface is given by

$$A = \frac{\mu_0}{4\pi} \int_S \frac{K\,ds}{R}$$

⇒ Due to volume is given by

$$A = \frac{\mu_0}{4\pi} \int_V \frac{J\,dv}{R}$$

## Forces due to Magnetic Fields:-

The force experienced by a magnetic field Can be Considered 3 different situations.

1. Force on a moving charge
2. Force on a Current elements.
3. Force between the 2 Current elements (or) ~~Eo~~ ampere Force Law.

## Force on a moving Charge:-

If the charge is stationary it experiences a force due to charges.

$F_e$, $F_m$.

Consider 2 point charges $Q_1$ & $Q_2$

$$Q_1 \xrightarrow[R]{} Q_2$$

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}$$

$$E = \frac{F}{Q}$$

$$F = F_e$$

$E = \frac{F_e}{Q}$

$\boxed{F_e = Q \cdot E}$, where

$F_e \rightarrow$ Force in Electrostatics

⇒ If Charge is moving it experience a force due to the Current & ~~magnitude~~ flux with a Velocity v. Then the force is $F_m$,

$\boxed{F_m = Q(V \times B)}$, where

V = velocity of charge

B = Magnetic flux density.

Total Force, $F = F_e + F_m$

$F = QE + Q(V \times B)$

$$\boxed{F = Q[E + (V \times B)]}$$

**Force on a Current Element:-**

The force on a current elements with a magnetic flux density is given by

$$\boxed{dF = I dL \times B}$$

$$F = \int_l I dl \times B$$

For Surface,

$$dF = K \cdot ds \times B$$

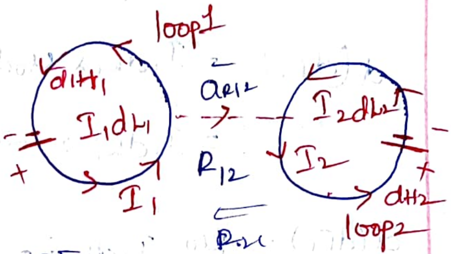
$$\boxed{F = \int_S K ds \times B}$$

For volume, $dF = J \cdot dv \times B$

$$\boxed{F = \int_V J \cdot dv \times B}$$

# Force between 2 Current Elements :-

* Consider a two conductors in the form of a loop which carries a currents $I_1$ & $I_2$.
* The two conductors are seperated with a distance $R_{12}$
* $dH_1$ is a differential magnetic field intensity due to conductor - 1,
* $dH_2$ is a differential magnetic field intensity due to conductor - 2.




From Biot - Savart's Law.

$$dH = \frac{I dL \sin\theta}{4\pi R^2}$$

In vector form, $dH = \frac{I dL \times \vec{a}_R}{4\pi R^2}$

then $dH_2 = \frac{I_1 dL_1 \times \bar{a}_{R_{12}}}{4\pi R_{12}^2} \rightarrow (1)$

$dF = I dL \times B$

$dF_2 = I_2 dL_2 \times B_2$

$d(dF_2) = I_2 dL_2 \times dB_2 \rightarrow (2)$

$B = \mu_0 H$

$B_2 = \mu_0 H_2$

$dB_2 = \mu_0 dH_2 \longrightarrow (3)$

Sub eqn (3) in eqn (2)

$d(dF_2) = I_2 dL_2 \times \mu_0 dH_2$

$= I_2 dL_2 \times \mu_0 \cdot \frac{I_1 dL_1 \times \vec{a}_{R_{12}}}{4\pi R_{12}^2}$

$$d(dF_2) = \frac{\mu_0}{4\pi} \cdot \frac{I_2 dL_2 \times I_1 dL_1 \times \vec{a}_{R_{12}}}{R_{12}^2}$$

$$F_{12} = \int_{L_1}\int_{L_2} \frac{\mu_0}{4\pi} \cdot \frac{I_2 dL_2 \times I_1 dL_1 \times \vec{a}_{R_{12}}}{R_{12}^2}$$

~~Cons~~ Consider $dH_1$ Calculate $F_1$ value.

$$dF = I dL \times B$$

$$dF_1 = I_1 dL_1 \times B_1$$

$$d(dF_1) = I_1 dL_1 \times dB_1 \longrightarrow (2)$$

$$B = \mu_0 H$$

$$B_1 = \mu_0 H_2 \longrightarrow (3)$$

$$dB_1 = \mu_0 dH_2.$$

Sub (3) in (2)

$$d(dF_1) = I_1 dL_1 \times \mu_0 dH_1$$

$$= I_1 dL_1 \times \mu_0 \frac{I_2 d_2 \times \bar{a}_{R21}}{4\pi R^2_{21}}$$

$$d(dF_1) = \frac{\mu_0}{4\pi} \frac{I_1 dL_1 \times I_2 dL_2 \times \bar{a}_{R21}}{R_{21}^2}$$

$$F_{21} = \int_{L_1}\int_{L_2} \frac{\mu_0}{4\pi} \frac{I_1 dL_1 \times I_2 dL_2 \times \bar{a}_{R21}}{R_{21}^2}$$

$$F_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \int_{L_1}\int_{L_2} \frac{dL_1 \times dL_2 \times \bar{a}_{R21}}{R_{21}^2}$$

**Force on a Current element:-**

$$\boxed{F = I\vec{L} \times \vec{B}}$$ Newtons.

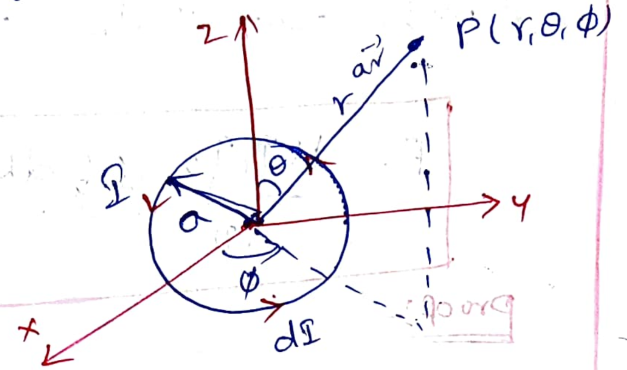
(or)

$$\boxed{F = BIl \sin\alpha}$$

# Magnetic Dipole:-

A bar magnet or a small filamentary current loop is usually referred to as a magnetic dipole.

Magnetic vector potential due to magnetic dipole is




$$\vec{A} = \frac{\mu \vec{m} \times \vec{a_r}}{4\pi r^2}$$

$$\vec{m} = I.S\,\vec{a_n}$$

$$\vec{m} = I.S\,\vec{a_y}$$

$S = \pi r^2$ ; radius $= a$ ; $S = \pi a^2$

$\vec{m}$ = Magnetic dipole moment.

$$\vec{m} = I\,(\pi a^2)\,\vec{a_y}$$

$$\vec{A} = \frac{\mu I\,(\pi a^2)\,\vec{a_y} \times \vec{a_r}}{4\pi r^2}$$

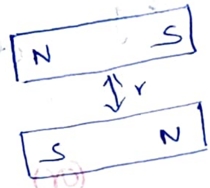
$$\vec{a_y} \times \vec{a_r} = |\vec{a_y}|\,|\vec{a_r}|\,\sin\theta\,\vec{a_n}$$

$$= \sin\theta\,\vec{a_\phi}$$

$$\boxed{\vec{A} = \frac{\mu I\,(\pi a^2)\,\sin\theta\,\vec{a_\phi}}{4\pi r^2}} \ \text{wb/m}$$

An arrangement of two equal & opposite magnetic poles seperated by a small distances.

Magnet dipole is a vector quantity.




Ex:- A bar magnet
A current carrying loop.

direction from S - to N - poles.

# Ampere's Force Law:

Ampere's force Law States that there exists a force between two Current elements $I_1 dL_1$ and $I_2 dL_2$ and it is given by

$$\boxed{F = \frac{\mu I_1 I_2}{4\pi} \oint \oint \frac{dL_2 \times (dL_1 \times a_r)}{r^2}}$$

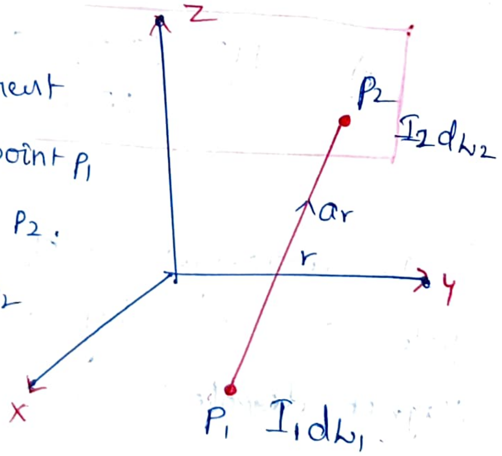
Proof:-

Here $\mu$ = permeability of the medium in which the Current elements are placed.

$a_r$ = distance between the elements.

Consider two differential Current elements $I_1 dL_1$ and $I_2 dL_2$ are Seperated by a distance r.

Let the differential Current element $I_1 dL_1$ be at point $P_1$ and $I_2 dL_2$ be at point $P_2$.




The magnetic field at $P_2$ due to $I_1 dL_1$ is given by

$$dH = \frac{I\, dL_1 \times a_r}{4\pi r^2}$$

(or)

$$dB = \frac{\mu I\, dL_1 \times a_r}{4\pi r^2}$$

This fields exerts a force the current element $I_2 dL_2$ at point $P_2$ and it is given by

$$d(dF) = I_2 dL_2 \times dB$$

This is the differential of a differential force on a differential current element due to a differential field, $dB$.- From the above expressions we get;

$$d(dF) = \frac{\mu I_2 dL_2 \times (I_1 dL_1 \times a_r)}{4\pi r^2}$$

$$= \frac{\mu I_1 I_2}{4\pi} \frac{dL_2 \times (dL_1 \times a_r)}{r^2}$$

$$\boxed{F = \frac{\mu I_1 I_2}{4\pi} \oint\oint \frac{dL_2 \times (dL_1 \times a_r)}{r^2}}$$ Hence proved

## Inductance and Magnetic Energy (or) Energy density in Magnetostatics:-

1. Inductance :-

Inductor:- An inductor is also called a coil, choke or reactor. It is a passive two-terminal electrical component that stores energy in a magnetic field when electric current flows through it. An inductor typically consists of an insulated wire wound into a coil.

Inductance:-

A circuit (or) closed conducting path carrying current $I$ produces a magnetic field $B$ that causes a flux $\Psi = \int B \cdot ds$. to pass through each turn of the circuit as shown in fig.

If the circuit has N identical turns, we define the flux linkage $\lambda$ as

$$\boxed{\lambda = N\phi}$$

Also, if the medium surrounding the circuit is linear, the flux linkage $\lambda$ is proportional to the current I producing it. that is

$$\lambda \propto I$$

(or)

$$\boxed{\lambda = LI}$$

L = Constant of proportionality called the inductance of the circuit.

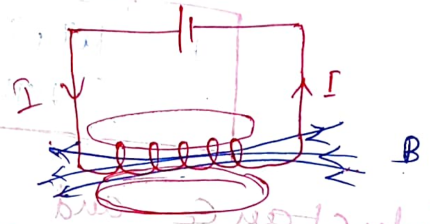



fig: Magnetic field B produced by circuit

## Self Inductance :-

It is defined as the ratio of magnetic flux linkage $\lambda$ to the current I through the inductor.

ie, $\boxed{L = \frac{\lambda}{I} = \frac{N\phi}{I}}$ Henry

**Note:-**

The Magnetic energy stored in a inductor is given by $\boxed{W_m = \frac{1}{2} I L^2 \text{ (Joules)}}$

## Mutual Inductance:-

The Mutual inductance between the two circuits is defined as the flux linkage of one circuit to the current in other circuit.

Consider the magnetic interaction between the two circuits as shown below figure.

Here, there are 4 fluxes. because of two coils are placed closely.

$\phi_{11}, \phi_{12}, \phi_{21}$ and $\phi_{22}$

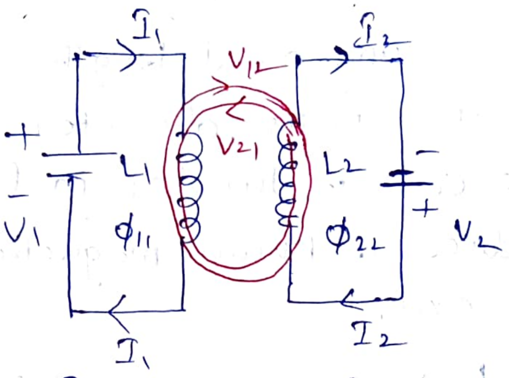



Fig:- Magnetic interaction between the two circuits.

where

$\phi_{11}$ = flux in coil 1 due to current $I_1$

$\phi_{12}$ = flux in coil 1 due to current $I_2$

$\phi_{21}$ = flux in coil 2 due to current $I_1$

$\phi_{22}$ = flux in coil 2 due to current $I_2$

$\phi_{11}, \phi_{22}$ are self inductances. $\phi_{21}$ & $\phi_{12}$ are mutual inductances. $M_{21}$ & $M_{12}$ respectively.

$$M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1\phi_{21}}{I_2}$$

$$M_{21} = \frac{N_2\phi_{12}}{I_1}$$

$$M_{12} = \frac{\text{Flux linkage of Circuit 1}}{\text{Current in Circuit 2}} = \frac{N_1\phi_{21}}{I_2}$$

$$M_{21} = \frac{\text{Flux linkage of Circuit 2}}{\text{Current in Circuit 1}} = \frac{N_2\phi_{12}}{I_1}$$

$$\boxed{\phi_1 = \phi_{11} + \phi_{12}} \qquad \boxed{\phi_2 = \phi_{22} + \phi_{21}}$$

# Self Inductance of Solenoid:

N- No of turns

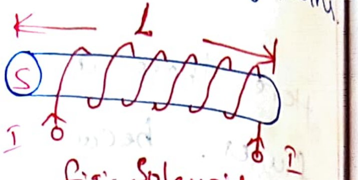



fig- Solenoid

It is a coil of wire which has a long axial length relative to its diameter.. The coil is tubular in form. It is used to produce a known magnetic flux density along its axis:

$$L = \frac{\lambda}{I} = \frac{N\phi}{I} \qquad \phi = \int_S \vec{B}.\vec{ds}$$

$$= B.\int_S 1.ds$$

$$\boxed{\phi = B.S}$$

$\therefore$ $\vec{H}$ due to Solenoid is

$$\vec{H} = \left(\frac{N}{l}\right) I \quad \vec{a_z} \ A/m$$

$$H = \left(\frac{N}{l}\right) I \ \ \& \ \text{—} \ (1)$$

$$B = \mu H$$

$$B = \mu \left(\frac{N}{l}\right) I \ \text{—} \ (2) \qquad \left[\frac{l \quad N\phi}{I}\right]$$

$$l$$

$$B = \frac{\phi}{S}$$

$$\phi =$$

$$\cancel{B} = \cancel{\mu} \frac{\left(\frac{N}{N\phi}\right) I}{I}$$

$$\phi = BS.$$

$$\phi = \frac{\frac{\mu N I}{N\phi}}{I} \cdot S$$

$$\phi = BS$$

$$\phi = \mu\left(\frac{N}{l}\right) I . S$$

$$\phi = \frac{\mu N I}{l} . S$$

$$l = \frac{N\left[\frac{\mu N I}{l}\right] . S}{I}$$

$$l = \frac{N^2 \mu \cancel{I}}{l} S$$

$$\boxed{L = \frac{N^2 \mu S}{l}}$$

## Self Inductance of Toroid:-

It consists of a coil wound on annular core. One side of each turn of the coil is threaded through the ring to form a Toroid.

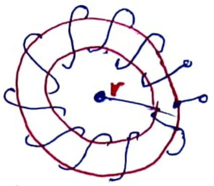


fig:- Toroid.

Inductance of Toroid $L = \frac{\mu_0 N^2 S}{2\pi r}$ H

N = Number of turns

r = average radius

S = Cross-sectional area.

**Note:-** For solenoid & Toroid Magnetic field intensity H is same.

## Energy density in Magnetostatics:-

Energ density in a Magnetic field

$$\boxed{W_m = \frac{1}{2}\mu H^2} \text{ (or) } \boxed{W_m = \frac{1}{2} BH} \text{ Joules.}$$

* Differential Equation:- $W_m = \frac{dW_m}{dV} = \frac{1}{2}\vec{B}\cdot\vec{H}$ J/m$^3$

* Integral Equation: $W_m = \frac{1}{2}\int_V \vec{B}\cdot\vec{H}\, dv$ J

## Energy stored in solenoid:-

The inductance of a solenoid is given by

$$L = \frac{\mu_0 N^2 S}{l}$$

N = number of turns

S = area of cross-section of a solenoid

l = length of the solenoid.

The energy stored in an inductor is given by

$$W_m = \frac{1}{2} L I^2$$

$$= \frac{1}{2} \mu_0 \frac{N^2 S}{l} \cdot I^2$$

$$= \frac{1}{2} \mu_0 \frac{N^2 I^2}{l^2} l s$$

$$= \frac{1}{2} \mu_0 \left(\frac{NI}{l}\right)^2 l s$$

Here : $ls$ = volume of space inside the coil

But, $H = \frac{NI}{l}$ in a solenoid.

Energy stored $= \frac{1}{2} \mu_0 H^2 (\underbrace{ls}_{V})$ Joules $= \boxed{W_m = \frac{1}{2} \int_V \mu H^2 dv}$

$\frac{W_m}{V} = \frac{1}{2} \mu_0 H^2 \ J/m^3$

(or) Energy density, $\boxed{W_m = \frac{1}{2} B \cdot H \text{ Joules}/m^3}$

#

## Energy stored in an Inductor:-

Energy stored in an Inductor.

$W_m = \frac{1}{2} B \cdot H = \frac{1}{2} \mu H^2$

energy stored $= \int_V \frac{1}{2} \mu H^2 dv = \frac{1}{2} \mu H^2 v$

but $v$ = volume of space inside the coil $= (ls)$, $m^3$.

$\therefore H = \frac{NI}{l}$ energy stored $= \frac{1}{2} \mu \left[\frac{NI}{l}\right]^2 ls$

$$= \frac{1}{2} \mu \left[\frac{N^2 I^2}{l^2}\right] ls$$

$$= \frac{1}{2} \underbrace{\frac{\mu N^2 s}{l}}_{L} I^2$$

$$\boxed{W_m = \frac{1}{2} I^2 L \text{ Joules}}$$

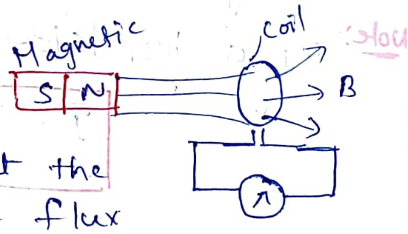
# Faraday's Law:- and Transformer e.m.f Inconsistency of Ampere's Law. and Displacement Current density:

## ① Faraday's Law:-

According to Faraday's experiments, a static magnetic field produces no current flow, but time varying magnetic field produces induced emf Causes a Current to flow.

Statements:- of Faraday's Law:-

① Faraday's First law states that " change of magnetic flux linked to a Coil, produces induce an emf across a coil."



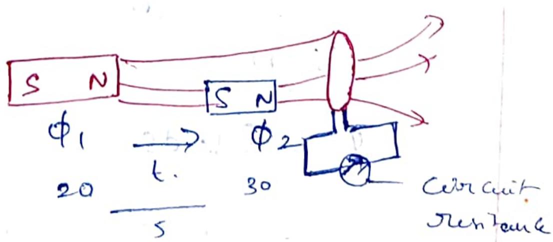

(or)

Faraday's Law states that the rate of change of magnetic flux induces an emf.

② Faraday's Second law states that " The induced emf across the Coil is equal to rate of change of flux in the coil.

$$e = -\frac{d\phi}{dt}.$$

Negative Sign shows that emf induced always oppose the change in flux.




$$\varepsilon_{mf} = \frac{\phi_2 - \phi_1}{t}$$

$$= \frac{30-20}{5}$$

$$\boxed{\varepsilon_{mf} = 2V}$$

I = Induced Current

R = meter resistance

E = Emf

$i = \frac{\varepsilon}{R}$ $\quad i = \frac{\varepsilon}{R}$ $\quad = -\frac{1}{R}\frac{d\phi}{dt}$ volts.

**Lenz's Law:-** According to lenz's law the polarity of Induced emf is such which oppose the change in the magnetic flux.

The variation of flux with time may be caused in three ways.

* By having a stationary loop in a time-varying field $\vec{B}$. transformer emf.
* By having a time-varying loop in a static $\vec{B}$ field motional emf
* By having a time-varying loop area in a time-varying $\vec{B}$ fields.

Note: If a loop / closed circuit has a 'R' and then

$$\boxed{I = \frac{V_{emf}}{R} \text{ Amp}}$$

## Transformer Emf (or) Stationary loop in time-varying $\vec{B}$ field.

From Faraday's Law.

$$Emf = -N\frac{d\phi}{dt} \rightarrow (1)$$

$$Emf = \int_l E \cdot dl \rightarrow (2)$$

$$\int_l E\,dl = -N\frac{d\phi}{dt}$$

if $N=1$

$$\phi = \int_s B \cdot ds$$

$$= \int_l E\,dl = -1 \cdot \frac{d}{dt}\int_s B \cdot ds$$

By Stoke's theorem,

$$\int_s (\nabla \times E) \, ds = \int_l E \, dl$$

$$\int_s (\nabla \times E) \, ds = -\frac{d}{dt} \int_s B \cdot ds$$

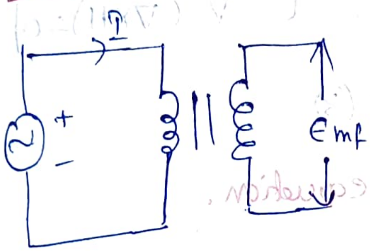
$$\nabla \times E = -\frac{dB}{dt}$$

$$D = \epsilon_0 E$$

$$B = \mu_0 H$$

$$\boxed{\nabla \times E = -\mu_0 \cdot \frac{dH}{dt}}$$

## Transformer EMF:-




From faraday's law change in flux which induces an emf.

* when a time varying current is applied primary winding of transformer which produces magnetic flux around it.

* The change in flux induces on emf in the Secondary winding which is known as transformer emf.

Note:- The induced emf is measured in volts & it's mathematical representation is

$$\boxed{emf = \int_l E \, dl}$$

* Inconsistency of ampere's Circuital law

(or) Maxwell-II eqn for time varying fields

W.K.T. from Ampere's Circuit Law.

$$\oint_l H \cdot dl = I \rightarrow ①$$

From the III - Maxwell eqn

$$\nabla \times H = J \rightarrow ②$$

Apply divergence on B.Sides.

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J \qquad [\because \nabla \cdot (\nabla \times H) = 0]$$

$$0 = \nabla \cdot J \rightarrow ③$$

From Continuity equation.

$$\nabla \cdot J = -\frac{d\rho_v}{dt} \rightarrow ④$$

From equation (4) $\rho_v$ which is varied w.r.to time but in eqn (3) there is no field which is not varied w.r.to time, So that Ampere Circuit law is not valid for all the Cases.

The James maxwell will do the modification to ampere's Circuited law

i.e, he will add one unknown parameter to the R.H.S of eqn (2)

$$\nabla \times H = J + K \rightarrow (5)$$

$K$ ↳ unknown parameter.

Apply divergence.

$$\nabla (\nabla \times H) = \nabla J + \nabla \cdot K$$

$$0 = \nabla J + \nabla \cdot K$$

$$\nabla J = -\nabla K$$

$$+\frac{\partial \rho_v}{\partial t} = +\nabla K$$

$$\frac{d\rho_v}{dt} = \nabla \cdot K$$

From Maxwell I $\Rightarrow \nabla \cdot D = \rho_v$

$$\frac{\partial}{\partial t}(\nabla \cdot D) = \nabla \cdot K$$

$$\boxed{\nabla \cdot \frac{dD}{dt} = \nabla \cdot K}$$

$$K = \frac{dD}{dt} \rightarrow (6)$$

Sub (6) in eqn (5)

$$\nabla \times H = J + \frac{dD}{dt} \longrightarrow 7$$

From Continuity eqn $J = \sigma E$

$$D = \epsilon_0 E$$

$$\boxed{\nabla \times H = \sigma E + \epsilon_0 \frac{dE}{dt}}$$

* Maxwell I & II eqn's for time varying fields

I - eqn $\nabla \times E = -\mu_0 \frac{dH}{dt}$

II - eqn $\nabla \times H = \sigma E + \epsilon_0 \frac{dE}{dt}$

## Displacement Current density (Jd):-

The ~~first~~ ~~term~~ in $\boxed{\nabla \times H = J + \frac{dD}{dt}}$ → 1 → point form,

Differential form of Maxwell - II eqn in time varying fields.

* The first term in equation (1) is conduction current density denoted by J.

* The Second term in equation ① represents Current density expressed in ampere per square meter. As this quantity is obtained from time varying electric flux density. This is also Called displacement density. Thus this is Called displacement Current density denoted by $J_D$. With these definitions we can write equation (1) as.

$$\boxed{\nabla \times \bar{H} = \bar{J}_c + \bar{J}_D}$$

$\frac{\partial D}{\partial t} = J_D$ = Displacement Current density

$J = J_c$ = conduction Current density.

$J = J_c$ Conduction Current density.

$J_D = \frac{\partial D}{\partial t}$ = Displacement Current density.

$J_c$ = Here attaching Sub Script C indicates that the current is due to the moving charges.

For Static electromagnetic fields, according to Ampere's Circuital law, we can write.

$$\nabla \times \bar{H} = \bar{J} \quad \text{——} (1)$$

Taking divergence on both sides

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J}$$

But, According to vector identity, "divergence of the Curl of any vector field is zero". Hence we can write,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} = 0 \quad \text{——} (2)$$

But the equation of Continuity is given by,

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{——} (3)$$

From equation (3) it is Clear that when

$\frac{\partial \rho_v}{\partial t} = 0$ then only equation (2) becomes true.

Thus equations (2) and (3) are not compatible for time varying fields. We must modify equation (1) by adding one unknown term say $\bar{N}$.

The equation (1) becomes

$$\nabla \times \vec{H} = \bar{J} + \bar{N} \qquad (4)$$

Again taking divergence on both the sides

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \bar{N} = 0$$

As $\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$, to get correct conditions we must write

$$\nabla \cdot \bar{N} = \frac{\partial \rho_v}{\partial t}$$

But according to Gauss law, or I Maxwell I equation

$$\rho_v = \nabla \cdot \bar{D}$$

Thus replacing $\rho_v$ by $\nabla \cdot D$.

$$\nabla \cdot \bar{N} = \frac{\partial}{\partial t}(\nabla \cdot \bar{D}) = \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

Comparing two sides of the equation,

$$\bar{N} = \frac{\partial \bar{D}}{\partial t} \qquad (5)$$

Now we can write Ampere's circuital law in point form as:

$$\boxed{\nabla \times H = \bar{J}_c + \frac{\partial \bar{D}}{\partial t}} \qquad (6)$$

$$\frac{\partial D}{\partial t} = J_D$$

$$\boxed{\nabla \times H = J_c + J_D}$$

# Maxwell's Equations for Time Varying fields:

## ① Maxwell's equation derived from Faraday's law:

### First equation:-

① $\nabla \times E = -\frac{dB}{dt}$

(or)

$\nabla \times E = -\mu_0 \frac{dH}{dt}$

**proof:-** From Faraday Law.

$EmF = -N\frac{d\phi}{dt}$

$EmF = \int_l E\,dl$

$\Rightarrow \int_l E\,dl = -N\frac{d\phi}{dt}$

if $N=1$

$\phi = \int_S B \cdot ds$

$\int_l E\,dl = -1\frac{d}{dt}\int_S B\cdot ds \rightarrow ①$

↳ Integral form of maxwell-I eqn.

By stoke's theorem.

$\int_S (\nabla \times E)\,ds = -\frac{d}{dt}\int_S B\cdot ds$

$\boxed{\nabla \times E = -\frac{d}{dt} B}$ ⇒ differential form I - eqn

↳ $B = \mu_0 H$.

$$\boxed{\nabla \times E = -\mu_0 \frac{dH}{dt}}$$

## Second equation:-

② $\nabla \times H = J + \frac{d \cdot D}{dt}$ (or) $\nabla \times H = \sigma \epsilon + \epsilon_0 \frac{d\epsilon}{dt}$

$\oint_l H \cdot dl = I \Rightarrow \nabla \times H = J$

$\nabla(\nabla \times H) = \nabla \cdot J$ $\qquad \nabla(\nabla \times H) = 0.$

$\nabla \cdot J = 0$ $\qquad$ ($\because$ from curl property)

From Continuity equation

$$\nabla \cdot J = -\frac{d\rho_v}{dt}$$

From James Maxwell equation

$$\nabla \times H = J + K$$

Apply divergence

$$\nabla(\nabla \times H) = \nabla J + \nabla \cdot K$$

$$0 = \nabla J + \nabla \cdot K$$

$$\nabla J = -\nabla \cdot K$$

$$-\frac{d\rho_v}{dt} = -\nabla \cdot K$$

$$\frac{d\rho_v}{dt} = \nabla \cdot K$$

From maxwell-I equation,

$$\nabla \cdot D = \rho_v$$

$$\frac{d}{dt}(\nabla \cdot D) = \nabla \cdot K$$

$$\nabla \cdot \frac{dD}{dt} = \nabla \cdot K$$

$$K = \frac{dD}{dt}$$

$$\nabla \times H = J + \frac{dD}{dt}$$

(or)

$(J = \sigma \epsilon)$ $(D = \epsilon_0 \epsilon)$

differential form

$$\nabla \times H = \sigma E + \epsilon_0 \frac{dE}{dt}$$

Apply Surface Integral

$$\int_S (\nabla \times H) ds = \int_S \left(\sigma E + \epsilon_0 \frac{dE}{dt}\right) ds$$

By Stoke's theorem

$$\oint_l H dl = \int_S (\nabla \times H) ds$$

Integral form of Maxwell - I equation.

$$\boxed{\oint_l H dl = \int_S \left(\sigma E + \epsilon_0 \frac{dE}{dt}\right) ds}$$

(3) Max well Third Equation.

$$\nabla \cdot D = \rho_v$$

Proof:-

From Gauss Law.

$$\psi = Q$$

$$D = \frac{\psi}{S} = \frac{Q}{S} \quad \psi = Q$$

$$\psi = \int_S D \cdot ds$$

$$D = \frac{d\psi}{ds} \quad d\psi = D ds$$

$$Q = \int_S D \cdot ds$$

$$\psi = \int_S D \cdot ds$$

$$Q = \int_V \rho_v dv$$

$$Q = \int_S D \cdot ds$$

$$\int_V \rho_v dv = \int_S D \cdot ds$$

from divergence theorem.

$$\int_V (\nabla \cdot D) dv = \int_S D \cdot ds$$

$$\int_V \rho_v dv = \int_V (\nabla \cdot D) dv$$

$$\boxed{\nabla \cdot D = \rho_v}$$

## (4) Max well Fourth equation

$$\nabla \cdot B = 0$$

proof :- $B = \frac{\phi}{S}$ ∴ $\frac{d\phi}{ds}$

$$d\phi = B \cdot ds$$

$$\phi = \int_S B \cdot ds \rightarrow (1)$$

$\phi = 0$ (As entering flux = leaving flux)

$$\int_S B \cdot ds = 0$$

From Divergence theorem

$$\int_V (\nabla \cdot B)\, dv = \int_S B \cdot ds$$

$$\int_V (\nabla \cdot B)\, dv = 0$$

$$\boxed{\nabla \cdot B = 0}$$

### Max well equations for Time varying fields.

| S.No. | Integral form | Differential form | Remark. |
|---|---|---|---|
| 1 | $\int_l E\,dl = \frac{-d}{dt}\int_S B \cdot ds$ | $\nabla \times E = \frac{-dB}{dt}$ (or) $\nabla \times E = \mu_0 \frac{dH}{dt}$ | Faraday's Law |
| 2 | $\int_l H\,dl = \int_S \sigma E + \epsilon_0 \frac{dE}{dt}$ (or) $\int_S \left(J + \frac{dD}{dt}\right) ds$ | $\nabla \times H = \sigma E + \epsilon_0 \frac{dE}{dt}$ (or) $\nabla \times H = J + \frac{dD}{dt}$ | Modified Ampere Circuital Law. |
| 3 | $\int_S D \cdot ds = \int_V \rho_v\, dv$ | $\nabla \cdot D = \rho_v$ | Gauss Law |
| 4 | $\int_S B \cdot ds = 0$ | $\nabla \cdot B = 0$ | Isolated Magnetic field |

# Maxwell's Equations in Different Final forms and word Statements.

① Word Statement:-

① $\nabla \cdot D = \rho_v$

$$\boxed{\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho_v \, dv} = Q_{enclosed} \rightarrow \text{Gauss law}$$

Statement:- The total electric flux density of a closed surface is equal to the total charge enclosed by a finite volume.

② $\vec{\nabla} \cdot \vec{B} = 0$.

$$\boxed{\oint_S \vec{B} \cdot d\vec{s} = 0}$$

Non-existance of monopole magnet (or) Gauss law for magnetic fields.

Statement:- The closed surface integral of magnetic flux density is always equal to zero.

③ $\nabla \times E = -\frac{\partial}{\partial t} \vec{B}$

$$\boxed{\oint_L \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}} \rightarrow \text{Faraday's Law.}$$

Statement:- The total electromotive force (EMF) induced in a closed path is equal to the negative surface integral of the rate of change of magnetic flux density with respect to time over an entire surface bounded by the same closed path.

④ $\nabla \times H = J + \frac{\partial D}{\partial t}$

(or) $\boxed{\oint_L H \cdot dl = \int_S \left(J + \frac{\partial D}{\partial t}\right) ds}$

Statement:- The total magnetomotive force around any closed path is equal to the surface integral of the sum/addition of conduction and displacement current densities over the entire surface bounded by the same closed path.

Different final forms:- For free space there is no conductivity. $\sigma = 0$; $\rho_v = 0$; $J = \sigma E = 0$

① Maxwell's equations for free space.

(A) point form.

① $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

② $\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t}$

③ $\nabla \cdot \bar{D} = 0$

④ $\nabla \cdot \bar{B} = 0$

(B) Integral form.

$\oint E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot d\bar{s}$

$\oint H \cdot d\bar{l} = \int_S \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s}$

$\oint_S \bar{D} \cdot d\bar{s} = 0$

$\oint_S B \cdot d\bar{s} = 0$

② Maxwell's equations for Good Conductors:-

$\rho_v = 0$; $\sigma \neq 0$, $\bar{J} \neq 0$ $\left[J >> \frac{\partial D}{\partial t}\right]$

(A) Point form

① $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

② $\nabla \times \bar{H} = \bar{J}$

③ $\nabla \cdot \bar{D} = 0$

④ $\nabla \cdot \bar{B} = 0$

(B) Integral form.

① $\oint \bar{E} \cdot d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$

② $\oint \bar{H} \cdot d\bar{L} = I = \int_S \bar{J} \cdot d\bar{s}$

③ $\oint_S \bar{D} \cdot d\bar{s} = 0$

④ $\oint_S \bar{B} \cdot d\bar{s} = 0$

# Maxwell's Equations for Harmonically Varying fields.

Let us Assume that the electric and magnetic fields are varying harmonically with time. the electric flux density can be written as,

$$\bar{D} = \bar{D}_0 e^{j\omega t}$$

Similarly the magnetic flux density can be written as

$$\bar{B} = \bar{B}_0 e^{j\omega t}$$

Taking partial derivative with respect to time, we can write,

$$\frac{\partial \bar{D}}{\partial t} = j\omega \bar{D}_0 e^{j\omega t} = j\omega \bar{D}$$

$$\frac{\partial \bar{B}}{\partial t} = j\omega \bar{B}_0 e^{j\omega t} = j\omega \bar{B}$$

(A) point form

(B) ~~Integral form.~~

$\oint \cancel{\bar{E}\cdot d\bar{l}} = \cancel{-\int_S j\omega \bar{B}\cdot ds}$

$\cancel{= -\int_S j\omega\mu \bar{H}\cdot ds}$

$\cancel{= j\omega \mu}$

(1) $\nabla \times \bar{E} = -j\omega \bar{B} = -j\omega\mu \bar{H}$

(2) $\nabla \times \bar{H} = \bar{J} + j\omega \bar{D} = \sigma \bar{E} + j\omega(\epsilon \bar{E})$
$= (\sigma + j\omega\epsilon)\bar{E}$

(3) $\nabla \cdot \bar{D} = \rho_v$

(4) $\nabla \cdot \bar{B} = 0$

## Integral form

(1) $\oint \bar{E}\cdot d\bar{L} = -\int_S j\omega \bar{B}\cdot d\bar{s} = -\int_S j\omega\mu \bar{H}\cdot d\bar{s} = -j\omega\mu \int_S \bar{H}\cdot d\bar{s}$

(2) $\oint \bar{H}\cdot d\bar{L} = I + \int_S j\omega \bar{D}\cdot d\bar{s} = \int_S \bar{J}\cdot d\bar{s} + \int_S j\omega\epsilon \bar{E}\cdot d\bar{s}$

$= \int_S \sigma \bar{E}\cdot d\bar{s} + \int_S j\omega\epsilon \bar{E}\cdot d\bar{s} = (\sigma + j\omega\epsilon)\int_S \bar{E}\cdot d\bar{s}$

(3) $\oint_S \bar{D}\cdot d\bar{s} = \oint_V \rho_v \, dv$

(4) $\int_S \bar{B}\cdot d\bar{s} = 0$

# Unit - III

## Boundary Conditions and uniform plane wave

### ① Boundary Conditions of Electromagnetic fields:- [E & D]

The boundary conditions are exists at boundaries in different medium's. i.e., when a field is moving from medium-1 to medium-2. Such Conditions are Called as boundary conditions.

* The boundary conditions are used to find the values of .

① Electric Field Intensity - E

② electric Flux Density - D

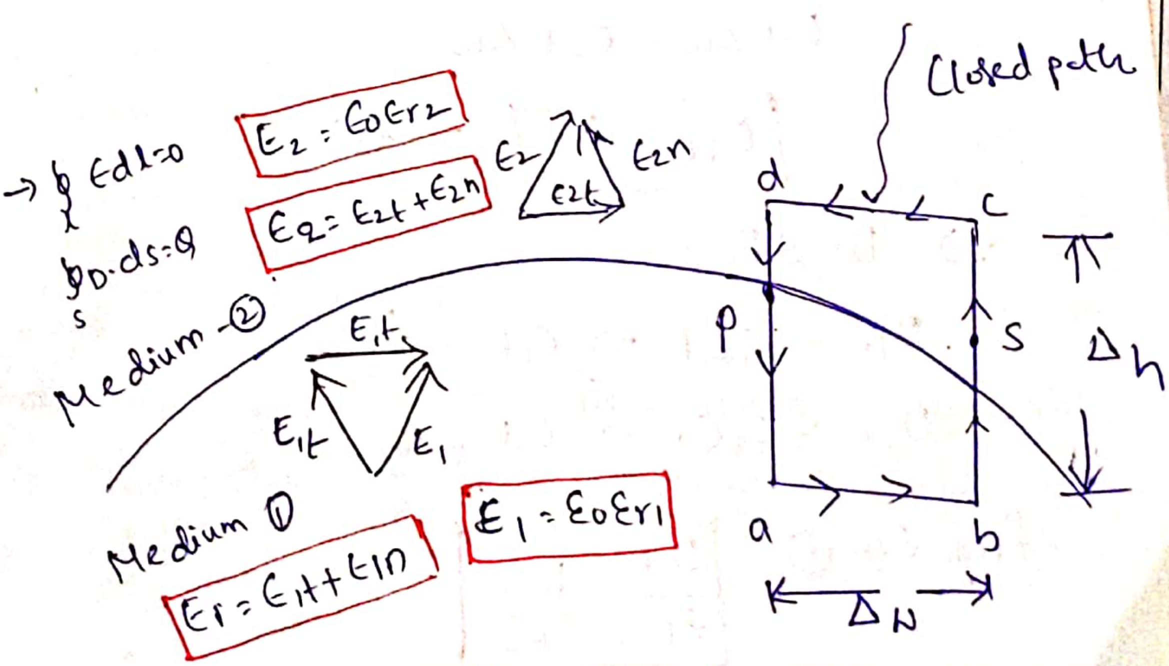
③ Magnetic Field Intensity - H

④ Magnetic Flux Density - B.

→ In order to Calculate the above 4 parameters we use the maxwell equations. i.e. $\oint_l E \cdot dl = 0$

$$\oint_s D \cdot ds = Q$$

### ① Dielectric - Dielectric Interface / Boundary Condition:



The Electric field $E_1$ which is moving from medium-1 to medium-2 (both the mediums are dielectrics.)

* The Electric field is decomped into 2 field components.

i.e.,

① Normal & ② Tangential.

* From the Maxwell-I eqn we consider a closed path named as abcd.

* Compare the normal & tangential components of $E_1$ & $E_2$ along a closed path.

* Compare tangential components of $E_1$ & $E_2$ with $a \to b$, & $c \to d$ respectively.

* Compare normal components of $E_1$ & $E_2$ with $b \to c$ & $d \to a$

$$\oint_l E \cdot dl = 0.$$

$$E dl = 0$$

$$E_{1t} \Delta w + E_{1n} \cancel{\frac{\Delta h}{2}} + E_{2n} \cancel{\frac{\Delta h}{2}} - E_{2t} \Delta w - E_{2n} \cancel{\frac{\Delta h}{2}} - E_{1n} \cancel{\frac{\Delta h}{2}} = 0$$

$$E_{1t} \Delta W - E_{2t} \Delta W = 0$$

$$E_{1t} \Delta W = E_{2t} \Delta W$$

$$\boxed{E_{1t} = E_{2t}} \longrightarrow ①$$

$\Rightarrow D = \varepsilon E$

$$D_{1t} = \varepsilon_1 E_{1t} \Rightarrow E_{1t} = \frac{D_{1t}}{\varepsilon_1}$$

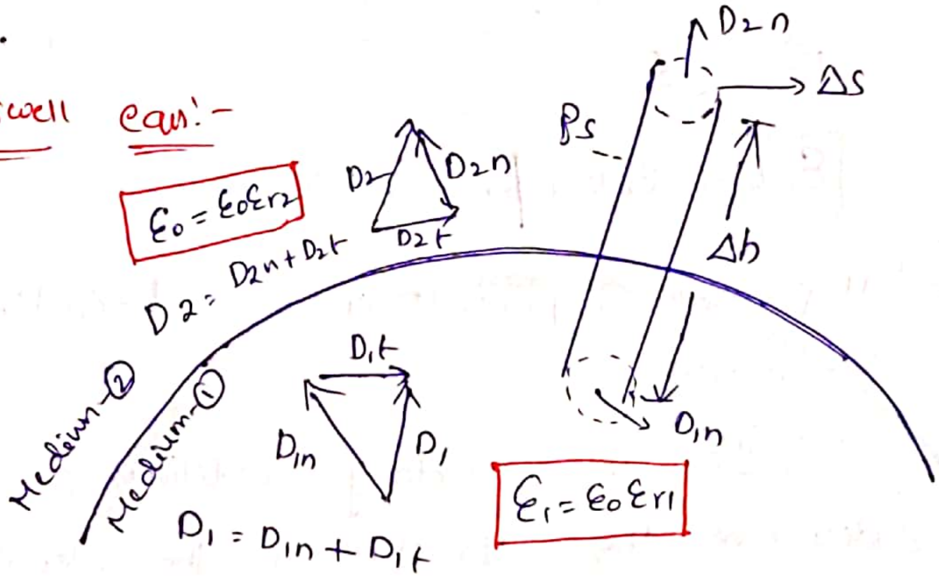
$$D_{2t} = \varepsilon_2 E_{2t} \Rightarrow E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

from eqn ① $\boxed{\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}}$

* When a field is moving from medium-1 to medium-2. it's normal Components are Cancelled each other & its tangential Components are equal.

* Similarly. when a field ie., D is moving from Med-1 to med-2 it's tangential components are equal but it depends on $\epsilon_1$ and $\epsilon_2$.

Consider II - Maxwell eqn:-




When a field is moving from medium-1 to medium-2 in a Closed Surface it's normal Components are equal to it's Surface charge density.

→ The direction of D is at top side & bottom side of the surfaces only.

$$\oint_S D \cdot ds = Q$$

$$\int_{s(top)} D \cdot ds + \int_{s(bottom)} D \cdot ds = \rho_s$$

$$D_{2n} \Delta s - D_{1n} \Delta s = \rho_s \Delta s$$

$$\boxed{D_{2n} - D_{1n} = \rho_s} \longrightarrow (3)$$

∴ eqn(3) describes $\vec{D}$ is directed from region-1 to region-2.

Similarly if $\vec{D}$ is directed from region-2 to region-1. then it is $\boxed{D_{1n} - D_{2n} = \rho_s}$

**Case-I:** If no free charge at the interface ie, $\rho_s = 0$

then eqn (5) becomes

$$\boxed{D_{1n} = D_{2n}} \longrightarrow (4)$$

**Case-II:-**

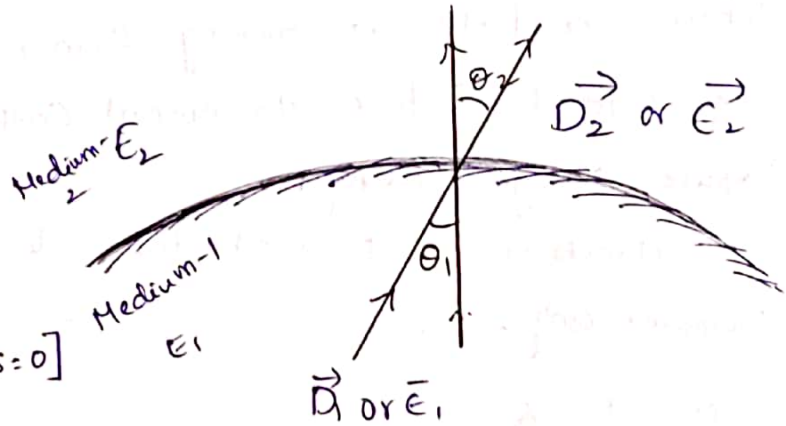
"Normal Components of $\vec{D}$ is Continuous across the interface."

$$\vec{D} = \vec{\varepsilon}\vec{E}$$

$$\boxed{\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}} \longrightarrow (5)$$

"Normal Components of $\vec{E}$ is discontinuous at the boundary or interface".

**Case-iii:** The boundary conditions can be used to determine the refraction of the electric field across the interface.




According to boundary Conditions.

$$\begin{cases} E_{1t} = E_{2t} \\ D_{1n} = D_{2n} \; [\text{at } \rho_s = 0] \end{cases}$$

$$E_{1t} = E_{2t}$$

$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \longrightarrow (6)$$

and $D_{1n} = D_{2n}$

$$D_1 \cos\theta_1 = D_2 \cos\theta_2$$

$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2 \longrightarrow (7)$$

Consider $\frac{(6)}{(7)}$

$$\frac{E_1 \sin\theta_1}{\varepsilon_1 E_1 \cos\theta_1} = \frac{E_2 \sin\theta_2}{\varepsilon_2 E_2 \cos\theta_2}$$

$$\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}} \longrightarrow (8)$$

eqn (8) is the law of refraction of the electric field at the boundary free of charge $[\rho_s = 0]$

## Dielectrics – Conductor Boundary Condition / Interfaces.

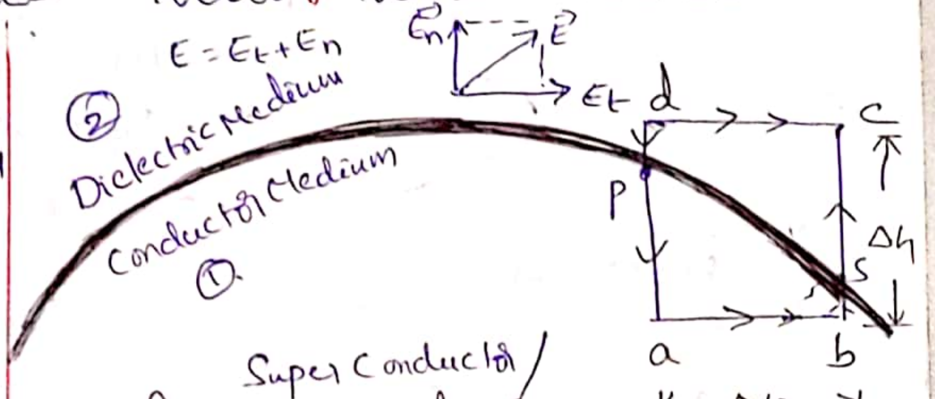
→ If the field is moving from Conductor to Dielectric it's tangential components equal to zero.

Apply Maxwell eqn – ①

Apply $\oint_L \vec{E}\, dl = 0$

$0 \cdot \Delta w + 0 \cdot \frac{\Delta h}{2} + E_n$

$E = E_t + E_n$




Super Conductor / for perfect conductor

$\sigma \rightarrow \infty \longrightarrow \vec{E} = 0$. [copper & silver]

* for a Super Conductor $[\sigma : J/E]$ it's conductivity is ($\sigma = \infty$) infinity. So, & Electric field is : 0

$$0 \cdot \Delta W + 0 \cdot \frac{\Delta h}{2} + \cancel{E_n \cdot \frac{\Delta h}{2}} - E_t \Delta W - \cancel{E_n \frac{\Delta h}{2}} + 0 \cdot \frac{\Delta h}{2} = 0$$

$$-E_t \Delta W = 0$$

$$\boxed{\therefore E_t = 0}$$

$\Rightarrow D = \varepsilon E$

$D_t = \varepsilon E_t$

$$\boxed{D_t = 0}$$

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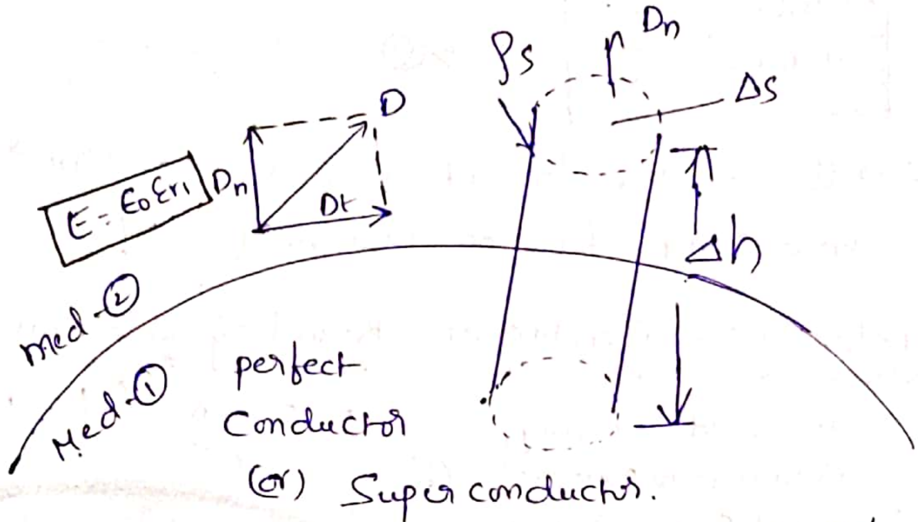
Consider $\oint_S D \cdot ds = Q$ :-




$\sigma = \infty, \quad E = 0 \quad \because \sigma = J/E$

$\oint_S D \cdot ds = Q$

$\oint_{S(top)} D \cdot ds + \oint_{S(bottom)} D \cdot ds = Q$

$D = \epsilon E$

$\downarrow$

$D = \epsilon_0 \epsilon_r \cdot E$

$D_n + 0 = \rho_s$

$\boxed{D_n = \rho_s} \longrightarrow \boxed{\epsilon_0 \epsilon_r E_n = \rho_s}$

## Conductor - Free Space Boundary Condition.

For Free Space $\epsilon_r = 1$

$\therefore$ From the Conductor - Dielectric boundary Conditions

$D_t = E_t = 0$ and $D_n = \epsilon_0 \epsilon_r E_n = \rho_s$

$\therefore$ Boundary Conditions of Conductor - free space are

$\boxed{D_t = D_t = 0}$ and

$\boxed{D_n = \epsilon_0 E_n = \rho_s}$

## Boundary Conditions on Magnetic Field $[\vec{B} \,\&\, \vec{H}]$

(or)

## Boundary Conditions on two different magnetic media.

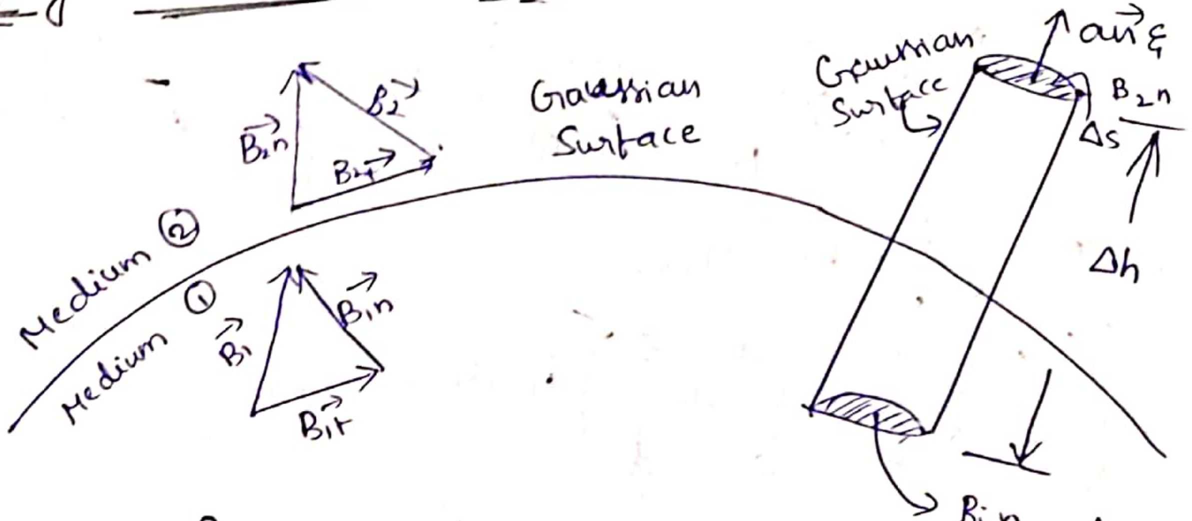



fig:- Boundary B/w two Magnetic Media.

The following Maxwell's equations are used to determine the boundary Conditions.

① $\oint_S \vec{B}\, ds = 0.$ ② $\oint_L \vec{H}\, d\vec{L} = + I_{enc}$

Apply $\oint \vec{B}\, d\vec{s} = 0$

$$\int_{top} \vec{B}\cdot d\vec{s} + \int_{Bottom} \vec{B}\cdot d\vec{s} + \cancel{\int_{side} \vec{B}\cdot d\vec{s}} = 0$$

[for Guassian Surface, we are not condier side]

$$B_{2n}\,\Delta s - B_{1n}\,\Delta s = 0$$

$$\boxed{B_{2n} = B_{1n}} \longrightarrow ①$$

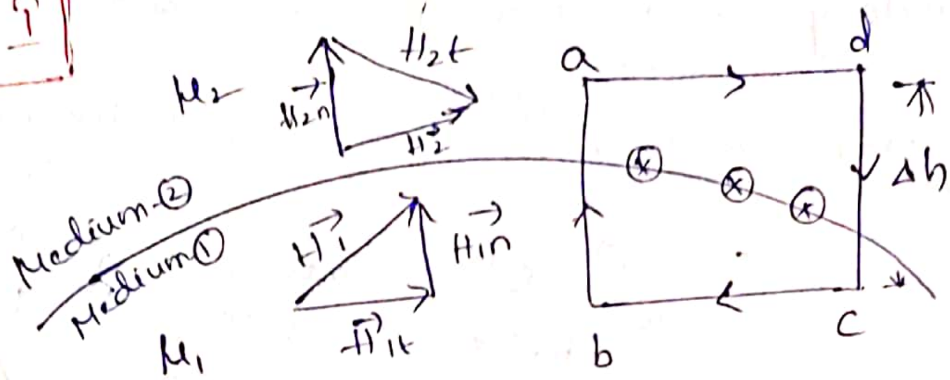
"Normal Components of $\vec{B}$ is Continuous at the Boundary".

$$\vec{B} = \mu \vec{H}$$

$$\mu H_{1n} = \mu_2 H_2 n.$$

"Normal Components of $\vec{H}$ is discontinuous at the Boundary.

② $\boxed{\oint H \, dl = I}$




Apply $\oint_c \vec{H} \, dl = I_{enc}$

$$\left(\int_a^b + \int_b^c + \int_c^d + \int_d^a\right) \vec{H} \cdot d\vec{l} = I_{enc}.$$

$$H_{2t}\Delta w - \cancel{H_{2n}\frac{\Delta h}{2}} - \cancel{H_{1n}\frac{\Delta h}{2}} - H_{1t}\Delta w + \cancel{H_{1n}\frac{\Delta h}{2}} + \cancel{H_{2n}\frac{\Delta h}{2}} = I_{enc}$$

$$H_{2t}\Delta w - H_{1t}\Delta w = K\Delta w.$$

$$(H_{2t} - H_{1t})\cancel{\Delta w} = K\cancel{\Delta w}$$

$$\boxed{H_{2t} - H_{1t} = K}$$

$$K = \frac{I}{\Delta w} \frac{(A)}{(m)}$$

$$I = K . d w (A)$$

" Tangential Components of $\vec{H}$ is also discontinuous in vector form. $\boxed{(H_{2t} - H_{1t}) \times \vec{a_{n_{21}}} = \vec{K}}$

Note:- If boundary is free of Current or media are not conductors i.e., $\vec{E} = 0$

$\therefore$ $\boxed{H_{1t} = H_{2t}}$ Tangential Components of $\vec{H}$ is Continuous at the boundary.

$\boxed{\begin{array}{l} B = \mu H \\ H = \frac{B}{\mu} \end{array}}$

and $\boxed{\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}}$ Tangential Components of $\vec{B}$ is discontinuous at the boundary.

## Law of Refraction:

$H \to$ Magnetic field Intensity

$B \to$ Magnetic flux density.

Consider the boundary conditions

$H_{1t} = H_{2t}$ and

$B_{1n} = B_{2n}$

$H_{1t} = H_{2t}$

$H_1 \sin\theta_1 = H_2 \sin\theta_2 \to (1)$

$B_{1n} = B_{2n}$

$B_1 \cos\theta_1 = B_2 \cos\theta_2$

$\mu_1 H_1 \cos\theta_1 = \mu_2 H_2 \cos\theta_2 \to (2)$

Divide (1) & (2)

$$\frac{H_1 \sin\theta_1}{\mu_1 H_1 \cos\theta_1} = \frac{H_2 \sin\theta_2}{\mu_2 H_2 \cos\theta_2}$$

$$\frac{\tan\theta_1}{\mu_1} = \frac{\tan\theta_2}{\mu_2}$$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}}$$

## Wave Equations for Conducting and perfect Dielectric Media

(1) EM wave characteristics:-

(1) EM wave:- A wave is said to be an EM wave "If it satisfies all the maxwell's equations."

EM wave in Different Media:-

(1) Free space (or) perfect dielectric

$(\sigma = 0, \varepsilon = \varepsilon_0, \mu = \mu_0)$

(2) Lossless (or) Good dielectric

$(\sigma = 0, \ \varepsilon = \varepsilon_0 \varepsilon_r, \mu = \mu_0 \mu_r \ \& \ \sigma << \omega\varepsilon)$

(3) Lossy dielectric

$(\sigma \neq 0, \ \varepsilon = \varepsilon_0 \varepsilon_r, \ \mu = \mu_0 \mu_r)$

(4) Good conductors (Helmholtz equations)

$(\sigma = \infty, \ \varepsilon = \varepsilon_0 \varepsilon_r, \ \mu = \mu_0 \mu_r \ \& \ \sigma >> \omega\varepsilon)$

* Wave Equations in free Space $[\rho_v = 0, \ \sigma = 0, \ \mu = \mu_0]$

$$\nabla \cdot \vec{D} = 0 \quad \text{--- (1)}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{--- (2)}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{--- (3)}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad \text{--- (4)}$$

By applying Curl on Both Sides to eqn (3)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$-\nabla^2 E = -\frac{\partial}{\partial t}(\nabla \times \mu_0 \vec{H})$$

$$+\nabla^2 E = \mu_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

$$\nabla^2 E = \mu_0 \frac{\partial}{\partial t}\left[\frac{\partial D}{\partial t}\right]$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}} \quad \text{--- (5)}$$

$$\boxed{\begin{array}{l} \nabla \cdot \vec{D} = 0 \\ \vec{D} = \varepsilon_0 \vec{E} \\ \nabla \cdot \vec{E} = 0 \\ \vec{B} = \mu_0 \vec{H} \end{array}}$$

The above equation is the wave equation in the free space interms of $\vec{E}$ only.

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* Apply curl on Both sides to eqn - ④

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \times \frac{d}{dt} \vec{D}$$

$$\vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{H}) - \vec{\nabla}^2 \vec{H} = \frac{d}{dt} (\vec{\nabla} \times \vec{D})$$

$$-\vec{\nabla}^2 \vec{H} = \varepsilon_0 \frac{d}{dt} \left[ -\frac{d\vec{B}}{dt} \right]$$

$$\boxed{\vec{\nabla}^2 \cdot \vec{H} = \varepsilon_0 \mu_0 \frac{d^2 \vec{H}}{dt^2}} \longrightarrow ⑥$$

The above equation is the wave equation in the free space interms of $\vec{H}$ only.

* The general wave equation, that travels with velocity 'v' is $\vec{\nabla}^2 \vec{F} - \left(\frac{1}{v^2}\right) \frac{d^2 \vec{F}}{dt^2} = 0$ — ⑦

* where "F" may be any vector field.

Consider eqn ⑥

$$\vec{\nabla}^2 H = \mu_0 \varepsilon_0 \frac{d^2 \vec{H}}{dt^2}.$$

$$\vec{\nabla}^2 \vec{H} - \mu_0 \varepsilon_0 \frac{d^2}{dt^2} \vec{H} = 0 \longrightarrow ⑧$$

By Comparing eqn ⑦ & ⑧

$$\frac{1}{v^2} = \mu_0 \varepsilon_0$$

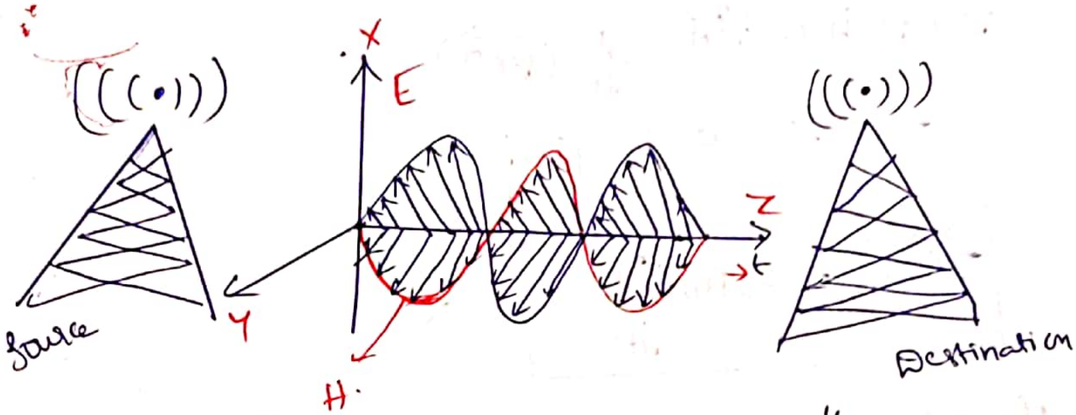
$$v^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$V = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.854 \times 10^{-12}}} = 2999.7$$

$$= \boxed{3 \times 10^8 \, m/s}$$

The velocity of light = $\boxed{3 \times 10^8 \, m/s.}$

## (ii) Wave equations for Conducting Medium:-




"propagation of EM wave."

"James Clerk Maxwell" was experimentally proved the existence of Electromagnetic wave.

→ From the above diagram the signal is generated from the source which is stored in a time varying magnetic field as well as electric field & propagated along z-direction ie.,

→ Electric field is travelled in x-direction, magnetic field is travelled in y-direction and the direction of signal propagation is along z-direction.

→ The EM wave will be travel with a speed, is equal to velocity of light in m/s.

We know that ① $\nabla \times H = J + \frac{dD}{dt}$

$$\nabla \times H = \sigma E + \varepsilon_0 \frac{dE}{dt}$$

② $\nabla \times E = -\frac{dB}{dt}$

$$= -\mu_0 \frac{dH}{dt}$$

③ $\nabla \cdot D = \rho_v$

④ $\nabla \cdot B = 0$

① From II. Maxwell eqn,

$$\nabla \times E = -\frac{\partial B}{\partial t} \longrightarrow (1)$$

where $B = \mu_0 H$

$$\nabla \times E = -\mu_0 \frac{\partial H}{\partial t} \longrightarrow (2)$$

$$\nabla \times (\nabla \times E) = -\mu_0 \left(\nabla \times \frac{\partial H}{\partial t}\right) \longrightarrow (3)$$

$$\nabla \times H = J + \frac{\partial D}{\partial t} \quad [\because \text{ from (1) eqn}]$$

$$\nabla \times H = \sigma E + \varepsilon_0 \frac{\partial E}{\partial t}$$

Apply derivative on both sides,

$$\nabla \times \frac{\partial H}{\partial t} = \frac{\partial}{\partial t}\left[\sigma E + \varepsilon_0 \frac{\partial E}{\partial t}\right]$$

$$\nabla \times \frac{\partial H}{\partial t} = \sigma \frac{\partial E}{\partial t} + \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \longrightarrow (4)$$

eqn (4) is Substitute in eqn (3)

$$\nabla \times (\nabla \times E) = -\mu_0\left[\sigma \frac{\partial E}{\partial t} + \varepsilon_0 \frac{\partial^2 E}{\partial t^2}\right]$$

$$\nabla \times \nabla \times E = -\sigma \mu_0 \frac{\partial E}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}]$$

$$\nabla \times \nabla \times A = \nabla \cdot (\nabla \cdot A) - \nabla^2 A.$$

Similarly

$$\nabla \times \nabla \times E = \nabla \cdot (\nabla \cdot E) - \nabla^2 E \longrightarrow (5)$$

$$D = \varepsilon_0 E$$

Apply divergence on both sides.

$$\nabla \cdot D = \varepsilon_0 \cdot \nabla \cdot E$$

$$\nabla \cdot D = \rho_v$$

$$\rho_v = \varepsilon_0 \cdot \nabla \cdot E$$

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$$\nabla \cdot E = \frac{\rho_v}{\varepsilon_0}$$

if $\rho_v = 0 \Rightarrow \nabla \cdot E = 0 \rightarrow (6)$

$$\nabla \times \nabla \times E = \nabla(0) - \nabla^2 E \rightarrow (7)$$

$$\cancel{-}\nabla^2 E = \cancel{-}\sigma \mu_0 \frac{dE}{dt} + \mu_0 \varepsilon_0 \frac{d^2 E}{dt^2}$$

$$\boxed{\nabla^2 E = \sigma \mu_0 \frac{dE}{dt} + \mu_0 \varepsilon_0 \frac{d^2 E}{dt^2}} \rightarrow (I)$$

(2) EM Wave eqn's in the form of H :-

from eqn (1) → eq$^n$ is $\nabla \times H = J + \frac{dD}{dt} \rightarrow (1)$

$$J = \sigma E$$

$$\nabla \times H = \sigma E + \frac{dD}{dt} \rightarrow (2)$$

$$D = \varepsilon_0 E$$

$$\nabla \times H = \sigma E + \varepsilon_0 \frac{dE}{dt} \rightarrow (2)$$

Apply curl on both sides.

$$\nabla \times \nabla \times H = \sigma(\nabla \times E) + \varepsilon_0 \left(\nabla \times \frac{dE}{dt}\right) \rightarrow (3)$$

Substitute these eqn's in eqn -(3)

$$\left\{ \begin{array}{ll} \nabla \times E = -\mu_0 \frac{dH}{dt} & \rightarrow \text{[From Maxwell eqn - (2)]} \\ \nabla \times \frac{\partial E}{dt} = -\mu_0 \frac{d^2 H}{dt^2} & \rightarrow \text{Multiply } \frac{d}{dt} \text{ on Both Sides.} \end{array} \right.$$

$$\nabla \times \nabla \times H = \sigma\left(-\mu_0 \frac{dH}{dt}\right) - \mu_0 \varepsilon_0 \frac{d^2 H}{dt^2} \rightarrow (4)$$

Now, $\nabla \times \nabla \times H = \nabla \cdot (\nabla \cdot H) - \nabla^2 H \rightarrow (5)$

$$B = \mu_0 H$$

$$\nabla \cdot B = \mu_0 \cdot \nabla \cdot H \quad [\because \nabla \cdot B = 0]$$

$\boxed{\nabla \cdot H = 0}$ → Substitute in eqn (5)

$$\nabla \times \nabla \times H = \nabla(0) - \nabla^2 H$$

$$\nabla \times \nabla \times H = -\nabla^2 H$$

From eqn (4).

$$-\nabla^2 H = -\sigma \mu_0 \frac{dH}{dt} - \mu_0 \varepsilon_0 \frac{d^2 H}{dt^2}$$

$$\boxed{\nabla^2 H = \sigma \mu_0 \frac{dH}{dt} + \mu_0 \varepsilon_0 \frac{d^2 H}{dt^2}} \rightarrow (II)$$

Representing eqn's (I) & eqn (II) are in phasor forms

**Wave eqn's in the form of matrix.**

$$\nabla^2 E = \sigma \mu_0 \frac{dE}{dt} + \mu_0 \varepsilon_0 \frac{d^2 E}{dt^2}$$

$$\nabla^2 D = \sigma \mu_0 \frac{dD}{dt} + \mu_0 \varepsilon_0 \frac{d^2 D}{dt^2}$$

$$\nabla^2 H = \sigma \mu_0 \frac{dH}{dt} + \mu_0 \varepsilon_0 \frac{d^2 H}{dt^2}$$

$$\nabla^2 B = \sigma \mu_0 \frac{dB}{dt} + \mu_0 \varepsilon_0 \frac{d^2 B}{dt^2}$$

$$\nabla^2 \begin{bmatrix} E \\ D \\ H \\ B \end{bmatrix} = \sigma \mu_0 \frac{d}{dt} \begin{bmatrix} E \\ D \\ H \\ B \end{bmatrix} + \mu_0 \varepsilon_0 \frac{d^2}{dt^2} \begin{bmatrix} E \\ D \\ H \\ B \end{bmatrix}$$

## III Wave eqn's for a perfect dielectric / lossless) Free space:

for a perfect dielectric..

No current flow $\boxed{I = 0}$

From $J = \frac{I}{S} \Rightarrow 0$

$\boxed{J = 0} \rightarrow$ Conduction current density.

* If $J = 0$, then Conductivity $= 0$

from $J = \sigma E$ or $\sigma = \frac{J}{E}$

$$\boxed{\sigma = 0}$$

~~Nowegn's for wave :-~~

Now wave eqn's are :-

$$\nabla^2 E = \mu_0 \varepsilon_0 \frac{d^2 E}{dt^2}$$

$$\nabla^2 D = \mu_0 \varepsilon_0 \frac{\partial^2 D}{\partial t^2}$$

$$\nabla^2 H = \mu_0 \varepsilon_0 \frac{\partial^2 H}{\partial t^2}$$

$$\nabla^2 B = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}.$$

## Uniform plane waves:

→ A wave is said to be plane wave, when the two fields [E & H] are lies in a same plane.

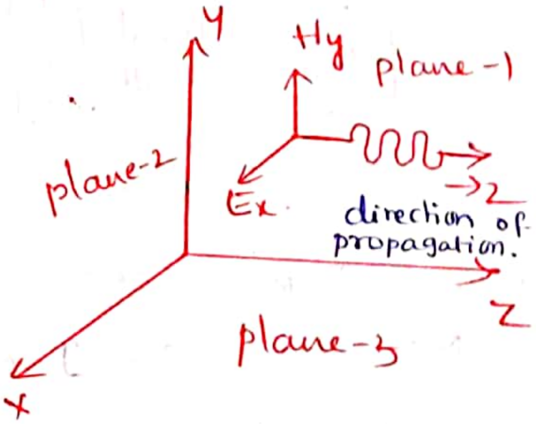
→ Let us consider an Electromagnetic wave which is travelled in free space along z-direction.

→ The plane wave is exist in x-y plane. shown in fig below.

→ A plane wave is said to be uniform plane wave when.

→ ① It should be a plane wave.

→ ② The plane wave having equal magnitudes & equal phase shifts.




We know that, wave equations for a free space is

$$\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \longrightarrow ①$$

$$\boxed{\nabla^2 E = \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}}$$

The wave is travelling along z-direction, so that the values of $\frac{\partial^2 E}{\partial x^2}$ & $\frac{\partial^2 E}{\partial y^2} = 0$.

Substitute $\nabla^2 E$ in eqn ①

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$0 + 0 + \frac{\partial^2 E}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 E}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 E}{\partial z^2}$$

* **Relation between E & H (or) Intrinsic Impedance (or) Wave Impedance (or) Characteristic Impedance ($\eta$) :-**

Consider the Maxwell equation $\longrightarrow$ ②

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

In free Space ($\sigma = 0$, $\vec{J} = \sigma \vec{E} \rightarrow J = 0$)

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \longrightarrow \text{Max well eqn} - ②$$

$$\vec{\nabla} \times \vec{H} = 0 + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial D}{\partial t}. \longrightarrow ③$$

where $D = \varepsilon_0 E$

$$\vec{\nabla} \times \vec{H} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \longrightarrow ④$$

$\therefore$ vector form.

equating 3 & 4

$$\begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \frac{\partial D}{\partial t}$$

$$\begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \varepsilon_0 \frac{\partial E}{\partial t}$$

Assume, the wave is travelling along x- direction, then $E_x, H_x$ Components are "zero" and also $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$

Then.

$$\begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & 0 & 0 \\ 0 & H_y & H_z \end{vmatrix} = \varepsilon_0 \frac{\partial}{\partial t}(\underbrace{E_x \vec{a_x}}_{0} + E_y \vec{a_y} + E_z \vec{a_z})$$

$\vec{E}$ field has 3 Components (x, y, z)

$$\vec{a_x}\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right] + \vec{a_y}\left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right] + \vec{a_z}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] = \varepsilon_0 \frac{\partial}{\partial t}\left[E_y \cdot \vec{a_y} + E_z \vec{a_z}\right]$$

$$a_x[0-0] + a_y\left[-\frac{\partial}{\partial x}H_z + 0\right] + a_z\left[\frac{\partial H_y}{\partial x} - 0\right] = \varepsilon_0 \frac{\partial}{\partial t}\left[E_y \vec{a_y} + E_z \vec{a_z}\right]$$

$$= -\frac{\partial H_z}{\partial x}\vec{a_y} + \frac{\partial H_y}{\partial x}\vec{a_z} = \varepsilon_0 \frac{\partial}{\partial t}\left[E_y \vec{a_y} + E_z \vec{a_z}\right]$$

(or)

for. $\nabla \times E$.

$$= -\frac{\partial E_z}{\partial x} a_y + \frac{\partial E_y}{\partial x} a_z = -\mu_0 \frac{\partial H_y}{\partial t} a_y - \mu_0 \frac{\partial H_z}{\partial t} a_z.$$

Equating the respective Components, we get.

$$-\frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{\partial E_y}{\partial t} \quad \text{—— ①}$$

$$\frac{\partial H_y}{\partial x} = \varepsilon_0 \frac{\partial E_z}{\partial t} \quad \text{—— ②}$$

$$\frac{\partial E_z}{\partial x} = \mu_0 \frac{\partial H_y}{\partial t} \longrightarrow ⑥$$

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t} \longrightarrow ⑦$$

Writing $E_y$ in the form of

$$E_y = f_1(x - V_0 t) \qquad V_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$E_y =$

Differentiating w.r. to "t" on Both sides.

$$\frac{d}{dt}\left(F_1(x - V_0 t)\right) = \frac{d}{dt} E_y$$

$$\frac{\partial E_y}{\partial t} = \frac{\partial f}{\partial (x - V_0 t)} \cdot \frac{\partial (x - V_0 t)}{\partial t} = -V_0 f' \quad \text{—— (3)}$$

$$f' = \frac{\partial f}{\partial (x - V_0 t)} \quad \text{—— (4)}$$

From eqn's (3) & (1),

We have

$$-\frac{\partial H_z}{\partial x} = \epsilon_0 \frac{d}{dt} E_y \qquad \frac{d}{dt} \epsilon_y = -V_0 f'$$

$$+\frac{\partial H_z}{\partial x} = +\epsilon_0 V_0 f' \longrightarrow \text{differential form}$$

$$H_z = \int \epsilon_0 V_0 f' dx \longrightarrow \text{Integration form.}$$

$$H_z = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \cdot \varepsilon_0 \int f' dx. \quad \text{—— (5)} \qquad V_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

eqn (4) substitute in eqn (5)

$$H_z = \frac{\sqrt{\varepsilon_0 \cdot \cancel{\varepsilon_0}}}{\sqrt{\mu_0 \cancel{\varepsilon_0}}} \int \frac{\partial f}{\partial (x - V_0 t)} \qquad \boxed{f' = \frac{\partial f}{\partial (x - V_0 t)}} \rightarrow H_z = \sqrt{\frac{\mu_0}{\varepsilon_0}} \int f' dx$$

$$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} \int \frac{\partial f}{\partial x} dx \rightarrow \because \text{differentiate } f'$$

$$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} \int \frac{df}{dx} dx + c$$ → Cancelled integration and differentiation.

$$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} f + c$$

f is the function of $E_y$.

$$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y.$$

$$\boxed{\frac{E_y}{H_z} = \sqrt{\frac{\mu_0}{\varepsilon_0}}}$$

$$\boxed{E_y = H_z \left[\frac{\mu_0}{\varepsilon_0}\right]}$$

Similarly, if we take.

$$E_z = f(x - V_0)t$$

$$\frac{dE_z}{dt} = -V_0 f' \longrightarrow (8)$$

from equations (2) and (8), we have

$$\frac{dH_y}{dx} = \epsilon_0 \frac{dE_z}{dt} = -\epsilon_0 V_0 f'$$

$$H_y = -\epsilon_0 V_0 \int f' dx$$

$$= -\epsilon_0 V_0 f$$ → f is the function of $E_z$

$$= -\epsilon_0 V_0 E_z.$$

$$\boxed{\frac{E_z}{H_y} = -\sqrt{\frac{\mu_0}{\varepsilon_0}}}$$

$$\boxed{E_z = -H_y \left[\frac{\mu_0}{\varepsilon_0}\right]}$$

$$\frac{E}{H} = \sqrt{\frac{E_y^2 + E_z^2}{H_y^2 + H_z^2}}$$ – (9) (Electric field components of y & z) : (Magnetic field Components of y & z)

Substitute $E_y$ & $E_z$ values in eqn ⑨

$$\frac{E}{H} = \sqrt{\frac{H_z^2\left[\frac{\mu_0}{\varepsilon_0}\right] + H_y^2\left[\frac{\mu_0}{\varepsilon_0}\right]}{H_y^2 + H_z^2}} \quad (\text{or}) \quad \sqrt{\frac{\left[\frac{\mu_0}{\varepsilon_0}\right]\left[H_z^2 + H_y^2\right]}{H_y^2 + H_z^2}}$$

$$= \frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{\frac{1}{36\pi \times 10^{9}}}} = 120\pi\,\Omega$$

$$\boxed{\eta = \frac{E}{H} = 120\pi\,\Omega} \quad (\text{or}) \quad \boxed{\eta = \frac{E}{H} = 376.8\,\Omega}$$

Hence proved.

The intrinsic impedance or characteristic impedance, $\eta_0$ is defined as $\boxed{\eta_0 = \frac{E}{H} = 120\pi\,\Omega \text{ (or) } 376.8\,\Omega}$ //

## Intrinsic (or) Impedance

When an EM wave travelling the ratio of Electric field to the magnetic field is known as Intrinsic impedance, which is a constant value.

$$= \eta = \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}$$

We know that $\mu = \mu_0 \mu_r$

$$\varepsilon = \varepsilon_0 \varepsilon_r \quad [\mu_r \,\&\, \varepsilon_r = 1]$$

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.854 \times 10^{-12}}}$$

$$\boxed{\eta = 367.819 \simeq 377\,\Omega}$$

## Sinusoidal variations (or) phasor form of wav equations:-

We Know that.

$$\nabla^2 E = \sigma \mu_0 \frac{dE}{dt} + \mu_0 \varepsilon_0 \frac{d^2 E}{dt^2} \quad \text{---} ①$$

Consider.

$$E = E_m e^{j\omega t} \quad \text{---} ②$$

∴ Sinusoidal form of E (or) phasor form of E.

$$\frac{dE}{dt} = E_m \frac{d}{dt} e^{j\omega t}$$

$$= E_m (j\omega) e^{j\omega t} \quad \text{---}$$

$$\frac{dE}{dt} = (j\omega) E \quad \text{---} ③$$

$$\Rightarrow \frac{d^2 E}{dt^2} = -\omega^2 E. \quad \text{---} ④$$

∴ eqn ③ & ④ Substitute in eqn ①

$$\nabla^2 E = \sigma \mu_0 (j\omega E) - \mu_0 \varepsilon_0 \omega^2 E$$

$$\nabla^2 E - j\sigma \mu_0 \omega E + \omega^2 \mu_0 \varepsilon_0 E = 0$$

$$\nabla^2 E - j\omega \mu_0 E [\sigma + j\omega \varepsilon_0] = 0$$

$$\nabla^2 E - j\omega \mu_0 [\sigma + j\omega \varepsilon_0] E = 0$$

$$\nabla^2 E = (j\omega \mu_0 \sigma - \omega^2 \mu_0 \varepsilon_0) E.$$

**Note:-**

∴ Time varying field represented in Sinusoidal as

$$\boxed{E = E_m e^{j\omega t}}$$

# Wave propagation in Lossless and Conducting Media:-

## ① Wave propagation in Lossless (or) perfect Dielectric Medium (or) In free space.

Consider the wave equations interms of $\vec{E}$ only for free Space.

$$\nabla^2 E = \mu_0 \varepsilon_0 \frac{d^2 E}{dt^2}$$ → represent this eqn in phasor form is

$$\frac{dE}{dt} = j\omega E$$

$$\frac{d^2E}{dt^2} = -\omega^2 E$$

$$\nabla^2 E = \mu_0 \varepsilon_0 (-\omega^2 E)$$

$$\nabla^2 E + \mu_0 \varepsilon_0 \omega^2 E = 0 \rightarrow \nabla^2 E + \beta^2 E = 0$$

if $\beta^2 = \omega^2 \mu_0 \varepsilon_0$

$$\boxed{\beta = \omega\sqrt{\mu_0 \varepsilon_0}} \text{ —(Rad/meters)}$$

where $\beta$ = phase shift Constant.

## Wave propagation in a Conducting Medium:-

We have.

$$\nabla^2 E = \sigma \mu_0 \frac{dE}{dt} + \mu_0 \varepsilon_0 \frac{d^2E}{dt^2} \quad \text{—①}$$

$$\frac{dE}{dt} = j\omega E$$ ∴ phasor form (or) Sinusoidal form of E

$$\frac{d^2E}{dt^2} = -\omega^2 E$$

∴ Represent the equation ① in phasor form is

$$\nabla^2 E = \sigma \mu_0 (j\omega E) - \mu_0 \varepsilon_0 \omega^2 E$$

$$\nabla^2 E - j\sigma \mu_0 \omega E + \omega^2 \mu_0 \varepsilon_0 E = 0$$

$$\nabla^2 E - j\mu_0 \omega (\sigma + j\omega \varepsilon_0) E = 0$$

$$\nabla^2 E - \gamma^2 E = 0 \longrightarrow ②$$

$\gamma$. is propagation Constant.

$\gamma = \alpha + j\beta$

$\alpha \rightarrow$ nepers. $\beta \rightarrow$ rad

where $\gamma^2 = j\mu_0\omega(\sigma + j\omega\varepsilon_0)$

$$\gamma = \sqrt{j\mu_0\omega\sigma - \omega^2\mu_0\varepsilon_0} \rightarrow ③$$

where ($\gamma = \alpha + j\beta$)

$$(\alpha + j\beta) = \sqrt{j\omega\mu_0\sigma - \omega^2\mu_0\varepsilon_0}$$

$$(\alpha + j\beta)^2 = j\omega\mu_0\sigma - \omega^2\mu_0\varepsilon_0$$

$$\alpha^2 - \beta^2 + 2j\alpha\beta = j\omega\mu_0\sigma - \omega^2\mu_0\varepsilon_0$$

$$\alpha^2 - \beta^2 = -\omega^2\mu_0\varepsilon_0 \rightarrow (4)$$

$$2\alpha\beta = \omega\mu_0\sigma \rightarrow (5)$$

$$(\alpha^2 + \beta^2)^2 = (\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2$$

$\boxed{(a+b)^2 = a^2 + b^2 + 2ab}$

[if $a = \alpha^2$
$b = \beta^2$
$(a+b)^2 = (\alpha^2 + \beta^2)^2$]

eqn 4 & 5 substitute in eqn ⑥

$$(\alpha^2 + \beta^2) = \sqrt{(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2} \quad — ⑥$$

$$= \sqrt{(-\omega^2\mu_0\varepsilon_0)^2 + \omega^2\mu_0^2\sigma^2}$$

$$\alpha^2 + \beta^2 = \sqrt{\omega^4\mu_0^2\varepsilon_0^2 + \omega^2\mu_0^2\sigma^2} \rightarrow ⑦$$

$$\alpha^2+\beta^2 = \sqrt{\omega^4\mu_0^2\varepsilon_0^2 + \omega^2\mu_0^2\sigma^2} \quad \text{--- (7)}$$

$$\alpha^2-\beta^2 = -\omega^2\mu_0\varepsilon_0. \quad \longrightarrow (4)$$

Addition (7 & 4 eqn)

$$2\alpha^2 = -\omega^2\mu_0\varepsilon_0 + \sqrt{\omega^4\mu_0^2\varepsilon_0^2+\omega^2\mu_0^2\sigma^2}$$

$$2\alpha^2 = -\omega^2\mu_0\varepsilon_0 + \sqrt{\omega^4\mu_0^2\varepsilon_0^2\left[1+\frac{\sigma^2}{\omega^2\varepsilon_0^2}\right]}$$

$$2\alpha^2 = -\omega^2\mu_0\varepsilon_0 + \omega^2\mu_0\varepsilon_0\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_0}\right)^2}$$

$$\alpha^2 = -\omega^2\mu_0\varepsilon_0 + \omega^2\mu_0\varepsilon_0\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_0}\right)^2}$$

$$= +\omega^2\mu_0\varepsilon_0\left[-1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_0}\right)^2}\right]$$

$$\boxed{\alpha = \sqrt{\frac{\omega^2\mu_0\varepsilon_0}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_0}\right)^2}-1\right]}}$$

To find $\beta$, Substract $(\alpha^2+\beta^2)$ & $(\alpha^2-\beta^2)$

$$\alpha^2+\beta^2 = \sqrt{\omega^4\mu_0^2\varepsilon_0^2+\omega^2\mu_0^2\sigma^2} \quad \text{--- (7)}$$

Subtract $(-)$ $\alpha^2-\beta^2 = -\omega^2\mu_0\varepsilon_0$ ——— (4)

$$2\beta^2 = \omega^2\mu_0\varepsilon_0 + \sqrt{\omega^4\mu_0^2\varepsilon_0^2+\omega^2\mu_0^2\sigma^2}$$

$$2\beta^2 = \omega^2\mu_0\varepsilon_0 + \sqrt{\omega^4\mu_0^2\varepsilon_0^2\left[1+\frac{\sigma^2}{\omega^2\varepsilon_0^2}\right]}$$

$$2\beta^2 = \omega^2\mu_0\varepsilon_0 + \omega^2\mu_0\varepsilon_0\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right.}\right.$$

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$$\beta^2 = \frac{\omega^2\mu_0\varepsilon_0 + \omega^2\mu_0\varepsilon_0\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_0}\right)^2}}{2}$$

$$\beta^2 = \frac{\omega^2\mu_0\varepsilon_0}{2}\left[1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_0}\right)^2}\right]$$

$$\boxed{\beta = \sqrt{\frac{\omega^2\mu_0\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_0}\right)^2}+1\right]}}$$

## Conductors & Dielectrics - characterization:-

Based on the Conductivity of a material, materials are classified into 3 - types.

① Conductors ($\sigma >> 1$)

② Insulators. ($\sigma << 1$)

③ Semiconductors. ($\sigma$ b/w Conductors & Insulators)

**Super Conductors** The Conductivity of material in Super conductors is infinity. $\sigma = \infty$.

**Good Conductor:-**

* A material is said to be Good Conductors,

when $\sigma >> \omega\varepsilon_0$ (or) $\boxed{\frac{\sigma}{\omega\varepsilon_0} >> 1}$

**Good Dielectric:-**

* A material is said to be Good dielectric, when

$\sigma << \omega\varepsilon_0$

$$\boxed{\frac{\sigma}{\omega\varepsilon_0} << 1}$$

Note:-

1) $\sqrt{\angle\theta} = \theta/2$

2) $\sqrt{\angle\theta_1 . \angle\theta_2} = \frac{\theta_1+\theta_2}{2}$

3) $\sqrt{\frac{\angle\theta_1}{\angle\theta_2}} = \frac{\theta_1-\theta_2}{2}.$

## Wave propagation in Good Conductors and Good Dielectrics

### Wave propagation in Good Conductors :-

We know that

$$\nabla^2 E - \gamma^2 E = 0$$

$$\gamma^2 = j\omega\mu_0(\sigma + j\omega\epsilon_0)$$

$$\gamma = \sqrt{j\omega\mu_0(\sigma + j\omega\epsilon_0)}$$

$$= \sqrt{j\omega\mu_0\sigma\Big(1+\underbrace{\frac{j\omega\epsilon_0}{\sigma}}_{\text{neglected.}}\Big)} \quad -① \qquad \frac{\sigma}{\omega\epsilon_0} >> 1. \text{ - Good Conductors}$$

But in eqn ① $\frac{\omega\epsilon_0}{\sigma} << 1$. - so, $j\frac{\omega\epsilon_0}{\sigma}$ is neglected.

Now,

$$\gamma = \sqrt{j\omega\mu_0\sigma} \qquad \boxed{\therefore \gamma = \alpha + j\beta}$$

$$\alpha + j\beta = \sqrt{\omega\mu_0\sigma}.\sqrt{j}$$

$$j = 0 + 1\cdot j$$

$$= 1\angle 90^\circ$$

$$\alpha + j\beta = \sqrt{\omega\mu_0\sigma}\cdot\sqrt{1\angle 90^\circ}$$

$$= \sqrt{\omega\mu_0\sigma}\angle 45^\circ$$

$$= \sqrt{\omega \mu_0 \sigma}\ (\cos 45^\circ + j \sin 45^\circ)$$

$$= \sqrt{\omega \mu_0 \sigma}\ \left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$$

$$\alpha + j\beta = \sqrt{\frac{\omega \mu_0 \sigma}{2}} + j\sqrt{\frac{\omega \mu_0 \sigma}{2}}$$

$$\boxed{\alpha = \sqrt{\frac{\omega \mu_0 \sigma}{2}}}\ ; \quad \boxed{\beta = \sqrt{\frac{\omega \mu_0 \sigma}{2}}}$$

## Wave propagation in a Good Dielectrics:-

$$\alpha = \frac{\omega^2 \mu_0 \varepsilon_0}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega \varepsilon_0}\right)^2} - 1\right]$$

where $\frac{\sigma}{\omega \varepsilon_0} << 1.$

$$\boxed{\text{for good dielectrics } \frac{\sigma}{\omega \varepsilon_0} << 1.}$$

$$\alpha = \sqrt{\frac{\omega^2 \mu_0 \varepsilon_0}{2}\left[\sqrt{1} - 1\right]}$$

$$\alpha = \sqrt{\frac{\omega^2 \mu_0 \varepsilon_0}{2}}\ [0]$$

$$\boxed{\alpha = 0}$$

$$\beta = \frac{\omega^2 \mu_0 \varepsilon_0}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega \varepsilon_0}\right)^2} + 1\right]$$

$$= \sqrt{\frac{\omega^2 \mu_0 \varepsilon_0}{2}\left[\sqrt{1} + 1\right]}$$

$$= \sqrt{\frac{\omega^2 \mu_0 \varepsilon_0}{\cancel{2}} \cdot \cancel{2}}$$

$$\boxed{\beta = \omega \sqrt{\mu_0 \varepsilon_0}}$$

# Polarization:-

The polarization of a plane wave can be defined as the orientation of the electric field vector as a function of time at a fixed point in space. polarization is mainly classified into 3 types.

① Linear polarization
② Elliptical polarization.
③ Circular polarization.

## ① Linear polarization:-

Def:- The electric field E has only x component and y component of E is 0. Then the wave is said to be linearly polarized in x- direction.

Consider an uniform plane wave travelling in z-direction, E & H is lying on x-y plane.

* If $E_y = 0$ & only $E_x$ is present, the wave is said to be polarized in the x- direction.

* Similarly, If $E_x = 0$ & only $E_y$ is present, the wave is said to be polarized in the y-direction.

* If both $E_x$ & $E_y$ are present and are in phase, the resultant $\vec{E}$ makes an angle $\tan^{-1}\left[\frac{E_y}{E_x}\right]$ as shown in fig.

* If $\vec{E}$ is always directed along the line, then the wave is said to be linearly polarized.

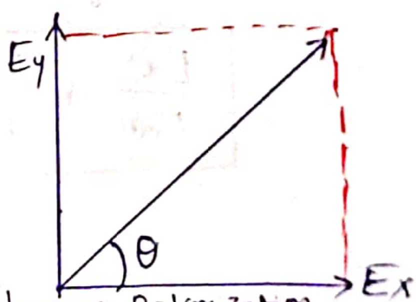



fig:- linear polarization

Note:- $E_x$ and $E_y$ Components are in phase with either equal or unequal amplitudes, for a uniform plane wave travelling in z-direction, the polarization is linear.

## Elliptical polarization

If Electric field of $E_x$ & $E_y$ Components have different magnitudes and 90° phase difference. then the Locus of the resultant $\vec{E}$ is an ellipse & The wave is said to be Elliptically polarized.

$$E(z) = E_a \vec{a_x} + j E_b \vec{a_y} \quad \text{— phasor form of } E.$$

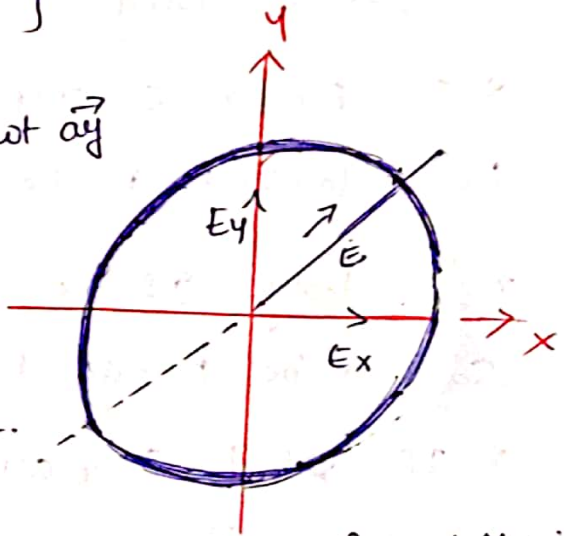
$$E(z,t) = Re\{\vec{E}(z)\, e^{j\omega t}\}$$

↳ time varying field.

direction

$$E(z,t) = Re\{[E_a a_x + jE_b \vec{a_y}]\, e^{j\omega t}\}$$

$$\vec{E}_{(z,t)} = E_a \cos\omega t\, \vec{a_x} + E_b \sin\omega t\, \vec{a_y}$$




$$E_x = E_a \cos\omega t$$

$$E_y = -E_b \sin\omega t.$$

$$\frac{E_x}{E_a} = \cos\omega t \; ; \; \frac{E_y}{E_b} = -\sin\omega t.$$

Squaring and adding the above terms.

$$\boxed{\frac{E_x^2}{E_a^2} + \frac{E_y^2}{E_b^2} = 1}$$

* Different Magnitudes

* 90° phase difference

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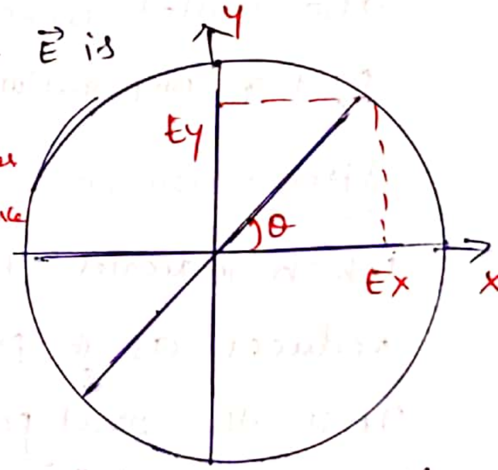
## ③ Circular polarization:

The amplitudes of Ex and Ey are same and the phase difference between the Ex & Ey is Exactly 90°. Then the locus of the resultant $\vec{E}$ is Circle. and the wave is said to be Circularly polarized. The resultant $\vec{E}$ is




$$\vec{E} = E_x \vec{a_x} + j E_y \vec{a_y}$$

* Equal Magnitudes
* 90° phase difference

$$E_x = E_y = E_a$$

$$\vec{E}_{(z)} = \vec{E} = E_a \vec{a_x} + j E_a \vec{a_y}$$

The time varying behaviour given as

$e^{j\omega t} = \cos\omega t + j\sin\omega t$.

$$\vec{E}_{(z,t)} = Re\{ \vec{E}_{(z)} e^{j\omega t} \}$$

$$\vec{E}(z,t) = Re\{ ( E_a \vec{a_x} + j E_a \vec{a_y} ) e^{j\omega t} \}$$

$$\vec{E}(z,t) = E_a \cos\omega t \vec{a_x} - E_a \sin\omega t \vec{a_y}$$

$$E_x = E_a \cos\omega t. \quad -①$$

$$E_y = -E_a \sin\omega t. \quad -②$$

Squaring and adding the above eqn's ① & ②

We will get.

$$\boxed{E_x^2 + E_y^2 = E_a^2}$$

## Definitions

**Phase Shift Constant ($\beta$):** It is defined as a measure of the phase shift in radians per unit length. $\beta$ is also called wave number and it is the imaginary part of propagation constant.

**Attenuation Constant ($\alpha$):-** It is defined as a constant which indicates the rate at which the wave amplitude reduces as it propagates from one point to another. It is the real part of propagation constant.

**Wave length ($\lambda$):-** It is defined as the distance between the corresponding points of 2 adjacent cycles of the wave form.

(or)

It is defined as that distance through which sinusoidal wave passes through a full cycle of $2\pi$ radians.

$$\boxed{\lambda = 2\pi/\beta} \quad \text{(or)} \quad \boxed{\lambda = \frac{2\pi}{\omega\sqrt{\mu_0\varepsilon_0}}\ (m)}$$

**Phase velocity ($V_p$):-** It is defined as the velocity of some point in the sinusoidal wave form. (or) The rate at which an EM wave travels in medium. $\boxed{V = \lambda F}$

**Intrinsic (or) characteristic Impedance:-** Intrinsic Impedance of a medium which has a finite value of conductivity is given by $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$

# Unit - IV

## Reflection and Refraction of plane waves

**Medium**:- A medium is the material through which a wave travels.

**Wave**:- Wave means transporting energy or information. Wave is a function of both space and time.

Free Space :- $\sigma = 0$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$.

lossless dielectrics :- $\sigma = 0$, $\varepsilon = \varepsilon_r \varepsilon_0$, $\mu = \mu_r \mu_0$, or $\sigma << \omega\varepsilon$

lossy dielectrics :- $\sigma \neq 0$, $\varepsilon = \varepsilon_r \varepsilon_0$, $\mu = \mu_r \mu_0$

Good Conductors: $\sigma \simeq \infty$, $\varepsilon = \varepsilon_0$, $\mu = \mu_r \mu_0$, $\sigma >> \omega\varepsilon$.

Examples for medium:- Gas, liquid, or solid.

## Reflection & Refraction of uniform planes:

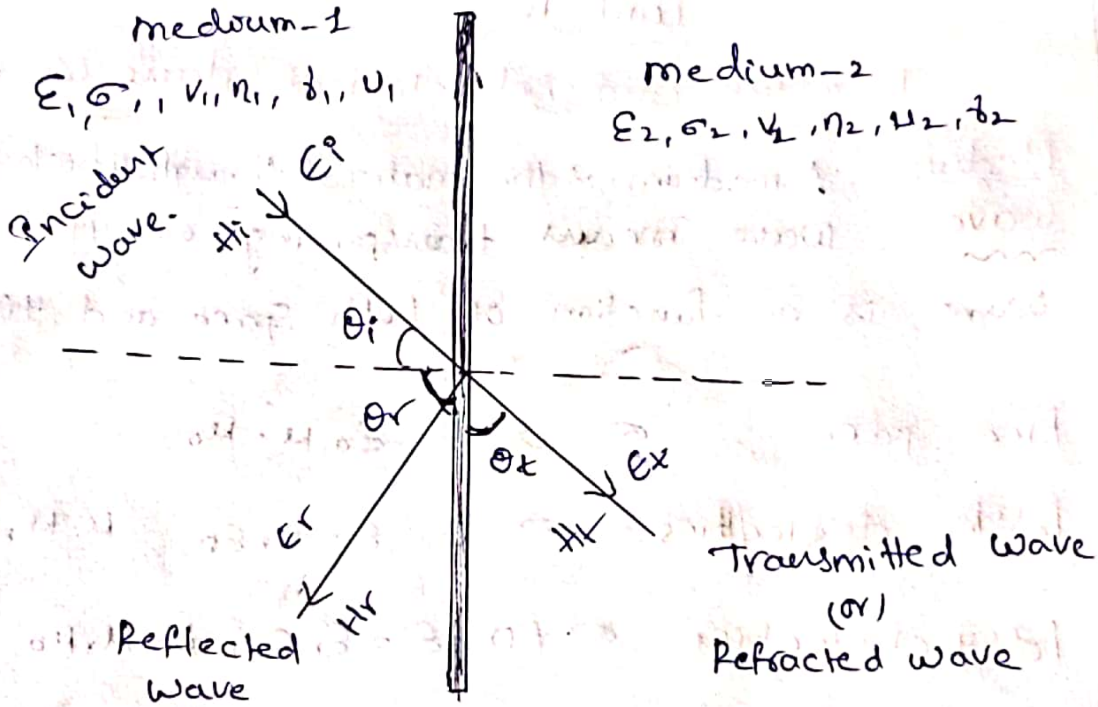
* Let us Consider a boundary which is seperated by 2 Mediums.

* The physical properties of a mediums are, permitivity, Conductivity, velocity, refractive index, permeability, and propagation constant.

* When a wave strikes a boundary or surfaces of the medium, Sometimes it reflected back to the same medium or it enters into the another medium.

**Reflection**:- If the wave is reflected back to the Same medium that mechanism is known as "reflection".

**Refraction**:- If the wave crosses the boundary & entered into the another medium is called "refraction" or "transmission".




Reflection Co-efficient [K or Γ] & Transmission Co-efficient (τ]:.

① Reflection Co-efficient [K or Γ]

The ratio of amplitudes of reflected wave to the incident wave.

$$\boxed{K = \frac{E_r}{E_i}}$$

② Transmission Co-efficient (τ]

The ratio of amplitudes of transmitted wave to the incident wave.

$$\boxed{\tau = \frac{E_t}{E_i}}$$

## Critical Angle :-

The angle at which the complete incident wave crosses the boundary & enter into medium-2 the refracted wave will travel with an angle of 90°.

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1} \text{ (or) } \frac{V_1}{V_2}$$

$$V = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$\rightarrow V_1 = \frac{1}{\sqrt{\mu_0\varepsilon_1}}$

$\rightarrow V_2 = \frac{1}{\sqrt{\mu_0\varepsilon_2}}$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\frac{\cancel{1}}{\sqrt{\cancel{\mu_0}\varepsilon_1}}}{\frac{\cancel{1}}{\sqrt{\cancel{\mu_0}\varepsilon_2}}} = \sqrt{\varepsilon_2/\varepsilon_1}$$

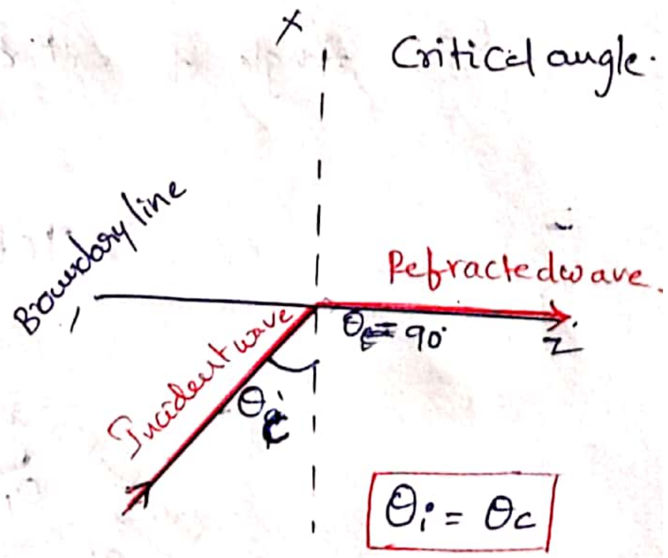



fig:- Critical Angle.

$\theta_1 > \theta_2$ ; $\theta_2 = 90°$

if $\theta_1 = \theta_c$ ; $\theta_2 = 90°$

$$\frac{\sin\theta_c}{\sin 90} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\frac{\sin\theta_c}{1} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\boxed{\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}$$

# Total Internal Reflection:-

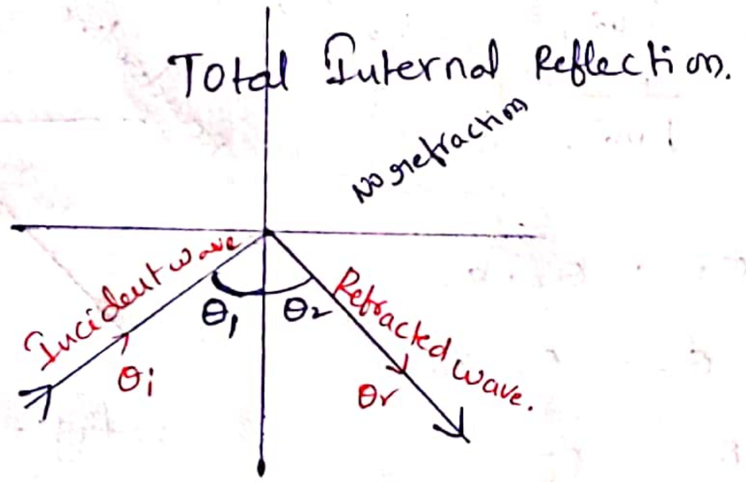
Defination-1

If the angle of incidence is greater than Critical angle then there is no refraction takes place, that means Complete incident wave is reflected back to the same medium.

$$\boxed{\theta_i > \theta_c}$$

$\theta_i$ : Angle of Incidence

$\theta_c$ : Angle of Critical (or) Critical Angle.




Defination - 2 : When wave is incident from the denser medium into racer medium at an angle equals or greater than the critical angle. Then the wave will be totally internally reflected back. This phenomenon is called total internal reflection.

If $\theta_i$ and $\theta_t$ are the incident and transmitted (Refracted) angles then from Snell's law.

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

when $\theta_t = 90°$ the transmitted wave travells along the boundary.

## Surface Impedance ($Z_s$)

The ratio of the tangential Component of the electric field to the Surface Current density at the Conductor Surface.

$$Z_s = \frac{E_{tan}}{J_s} \quad \text{—— (1)}$$

With the Surface at $z=0$, plane, then the Current distribution in z- direction is given by,

$$J = J_o e^{-\gamma z} \quad \text{—— (2)}$$

Linear Current density is given by,

$$J_s = \int_0^\infty J_o e^{-\gamma z} dz.$$

$$J_s = \frac{J_o}{\gamma}$$

But we know that, $J_o = \sigma E_{tan}$

$$J_s = \frac{\sigma E_{tan}}{\gamma} \rightarrow \frac{\sigma E_{tan}}{\gamma J_s} = 0. \quad \frac{\sigma}{\gamma} \cdot Z_s = 0 \rightarrow Z_s = \frac{\gamma}{\sigma}$$

$$Z_s = \frac{\gamma}{\sigma}$$

The propagation Constant '$\gamma$' is given by,

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

For Conducting medium, $\sigma >> \omega\varepsilon$

$$\gamma = \sqrt{j\omega\mu\sigma}$$

$$Z_S = \frac{\sqrt{j\omega\mu\sigma}}{\sigma}$$

$$Z_S = \sqrt{\frac{j\omega\mu}{\sigma}} = \eta \qquad \left(\because \eta = \sqrt{\frac{j\omega\mu}{\sigma}}\right)$$

⇒ For a good conductor, the surface impedance of a plane conductor with thickness greater than the skin depth of a conductor is equal to the characteristic impedance of the conductor.

$$\boxed{Z_S = \eta}$$

## Poynting vector & poynting theorem:-

The energy stored in an electric field and magnetic field is transmitted at a certain rate of energy flow which can be calculated with the help of poynting theorem.

① Poynting vector:- Poynting vector is a product of electric field and magnetic fields.

electric field → V/m

Magnetic field → A/m.

⇒ then poynting vector (or) power density $\vec{P}$ is given by. $\boxed{\vec{P} = \vec{E} \times \vec{H}}$ VA/m² (or) watts/m².

where $\bar{P}$ is called poynting vector. The direction of $\bar{P}$ indicates instantaneous power flow at that point.

## Poynting theorem:-

"The net power flowing out of a given volume 'v' is equal to the time rate of decrease in the energy stored within volume 'v' minus the ohmic power dissipated.

$\bar{E} = E_x$

From Maxwell equations for time varying fields.

$$\nabla \times E = -\mu_0 \frac{dH}{dt} \rightarrow (1)$$

$$\nabla \times H = J + \frac{dD}{dt}$$

$$\nabla \times H = \sigma E + \varepsilon_0 \left[\frac{dE}{dt}\right] - (2)$$

$$\boxed{\begin{aligned} \nabla \times E &= -\frac{dB}{dt} - (1) \\ \nabla \times H &= J + \frac{dD}{dt} - (2) \end{aligned}}$$

$B = \mu_0 H \quad J = \sigma E$

$D = E\varepsilon_0$

* Multiply with E on both sides, on eqn (2).

We get,

$$E(\nabla \times H) = \sigma E^2 + \varepsilon_0 E . \frac{dE}{dt} \rightarrow (3)$$

According to Identity theorem.

$$\left.\begin{aligned} \nabla.(A \times B) &= B(\nabla \times A) - A.(\nabla \times B) \\ A.(\nabla \times B) &= B.(\nabla \times A) - \nabla.(A \times B) \end{aligned}\right\} \text{Vector Identity,}$$

from Identity theorem, $A = E$, $B = H$.

If substitute $A = E$, $B = H$. in Identity theorem we will get

$$E(\nabla \times H) = H(\nabla \times E) - \nabla(E \times H)$$

Where $P = E \times H$.

$$E(\nabla \times H) = H(\nabla \times E) - \nabla P$$

$$H(\nabla \times E) - \nabla P = E(\nabla \times H)$$

$$H(\nabla \times E) - \nabla P = \sigma E^2 + \epsilon_0 E \frac{dE}{dt} \longrightarrow (4)$$

$$\boxed{\text{Where } E(\nabla \times H) = \sigma E^2 + \epsilon_0 E \cdot \frac{dE}{dt}}$$

$$H(\nabla \times E) = H\left(-\mu_0 \frac{dH}{dt}\right)$$

$$\boxed{\nabla \times E = -\mu_0 \frac{dH}{dt}}$$

$$= -\mu_0 \left[H \cdot \frac{dH}{dt}\right)$$

$$= -\mu_0 \left[\frac{dH^2}{dt}\right]$$

$$= -\mu_0 \left[2H \frac{dH}{dt}\right]$$

$$H \frac{dH}{dt} = \frac{1}{2} \frac{dH^2}{dt}$$

$$= \frac{-\mu_0}{2} \cdot \frac{dH^2}{dt}$$

$$E \frac{dE}{dt} = \frac{1}{2} \frac{dE^2}{dt}$$

$$H(\nabla \times E) = -\frac{\mu_0}{2} \cdot \frac{dH^2}{dt} \quad \text{--- } (5)$$

Substitute eqn (5) in eqn (4)

$$-\nabla \cdot P = \sigma E^2 + \frac{\epsilon_0}{2} \frac{dE^2}{dt} + \frac{\mu_0}{2} \frac{dH^2}{dt}$$

$$-\nabla \cdot P = \sigma E^2 + \frac{1}{2} \frac{d}{dt} \left[E^2 \epsilon_0 + \mu_0 H^2\right]$$

$$\nabla \cdot P = -\sigma E^2 - \frac{1}{2} \frac{d}{dt} \left[\epsilon_0 E^2 + \mu_0 H^2\right]$$

Apply volume integral on Both Sides

$$\underbrace{\int_S \nabla(P) ds}_{\text{I}} = \underbrace{-\int_V \sigma E^2 dv}_{\text{II}} \underbrace{-\int_V \frac{d}{dt}\left(\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2\right] dv}_{\text{III}} \quad (6)$$

In the above ~~the~~ term / equation

I st term : The net power flow per unit area.

II nd term : ohmic power dissipation (or) Conduction losses.

(3) III - term → Time rate of decrease of energy stored.

Hence, It is proved.

## Applications of poynting theorem

* Electromagnetic energy transfer between the magnetosphere and Ionosphere.
* EM waves travel from one point to another, there will be energy flow across the surface involved

## * power Loss in a plane Conductor

The power loss in a conductor is nothing but the power flow per unit area through the surface.

Let $\bar{E}_{tan}$ and $\bar{H}_{tan}$ be the tangential components of the electric field $\bar{E}$ and the magnetic field $\bar{H}$ at the surface of the conductor respectively. The tangential component of the electric field $\bar{E}$ is obtained from the relationship given by,

$$\eta = \frac{E_{tan}}{H_{tan}} \quad \text{---(1)}$$

where

$\eta$: Intrinsic impedance of the conductor.

$$E_{tan} = \eta H_{tan} \quad \text{---(2)}$$

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The average power flow per unit area normal to the Surface is given by

$$P_n = \frac{1}{2} Re(\bar{E}_{tan} \times \bar{H}_{tan}) \quad \text{—— (3)}$$

The tangential Components of $\bar{E}$ and $\bar{H}$ are at right angles to each other. But for a good Conductor $E_{tan}$ leads $H_{tan}$ by $45^\circ$, eqn (3) Can be modified as,

$$P_n = \frac{1}{2} |E_{tan}||H_{tan}| \cos 45^\circ \longrightarrow (4)$$

$$= \frac{1}{2} |E_{tan}||H_{tan}| \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} |\eta||H_{tan}||H_{tan}|$$

$$= \frac{1}{2\sqrt{2}} |\eta||H_{tan}|^2 \qquad \eta = \frac{E_{tan}}{H_{tan}}$$

$$= \frac{1}{2\sqrt{2}} \cancel{|\eta|} \frac{|E_{tan}|^2}{|\eta|^{\cancel{2}}} \qquad E_{tan} = \eta H_{tan}$$

$$H_{tan} = \frac{E_{tan}}{\eta}$$

$$P_n = \frac{1}{2\sqrt{2}} \frac{|E_{tan}|^2}{|\eta|}$$

As we have assumed that the thickness of the Conductor is much greater than the skin depth, it can be considered that surface impedance $Z_s$ is equal to the intrinsic impedance $\eta$ of the Conductor. The equation (4) can be written as,

$$P_n = \frac{1}{2\sqrt{2}} |Z_s||H_{tan}|^2$$

$$P_n = \frac{1}{2\sqrt{2}} \frac{|E_{tan}|^2}{|z_s|^2} |z_s|$$

$$P_n = \frac{1}{2\sqrt{2}} \frac{|E_{tan}|^2}{|z_s|^2} |z_s| \quad \text{——— (5)}$$

The current density $\bar{J}_s$ in a conductor is proportional to the magnetic field strength at the surface.

$$P_n = \frac{1}{2\sqrt{2}} |z_s| |J_s|^2 \quad \text{——— (6)}$$

$$\boxed{J_s = \frac{E_{tan}}{z_s}}$$

$$\boxed{Z_s = \frac{E_{tan}}{J_s}}$$

In the equations (4), (5) and (6) the values of $E_{tan}$, $H_{tan}$ and $J_s$ are assumed to be the maximum values.

If these values are expressed as effective or r.m.s values then the equations are given by,

$$P_{n(eff)} = \frac{1}{\sqrt{2}} \frac{|E_{tan(eff)}|^2}{|z_s|} |z_s| = \frac{1}{\sqrt{2}} |z_s| |J_{s(eff)}|^2 \quad \text{——— (7)}$$

Equations (4), (5), (6) and (7) represent the normal component of the poynting vector.

i.e., the power loss in a conductor.

**Brewster Angle:-** when an incident wave strikes a boundary. the angle of incidence at which there is no reflection takes place that angle is called as "Brewster Angle."

⇒ Mathematically Brewster Angle is expressed as

$$\boxed{\theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}$$

We have equation from parallel polarization

$$\frac{E_r}{E_i} = \frac{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_i - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i}}{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i}}$$

∴ According to definition of brewster Angle

$$\frac{E_r}{E_i} = 0$$

$$\therefore \frac{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_i - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i}}{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i}} = 0$$

$$\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_i - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i} = 0$$

$$\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_i = \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i}$$

Squaring on Both sides

$$\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\cos^2\theta_i = \frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i \qquad \left[\cos\theta_i = \sqrt{1-\sin^2\theta_i}\right]$$

$$\left[\frac{\varepsilon_2}{\varepsilon_1}\right]^2\left[1-\sin^2\theta_i\right] = \frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i$$

$$\left[\frac{\epsilon_2}{\epsilon_1}\right]^2 - \left[\frac{\epsilon_2}{\epsilon_1}\right]^2 \sin^2\theta_i = \frac{E_2}{E_1} - \sin^2\theta_i$$

$$\sin^2\theta_i - \left[\frac{E_2}{E_1}\right]^2 \sin^2\theta_i = \frac{E_2}{E_1} - \left[\frac{E_2}{E_1}\right]^2$$

$$\sin^2\theta_i \left[1 - \left[\frac{\epsilon_2}{E_1}\right]^2\right] = \frac{\epsilon_2}{E_1}\left[1 - \frac{\epsilon_2}{E_1}\right]$$

$$\sin^2\theta_i = \frac{\frac{E_2}{E_1}\left[1 - \frac{E_2}{E_1}\right]}{\left[1^2 - \left[\frac{\epsilon_2}{E_1}\right]^2\right]}$$

$$\sin^2\theta_i = \frac{\frac{E_2}{E_1}\left[\cancel{1 - \frac{E_2}{E_1}}\right]}{\left[1 + \frac{E_2}{E_1}\right]\left[\cancel{1 - \frac{E_2}{E_1}}\right]}$$

$$\sin^2\theta_i = \frac{\frac{E_2}{E_1}}{\left[1 + \frac{E_2}{E_1}\right]}$$

$$\sin^2\theta_i = \frac{\frac{E_2}{\cancel{E_1}}}{\frac{E_1 + E_2}{\cancel{E_1}}} = \frac{E_2}{E_1 + E_2}$$

$$\boxed{\sin^2\theta_i = \frac{E_2}{E_1 + E_2}} \quad \text{—— } (1)$$

$$\cos^2\theta_i = 1 - \sin^2\theta_i$$

$$= 1 - \frac{E_2}{E_1 + E_2}$$

$$\cos^2\theta_i = \frac{E_1 + \cancel{E_2} - \cancel{E_2}}{E_1 + E_2} = \frac{E_1}{E_1 + E_2}$$

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$$\boxed{\cos^2\theta_i = \frac{E_1}{E_1+E_2}} \longrightarrow (2)$$

$$\frac{\sin^2\theta_i}{\cos^2\theta_i} = \frac{\dfrac{E_2}{\cancel{E_1+E_2}}}{\dfrac{E_1}{\cancel{E_1+E_2}}}$$

$$\tan^2\theta_i = \frac{E_2}{E_1}$$

$$\tan\theta_i = \sqrt{\frac{E_2}{E_1}}$$

$$\theta_i = \tan^{-1}\left[\sqrt{\frac{E_2}{E_1}}\right]$$

$$\boxed{\theta_B = \tan^{-1}\left[\sqrt{\frac{E_2}{E_1}}\right]}$$

Derivation of Brewster angle for perpendicular polarization

We know that

$$\frac{E_r}{E_i} = \frac{\cos\theta_i - \sqrt{\frac{E_2}{E_1} - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\frac{E_2}{E_1} - \sin^2\theta_i}}$$

According to Brewster angle definition.

$$\frac{E_r}{E_i} = 0$$

$$\cos\theta_i - \sqrt{\frac{E_2}{E_1} - \sin^2\theta_i} = 0$$

$$\cos\theta_i = \sqrt{\frac{\epsilon_2}{\epsilon_1} - \sin^2\theta_i}$$

Squaring on Both Sides.

$$\cos^2\theta_i = \frac{\epsilon_2}{\epsilon_1} - \sin^2\theta_i$$

$$\sin^2\theta_i + \cos^2\theta_i = \frac{\epsilon_2}{\epsilon_1}$$

$$\therefore \quad 1 = \frac{\epsilon_2}{\epsilon_1}$$

$$\boxed{\epsilon_1 = \epsilon_2}$$

This indicate that, there is no brewster angle in Perpendicular polarization.

## Normal and oblique Incidences for both perfect Conductor and perfect Dielectrics.

Definitions:-

**Normal Incidence:-**

when a uniform plane wave is incidences normally to the boundary between the two media, then it is known as "Normal incidence".

**Oblique Incidence:-** when a uniform plane wave is incidences with an angle (oblique) to the boundary between the two media's, then it is known as oblique Incidence

* when the Em wave travelling from one medium to another medium. The wave will be partially refracted or reflected. It depends upon the type of wave incidence. Wave Incidence classified as 2 types. ① Normal Incidence ② oblique Incidence.

## ① Normal Incidence

### Wave Incident Normally on a perfect Conductor:-

When a uniform plane wave incident normally to the Surface of a perfect Conductor, it is reflected Completly.

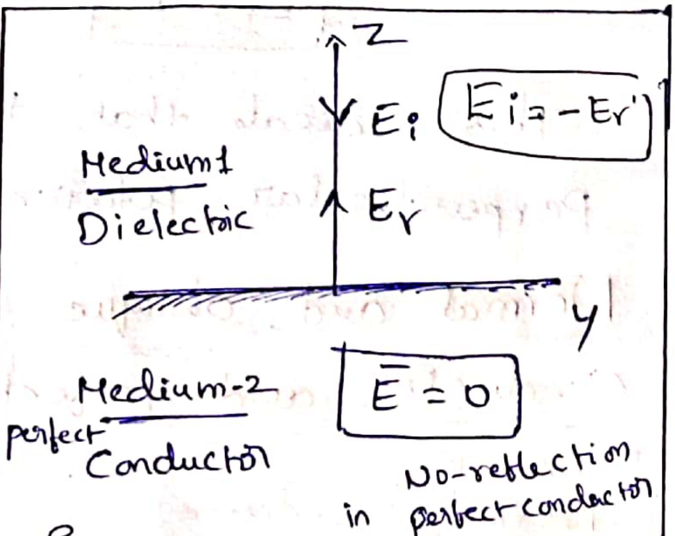



fig:- wave incident normally on a perfect conductor.

Let's the electric field of incident wave is given by

$E_i = E_i e^{-j\beta z}$. (for Incident wave)

$E_r = E_r e^{j\beta z}$ (for reflected Wave)

The resultant Electric field is given by.

$$E(z) = E_i + E_r$$

$$E(z) = \left[E_i e^{-j\beta z} + E_r e^{j\beta z}\right]$$

but $E_i = -E_r$. (from fig)

$$E(z) = \left[E_i e^{-j\beta z} - E_r e^{j\beta z}\right]$$

$$E(z) = -E_i \left[e^{j\beta z} - e^{-j\beta z}\right]$$

$$E(z) = -2jE_i \frac{\left[e^{j\beta z} - e^{-j\beta z}\right]}{2j}$$

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta.$$

if $\theta = \beta z$

$$\boxed{E_{(z)} = -2jE_i \sin\beta z}$$

The time varying behaviour of above eqn is

$$E(z,t) = Re\left(E(z)\, e^{j\omega t}\right)$$

$$= Re\left\{\left(-2j\, E_i \sin(\beta z)\right) e^{j\omega t}\right\}$$

$$\cancel{E(z,t) = -2jj\, E_i \sin\beta z}$$

$$E(z,t) = Re\left(-2j\, E_i \sin(\beta z)\left(\cos\omega t + j\sin\omega t\right)\right)$$

$$e^{j\theta} = j\sin\theta + \cos\theta = \cos\theta + j\sin\theta.$$

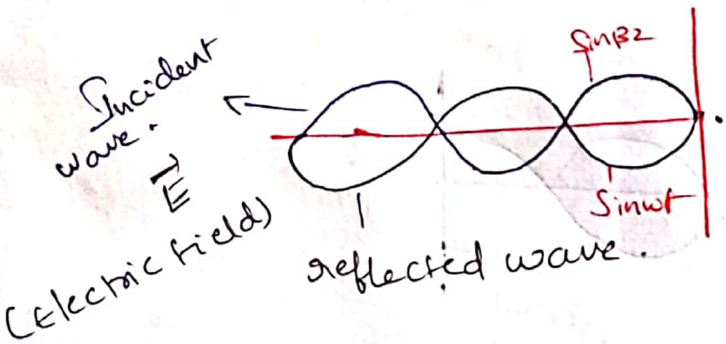
$$E(z,t) = -2j\, E_i \sin\beta z \cos\omega t - 2j^2 E_i \sin\omega t \sin\beta z$$

$$= \underbrace{-2j\, E_i \sin\beta z \cos\omega t}_{\text{Imaginary term}} + 2E_i \sin\omega t \sin\beta z \qquad \boxed{e^{j\omega t} = \cos\omega t + j\sin\omega t}$$

$$=$$

$$\boxed{E(z,t) = 2E_i \sin\beta z . \sin\omega t}$$

The above equation shows that the incident and reflected wave combines and gives a standing wave ratio.




In order to maintain the reversal of energy of propagation, the magnetic field must be reflected without reversal of phase.

i.e., $\boxed{H_i = H_r}$

$$H_i = H_i e^{-j\beta z}$$

$$H_r = H_r e^{j\beta z}$$

$$E( = H_i + H_r$$

$$= \left[H_i e^{-j\beta z} + H_r e^{+j\beta z}\right] \qquad \boxed{H_i = H_r}$$

$$= H_i\left[e^{-j\beta z} + e^{j\beta z}\right]$$

$$= 2H_i \left[\frac{e^{-j\beta z} + e^{j\beta z}}{2}\right]$$

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta$$

$$\boxed{E(z) = 2H_i \cos(\beta z)}$$

The time varying behaviour of above equation is

$$E(z,t) = Re\,(E(z)\, e^{j\omega t})$$

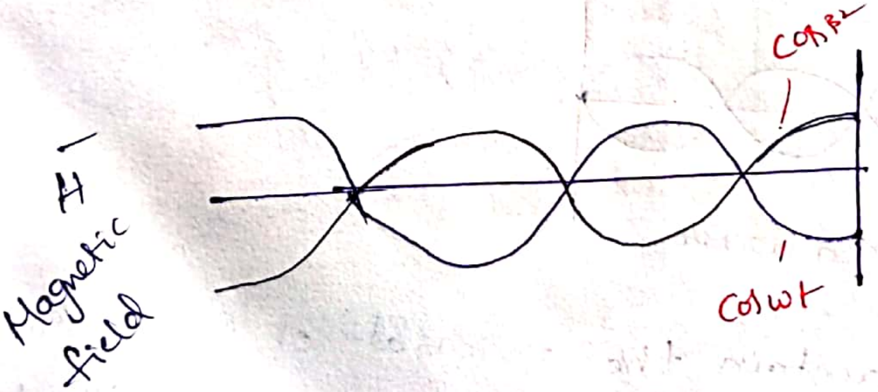
$$= Re\{(2H_i \cos(\beta z))\, e^{j\omega t}\}$$

$$e^{j\omega t} = \cos\omega t + j\sin\omega t.$$

$$E(z,t) = 2H_i \cos(\beta z)\,\cos\omega t + j\sin\omega t.$$

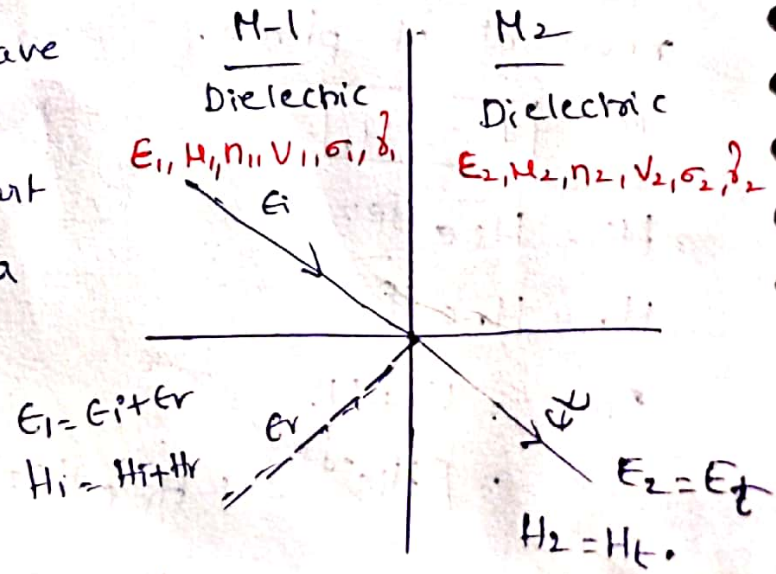
$$= 2H_i \cos\beta z \cos\omega t + \underbrace{2H_i \cos\beta z\, j\sin\omega t}_{\text{Imaginary}}$$

$$\boxed{\tilde{E}(z,t) = 2H_i \cos\beta z.\cos\omega t}$$




## II. Wave Incident normally on a perfect Dielectric

When a uniform plane wave is incident on the surface of a perfect dielectric a part of energy transmitted and a part of



Consider 2 perfect dielectric media seperated by a boundary as shown in the above figure.

We know that $\boxed{\eta = \frac{E}{H}}$

$$E_i = \eta_1 H_1 \qquad H_i = \frac{E_i}{\eta_i}$$

$$E_r = -E_i = -\eta_1 H_r \qquad H_r = -\frac{E_r}{\eta_1}$$

$$E_t = \eta_2 H_t \qquad H_t = \frac{E_t}{\eta_2}$$

According to the boundary condition [Dielectric - Dielectric] The tangential of E or H is continuous across the boundary.

$$H_{1t} = H_{2t} \; ; \quad E_{1t} = E_{2t}.$$

$$\therefore \; E_i + E_r = E_t \quad \text{——} (1)$$

$$\therefore \; H_i + H_r = H_t \quad \text{——} (2)$$

$$H_t = H_i + H_r$$

$$H_t = \frac{E_i}{\eta_1} - \frac{E_r}{\eta_1}$$

$$H_t = \frac{1}{\eta_1}[E_i - E_r] \quad \text{——} (3)$$

$$H_t = \frac{E_t}{\eta_2} = \frac{E_i + E_r}{\eta_2}$$

$$H_t = \frac{1}{\eta_2}[E_i + E_r] \quad \text{——} (4)$$

$$\therefore \; \frac{1}{\eta_1}[E_i - E_r] = \frac{1}{\eta_2}[E_i + E_r]$$

$$\eta_2 E_i - \eta_2 E_r = \eta_1 E_i + \eta_1 E_r$$

$$\eta_2 E_i - \eta_1 E_i = \eta_1 E_r + \eta_2 E_r$$

$$E_i[\eta_2 - \eta_1] = E_r[\eta_1 + \eta_2]$$

$$\boxed{K = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}$$ → this equation represents the reflection co-efficient in term of $\vec{E}$

## Transmission Co-efficient:-

$$\frac{E_t}{E_i} = \frac{E_i + E_r}{E_i} = 1 + \frac{E_r}{E_i}$$

$$= 1 + \left[\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right]$$

$$= \frac{\eta_2 + \cancel{\eta_1} + \eta_2 - \cancel{\eta_1}}{\eta_2 + \eta_1}$$

$$\boxed{\frac{E_t}{E_i} = \frac{2\eta_2}{\eta_2 + \eta_1}}$$

From equation ① & ②

$E_i + E_r = E_t =$

$\eta = \frac{E}{H}$

$E = \eta H$

$H = \frac{E}{\eta}$

$E_i + E_r = E_t \longrightarrow$ ①

$\eta_1 H_i - \eta_1 H_r = E_t$

$\eta_1 [H_i - H_r] = E_t$ — } equating this 2 eqn's

$E_t = \eta_2 (H_i + H_r)$ —

$\eta_1 [H_i - H_r] = \eta_2 [H_i + H_r]$

$\eta_1 H_i - \eta_1 H_r = \eta_2 H_i + \eta_2 H_r$

$\eta_1 H_i - \eta_2 H_i = \eta_2 H_r + \eta_1 H_r$

$H_i (\eta_1 - \eta_2) = H_r (\eta_2 + \eta_1)$

$$\boxed{\frac{H_r}{H_i} = \frac{\eta_1 - \eta_2}{\eta_2 + \eta_1}}$$

The above equation is the reflection co-efficient interms of $\vec{H}$. (Magnetic field)

## Transmission Co-efficient:-

$$\frac{H_t}{H_i} = \frac{H_i + H_r}{H_i}$$

$$= 1 + \frac{H_r}{H_i}$$

$$= 1 + \frac{\eta_1 - \eta_2}{\eta_2 + \eta_1}$$

$$\frac{H_t}{H_i} = \frac{\cancel{\eta_2} + \eta_1 + \eta_1 - \cancel{\eta_2}}{\eta_2 + \eta_1}$$

$$\boxed{\frac{H_t}{H_i} = \frac{2\eta_1}{\eta_1 + \eta_2}}$$

## OBLique Incidence: of A Plane wave on a boundary Plane:-

Reflection and transmission of a wave depend on

1. The type of polarization of a wave. (General polarizations, namely parallel and perpendicular polarizations are Considered.

2. The medium of the boundary.

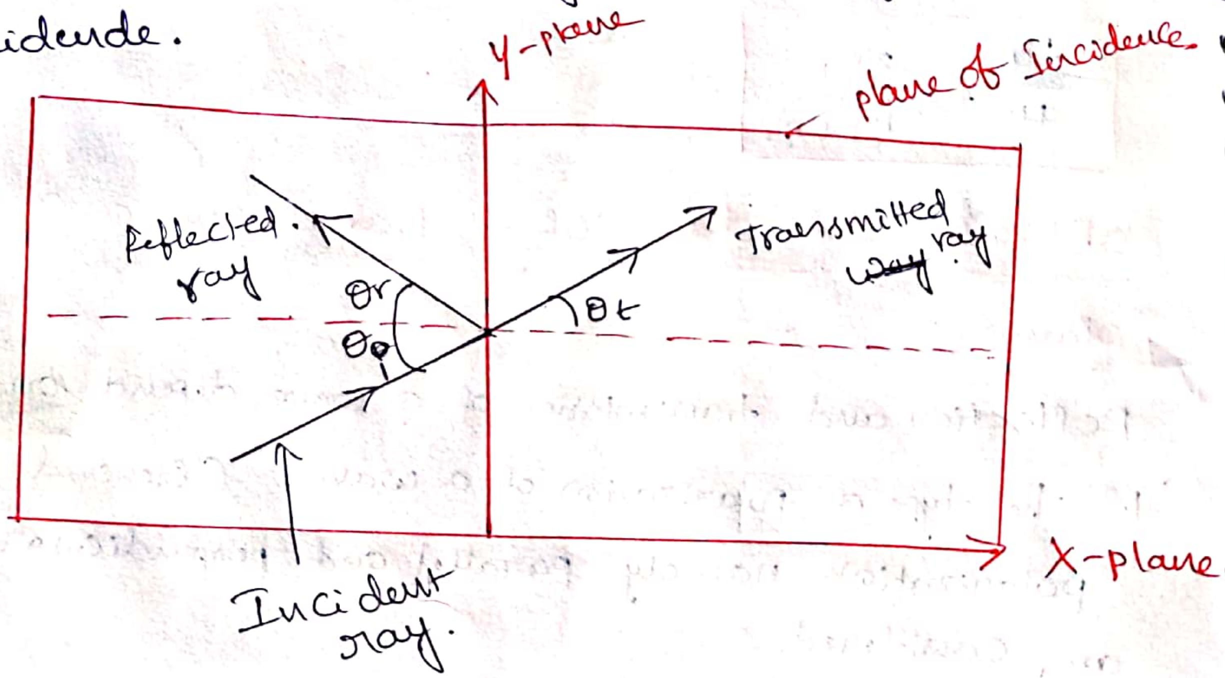
① **parallel polarization:-** It is defined as the polarization in which the electric field of the wave is parallel to the plane of incidence. parallel polarization is also called vertical polarization.

## Perpendicular polarization:-

It is defined as the polarization in which the electric field of the wave is perpendicular to the plane of incidence. perpendicular polarization is also called horizontal polarization.

**Plane of Incidence:-** It is a plane which contains the incident, reflected and transmitted rays and is normal to the boundary.

It is described below fig. which is x-y plane of Incidende.



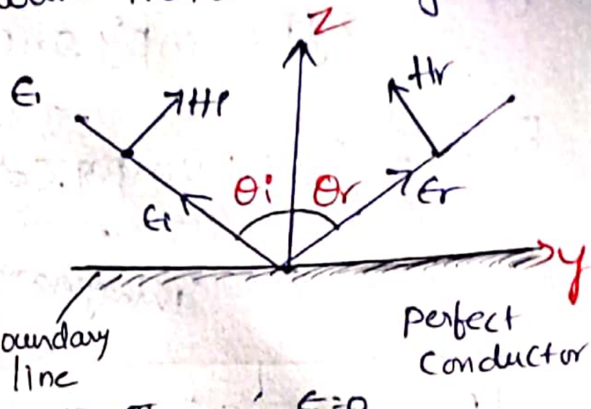

### ① wave Incident obliquely on a perfect Conductor:-

when a wave is incident obliquely on a perfect Conductor, it is necessary to consider 2 special Cases.

**Case I:- Perpendicular polarization:-**

The $\vec{E}$ is perpendicular to the plane of incidence, then this case is known as perpendicular polarization as in the below figure.

Let's the incident & reflected wave makes an angle $\theta_i = \theta_r = \theta$.




Assume the incident $\vec{E}$ will be

$$E = E_i e^{-j\beta(\vec{a_n}\cdot\vec{r})}$$

$$(a_n \cdot r) = x\cos\left[\frac{\pi}{2} - \theta\right] + y\cos(\pi - \theta) + z\cos\frac{\pi}{2}$$

$$a_n \cdot r = y\sin\theta - z\cos\theta.$$

$$E_i = E_i e^{+j\beta}(y\sin\theta - z\cos\theta)$$

Similarly

$$E_r = E_r e^{+j\beta}(y\sin\theta - z\cos\theta)$$

$$E = E_i + E_r$$

$$E = E_i e^{-j\beta(y\sin\theta - z\cos\theta)} + E_r e^{+j\beta(y\sin\theta - z\cos\theta)}$$

$$= E_i e^{-j\beta(y\sin\theta - z\cos\theta)} - E_i e^{j\beta(y\sin\theta - z\cos\theta)}$$

$$= -E_i\left[e^{j\beta(y\sin\theta - z\cos\theta)} - e^{-j\beta(y\sin\theta - z\cos\theta)}\right]$$

$$E = \frac{-2jE_i\left[e^{j\beta(y\sin\theta - z\cos\theta)} - e^{-j\beta(y\sin\theta - z\cos\theta)}\right]}{2j}$$

$$\boxed{E = -2jE_i \sin\beta(y\sin\theta - z\cos\theta)}$$

$$\boxed{\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta.}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

**Time - varying Behaviour:-**

$$\bar{E} = \mathrm{Re}\left\{-2jE_i \sin\beta(y\sin\theta - z\cos\theta)\, e^{j\omega t}\right\}$$

$$\bar{E} = -2jE_i \sin\beta(y\sin\theta - z\cos\theta)\cos\omega t + j\sin\omega t.$$

$$E = -2jE_i \sin\beta\, y\sin\theta - z\cos\theta \cos\omega t + (-2j)\cdot jE_i \sin\beta \sin\omega t.\ (y\sin\theta - z\cos\theta)$$

$E = -2j E_i \sin\beta(y\sin\theta - z\cos\theta)\cos\omega t + 2E_i \sin\beta t \sin\omega t (y\sin\theta - z\cos\theta)$

$\dot{E} = \underbrace{-2jE_i \sin\beta y . \cos\omega t(y\sin\theta - z\cos\theta)}_{\text{Imaginary part}} + 2E_i \sin\beta t . \sin\omega t (y\sin\theta - z\cos\theta)$

$$\boxed{E = 2E_i \sin\beta t . \sin\omega t (y\sin\theta - z\cos\theta)}$$

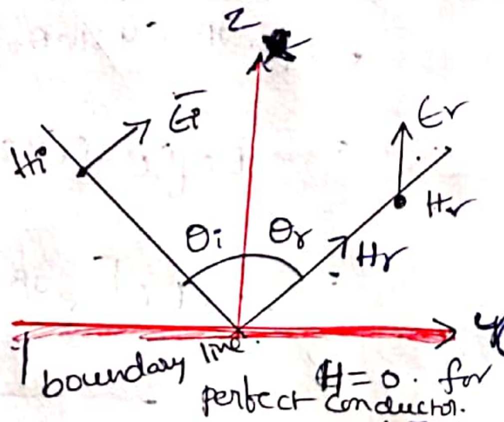
## Parallel polarization :-

The $\vec{E}$ is parallel to the plane of incident then that case is known as parallel (or) vertical polarization. shown in the below figure. The incident magnetic field is given by

$H_i = H_i e^{-j\beta(\bar{a}_n . \bar{r})}$

$a_n$ = unit vector normal to the plane.

$r = (x, y, z)$ is a radius vector on the plane.

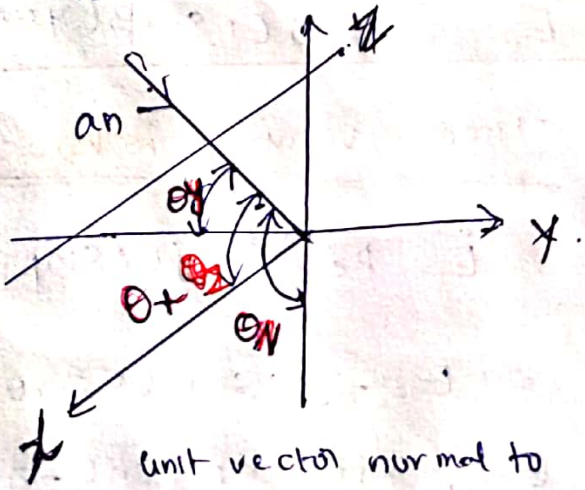



$a_n . r = x\cos\theta_x + y\cos\theta_y + z\cos\theta_z$

$\theta_y = \left[\frac{\pi}{2} - \theta_i\right]$ $\theta_z = (\pi - \theta_i)$ $\theta_x = \frac{\pi}{2}$

$\theta_x, \theta_y$ and $\theta_z$ are the angles made by a unit vector normal to the plane with x, y and z- axes.

$$\boxed{a_n . r = y\sin\theta - z\cos\theta}$$




unit vector normal to the plane.

$H_i = H_i e^{-j\beta(y\sin\theta - z\cos\theta)}$

Similarly

$H_r = H_r e^{-j\beta(y\sin\theta - z\cos\theta)}$

$H = H_i + H_r$ $\boxed{H_i = H_r}$

$H = H_i e^{-j\beta(y\sin\theta - z\cos\theta)} + H_r e^{-j\beta(y\sin\theta - z\cos\theta)}$

$= H_i e^{-j\beta(y\sin\theta - z\cos\theta)} + H_i e^{-j\beta(y\sin\theta - z\cos\theta)}$

$= H_i [e^{-j\beta(y\sin\theta - z\cos\theta)} + e^{-j\beta(y\sin\theta - z\cos\theta)}]$

$$H = 2\cancel{\beta} \cdot H_i \frac{[e^{-j\beta(y\sin\theta - z\cos\theta)} + e^{-j\beta(y\sin\theta - z\cos\theta)}]}{2\cancel{\beta}}$$

$$\boxed{H = 2H_i \cos_\beta(y\sin\theta - z\cos\theta)} \rightarrow \boxed{H = 2H_i\cos\beta(y\sin\theta - z\cos\theta)}$$

## Time Varying Behaviour

$H = \mathrm{Re}\{2H_i \cos(y\sin\theta - z\cos\theta) e^{j\omega t}$

where $e^{j\omega t} = \cos\omega t + j\sin\omega t$

$H = 2H_i \cos(y\sin\theta - z\cos\theta) \cos\omega t + j\sin\omega t$

$H = 2H_i \cos\beta y\sin\theta - z\cos\theta \cos\omega t + \underbrace{2H_i\cos\beta(y\sin\theta - z\cos\theta) j\sin\omega t}_{\text{Imagnery}}$

$$\boxed{H = 2H_i\cos\beta(y\sin\theta - z\cos\theta)\cos\omega t}$$

## Dielectrics:

* For all most all dielectrics $\boxed{\mu_1 = \mu_2 \approx \mu_0}$

* Let's the incident wave makes an angle $\theta_i$, the reflected wave makes an angle $\theta_r$ & the transmitted wave makes an angle $\theta_t$ then according to Snell's law: $\boxed{\dfrac{\sin\theta_i}{\sin\theta_t} = \sqrt{\dfrac{\varepsilon_{r_2}}{\varepsilon_{r_1}}}}$

## Wave incident obliquely on a perfect Dielectrics

### ① parallel polarization (vertical polarization):-

When a wave is incident on a dielectric, a part of it is reflected and a part of it is transmitted through the dielectric. If $\theta_i$, $\theta_r$ and $\theta_t$ are the angles of the incident, reflected and transmitted rays $\theta_i = \theta_r$. The angles $\theta_i$ and $\theta_t$ are related by Snell's law, that is

$$\boxed{\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}}$$

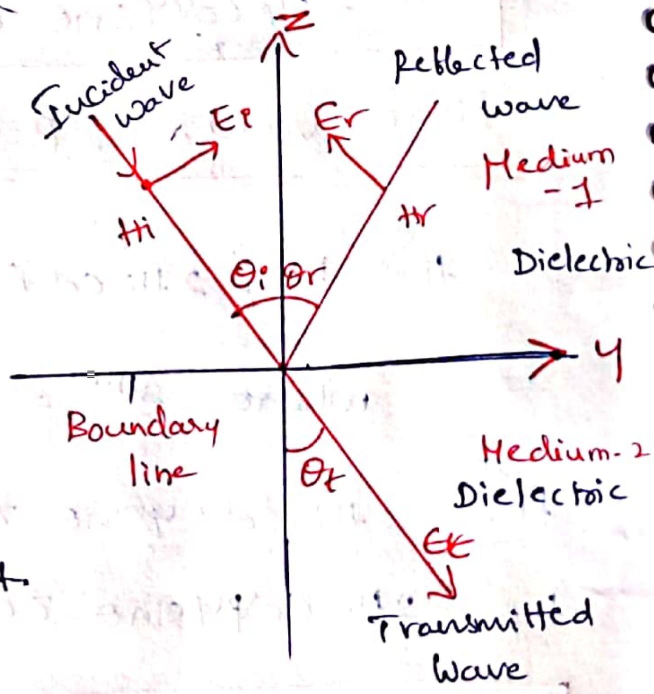



fig:- Incident, reflected and transmitted rays.

④ Consider fig.

The boundary condition on E is

$$E_{tan1} = E_{tan2} \quad \text{—①}$$

$$(E_i - E_r)\cos\theta_i = E_t \cos\theta_t.$$

Dividing both sides by $E_i$, we get

$$\left[1 - \frac{E_r}{E_i}\right]\cos\theta_i = \frac{E_t}{E_i}\cos\theta_t$$

$$\frac{E_t}{E_i} = \left[1 - \frac{E_r}{E_i}\right]\frac{\cos\theta_i}{\cos\theta_t} \quad \text{—②}$$

By the law of Conservation of energy, incident energy is equal to the sum of the ~~refle~~ reflected and transmitted energies, that is

$$\left[\text{Average power} = \frac{E^2}{2\eta}\right]$$

$$P_i = P_r + P_t$$

$$P_i - P_r = P_t.$$

$\therefore P_i = P_r + P_t$

$$\frac{E_i^2}{n_1} = \frac{E_r^2}{n_1} + \frac{E_t^2}{n_2}$$

$$\frac{E_i^2}{n_1}\cos\theta_i = \frac{E_r^2}{n_1}\cos\theta_r + \frac{E_t^2}{n_2}\cos\theta_t$$

dividing $\frac{E_i^2}{n_1}\cos\theta_i$ on Both sides

$$1 = \frac{\frac{E_r^2}{\cancel{n_1}}\cancel{\cos\theta_i}}{\frac{E_i^2}{\cancel{n_1}}\cancel{\cos\theta_i}} + \frac{\frac{E_t^2}{n_2}\cos\theta_t}{\frac{E_i^2}{n_1}\cos\theta_i}$$

$$1 = \frac{E_r^2}{E_i^2} + \frac{\frac{E_t^2}{n_2}\cos\theta_t}{\frac{E_i^2}{n_1}\cos\theta_i}$$

$$\boxed{\frac{E_r^2}{E_i^2} = 1 - \frac{n_1}{n_2}\cdot\frac{E_t^2}{E_i^2}\cdot\frac{\cos\theta_t}{\cos\theta_i}} \longrightarrow (3)$$

Substitute eqn (2) in eqn (3)

$$\frac{E_r^2}{E_i^2} = 1 - \frac{n_1}{n_2}\left[1-\frac{E_r}{E_i}\right]^2\left[\frac{\cos\theta_i}{\cos\theta_t}\right]^2\left[\frac{\cancel{\cos\theta_t}}{\cancel{\cos\theta_i}}\right]$$

$$\frac{E_r^2}{E_i^2} = 1 - \frac{n_1}{n_2}\left[1-\frac{E_r}{E_i}\right]^2\left[\frac{\cos\theta_i}{\cos\theta_t}\right]$$

$$\frac{E_r^2}{E_i^2} - 1 = -\frac{n_1}{n_2}\left(1-\frac{E_r}{E_i}\right)^2\left[\frac{\cos\theta_i}{\cos\theta_t}\right] \rightarrow \frac{E_r^2}{E_i^2} - 1^2 = -\frac{n_1}{n_2}\left[\frac{E_r}{E_i}-1\right]^2\left[\frac{\cos\theta_i}{\cos\theta_t}\right]$$

$$\left[\frac{E_r}{E_i}+1\right]\left[\cancel{\frac{E_r}{E_i}-1}\right] = -\frac{n_1}{n_2}\left[\frac{E_r}{E_i}-1\right]^{\cancel{2}}\left[\frac{\cos\theta_i}{\cos\theta_t}\right]$$

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$$\frac{E_r}{E_i} + \frac{n_1}{n_2}\left[\frac{E_r}{E_i} - 1\right]\left[\frac{\cos\theta_i}{\cos\theta_t}\right] = -1$$

$$\frac{E_r}{E_i} + \frac{n_1}{n_2}\frac{E_r}{E_i}\frac{\cos\theta_i}{\cos\theta_t} - \frac{n_1 \cos\theta_i}{n_2 \cos\theta_t} = -1$$

$$\frac{E_r}{E_i}\left[1 + \frac{n_1}{n_2}\frac{\cos\theta_i}{\cos\theta_t}\right] - \frac{n_1}{n_2}\frac{\cos\theta_i}{\cos\theta_t} = -1$$

$$\frac{E_r}{E_i} = \frac{-1 + \frac{n_1}{n_2}\frac{\cos\theta_i}{\cos\theta_t}}{1 + \frac{n_1}{n_2}\frac{\cos\theta_i}{\cos\theta_t}}$$

$$\frac{E_r}{E_i} = \frac{\frac{-1 n_2 \cos\theta_t + n_1 \cos\theta_i}{n_2 \cancel{\cos\theta_t}}}{\frac{n_2 \cos\theta_t + n_1 \cos\theta_i}{n_2 \cancel{\cos\theta_t}}}$$

$$\boxed{\frac{E_r}{E_i} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}} \longrightarrow (5)$$

$$n_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_0}{\varepsilon_1}}$$

$$n_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\mu_0}{\varepsilon_2}} \quad \therefore \boxed{\mu_1 = \mu_2 = \mu_0}$$

Substitute in eqn (5)

$$\frac{E_r}{E_i} = \frac{\sqrt{\frac{\mu_0}{\varepsilon_1}}\cos\theta_i - \sqrt{\frac{\mu_0}{\varepsilon_2}}\cos\theta_t}{\sqrt{\frac{\mu_0}{\varepsilon_1}}\cos\theta_i + \sqrt{\frac{\mu_0}{\varepsilon_2}}\cos\theta_t}$$

$$\frac{E_r}{E_i} = \frac{\frac{1}{\sqrt{\varepsilon_1}}\cos\theta_i - \frac{1}{\sqrt{\varepsilon_2}}\cos\theta_t}{\frac{1}{\sqrt{\varepsilon_1}}\cos\theta_i + \frac{1}{\sqrt{\varepsilon_2}}\cos\theta_t}.$$

$$= \frac{\sqrt{\varepsilon_2}\cos\theta_i - \sqrt{\varepsilon_1}\cos\theta_t / \sqrt{\varepsilon_1}\cdot\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_2}\cos\theta_i + \sqrt{\varepsilon_1}\cos\theta_t / \sqrt{\varepsilon_1}\cdot\sqrt{\varepsilon_2}}$$

$$= \frac{\sqrt{\varepsilon_2}\cos\theta_i - \sqrt{\varepsilon_1}\cos\theta_t}{\sqrt{\varepsilon_2}\cos\theta_i + \sqrt{\varepsilon_1}\cos\theta_t}.$$

$$= \frac{\sqrt{\varepsilon_2}\cos\theta_i - \sqrt{\varepsilon_1}\left(\sqrt{1-\sin^2\theta_t}\right)}{\sqrt{\varepsilon_2}\cos\theta_i + \sqrt{\varepsilon_1}\left[\sqrt{1-\sin^2\theta_t}\right]} \quad \text{—— (6)}$$

$\cos\theta = \sqrt{1-\sin^2\theta}.$

By Snell's law.

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\sin\theta_t = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\,\sin\theta_i$$

Squaring and adding (1 −) on Both sides

$$1 - \sin^2\theta_t = 1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i$$

$$1 - \sin^2\theta_t = \frac{\varepsilon_2 - \varepsilon_1\sin^2\theta_i}{\varepsilon_2} \longrightarrow (7).$$

Substitute eqn (7) in eqn (6)

$$= \frac{\sqrt{\varepsilon_2}\cos\theta_i - \sqrt{\varepsilon_1}\left[\sqrt{\frac{\varepsilon_2-\varepsilon_1\sin^2\theta_i}{\varepsilon_2}}\right]}{\sqrt{\varepsilon_2}\cos\theta_i + \sqrt{\varepsilon_1}\left[\sqrt{\frac{\varepsilon_2-\varepsilon_1\sin^2\theta_i}{\varepsilon_2}}\right]}$$

$$\frac{E_r}{E_i} = \frac{\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\cos\theta_i - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i})}{\frac{\sqrt{\varepsilon_2}}{\varepsilon_1}\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i}]}$$

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Multiply and divide with $\left[\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}\right]$

$$\boxed{\frac{E_r}{E_i} = \frac{\frac{\epsilon_2}{\epsilon_1}\cos\theta_i - \left[\sqrt{\frac{\epsilon_2}{\epsilon_1} - \sin^2\theta_i}\right]}{\frac{\epsilon_2}{\epsilon_1}\cos\theta_i + \left[\sqrt{\frac{\epsilon_2}{\epsilon_1} - \sin^2\theta_i}\right]}}$$

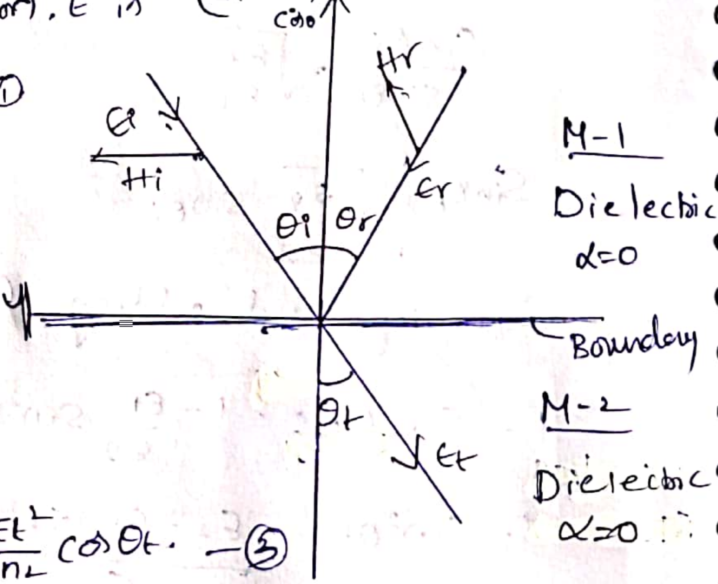
## ② perpendicular (or) Horizontal polarization:-

$E_{tan1} = E_{tan2}$

$(E_i + E_r)\cos\theta = E_t \cos\theta_t$

From the boundary condition, E is

$$E_i + E_r = E_t \quad \text{——— ①}$$




divide with $E_i$ on Both Sides.

$$1 + \frac{E_r}{E_i}\cdot\frac{\cos\theta_i}{\cos\theta_t} = \frac{E_t}{E_i} \quad \text{——— ②}$$

$$P_i = P_r + P_t \qquad \boxed{P = \frac{E^2}{n}}$$

$$\frac{E_i^2}{n_1}\cos\theta_i = \frac{E_r^2}{n_1}\cos\theta_i + \frac{E_t^2}{n_2}\cos\theta_t \quad \text{——— ③}$$

divide with $\frac{E_i^2}{n_1}\cos\theta_i$ on Both Sides.

$$1 = \frac{\frac{E_r^2 \cos\theta_i}{\cancel{n_1}}}{\frac{E_i^2 \cos\theta_i}{\cancel{n_1}}} + \frac{\frac{E_t^2 \cos\theta_t}{n_2}}{\frac{E_i^2 \cos\theta_i}{n_1}}$$

$$1 = \frac{E_r^2 \cos\theta_i}{E_i^2} + \frac{n_1}{n_2}\frac{E_t^2}{E_i^2}\frac{\cos\theta_t}{\cos\theta_i}$$

$$\frac{E_r^2}{E_i^2} = 1 - \frac{n_1}{n_2}\frac{E_t^2}{E_i^2}\frac{\cos\theta_t}{\cos\theta_i} \longrightarrow ④$$

Substitute eqn ② in eqn ④

$$\frac{E_r^2}{E_i^2} = 1 - \frac{n_1}{n_2}\left[1 + \frac{E_r}{E_i}\right]^2 \left[\frac{\cos\theta_i}{\cos\theta_t}\right]^{\cancel{2}} \frac{\cos\theta_t}{\cos\theta_t}$$

$$\frac{E_r^2}{E_i^2} - 1^2 = -\frac{n_1}{n_2}\left[\frac{E_r}{E_i} + 1\right]^2 \frac{\cos\theta_i}{\cos\theta_t}$$

$$\cancel{\left[\frac{E_r}{E_i} + 1\right]}\left[\frac{E_r}{E_i} - 1\right] = -\frac{n_1}{n_2}\left[\frac{E_r}{E_i} + 1\right]^{\cancel{2}} \frac{\cos\theta_i}{\cos\theta_t}$$

$$\frac{E_r}{E_i} - 1 = -\frac{n_1}{n_2}\left[\frac{E_r}{E_i} + 1\right] \frac{\cos\theta_i}{\cos\theta_t}$$

$$\frac{E_r}{E_i} + \frac{n_1}{n_2}\left[\frac{E_r}{E_i} + 1\right] \frac{\cos\theta_i}{\cos\theta_t} = 1$$

$$\frac{E_r}{E_i} + \frac{n_1}{n_2}\frac{E_r}{E_i}\frac{\cos\theta_i}{\cos\theta_t} + \frac{n_1}{n_2}\frac{\cos\theta_i}{\cos\theta_t} = 1$$

$$\frac{E_r}{E_i}\left[1 + \frac{n_1}{n_2}\frac{\cos\theta_i}{\cos t}\right] = 1 - \frac{n_1}{n_2}\frac{\cos\theta_i}{\cos\theta_t}$$

$$\frac{E_r}{E_i} = \frac{1 - \frac{n_1}{n_2}\frac{\cos\theta_i}{\cos t}}{1 + \frac{n_1}{n_2}\frac{\cos_i}{\cos t}}$$

$$\frac{E_r}{E_i} = \frac{\dfrac{n_2\cos\theta_i - n_1\cos\theta_t}{\cancel{n_2\cos\theta_t}}}{\dfrac{n_2\cos\theta_i + n_1\cos\theta_t}{\cancel{n_2\cos\theta_t}}}$$

$$\boxed{\frac{E_r}{E_i} = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t}} \longrightarrow (5)$$

$\therefore \boxed{\mu_1 = \mu_2 = \mu_0}$

$$\left.\begin{aligned} n_1 &= \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_0}{\varepsilon_1}} \\ n_2 &= \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\mu_0}{\varepsilon_1}} \end{aligned}\right\}$$ Substitute in eq,n (5)

$$= \frac{\sqrt{\frac{\mu_0}{\varepsilon_2}} \cos\theta_i - \sqrt{\frac{\mu_0}{\varepsilon_1}} \cos\theta_t}{\sqrt{\frac{\mu_0}{\varepsilon_2}} \cos\theta_i + \sqrt{\frac{\mu_0}{\varepsilon_1}} \cos\theta_t}$$

$$= \frac{\frac{1}{\sqrt{\varepsilon_2}} \cos\theta_i - \frac{1}{\sqrt{\varepsilon_1}} \cos\theta_t}{\frac{1}{\sqrt{\varepsilon_2}} \cos\theta_i + \frac{1}{\sqrt{\varepsilon_1}} \cos\theta_t}$$

$$= \frac{\dfrac{\sqrt{\varepsilon_1} \cos\theta_i - \sqrt{\varepsilon_2} \cos\theta_t}{\cancel{\sqrt{\varepsilon_2}\sqrt{\varepsilon_1}}}}{\dfrac{\sqrt{\varepsilon_1} \cos\theta_i + \sqrt{\varepsilon_2} \cos\theta_t}{\cancel{\sqrt{\varepsilon_2}\sqrt{\varepsilon_1}}}}$$

$$= \frac{\sqrt{\varepsilon_1} \cos\theta_i - \sqrt{\varepsilon_2} \cos\theta_t}{\sqrt{\varepsilon_1} \cos\theta_i + \sqrt{\varepsilon_2} \cos\theta_t}$$

divide with '$\sqrt{\varepsilon_2}$'

$$= \frac{\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} \cos\theta_i - \frac{\cancel{\sqrt{\varepsilon_2}}}{\cancel{\sqrt{\varepsilon_2}}} \cos\theta_t}{\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} \cos\theta_i + \frac{\cancel{\sqrt{\varepsilon_2}}}{\cancel{\sqrt{\varepsilon_2}}} \cos\theta_t}$$

$$= \frac{\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} \cos\theta_i - \cos\theta_t}{\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} \cos\theta_i + \cos\theta_t}$$

By using Snell's law.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos\theta = \sqrt{1-\sin^2\theta}.$$

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\sin\theta_t = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i$$

Squaring & adding (-1) on Both sides.

$$1 - \sin^2\theta_t = 1 - \frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i$$

$$1 - \sin^2\theta_t = \frac{\epsilon_2 - \epsilon_1\sin^2\theta_i}{\epsilon_2} \quad \text{——} (6)$$

$$= \frac{\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}}\cos\theta_i - \sqrt{1-\sin^2\theta_t}}{\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}}\cos\theta_i + \sqrt{1-\sin^2\theta_t}} \quad \longrightarrow (7)$$

$\rightarrow$ Substitute eqn (7) in eqn (6)

$$\frac{E_r}{E_i} = \frac{\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}}\cos\theta_i - \left[\sqrt{\frac{\varepsilon_2-\varepsilon_1\sin^2\theta_i}{\varepsilon_2}}\right]}{\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}}\cos\theta_i + \left[\sqrt{\frac{\epsilon_2-\epsilon_1\sin^2\theta_i}{\epsilon_2}}\right]}$$

Multiply with $\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}$ on both Numerator & Dinominator.

$$\frac{E_r}{E_i} = \frac{\left(\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\right)\cdot\left(\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}}\right)\cos\theta_i - \left[\sqrt{\frac{\left(\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\sin^2\theta_i\right)\varepsilon_1}{\varepsilon_2}}\right]\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}}{\left(\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\right)\left[\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}}\right]\cos\theta_i + \left[\sqrt{\frac{\left[\frac{\varepsilon_2}{\varepsilon_1}-\sin^2\theta_i\right]\varepsilon_1}{\varepsilon_2}}\right]\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}}$$

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$$\frac{E_r^{\perp}}{E_i} = \boxed{\frac{\cos\theta_i - \sqrt{\left[\frac{\varepsilon_2}{\varepsilon_1}\right] - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\left[\frac{\varepsilon_2}{\varepsilon_1}\right] - \sin^2\theta_i}}}$$

## Problems

① A perpendicularly polarised wave is incident at an angle of $\theta_i = 15°$. It is propagating from medium 1 to medium 2. Medium 1 is defined by $\varepsilon_{r_1} = 8.5, \mu_{r_1} = 1$, $\sigma_1 = 0$ and medium 2 is free space. If $E_i = 1.0\ mV/m$, determine $E_r, H_i, H_r$.

**Solution:-**

The intrinsic impedance of medium 1 is given by.

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \times \sqrt{\frac{\mu_{r_1}}{\varepsilon_{r_1}}} = 377 \times \sqrt{\frac{1}{8.5}} = \cancel{8}$$

$$\boxed{\eta_1 = 129\,\Omega}$$

As medium 2 is free space,

$$\eta_2 = 377\,\Omega$$

By Snell's law

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \sqrt{\frac{\varepsilon_0}{\varepsilon_0 . 8.5}}$$

$$\boxed{\theta_t = 49°}$$

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The reflection Co-efficient for electric field is

$$\frac{E_r}{E_i} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_i} = 0.490$$

$$E_r = 0.490 \times 1.0 \times 10^{-3} = 0.490 \, mV/m$$

$$E_r = 0.490 \, mV/m$$

Now. $\frac{E_i}{H_i} = \eta_1$

(or) $H_i = \frac{E_i}{\eta_1} = \frac{1.0 \times 10^{-3}}{129}$

$$H_i = 7.75 \mu A/m$$

and Similarly. $\frac{E_r}{H_r} = \eta_1$

$$H_r = \frac{E_r}{\eta_1} = 3.798 \mu \, A/m.$$

# UNIT-V

## Transmission Lines

A transmission line basically consists of two or more parallel conductors used to connect a source to a load. The source may be a generator, a transmitter, or an oscillator; the load may be a factory, an antenna, or an oscilloscope, respectively.
Transmission lines are commonly used in power distribution, communications, electrical laboratories and transmission lines such as the twisted-pair and coaxial cables are used in computer networks such as the ethernet and internet.
Transmission line problems are usually solved using EM field theory and electric circuit theory, the two major theories on which electrical engineering is based.

**Types of Transmission lines:**

a) Two-wire line
b) Coaxial line
c) Planar line
d) Wire above conducting plane
e) Micro strip line
f ) Waveguides
g ) Optical cable

**a) Two-wire line:** This transmission line consists of a pair of parallel conducting wires separated by a uniform distance .These are used in power systems or telephones lines.

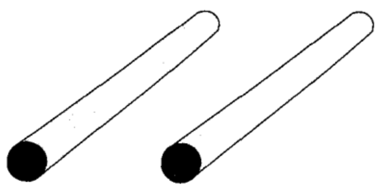


Fig. Two-wire line.

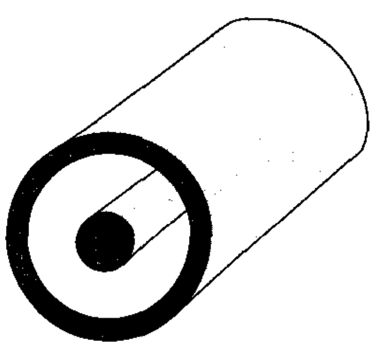
**Merits:**

1. The cost of two wire transmission line is very low as compared to other types of lines.
2. To design the open two line transmission line is quite simple and easy too.
3. Open two wire lines are capable of handling high power.

**Demerits:**

1. The external interference of the signal in open two wire lines is more.
2. Due to external interference the output at the load end of two wire transmission line will be noisy.
3. To use the two wire transmission lines in the twisty paths is quite difficult.
4. It cannot be used on very high frequencies because it will generate skin effect.

**b) Coaxial line:** The co-axial cable can be constructed by placing a solid conductor inside a hallow cylindrical coaxially and these two are separated by a dielectric. They are used as TV cables, telephone cables and power cables.



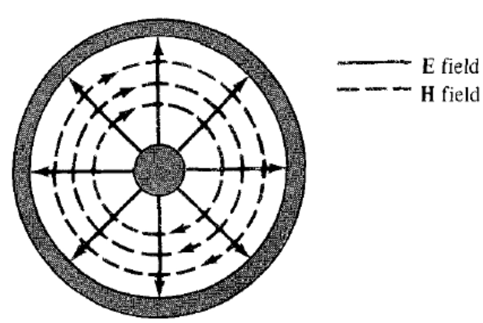



Fig.(a) Coaxial cable (b).E and H fields in coaxial cable

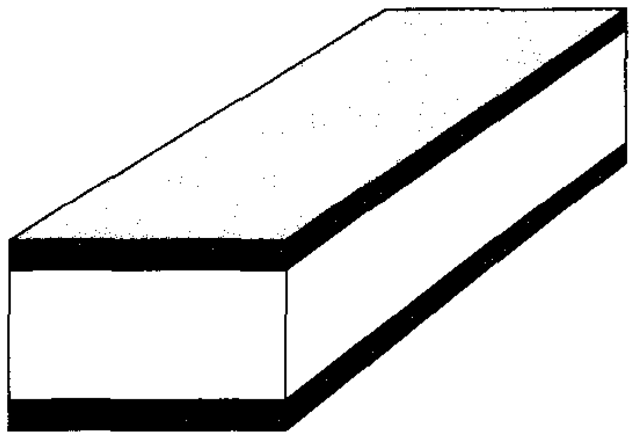
**Merits:**

1. As the outer conductor (braded wire) is grounded, therefore the possibility of external interference is minimized. The output of the load end will be less noised.
2. The coaxial cable is used for high frequencies transmission.
3. This type of transmission cables can be easily used if the path of energy from source to load is twisty or complicated.
4. Coaxial cable occupies less space as compared to two wire lines.
5. The conductor which carries the energy from source to load is protected from dust, rust etc. due to proper insulation.

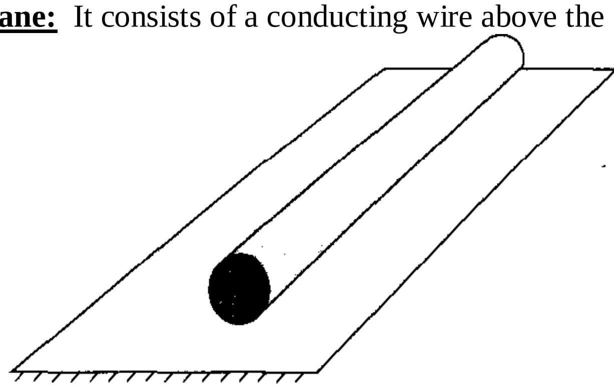
**Demerits:**

1. This type of transmission line is costly with respect to two wire lines.
2. Designing of coaxial cable is difficult as compared to two wire lines.
3. This type of transmission lines handles low power transmissions.

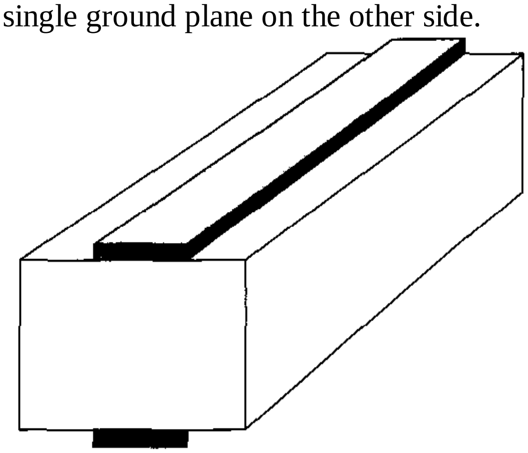
**(c) Planar line:** It consists of two parallel conductors separated by dielectric medium as shown in fig. It supports TE and TM waves.



**d) Wire above conducting plane:** It consists of a conducting wire above the ground plane as shown in fig.



**c) Micro strip line:** The Micro strip line is transmission line geometry with a single conductor trace on one side of a dielectric substrate and a single ground plane on the other side.



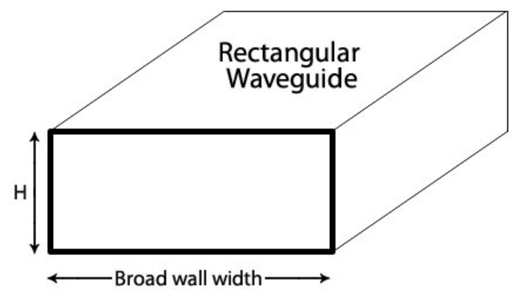
**Merits:**

1. Very high frequency.
2. Small size
3. Low weight.
4. Losses are minimum.
5. This type of transmission line is used for very high frequency.
6. Micro strip lines are used in integrated circuits where distance between load and source is very short.
7. As the path of energy is made of very good conductor like gold, therefore the losses of energy are minimum possible.
8. The weight of micro strip line is low.

**Demerits:**

1. The cost of micro strip is very high as compared to coaxial and two wire line.
2. The micro strip line cannot be used as a transmission line when the distance between source and load is long.
3. This type of transmission line cannot be used in twisty paths between source and load.

**d) Wave guides:** The wave guides are hallowed or dielectric filled conductor used to transmit the electromagnetic energy at micro wave frequency ranges.
The wave propagates in TE ,TM,TEM modes.



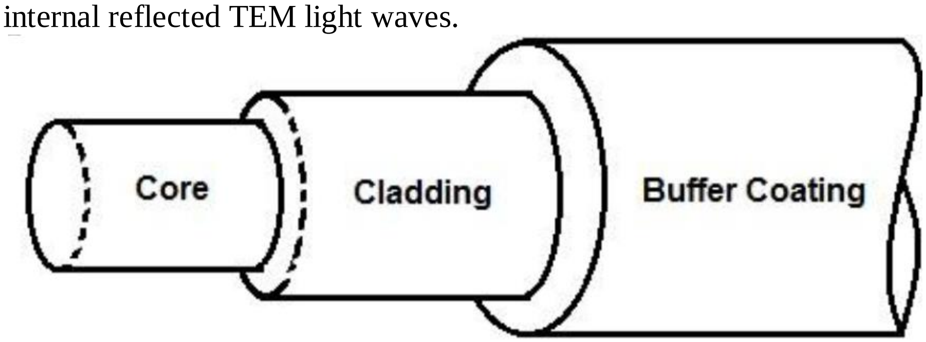

**Merits:**

1. The large surface area of waveguides greatly reduces copper ($I^2R$) losses.
2. Dielectric losses are also lower in wave guides than in two-wire and coaxial transmission line

**Demerits:**

1. Physical size is the primary lower-frequency limitation of waveguides. The width of a waveguide must be approximately a half wavelength at the frequency of the wave to be transported
2. Waveguides are difficult to install because of their rigid, hollow-pipe shape. Special couplings at the joints are required to assure proper operation.
3. The inside surfaces of waveguides are often plated with silver or gold to reduce skin effect losses. These requirements increase the costs and decrease the practicality of waveguide systems at any other than microwave frequencies.

**e) Optical fibers transmission line:** It consists of core and cladding. Information passes through the core in the form of totally internal reflected TEM light waves.




**Merits:**

1. The fiber optics offers the high bandwidth.
2. Fiber immune to electromagnetic interference, Fiber has a very low rate of bit error, Fiber-optic transmission is virtually noise free.

3. Fiber provides an extremely secure transmission medium.
4. When high freq signal are propagated through the optical fiber the loss is very low.
5. Because of very small size and light in weight and large flexibility, it is easy to install and compatibility with digital technology.
6. As optical fiber has no electrical conductivity, therefore grounding and protection are not necessary.
7. Lack of electrical signals in the fiber, so it cannot shock or other hazards. This makes optical fibers suitable for work in explosive atmospheres.

**Demerits:**

1. Installing fiber optic cabling is still relatively costly.
2. Equipment used in the fiber optics is expensive, specialized optical test equipment is needed in testing of optical fiber.
3. Fiber is a small and compact cable, and it is highly susceptible to becoming cut or damaged during installation or construction activities.
4. Damage to Fiber Optic Cables from birds, ants, Sharks etc
5. Even though the raw material for making optical fibers, sand, is cheap, optical fibers are still more expensive per meter than copper.
6. The glass can be affected by various chemicals including hydrogen gas (a problem in underwater cables).
7. Optical fiber cannot be joined together as a easily as copper cable and requires additional training of personnel and expensive precision splicing and measurement equipment.
8. As optical fibers have no electrical conductivity, therefore additional copper cable is not used with optical fiber to provide power supply to the repeaters.

**Transmission Line Parameters:**

The transmission line is described in terms of its line parameters, which are its resistance per unit length R, inductance per unit length L, conductance per unit length G, and capacitance per unit length C. These are also called primary parameters. These are independent of frequency. These parameters are not lumped but distributed that means these are uniformly distributed along the entire length of the line.

Resistance(R) : A series resistance is due to the internal resistance of the conductors of a transmission line. It depends on the conductivity and cross-sectional area of the conductors. But at high frequencies, it depends on skin depth. It is measured as loop resistance per unit length of the line. Its units are Ω/m.

Inductance (L):A series inductance is due to the magnetic flux produced around the conductors of a transmission line. The flux linkage per unit current gives the inductance of the line. It is measured as loop inductance per unit length of the line. Its units are H/m.

Capacitance(C): Two conductors of a transmission line separated by a dielectric form a capacitor. Thus a shunt capacitance is formed due to the electric field between the conductors. It is measured as shunt capacitance per unit length of the line. Its units are F/m.

Conductance (G): A shunt conductance is due to the leakage current between the conductors of a line since the dielectric medium between the conductors is not perfect. It is measured as shunt conductance per unit length of the line. Its units are ℧/m.

The series impedance Z and shunt admittance Y of the line per unit length can be expressed as

$$Z = R + j\omega L$$
$$Y = G + j\omega C$$

The equivalent circuit of the transmission line is shown in fig

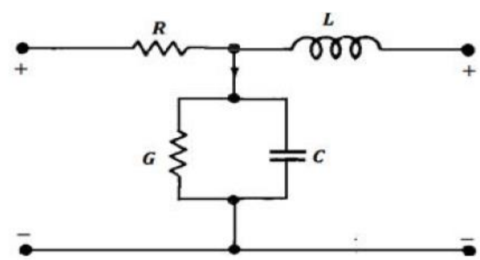



Fig. Equivalent circuit of transmission line

Transmission Line Equations: consider a transmission line with two parallel conductors. Let R, L, C and G be the primary parameters. Consider a point P on the line at a distance x from the source as shown in fig.

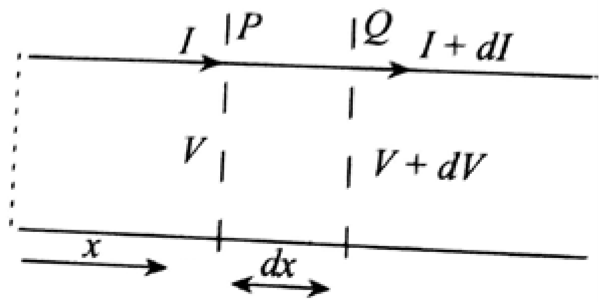



Fig. Voltages and currents on the transmission line

Let Q be the another point at distance dx from the P.

Let V and I be the voltage and current at point P respectively. Let V+dV and I+dI be the voltage and current at point Q respectively.

For a small length dx of the line, the series impedance is $(\mathrm{R}+\mathrm{j}\omega\mathrm{L})\mathrm{dx}$ and shunt admittance is $(\mathrm{G}+\mathrm{j}\omega\mathrm{C})\mathrm{dx}$ .

The potential difference between P and Q is

$$\mathrm{V}-(\mathrm{V}+\mathrm{dV})=\mathrm{I}(\mathrm{R}+\mathrm{j}\omega\mathrm{L})\mathrm{dx}$$

$$-\frac{\mathrm{dV}}{\mathrm{dx}}=(\mathrm{R}+\mathrm{j}\omega\mathrm{L})\mathrm{I} \quad ----(1)$$

The current difference between P and Q is

$$\mathrm{I}-(\mathrm{I}+\mathrm{dI})=\mathrm{V}(\mathrm{G}+\mathrm{j}\omega\mathrm{C})\mathrm{dx}$$

$$-\frac{\mathrm{dI}}{\mathrm{dx}}=(\mathrm{G}+\mathrm{j}\omega\mathrm{C})\mathrm{V} \quad ----(2)$$

Taking differentiation of eq.(1)

$$-\frac{\mathrm{d}^2\mathrm{V}}{\mathrm{dx}^2}=(\mathrm{R}+\mathrm{j}\omega\mathrm{L})\frac{\mathrm{dI}}{\mathrm{dx}} \quad -----(3)$$

Substitute the eq.(2) in eq.(3)

$$\frac{\mathrm{d}^2\mathrm{V}}{\mathrm{dx}^2}=(\mathrm{R}+\mathrm{j}\omega\mathrm{L})(\mathrm{G}+\mathrm{j}\omega\mathrm{C})\mathrm{V}$$

Similarly differentiate eq.(2) and substitute eq.(1) we get

$$\frac{\mathrm{d}^2\mathrm{I}}{\mathrm{dx}^2}=(\mathrm{R}+\mathrm{j}\omega\mathrm{L})(\mathrm{G}+\mathrm{j}\omega\mathrm{C})\mathrm{I}$$

The propagation constant

$$\gamma=\alpha+\mathrm{j}\beta=\sqrt{(\mathrm{R}+\mathrm{j}\omega\mathrm{L})(\mathrm{G}+\mathrm{j}\omega\mathrm{C})}$$

Where α=attenuation constant and β=phase shift constant

Therefore

$$\frac{\mathrm{d}^2\mathrm{V}}{\mathrm{dx}^2}=\gamma^2\mathrm{V} \quad ----(4)$$

$$\frac{\mathrm{d}^2\mathrm{I}}{\mathrm{dx}^2}=\gamma^2\mathrm{I} \quad ----(5)$$

The equations (4) and (5) are the second order differential equations in terms of voltage and current whose solutions are given by

$$\mathrm{V}=\mathrm{a}e^{\gamma\mathrm{x}}+\mathrm{b}e^{-\gamma\mathrm{x}} \quad ---(6)$$
$$\mathrm{I}=\mathrm{c}e^{\gamma\mathrm{x}}+\mathrm{d}e^{-\gamma\mathrm{x}} \quad ----(7)$$

Where a, b, c and d are the constants.

In terms of hyperbolic functions the above equations become

$$\mathrm{e}^{\gamma\mathrm{x}}=\cosh\gamma x+\sinh\gamma x$$
$$\mathrm{e}^{-\gamma\mathrm{x}}=\cosh\gamma x-\sinh\gamma x$$

$$V = A\cosh\gamma x + B\sinh\gamma x \quad --(8)$$
$$I = C\cosh\gamma x + D\sinh\gamma x \quad --(9)$$
Instead of four constants A,B,C and D the above equations can simplified to only two constants, by substituting the value of V from eq.(8) in eq.(1)
$$-\frac{d}{dx}(A\cosh\gamma x + B\sinh\gamma x) = (R + j\omega L)I$$
$$-(\gamma A\sinh\gamma x + B\gamma\cosh\gamma x) = (R + j\omega L)I$$
$$I = \frac{-\gamma}{(R + j\omega L)}(A\sinh\gamma x + B\cosh\gamma x)$$
$$I = -\sqrt{\frac{G + j\omega C}{R + j\omega L}}(A\sinh\gamma x + B\cosh\gamma x)$$
$$I = -\frac{1}{Z_0}(A\sinh\gamma x + B\cosh\gamma x)$$
Where
$$\text{Characteristic impedance } Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$
$$V = A\cosh\gamma x + B\sinh\gamma x$$
$$I = -\frac{1}{Z_0}(A\sinh\gamma x + B\cosh\gamma x)$$
The constants A and B can be obtained by using initial conditions.
Let $V_s$ and $I_s$ be the source voltage and current respectively. At source end, x=0, the voltage V= $V_s$ and current I= $I_s$
Then
$$V_s = A\cosh\gamma(0) + B\sinh\gamma(0)$$
Therefore $$V_s = A$$
$$I_s = -\frac{1}{Z_0}(A\sinh\gamma(0) + B\cosh\gamma(0))$$
Therefore $$B = -I_s Z_0$$
Substituting the constants A and B in above equations
$$V = V_s\cosh\gamma x - I_s Z_0\sinh\gamma x$$
$$I = I_s\cosh\gamma x - \frac{V_s}{Z_0}\sinh\gamma x$$
These are called transmission line equations. They give voltage and current at a point of distance x from the sending end in terms of source voltage and current.
**Infinite line:** A line is said to be infinite if all the input signals are consumed by the line and there is no reflected signal.
$$I = ce^{\gamma x} + de^{-\gamma x} \quad ---(1)$$
When x=0, the current at the sending end is $I=I_{Si}$
Substitute x=0 in Eq.(1)
$$I_{Si} = c + d$$
When x=∞, the current at receiving end is I=0
Substituting x=∞ in Eq.(1)
$$0 = ce^{\gamma\infty} + de^{-\gamma\infty}$$
$$0 = c\infty$$
$$c=0$$
Therefore
$$d = I_{Si}$$

The current at any point on the infinite line is given by

$$I = I_{Si}e^{-\gamma x}$$

Similarly the voltage at any point the infinite line is given by

$$V = V_{Si}e^{-\gamma x}$$

**Secondary Constants:** The propagation constant γ and the characteristic impedance $Z_0$ are referred as secondary constants.

**1.Propagation Constant(γ) :** The propagation constant γ is a complex quantity.

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

Where α is the attenuation constant (in nepers per meter or decibels per meter), and β is the phase constant (in radians per meter).

**2.Characteristic impedance($Z_0$) :** The characteristic impedance $Z_0$ of the line is the ratio of positively traveling voltage wave to current wave at any point on the line. The characteristic impedance is also defined as the input impedance of an infinite line.

We know that

$$-\frac{dV}{dx} = (R + j\omega L)I$$

For infinite line

$$V = V_S e^{-\gamma x}$$
$$I = I_S e^{-\gamma x}$$
$$-\frac{d}{dx}(V_S e^{-\gamma x}) = (R + j\omega L)I_S e^{-\gamma x}$$
$$\gamma V_S e^{-\gamma x} = (R + j\omega L)I_S e^{-\gamma x}$$
$$\gamma V_S = (R + j\omega L)I_S$$
$$\frac{V_S}{I_S} = \frac{R + j\omega L}{\gamma}$$

The input impedance of infinite line is given by

$$Z_0 = \frac{V_S}{I_S}$$

$$\text{Characteristic impedance } Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

The propagation constant γ and the characteristic impedance Zo are important properties of the line because they both depend on the line parameters *R, L, G,* and *C* and the frequency of operation.

**Attenuation and phase constants:**

The propagation constant γ is a complex quantity.

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

Squaring the magnitude of γ we get

$$\alpha^2 + \beta^2 = \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} \quad ---(1)$$

Squaring γ on the both sides

$$(\alpha + j\beta)^2 = (R + j\omega L)(G + j\omega C)$$
$$\alpha^2 - \beta^2 + j2\alpha\beta = RG - \omega^2 LC + j(\omega LG + \omega RC)$$

Equating the real parts we get

$$\alpha^2 - \beta^2 = RG - \omega^2 LC \quad ----(2)$$

Adding Eq.(1) and Eq.(2)

$$2\alpha^2 = \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} + RG - \omega^2 LC$$

The real part of γ is called attenuation constant. It determines the reduction or attenuation in voltage and current along the line. Its unit is neper per km. 1neper=8.686dB.

$$\alpha = \sqrt{\frac{1}{2}\left[\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} + (RG - \omega^2 LC)\right]} \quad NP/km$$

Subtracting Eq.(2) from Eq.(1)

$$\beta = \sqrt{\frac{1}{2}\left[\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} - (RG - \omega^2 LC)\right]} \quad rad/km$$

The imaginary part of γ is called the phase constant. It determines the variation in phase position of voltage and current along the line. Its unit is radians per km.

**Lossless Line *(R = 0 = G):*** A transmission line is said to be loss less if the conductors of the line are perfect($\sigma_c \approx \infty$) and the dielectric medium separating them is lossless ($\sigma_d \approx 0$).
Condition for losses less line is, R = 0 = *G.*
The high frequency lines are termed as lossless lines because the ω=2πf in the series impedance(R+jωL) and shunt admittance (G+jωC) becomes very large due to the high frequency.
Therefore the real part of the series impedance can be neglected.
The propagation constant and characteristic impedance for losses line are given by

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Substitute R = 0 = *G*

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}$$

Therefore

$$\alpha = 0 \text{ and } \beta = \omega\sqrt{LC}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

The phase velocity is

$$v_0 = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

**Distortion less Line*(R/L = G/C):*** A transmission line is said to be distortion less if the attenuation constant 'α' is frequency independent while the phase constant 'β' is linearly dependent on frequency.
The condition for the distortion less line is *R/L = G/C***.**
The propagation constant for distortion less line is given by

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{RG\left(1 + \frac{j\omega L}{R}\right)\left(1 + \frac{j\omega C}{G}\right)}$$

For the distortion less line

$$\frac{R}{L} = \frac{G}{C}$$

Therefore

$$\gamma = \sqrt{RG}\left(1 + \frac{j\omega C}{G}\right)$$

$$\alpha + j\beta = \sqrt{RG} + j\omega\sqrt{LC}$$

$$\alpha = \sqrt{RG} \quad \text{and} \quad \beta = \omega\sqrt{LC}$$

The characteristic impedance for distortion less line is given by

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{R(1 + j\omega L/R)}{G(1 + j\omega C/G)}} = \sqrt{\frac{R}{G}}$$

$$R_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}} \qquad \text{and } X_0 = 0$$

The phase velocity is

$$v_0 = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

**Note:**

1. The phase velocity is independent of frequency.
2. The phase velocity and characteristic impedance are same for both lossless and distortionless lines.
3. A lossless line is also a distortionless line, but a distortionless line is not necessarily lossless. Although lossless lines are desirable in power transmission, telephone lines are required to be distortionless.

**Input Impedance:**

A transmission line terminated with any load impedance $Z_R$ at x=l is shown in fig.

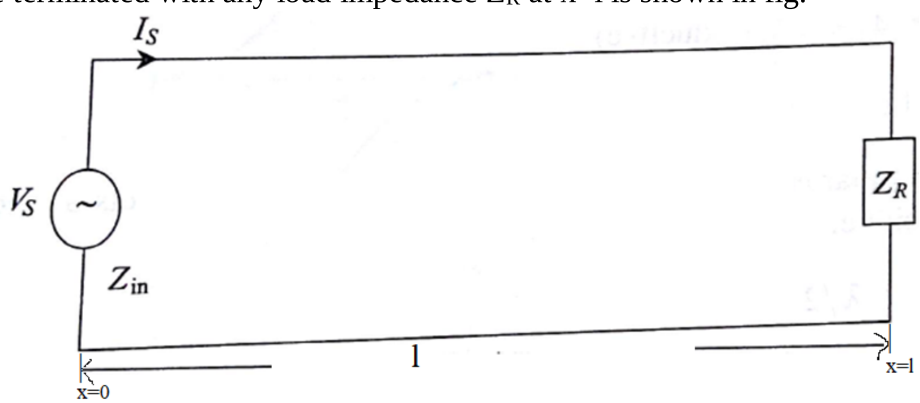



Fig . Input impedance of transmission line

The V and I equations at a point of distance x from the sending end in terms of source voltage and current are given by.

$$V = V_s \cosh \gamma x - I_s Z_0 \sinh \gamma x$$

$$I = I_s \cosh \gamma x - \frac{V_s}{Z_0} \sinh \gamma x$$

The voltage and currents at load end (x=l) are given by

$$V_R = V_s \cosh \gamma l - I_s Z_0 \sinh \gamma l$$

$$I_R = I_s \cosh \gamma l - \frac{V_s}{Z_0} \sinh \gamma l$$

The load impedance $Z_R$ is given by

$$Z_R = \frac{V_R}{I_R}$$

$$Z_R = \frac{V_s \cosh \gamma l - I_s Z_0 \sinh \gamma l}{I_s \cosh \gamma l - \frac{V_s}{Z_0} \sinh \gamma l}$$

$$Z_R \left( I_s \cosh \gamma l - \frac{V_s}{Z_0} \sinh \gamma l \right) = V_s \cosh \gamma l - I_s Z_0 \sinh \gamma l$$

$$Z_R I_s \cosh \gamma l - Z_R \frac{V_s}{Z_0} \sinh \gamma l = V_s \cosh \gamma l - I_s Z_0 \sinh \gamma l$$

$$\mathrm{Z_R I_s} \cosh \gamma l + \mathrm{I_s Z_0} \sinh \gamma l = \mathrm{V_s} \cosh \gamma l + \mathrm{Z_R} \frac{\mathrm{V_s}}{\mathrm{Z_0}} \sinh \gamma l$$

$$\mathrm{I_s}(\mathrm{Z_R} \cosh \gamma l + \mathrm{Z_0} \sinh \gamma l) = \mathrm{V_s} \left( \cosh \gamma l + \frac{\mathrm{Z_R}}{\mathrm{Z_0}} \sinh \gamma l \right)$$

$$\frac{\mathrm{V_s}}{\mathrm{I_s}} = \frac{\mathrm{Z_R} \cosh \gamma l + \mathrm{Z_0} \sinh \gamma l}{\cosh \gamma l + \frac{\mathrm{Z_R}}{\mathrm{Z_0}} \sinh \gamma l}$$

The input impedance of the line is given by

$$Z_{in} = \frac{\mathrm{V_s}}{\mathrm{I_s}}$$

$$Z_{in} = \mathrm{Z_0} \left[ \frac{\mathrm{Z_R} \cosh \gamma l + \mathrm{Z_0} \sinh \gamma l}{\mathrm{Z_0} \cosh \gamma l + \mathrm{Z_R} \sinh \gamma l} \right]$$

$$Z_{in} = \mathrm{Z_0} \left[ \frac{\mathrm{Z_R} + \mathrm{Z_0} \tanh \gamma l}{\mathrm{Z_0} + \mathrm{Z_R} \tanh \gamma l} \right]$$

For lossless line $\gamma = j\beta$

$$Z_{in} = \mathrm{Z_0} \left[ \frac{\mathrm{Z_R} + \mathrm{jZ_0} \tan \beta l}{\mathrm{Z_0} + \mathrm{jZ_R} \tan \beta l} \right]$$

**Voltage and Current at any point on the transmission line:**

Consider the transmission line of length $l$ terminating with an impedance $Z_R$. Let $V_R$ and $I_R$ be the voltage and current at the load $Z_R$.

The voltage and current at a point of distance x from the sending end in terms of source voltage and current are given by

$$\mathrm{V} = \mathrm{A} \cosh \gamma \mathrm{x} + \mathrm{B} \sinh \gamma x \quad --- (1)$$

$$\mathrm{I} = -\frac{1}{\mathrm{Z_0}} (\mathrm{A} \sinh \gamma \mathrm{x} + \mathrm{B} \cosh \gamma x) \; --- (2)$$

Where A and B are constants.

At $x=l$    $V = V_R$ and $I = I_R$

Substituting these values in eq.(1) and eq.(2)

$$\mathrm{V_R} = \mathrm{A} \cosh \gamma l + \mathrm{B} \sinh \gamma l \; ----(3)$$

$$\mathrm{I_R} = -\frac{1}{\mathrm{Z_0}} (\mathrm{A} \sinh \gamma l + \mathrm{B} \cosh \gamma l) \; --- (4)$$

Multiplying Eq.(3) with $\cosh\gamma l$ and Eq.(4) with $\sinh\gamma l$

$$\mathrm{V_R} \cosh \gamma l = \mathrm{A} \cosh^2 \gamma l + \mathrm{B} \sinh \gamma l \cosh \gamma l \; ----(5)$$

$$\mathrm{Z_0 I_R} \sinh \gamma l = -\mathrm{A} \sinh^2 \gamma l - \mathrm{B} \cosh \gamma l \sinh \gamma l \quad --- (6)$$

Adding Eq.(5) and Eq.(6)

$$\mathrm{V_R} \cosh \gamma l + \mathrm{Z_0 I_R} \sinh \gamma l = \mathrm{A}(\cosh^2 \gamma l - \sinh^2 \gamma l)$$

Therefore

$$\mathrm{A} = \mathrm{V_R} \cosh \gamma l + \mathrm{Z_0 I_R} \sinh \gamma l$$

Multiplying Eq.(3) with $\sinh\gamma l$ and Eq.(4) with $\cosh\gamma l$

$$\mathrm{V_R} \sinh \gamma l = \mathrm{A} \cosh \gamma l \sinh \gamma l + \mathrm{B} \sinh^2 \gamma l \quad ----(7)$$

$$\mathrm{Z_0 I_R} \cosh \gamma l = -\mathrm{A} \sinh \gamma l \cosh \gamma l - \mathrm{B} \cosh^2 \gamma l \quad --- (8)$$

Adding Eq.(7) and Eq.(8)

$$\mathrm{V_R} \sinh \gamma l + \mathrm{Z_0 I_R} \cosh \gamma l = -(\cosh^2 \gamma l - \sinh^2 \gamma l)$$

$$\mathrm{B} = -(\mathrm{V_R} \sinh \gamma l + \mathrm{Z_0 I_R} \cosh \gamma l)$$

Substituting the values of A and B in Eq.(1)

$$\mathrm{V} = (\mathrm{V_R} \cosh \gamma l + \mathrm{Z_0 I_R} \sinh \gamma l) \cosh \gamma \mathrm{x} - (\mathrm{V_R} \sinh \gamma l + \mathrm{Z_0 I_R} \cosh \gamma l) \sinh \gamma x$$

$$\mathrm{V} = (\mathrm{V_R} \cosh \gamma l \cosh \gamma \mathrm{x} + \mathrm{Z_0 I_R} \sinh \gamma l \cosh \gamma \mathrm{x}) - (\mathrm{V_R} \sinh \gamma l \sinh \gamma x + \mathrm{Z_0 I_R} \cosh \gamma l \sinh \gamma x)$$

$$V = V_R(\cosh \gamma l \cosh \gamma x - \sinh \gamma l \sinh \gamma x) + Z_0 I_R(\sinh \gamma l \cosh \gamma x - \cosh \gamma l \sinh \gamma x)$$
$$V = V_R \cosh \gamma(l - x) + Z_0 I_R \sinh \gamma(l - x)$$
Similarly Substituting the values of A and B in Eq.(2)
$$I = I_R \cosh \gamma(l - x) + + \frac{V_R}{Z_0} \sinh \gamma(l - x)$$
If y=*l*-x then
$$V = V_R \cosh \gamma y + Z_0 I_R \sinh \gamma y$$
$$I = I_R \cosh \gamma y + + \frac{V_R}{Z_0} \sinh \gamma y$$
These are the voltage and current equations at a point of distance y from the load end in terms of terminal voltage and current.

**Transmission line terminated with characteristic impedance:**

Consider a line of length *l* which is terminated with characteristic impedance $Z_0$ as shown in fig.

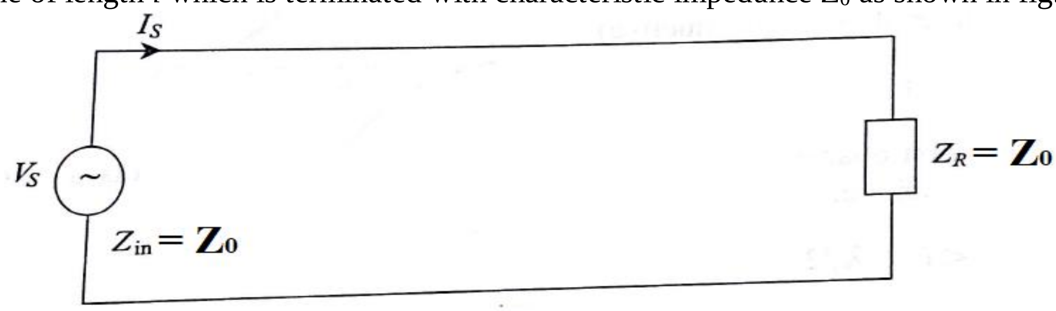



Fig. Transmission line terminated with $Z_0$

The input impedance of the line is given by
$$Z_{in} = Z_0 \left[\frac{Z_R + Z_0 \tanh \gamma l}{Z_0 + Z_R \tanh \gamma l}\right]$$
Here $Z_R = Z_0$
$$Z_{in} = Z_0 \left[\frac{Z_0 + Z_0 \tanh \gamma l}{Z_0 + Z_0 \tanh \gamma l}\right]$$
$$Z_{in} = Z_0$$
Hence the input impedance of a finite line terminated with characteristic impedance $Z_0$ is equal to characteristic impedance $Z_0$.

Therefore a finite line terminated with characteristic impedance $Z_0$ is equal to infinite line.

A line terminated with characteristic impedance $Z_0$ is called matched line.

**Standing Waves In Transmission Lines:** Signal energy is transmitted through a transmission line from the source to the load in the form of voltage and current waves. When the terminated load impedance is different from the characteristic impedance of the line, then the some part of the transmitted signal is returns back and there exists a reflected wave .These incident and reflected travelling waves create a standing waves.

The voltage and current equations along the transmission line equation are given by
$$V = V_i e^{-\gamma x} + V_r e^{\gamma x}$$
$$I = I_i e^{-\gamma x} + I_r e^{\gamma x}$$

In the above equations 1st term is called incident wave and 2nd term is called reflected wave.

**Reflection Coefficient:**

The reflection coefficient is defined as the ratio of reflected voltage to incident voltage. The reflection coefficient is also defined as the ratio of reflected current to incident current. It is denoted by K.

Let $V_i$ and $V_r$ are the incident and reflected voltages respectively.
$$K = \frac{V_r}{V_i}$$

Let $I_i$ and $I_r$ are the incident and reflected currents respectively.

$$K = -\frac{I_r}{I_i}$$

The voltage at the load $Z_R$ is given by

$$V_R = V_i + V_r \quad ---(1)$$
$$I_R = I_i + I_r \quad ---(2)$$

But characteristic impedance $Z_0$ is $\quad Z_0 = \frac{V_i}{I_i} = -\frac{V_r}{I_r} \quad ----(3)$

Substitute the Eq.(3) in Eq.(2), we get

$$I_R = \frac{V_i}{Z_0} - \frac{V_r}{Z_0}$$
$$Z_0 I_R = V_i - V_r \quad ---(4)$$

Divide the Eq.(1) by Eq.(4), we get

$$\frac{V_R}{Z_0 I_R} = \frac{V_i + V_r}{V_i - V_r}$$

We know that

$$Z_R = \frac{V_R}{I_R}$$
$$\frac{Z_R}{Z_0} = \frac{1 + \frac{V_r}{V_i}}{1 - \frac{V_r}{V_i}}$$

We know that the reflection coefficient K is

$$K = \frac{V_r}{V_i}$$

$$\frac{Z_R}{Z_0} = \frac{1 + K}{1 - K}$$

$$Z_R(1 - K) = Z_0(1 + K)$$
$$(Z_R - Z_R K) = (Z_0 + Z_0 K)$$
$$Z_R - Z_0 = Z_R K + Z_0 K$$
$$K(Z_R + Z_0) = Z_R - Z_0$$

The reflection coefficient K is

$$\boxed{K = \frac{Z_R - Z_0}{Z_R + Z_0}}$$

The range of reflection coefficient K is $\quad -1 \leq |K| \leq 1$

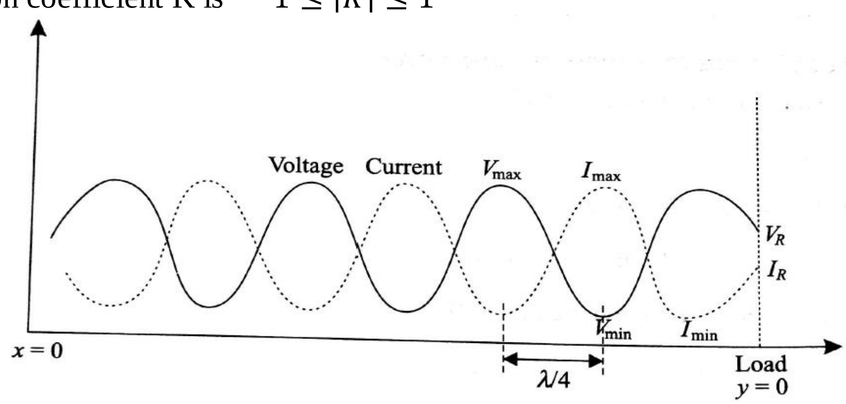



Fig. Standing wave pattern on transmission line

**Voltage standing wave ratio (VSWR):** The VSWR is defined as the ratio of the maximum voltage to the minimum voltage on the line having standing waves.

$$VSWR = S = \frac{V_{max}}{V_{min}} = \frac{|V_i| + |V_r|}{|V_i| - |V_r|} = \frac{1+|K|}{1-|K|}$$

$$\boxed{S = \frac{1+|K|}{1-|K|}}$$

Similarly the current standing wave ratio can be defined as the ratio between maximum current to the minimum current.

The range of VSWR is $1 \leq S \leq \infty$

$$(Z_{in})_{max} = \frac{V_{max}}{I_{min}} = SZ_o$$

$$(Z_{in})_{min} = \frac{V_{min}}{I_{max}} = \frac{Z_o}{S}$$

**Short circuited line ($Z_R$ = 0):** If transmission line is terminated with short circuit, then the line is called short circuited line.

The input impedance of the short circuited line can be obtained by substituting the $Z_R = 0$ in the input impedance of the transmission line.

$$Z_{in} = Z_0 \left[\frac{Z_R + Z_0 \tanh \gamma l}{Z_0 + Z_R \tanh \gamma l}\right]$$

If $Z_R$=0 then

$$Z_{sc} = Z_0 \tanh \gamma l$$

The input impedance of the short circuited lossless transmission line is given by

$$Z_{sc} = jZ_0 \tan \beta l$$

The reflection coefficient of the short circuited transmission line is K= -1.

The VSWR of the short circuited transmission line is S=∞.

The variation of input impedance of SC line is shown in fig.

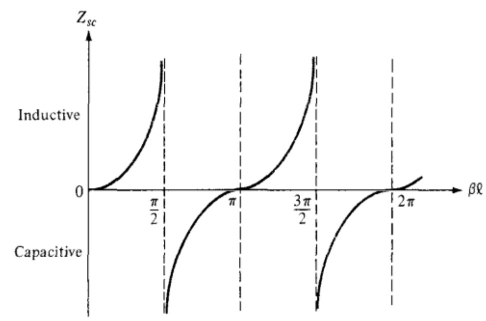



Fig . Input impedance of SC

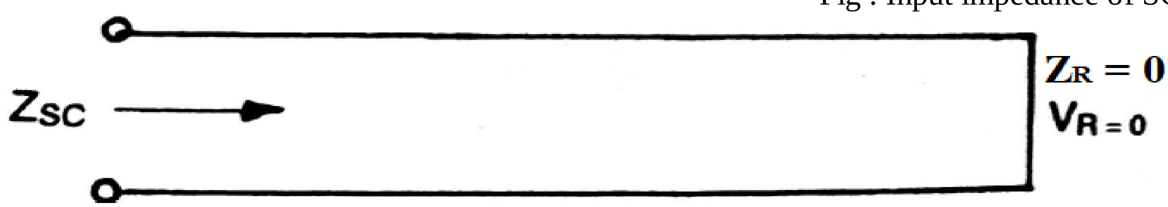



Fig. Short circuit transmission line

**Open circuited line ($Z_R$ = ∞):** If transmission line is terminated with open circuit, then the line is called open circuited line.

The input impedance of the open circuited line can be obtained by substituting the $Z_R$ =∞ in the input impedance of the transmission line.

$$Z_{in} = Z_0 \left[\frac{Z_R + Z_0 \tanh \gamma l}{Z_0 + Z_R \tanh \gamma l}\right]$$

If $Z_R$=∞ then

$$Z_{oc} = Z_0 \coth \gamma l$$

The input impedance of the open circuited line lossless transmission line is given by

$$Z_{oc} = -jZ_0 \cot \beta l$$

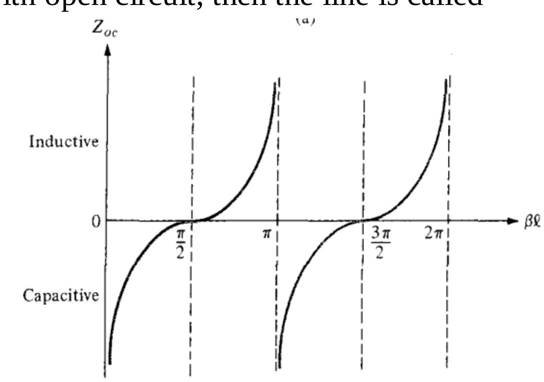



Fig. Input impedance of OC

The reflection coefficient of the open circuited line transmission line is K= 1. The VSWR of the open circuited line transmission line is S=∞. The variation of input impedance of OC line is shown in fig.

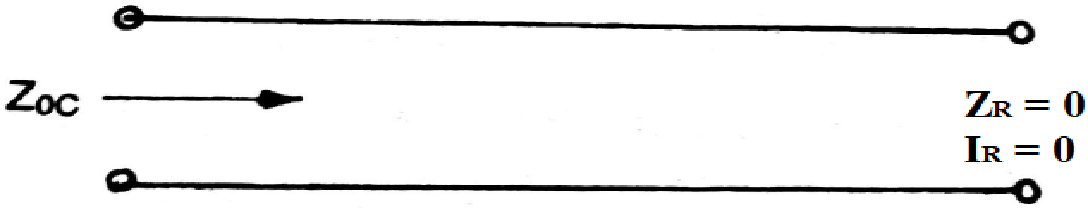



Fig. Open circuit transmission line

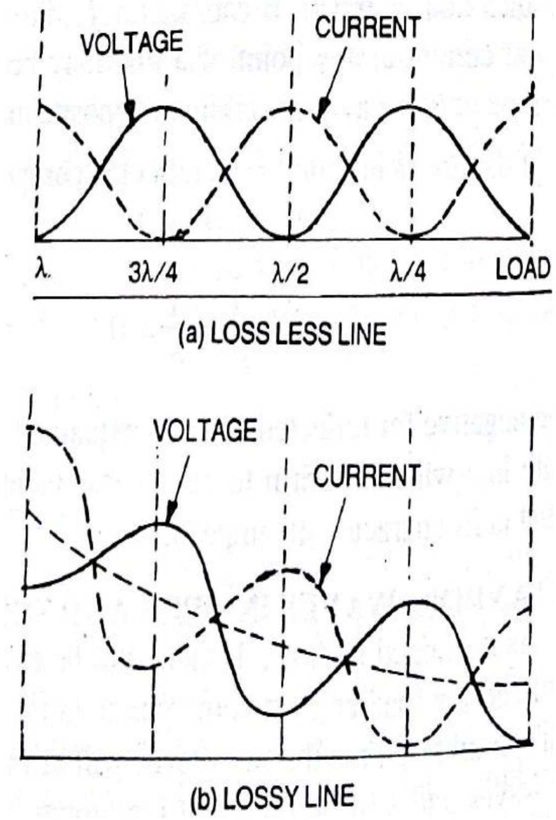



Fig. Voltage and current distribution on SC TL

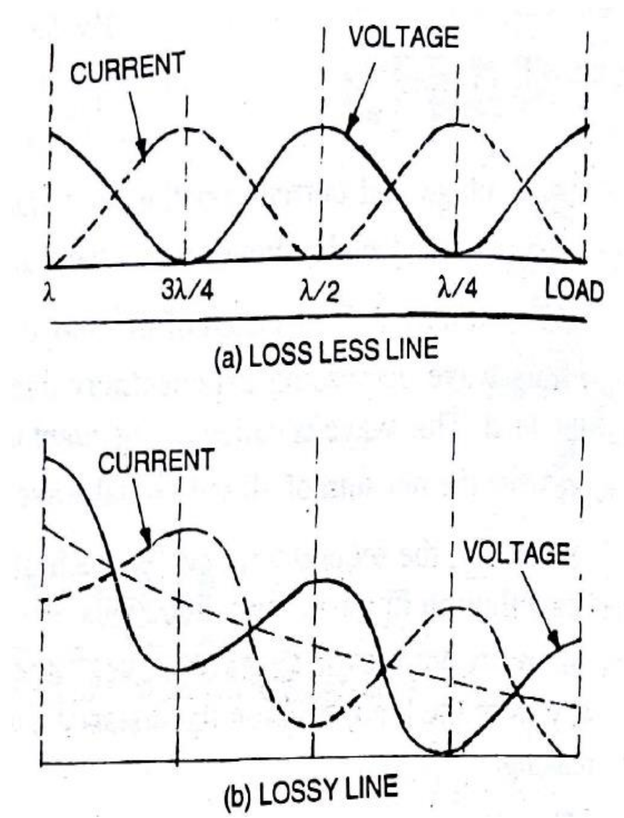



Fig. The variation of input impedance of OC line

**<u>λ/8- line:</u>** If length of the line is λ/8 then it is called λ/8 line or eight-wave line.

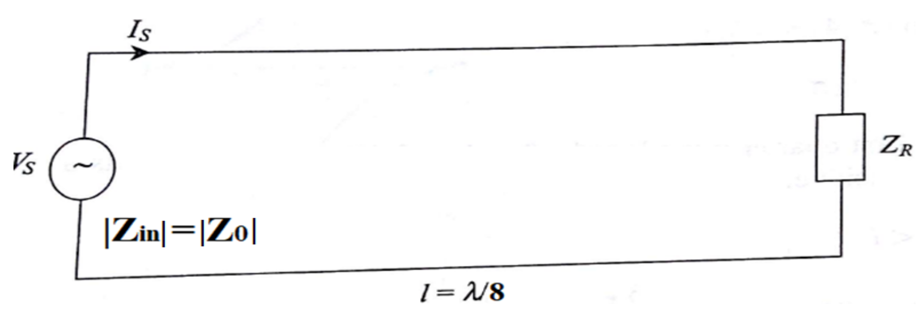



Fig. λ/8 line

The input impedance of the lossless transmission line is given by

$$Z_{in} = Z_0\left[\frac{Z_R + jZ_0\tan\beta l}{Z_0 + jZ_R\tan\beta l}\right]$$

$$\beta l = \frac{2\pi}{\lambda}\frac{\lambda}{8} = \frac{\pi}{4}$$

$$Z_{in} = Z_0\left[\frac{Z_R + jZ_0\tan\frac{\pi}{4}}{Z_0 + jZ_R\tan\frac{\pi}{4}}\right]$$

$$Z_{in} = Z_0 \left[\frac{Z_R + jZ_0}{Z_0 + jZ_R}\right]$$
$$|Z_{in}| = |Z_0|$$
The λ/8 line is used to transform any load impedance $Z_R$ to input impedance $Z_{in}$ whose magnitude is equal to magnitude of $Z_0$.

**<u>Quarter-wave line:</u>** If length of the line is λ/4 then it is called λ/4 line or quarter-wave line.

The input impedance of the lossless transmission line is given by
$$Z_{in} = Z_0 \left[\frac{Z_R + jZ_0 \tan \beta l}{Z_0 + jZ_R \tan \beta l}\right]$$
$$\beta l = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$$
$$Z_{in} = Z_0 \left[\frac{Z_R + jZ_0 \tan \frac{\pi}{2}}{Z_0 + jZ_R \tan \frac{\pi}{2}}\right]$$
$$Z_{in} = Z_0 \left[\frac{\frac{Z_R}{\tan \frac{\pi}{2}} + jZ_0}{\frac{Z_0}{\tan \frac{\pi}{2}} + jZ_R}\right]$$
$$Z_{in} = \frac{{Z_0}^2}{Z_R}$$

The λ/4 line can transform low impedance into high impedance and vice versa, thus it can be considered as an impedance inverter.

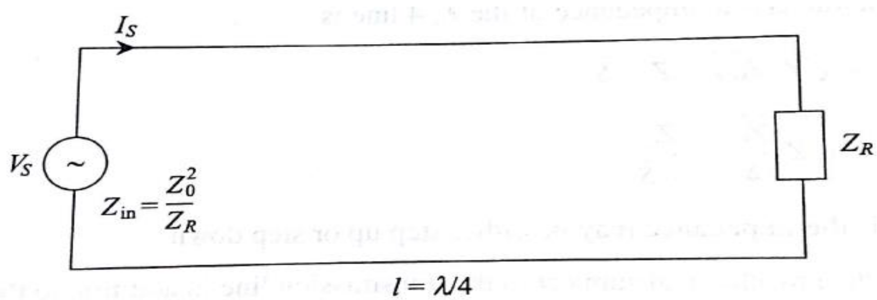


Fig. Quarter-wave line

The quarter wave line may be used as an impedance transformer for matching of load impedance $Z_R$ with input impedance $Z_{in}$. For impedance matching $Z_R$ and $Z_{in}$ , the line characteristic impedance $Z_o$ may be selected as $Z_o = \sqrt{Z_R Z_{in}}$

**<u>λ/2- line(half-wave line):</u>** If length of the line is λ/2 then it is called λ/2 line or half-wave line.

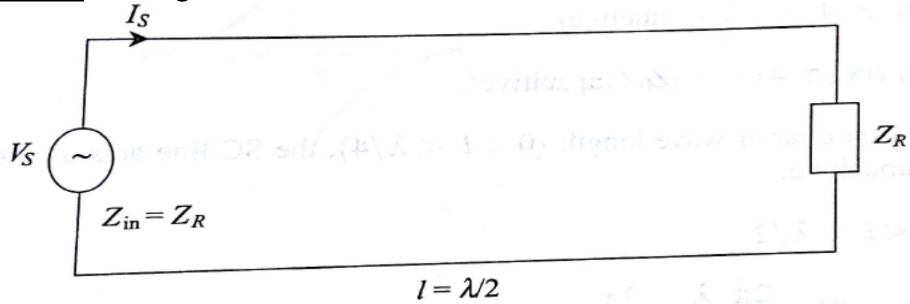


Fig. Half-wave line.

The input impedance of the lossless transmission line is given by
$$Z_{in} = Z_0 \left[\frac{Z_R + jZ_0 \tan \beta l}{Z_0 + jZ_R \tan \beta l}\right]$$

$$\beta l = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi$$

$$Z_{in} = Z_0\left[\frac{Z_R + jZ_0 \tan\pi}{Z_0 + jZ_R \tan\pi}\right]$$

$$Z_{in} = Z_R$$

The input impedance $Z_{in}$ of the λ/2 line is equal to load impedance $Z_R$.

**Transmission lines of various lengths can be used as circuit elements:** Different lengths of transmission lines can be used as circuit elements as discussed below.

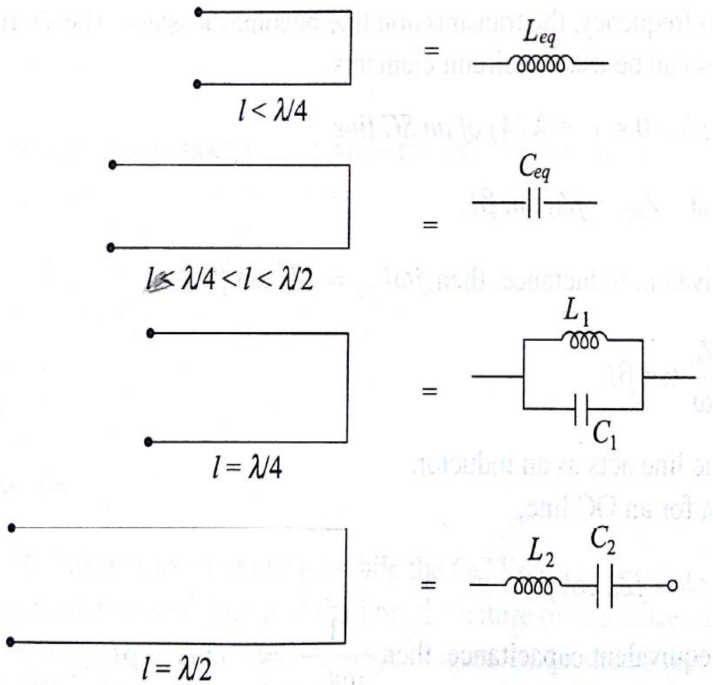



Fig. SC lines at different lengths with equivalent circuits

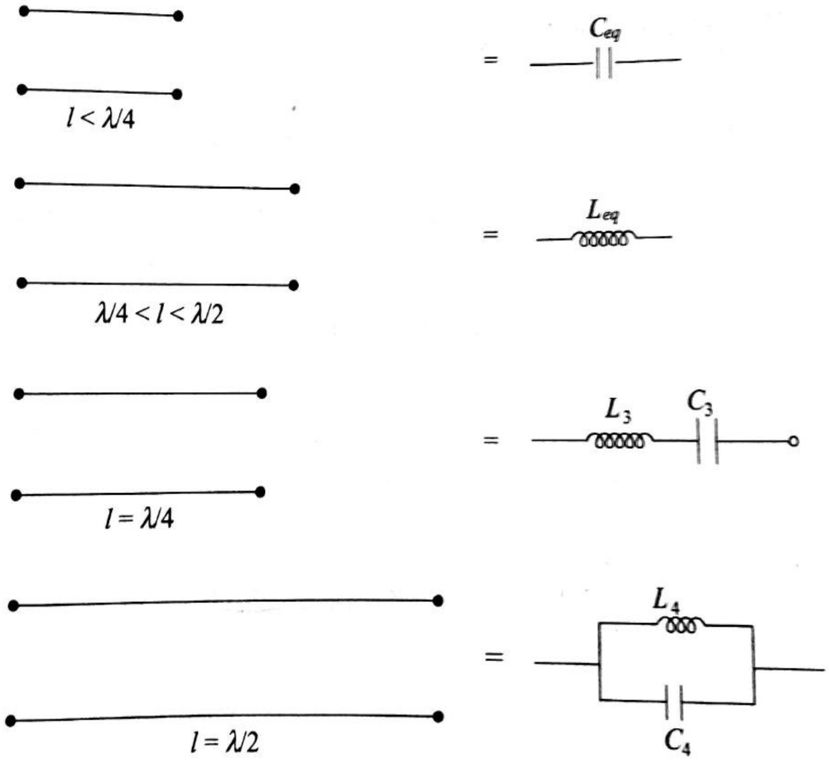



Fig. OC lines at different lengths with equivalent circuits

The input impedance of the short circuited lossless transmission line is given by

$$Z_{sc} = j\mathrm{Z}_0 \tan \beta l \quad --- (1)$$

The input impedance of the open circuited lossless transmission line is given by

$$Z_{oc} = -j\,\mathrm{Z}_0 \cot \beta l \quad --- (2)$$

The Eq.s(1) and (2) shows that input impedance of an open and short circuited lossless line is a pure reactance. Desired value of the reactance is obtained by varying the electrical length βl of the stubs.
If length of the short circuited line is less than λ/4, It will act as inductance. If length of the short circuited line is greater than λ/4 and less then λ/2, It will act as capacitance. If length of the short circuited line is equal to λ/4, It will act as parallel resonance circuit with high impedance. If length of the short circuited line is equal to λ/2, It will act as series resonance circuit with low impedance.

If length of the open circuited line is less than λ/4, It will act as capacitance. If length of the open circuited line is greater than λ/4 and less then λ/2, It will act as inductance. If length of the open circuited line is equal to λ/4, It will act as series resonance circuit with low impedance. If length of the open circuited line is equal to λ/2, It will act as parallel resonance circuit with high impedance.

**The Smith Chart:**

The Smith chart is the most commonly used graphical techniques in solving transmission problems in simple way. It is basically a graphical indication of the impedance and VSWR of a transmission line as one moves along the line.
The construction of the chart is based on the reflection coefficient

$$K = \frac{Z_R - Z_0}{Z_R + Z_0}$$

Instead of having separate Smith charts for transmission lines with different characteristic impedances such as Zo = 60,100, and 120. A normalized chart can be used for any line in which all impedances are normalized with respect to the characteristic impedance Zo of the particular line under consideration.

$$K = \frac{\frac{Z_R}{Z_0} - 1}{\frac{Z_R}{Z_0} + 1} = \frac{z_r - 1}{z_r + 1}$$

Where $z_r$ is the normalized impedance

$$z_r = \frac{1 + K}{1 - K}$$

Since $z_r$ and K both are complex quantities, we have

$$R + jX = \frac{1 + K_r + jK_x}{1 - (K_r + jK_x)}$$

$$R + jX = \frac{(1 + K_r) + jK_x}{(1 - K_r) - jK_x}$$

Rationalizing on the right hand side, we get

$$R + jX = \frac{(1 + K_r) + jK_x}{(1 - K_r) - jK_x} \times \frac{(1 - K_r) + jK_x}{(1 - K_r) + jK_x}$$

$$R + jX = \frac{1 - K_r^2 - K_x^2 + 2jK_x}{(1 - K_r)^2 + K_x^2}$$

Equating real and imaginary parts on both sides, we get

$$R = \frac{1 - K_r^2 - K_x^2}{(1 - K_r)^2 + K_x^2} \quad ----(1)$$

$$X = \frac{2K_x}{(1 - K_r)^2 + K_x^2} \quad --- (2)$$

Equations (1) and (2) will yield two set of orthogonal circles when solved separately. Eq.(1) will results in family of circle called R-circle while Eq.(2) will results in family of circle called X-circle.

(i) The constant R-circle:

$$R\{1+K_r^2-2K_r+K_x^2\}=1-K_r^2-K_x^2$$
$$R+RK_r^2-2RK_r+RK_x^2=1-K_r^2-K_x^2$$
$$K_r^2(R+1)+K_x^2(R+1)-2RK_r=1-R$$
$$K_r^2+K_x^2-\frac{2R}{1+R}K_r=\frac{1-R}{1+R}$$

Adding $\frac{R^2}{(1+R)^2}$ on both sides, we get

$$K_r^2-\frac{2R}{1+R}K_r+\frac{R^2}{(1+R)^2}+K_x^2=\frac{1-R}{1+R}+\frac{R^2}{(1+R)^2}$$
$$\left(K_r-\frac{R}{1+R}\right)^2+K_x^2=\left(\frac{1}{1+R}\right)^2$$

This equation represents a family of circles on the reflection co-efficient plane. These circles are called constant –R circles with center $\left(\frac{R}{1+R},0\right)$ and radius $\frac{1}{1+R}$ .

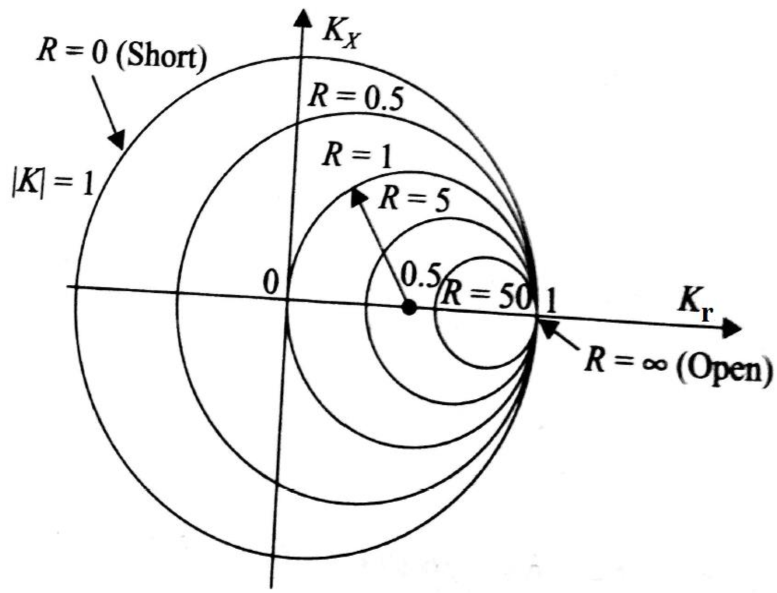



Fig. Family of constant R circles

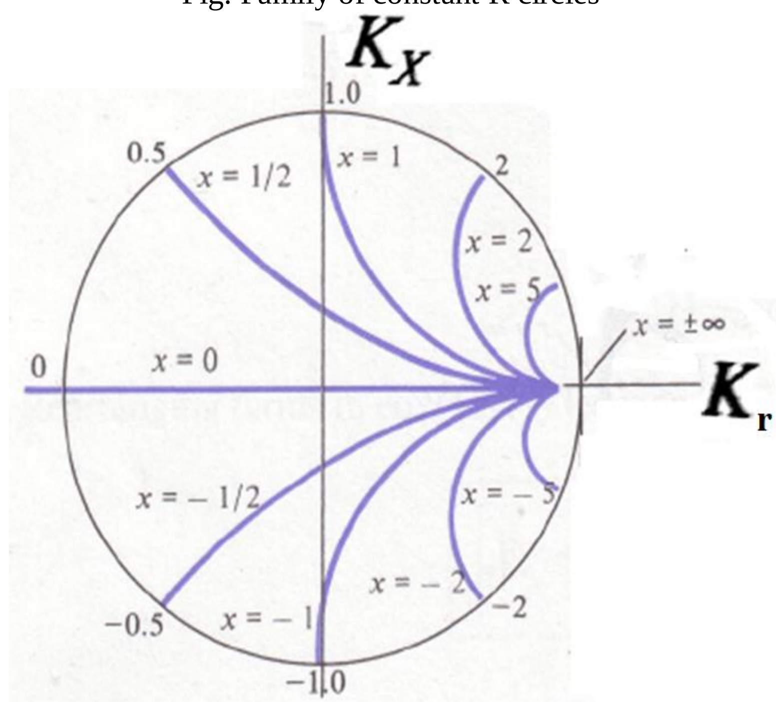



Fig. Family of constant X circles

(i) The constant X-circle:

$$X = \frac{2K_x}{(1-K_r)^2 + K_x^2}$$
$$(1-K_r)^2 + K_x^2 = \frac{2K_x}{X}$$
$$(1-K_r)^2 + K_x^2 - \frac{2K_x}{X} = 0$$

Adding $\left(\frac{1}{X}\right)^2$ on both sides, we get

$$(1-K_r)^2 + K_x^2 - \frac{2K_x}{X} + \left(\frac{1}{X}\right)^2 = \left(\frac{1}{X}\right)^2$$
$$(K_r-1)^2 + \left(K_x - \frac{1}{X}\right)^2 = \left(\frac{1}{X}\right)^2$$

This equation represents a family of circles on the reflection co-efficient plane. These circles are called constant –X circles with center $\left(1, \frac{1}{X}\right)$ and radius $\frac{1}{X}$.

The complete smith chart can be obtained by the superposition of the family of the constant-R circles and constant-X circles on K plane as shown in fig.

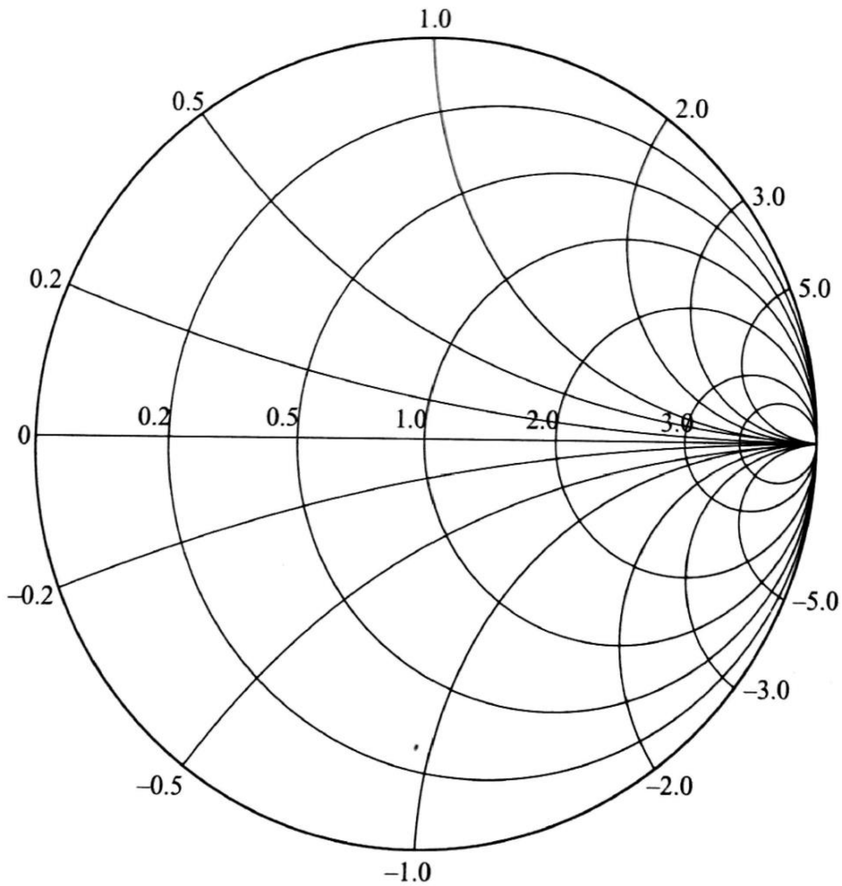



Fig. Smith chart

**Properties of smith chart:**

1. The smith chart can be used for impedance as well as for admittance.
2. The smith chart consists of constant –R circles and constant –X circles.
3. The point $P_{sc}$ on the chart represents short circuit or zero impedance and the point $P_{oc}$ on the chart represents open circuit or infinite impedance.
4. A complete revolution of 360° around the Smith chart represents a distance of λ/2 on the line.
5. Clockwise movement on the chart is regarded as moving toward the generator from the load on the transmission line. Similarly, counterclockwise movement on the chart corresponds to moving toward the load from the generator.

6. There are three scales around the periphery of the Smith chart as illustrated in Figure. The outermost scale is used to determine the distance on the line from the generator end in terms of wavelengths, and the next scale determines the distance from the load end in terms of wavelengths. The innermost scale is used to determine angle of reflection coefficient in degrees.
7. Since a $\lambda/2$ distance on the line corresponds to a movement of 360° on the chart, $\lambda$ distance on the line corresponds to a 720° movement on the chart.
8. $V_{max}$ occurs where $Z_{in(max)}$ located on the chart while $V_{min}$ occurs where $Z_{in(min)}$ located
9. The distance between $Z_{sc}$ and $Z_{oc}$ is $\lambda/4$.

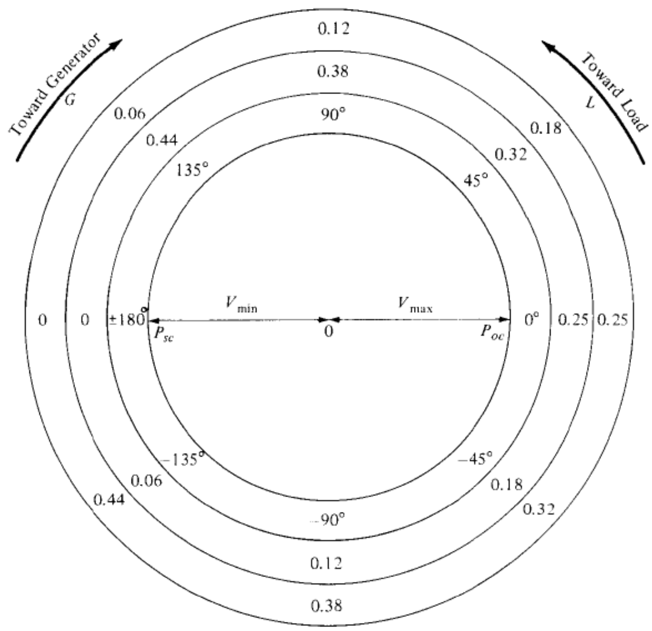



Fig. Smith chart illustrating scales around the periphery and movements around the chart line.

**Applications of smith chart:**

1. Plotting impedance
2. Measurement of VSWR
3. Measurement of reflection coefficient K
4. Measurement of input impedance of the transmission line.
5. Impedance to admittance conversion
6. Finding voltage maximum and voltage minimum locations
7. Stub matching

**Microstrip transmission line:**
A microstrip line consists of a single ground plane and an open strip conductor separated by dielectric substrate as shown in Figure.

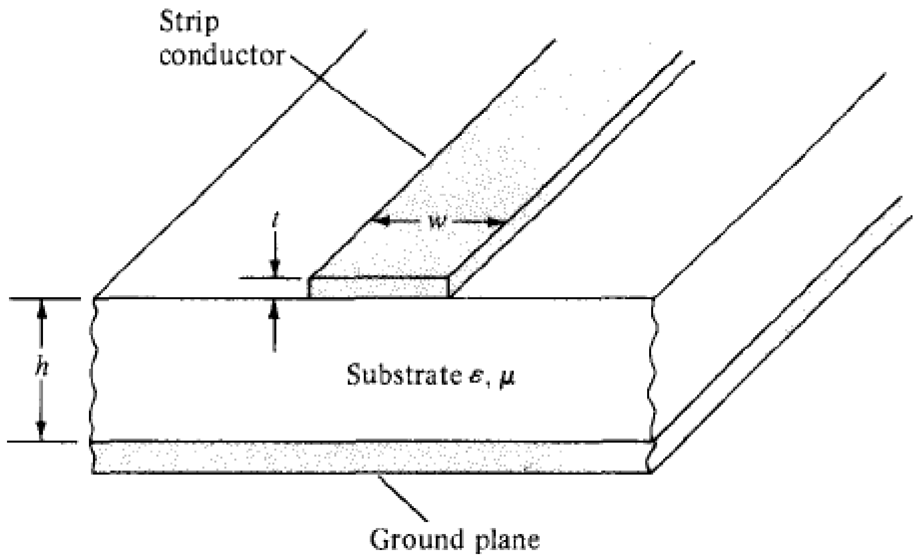


Fig. Microstrip line

The characteristic impedance of microstrip line is given by the following approximate formula

$$Z_o = \begin{cases} \dfrac{60}{\sqrt{\varepsilon_{eff}}} \ln\left(\dfrac{8h}{w} + \dfrac{w}{h}\right), & w/h \leq 1 \\ \dfrac{1}{\sqrt{\varepsilon_{eff}}} \dfrac{120\pi}{[w/h + 1.393 + 0.667 \ln(w/h + 1.444)]}, & w/h \geq 1 \end{cases}$$

$$\varepsilon_{eff} = \frac{(\varepsilon_r + 1)}{2} + \frac{(\varepsilon_r - 1)}{2\sqrt{1 + 12h/w}}$$

The input impedance of the lossless microstrip line is given by

$$Z_{in} = Z_0 \left[\frac{Z_R + jZ_0 \tan\beta l}{Z_0 + jZ_R \tan\beta l}\right]$$

**Impedance matching techniques:**
When the transmission line is terminated with a load impedance which is not equal to the characteristic impedance of the line, mismatch occurs and reflected wave exists on the line. Mismatch reduces efficiency and increases power loss. To avoid mismatching, it is necessary to add impedance matching device between the load and the line.
The following are the impedance matching techniques
1. Quarter-wave impedance matching or quarter-wave transformer
2. Stub matching

**Quarter wave impedance matching or quarter wave transformer:**

When the transmission line is terminated with a load impedance which is not equal to the characteristic impedance of the line, mismatch occurs and reflected wave exists on the line. Mismatch reduces efficiency and increases power loss. Quarter wave line is inserted between the line and load to match load impedance to the line as shown in Fig.

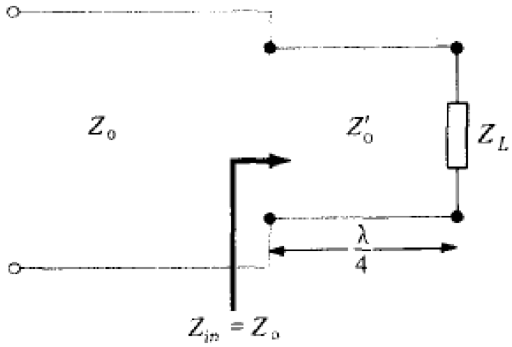



Fig. Quarter wave transformer

$Z_0^1$ is selected such that $Z_{in} = Z_0$

$$Z_0^1 = \sqrt{Z_0 Z_L}$$

The λ/4 line is also called Quarter wave transformer.

Advantages:

1. Simple to design

Disadvantages:

1. We have to cut the line to insert a quarter wave transformer in between the line and the load.
2. It is frequency sensitive

**Stub matching:**

A piece of transmission line is called stub. The stub may be open or short circuited. A small section of open or short circuited stub is used as impedance matching device, which can be connected in parallel to the line at a certain distance from the load. This matching device is called stub matching.

The stub has the same characteristic impedance as the main line. It is more difficult to use a series stub although it is theoretically feasible. An open-circuited stub radiates some energy at high frequencies. Consequently, short-circuited parallel stubs are preferred.

There two types of stub matching methods

1. Single stub matching
2. Double stub matching

**1. Single stub matching:**

In this method, to achieve impedance matching, a short circuited stub is connected in parallel to the line at a certain distance from the load as shown in Fig. Since the stub is connected in parallel, it is easy to use admittance instead of impedance for analysis.

When load admittance $Y_R$ is connected to the line and if it is not equal to the characteristic admittance $Y_0$, mismatch occurs and reflected wave exists on the line.

The input admittance at point 1 looking towards load is given by

$$Y_{in} = Y_0 \pm jB$$

This is the admittance at point 1 before stub is connected. The point 1 is selected such that at point 1, $Y_0=1/Z_0$. The short circuited stub is connected at point 1 in parallel with the main line. The length of the stub is selected such that its input succeptance is $\mp jB$.

The total input admittance at 1 is

$$Y_{in} = Y_0 \pm jB \mp jB = Y_0$$

Thus the input impedance looking towards load is

$$Z_{in} = Z_0$$

Therefore the line terminated with $Z_0$ at point 1, hence the line is said be matched.

Length and location stub:
Consider a transmission line terminated with load admittance $Y_R$. Let a short circuited stub of length $l_t$ is connected to the main line at a distance $l_s$ from the load as shown in Fig.

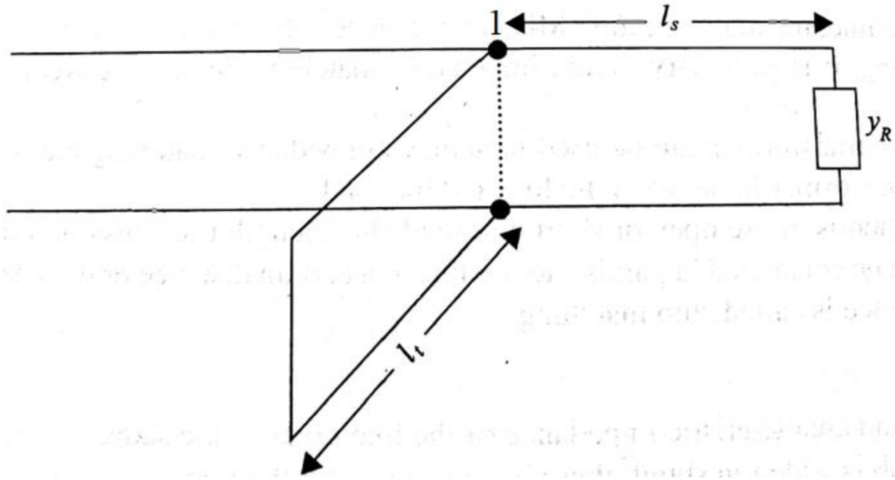



Fig. Single stub matching

We know that the input impedance at any point on the line is

$$Z_{in} = Z_0 \left[\frac{Z_R + jZ_0 \tan \beta l}{Z_0 + jZ_R \tan \beta l}\right]$$

The corresponding input admittance is given by

$$Y_{in} = \frac{1}{Z_{in}}$$

$$Y_{in} = Y_0 \left[\frac{Y_R + jY_0 \tan \beta l}{Y_0 + jY_R \tan \beta l}\right]$$

The normalized input admittance is given by

$$y_{in} = \frac{Y_{in}}{Y_0}$$

$$y_{in} = \left[\frac{\frac{Y_R}{Y_0} + j \tan \beta l}{1 + j\frac{Y_R}{Y_0} \tan \beta l}\right]$$

$$y_{in} = \left[\frac{y_R + j \tan \beta l}{1 + jy_R \tan \beta l}\right]$$

Where the normalized load admittance is

$$y_R = \frac{Y_R}{Y_0}$$

Separating the real and imaginary terms

$$y_{in} = \left(\frac{y_R + j \tan \beta l}{1 + jy_R \tan \beta l}\right)\left(\frac{1 - jy_R \tan \beta l}{1 - jy_R \tan \beta l}\right)$$

$$y_{in} = \frac{y_R + y_R \tan^2 \beta l + j \tan \beta l - jy_R^2 \tan \beta l}{1 + y_R^2 \tan^2 \beta l}$$

$$y_{in} = \frac{y_R(1 + \tan^2 \beta l)}{1 + y_R^2 \tan^2 \beta l} + j\frac{(1 - y_R^2) \tan \beta l}{1 + y_R^2 \tan^2 \beta l}$$

For no reflection, at distance $l = l_s$, the real part of normalized input impedance is equal to unity.

Therefore

$$\frac{y_R(1+\tan^2\beta l_s)}{1+y_R^2\tan^2\beta l_s}=1$$

$$y_R(1+\tan^2\beta l_s)=1+y_R^2\tan^2\beta l_s$$

$$y_R+y_R\tan^2\beta l_s=1+y_R^2\tan^2\beta l_s$$

$$(y_R-y_R^2)\tan^2\beta l_s=1-y_R$$

$$\tan^2\beta l_s=\frac{1-y_R}{y_R(1-y_R)}$$

$$\tan^2\beta l_s=\frac{1}{y_R}$$

$$\tan\beta l_s=\frac{1}{\sqrt{y_R}}=\sqrt{\frac{Y_0}{Y_R}}$$

$$l_s=\frac{1}{\beta}\tan^{-1}\sqrt{\frac{Y_0}{Y_R}}$$

Therefore the location of stub is given by

$$l_s=\frac{\lambda}{2\pi}\tan^{-1}\sqrt{\frac{Z_R}{Z_0}}$$

At this location the imaginary part of $y_{in}$ is

$$b_s=\frac{(1-y_R^2)\tan\beta l_s}{1+y_R^2\tan^2\beta l_s}$$

But

$$\tan\beta l_s=\frac{1}{\sqrt{y_R}}$$

$$b_s=\frac{(1-y_R^2)\frac{1}{\sqrt{y_R}}}{1+y_R^2\frac{1}{y_R}}$$

$$b_s=\frac{(1-y_R^2)}{\sqrt{y_R}(1+y_R)}$$

$$b_s=\frac{(1-y_R)}{\sqrt{y_R}}$$

$$b_s=\frac{\left(1-\frac{Y_R}{Y_0}\right)}{\sqrt{\frac{Y_R}{Y_0}}}$$

$$b_s=\frac{(Y_0-Y_R)}{\sqrt{Y_RY_0}}$$

The input impedance of the short circuited stub is given by

$$Z_{sc}=jZ_0\tan\beta l_t$$

The normalized input admittance is

$$y_{sc}=-j\cot\beta l_t$$

The normalized succeptance of stub is

$$b_{ss}=-\cot\beta l_t$$

At point 1 the total succeptance is zero

$$b_s + b_{ss} = 0$$

$$\frac{(Y_0 - Y_R)}{\sqrt{Y_R Y_0}} - \cot\beta l_t = 0$$

$$\cot\beta l_t = \frac{(Y_0 - Y_R)}{\sqrt{Y_R Y_0}}$$

$$\tan\beta l_t = \frac{\sqrt{Y_R Y_0}}{(Y_0 - Y_R)}$$

$$\tan\beta l_t = \frac{\sqrt{Z_R Z_0}}{(Z_R - Z_0)}$$

$$l_t = \frac{1}{\beta}\tan^{-1}\left(\frac{\sqrt{Z_R Z_0}}{(Z_R - Z_0)}\right)$$

Therefore the length of the stub is

$$l_t = \frac{\lambda}{2\pi}\tan^{-1}\left(\frac{\sqrt{Z_R Z_0}}{(Z_R - Z_0)}\right)$$

**Disadvantages of single stub matching:**

1. The location and length of the stub are frequency dependent. If frequency of the wave changes, the location and length of the stub should be changed.
2. In practical cases, the location of stub has to be moved along the line for fine adjustment.

**Double stub matching:**

To overcome the drawbacks of the single stub matching, two stubs are used at different locations. This is called double stub matching.

Consider a double stub matching system consisting of two short circuited stubs connected in parallel to the main line near the load as shown in Fig.

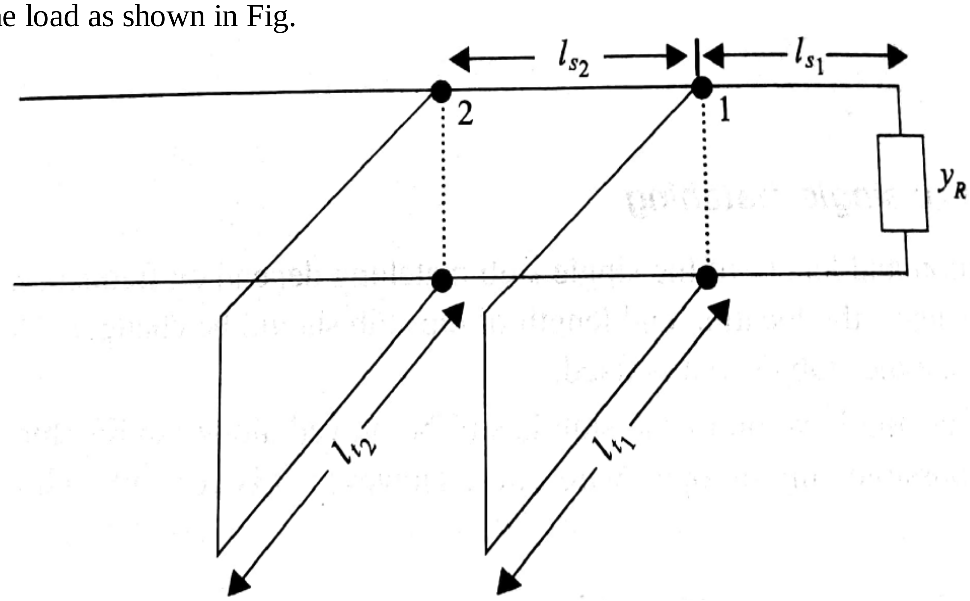



Fig. double stub matching

The characteristic admittance of the stubs should be equal to the characteristic admittance of the line. In double stub matching the locations of the stubs are arbitrary, but the spacing between two stubs must be equal to λ/8, λ/4 or 3λ/8.

Therefore, in the design of double stub matching, it is better to keep the locations of stubs fixed. Impedance matching is done by finding the lengths of the stubs. When the frequency changes, the stub lengths can be adjusted to achieve impedance matching.

**Design of double stub matching:**

1. The normalized admittance at the location of stub1 from load is

$$y_A = g_A + jb_A$$

2. Connect a short circuited stub1 having succepatance $\pm jb_1$ to the line at point 1

The normalized admittance at point 1 after stub connection is

$$y_A = g_A + j(b_A \pm b_1)$$

3. For the impedance matching, the normalized admittance at the location of stub2 is

$$y_B = 1 \pm jb_B$$

4. The spacing between two stubs is selected such the real part of normalized admittance at the location of stub2 is unity.

5. Connect second short circuited stub having succeptance of $\mp jb_B$to the line at 2

Now the normalized input admittance at point 2 after stub2 connection

$$y_{in} = 1 \pm jb_B \mp jb_B = 1$$

$$y_{in} = 1$$

$$Z_{in} = Z_0$$

Therefore the line terminated with $Z_0$ at point 2, hence the line is said be matched.

**Loading:**

The processes of increasing the inductance L of the transmission line artificially is called loading. And such a line is called loaded line.

There are three types of loading methods

1. Patched loading
2. Continuous loading
3. Lumped loading