JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY ANANTAPUR
(Established by Govt. of A.P., ACT No.30 of 2008)
ANANTHAPURAMU – 515 002 (A.P) INDIA

Computer Science & Engineering

<table>
<tr><td>Course Code</td><td colspan="2">Discrete Mathematics & Graph theory</td><td>L</td><td>T</td><td>P</td><td>C</td></tr>
<tr><td>20A54304</td><td colspan="2">(Common to CSE, IT, CSE(DS), CSE (IoT), CSE (AI), CSE (AI & ML) and AI & DS)</td><td>3</td><td>0</td><td>0</td><td>3</td></tr>
<tr><td>Pre-requisite</td><td>Basic Mathematics</td><td>Semester</td><td colspan="4">III</td></tr>
<tr><td colspan="7"></td></tr>
<tr><td colspan="7">Course Objectives:</td></tr>
<tr><td colspan="7">Introduce the concepts of mathematical logic and gain knowledge in sets, relations and functions and Solve problems using counting techniques and combinatorics and to introduce generating functions and recurrence relations. Use Graph Theory for solving real world problems</td></tr>
<tr><td colspan="7">Course Outcomes (CO):</td></tr>
<tr><td colspan="7">After completion of the course, students will be able to
Apply mathematical logic to solve problems.
Understand the concepts and perform the operations related to sets, relations and functions.
Gain the conceptual background needed and identify structures of algebraic nature.
Apply basic counting techniques to solve combinatorial problems.
Formulate problems and solve recurrence relations.
Apply Graph Theory in solving computer science problems</td></tr>
<tr><td>UNIT - I</td><td colspan="2">Mathematical Logic</td><td colspan="4">8 Hrs</td></tr>
<tr><td colspan="7">Introduction, Statements and Notation, Connectives, Well-formed formulas, Tautology, Duality law, Equivalence, Implication, Normal Forms, Functionally complete set of connectives, Inference Theory of Statement Calculus, Predicate Calculus, Inference theory of Predicate Calculus.</td></tr>
<tr><td>UNIT - II</td><td colspan="2">Set theory</td><td colspan="4">9 Hrs</td></tr>
<tr><td colspan="7">Basic Concepts of Set Theory, Relations and Ordering, The Principle of Inclusion-Exclusion, Pigeon hole principle and its application,Functions composition of functions, Inverse Functions, Recursive Functions, Lattices and its properties. Algebraic structures: Algebraic systems-Examples and General Properties, Semi groups and Monoids, groups, sub groups, homomorphism, Isomorphism.</td></tr>
<tr><td>UNIT - III</td><td colspan="2">Elementary Combinatorics</td><td colspan="4">8 Hrs</td></tr>
<tr><td colspan="7">Basics of Counting, Combinations and Permutations, Enumeration of Combinations and Permutations, Enumerating Combinations and Permutations with Repetitions, Enumerating Permutations with Constrained Repetitions, Binomial Coefficients, The Binomial and Multinomial Theorems.</td></tr>
<tr><td>UNIT - IV</td><td colspan="2">Recurrence Relations</td><td colspan="4">9 Hrs</td></tr>
<tr><td colspan="7">Generating Functions of Sequences, Calculating Coefficients of Generating Functions, Recurrence relations, Solving Recurrence Relations by Substitution and Generating functions, The Method of Characteristic roots, Solutions of Inhomogeneous Recurrence Relations.</td></tr>
<tr><td>UNIT - V</td><td colspan="2">Graphs</td><td colspan="4">9 Hrs</td></tr>
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Basic Concepts, Isomorphism and Subgraphs, Trees and their Properties, Spanning Trees, Directed Trees, Binary Trees, Planar Graphs, Euler's Formula, Multigraphs and Euler Circuits, Hamiltonian Graphs, Chromatic Numbers, The Four Color Problem
Textbooks:
1. Joe L. Mott, Abraham Kandel and Theodore P. Baker, Discrete Mathematics for Computer Scientists & Mathematicians, 2nd Edition, Pearson Education. 2. J.P. Tremblay and R. Manohar, Discrete Mathematical Structures with Applications to Computer Science, Tata McGraw Hill, 2002.
Reference Books:
1. Kenneth H. Rosen, Discrete Mathematics and its Applications with Combinatorics and Graph Theory, 7th Edition, McGraw Hill Education (India) Private Limited. 2. Graph Theory with Applications to Engineering and Computer Science byNarsinghDeo.
Online Learning Resources:
http://www.cs.yale.edu/homes/aspnes/classes/202/notes.pdf

Statements and Notation, Connectives

Statements (propositions) :- A statement is a collection of declarative sentense that has either true (or) false but not both. A propositional Consists of propositional variables and connectives. we denote the propositional variables by $p, q, r, s, \ldots$ etc.

The truth value of a statement is true, denoted by 'T' if it is a true statement and false denoted by 'F' if it is a false statement.

EX :- 1. Amaravathi is in Andhra pradesh (TRUE)

2. $12+9 = 3-2$ (FALSE)

3. $2+3 = 5$ (TRUE)

EX :- 1) $x+y = 2$ 2) "A is Less than 2"

The above are not statements because they are neither true nor false.

Connectives :- compound statements consist of two or more statements connected with what are called "Connectives".

EX: "The sky is clear and it is raining" is a statement consisting of two statements connected with the word "and" other connectives include "or" "if .... then" "if and only if" "NAND" "NOR" "Exclusive OR" etc.....

More Examples :-

1. I have a cold or I have a fever.
2. If it is cold out then I will need a jacket

Negation :- Let 'p' be a proposition, the negation of 'p' denoted by '$\sim p$' (or) '$\neg p$' (or) $\overline{p}$ is the statement. The proposition '$\sim p$' is read "not p". The truth values of the negation of 'p', '$\sim p$' is the opposite of the truth values of p.

The truth table for the Negation of a statement

p	$\sim p$
T	F
F	T

Conjunction :- Let 'p' and 'q' be propositions, the conjunction of 'p' and 'q' is denoted by '$p \wedge q$' is the proposition 'p' and 'q'. The conjunction '$p \wedge q$' is true when both 'p' and 'q' are true and is false otherwise.

The truth table for the conjunction of two statements is given by

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction :- Let 'p' and 'q' be propositions the disjunction of 'p' and 'q' is denoted by '$p \vee q$' is the proposition "p or q". The disjunction $p \vee q$ is false when both 'p' and 'q' are false and is true otherwise

The truth table for the disjunction of two statements is given by

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional :- Let 'p' and 'q' be propositions, the conditional statement $p \to q$ is the proposition 'if p then q'. The conditional statement $p \to q$ is false when 'p' is true and 'q' is false and true otherwise.

In the conditional statement $p \to q$, 'p' is called the hypothesis (or) antecedent (or) premise and 'q' is called the Conclusion (or) Consequence.

The truth table for the conditional statement is given by

(Implication)

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional :- Let 'p' and 'q' be propositions the biconditional statement $p \leftrightarrow q$ is the proposition 'p if and only if q". The biconditional statement $p \leftrightarrow q$ is true when 'p' and 'q' have the same truth values and is false otherwise. Biconditional statement is also called Bi implications.

The truth table for the biconditional statement is given by

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(Bi implication)

Exclusive or :- Let 'p' and 'q' be propositions the exclusive or of p and q denoted by $p \oplus q$ is the proposition that is true when exactly one of 'p' and q is true and is false otherwise.

The truth table for the exclusive or of two statements is given by

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Order of precedence for logical connectives :-

Connectives	precedence
$\sim$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Converse, Inverse and Contrapositive of an implication:

The converse of the implication $p \rightarrow q$ is $q \rightarrow p$.

The inverse of the implication $p \rightarrow q$ is $\sim p \rightarrow \sim q$

The contrapositive of the implication $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

Ex: Let $p \to q$; if ΔABC is equilateral then it is isosceles. find the inverse, converse and contrapositive. ③

sol: Converse :- $q \to p$: if ΔABC is isosceles then it is equilateral.

Inverse :- $\sim p \to \sim q$: If ΔABC is not Equilateral then it is not isosceles.

contrapositive :- $\sim q \to \sim p$: If ΔABC is not isosceles then it is not equilateral.

other connectives :-

1) NAND :- The word "NAND" is a combination of 'NOT' and "AND". where 'NOT' stands for negation and 'AND' for the conjunction. It is denoted by the symbol $\uparrow$

$$p \uparrow q \leftrightarrow \sim(p \wedge q)$$

2) NOR :- The word "NOR" is a combination of 'NOT' and 'OR'. where 'NOT' stands for negation and 'OR' for the disjunction. It is denoted by the symbol $\downarrow$

$$p \downarrow q \leftrightarrow \sim(p \vee q)$$

Statement formulas :-

1. Idempotent Law : $p \vee p = p$, $p \wedge p = p$

2. Commutative Law : $p \vee q = q \vee p$, $p \wedge q = q \wedge p$.

3. Associative Laws :

$$p \vee (q \vee r) = (p \vee q) \vee r.$$
$$p \wedge (q \wedge r) = (p \wedge q) \wedge r.$$

4. Distributed Laws :

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

5. Absorption Laws :

$$p \vee (p \wedge q) = p \quad , \quad p \wedge (p \vee q) = p.$$

6. De Morgan's Laws :

$$\sim(p \vee q) = (\sim p) \wedge (\sim q) \quad , \quad \sim(p \wedge q) = (\sim p) \vee (\sim q)$$

7. Double Negation : $\sim(\sim p) = p$

8. $(p \vee \sim p) = T$, $(p \wedge \sim p) = F$

9. $(p \rightarrow q) \wedge (p \rightarrow \sim q) = \sim p.$

10. Contrapositive : $p \rightarrow q = \sim q \rightarrow \sim p.$

Truth Tables :- A truth table displays the relationship between the truth values of propositions.

EX: Truth table for Expression $\sim(p \wedge \sim q)$

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

① construct a truth table for each of the following compound propositions.

i) $(p \vee q) \rightarrow (p \wedge q)$.

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

ii) $(p \to q) \to (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \to (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

iii) $(q \to \sim p) \leftrightarrow (p \leftrightarrow q)$

p	q	$\sim p$	$q \to \sim p$	$p \leftrightarrow q$	$(q \to \sim p) \leftrightarrow (p \leftrightarrow q)$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

iv) $(p \leftrightarrow q) \leftrightarrow [(p \wedge q) \vee (\sim p \wedge \sim q)]$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \wedge \sim q$	$p \leftrightarrow q$	$(p \wedge q) \vee (\sim p \wedge \sim q)$	statement.
T	T	F	F	T	F	T	T	T
T	F	F	T	F	F	F	F	T
F	T	T	F	F	F	F	F	T
F	F	T	T	F	T	T	T	T

v) $(\sim p \leftrightarrow \sim q) \leftrightarrow (p \leftrightarrow q)$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$	$p \leftrightarrow q$	Statement.
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

② Construct a truth table for the proposition

$\sim[p \vee (q \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \rightarrow r)]$

Sol:

p	q	r	$q \wedge r$	$p \vee (q \wedge r) = a$	$\sim a$	$p \vee q$	$p \rightarrow r$	$(p \vee q) \wedge (p \rightarrow r) = b$	$a \leftrightarrow b$
T	T	T	T	T	F	T	T	T	F
T	T	F	F	T	F	T	F	F	T
T	F	T	F	T	F	T	T	T	F
T	F	F	F	T	F	T	F	F	T
F	T	T	T	T	F	T	T	T	F
F	T	F	F	F	T	T	T	T	T
F	F	T	F	F	T	F	T	F	F
F	F	F	F	F	T	F	T	F	F

③ calculate the truth table for the composition

$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

p	q	r	$p \rightarrow q$	$p \rightarrow r$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r) = a$	$(p \rightarrow q) \rightarrow (p \rightarrow r) = b$	$a \rightarrow b$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

④ obtain the truth table for $(p \vee q) \wedge (p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \vee q$	$p \rightarrow q$	$(p \vee q) \wedge (p \rightarrow q)$	$q \rightarrow p$	statement
T	T	T	T	T	T	T
T	F	T	F	F	T	F
F	T	T	T	T	F	F
F	F	F	T	F	T	F

Well formed formulas (wff) :

Mathematical Expressions are represented in well defined from using paranthesis, brackets and square brackets to avoid the ambiquity. In logic a statement which is collection of variables or statements represented in well-defined form using parenthesis according to priority of operation. These statements are known as well-formed formula. (wff).

A well formed formula is defined as

1. If 'p' is propositional variable then it is wff.
2. If 'α' is wff then $\sim\alpha$ is also a wff.
3. If 'α' and 'β' are wff then $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ are wff.
4. If any of the parenthesis is missing then the statement is not wff.

EX: Following are well formed formulas.

1. $\sim (P \vee Q)$
2. $\sim (P \wedge Q)$
3. $(P \rightarrow (P \wedge Q))$
4. $(P \rightarrow (P \vee Q))$
5. $((P \rightarrow Q) \leftrightarrow R)$ etc.

Following are not well formed formulas.

1. $(P \rightarrow Q) \rightarrow (\wedge Q)$ This is not wff because $\wedge Q$ is not
2. $(P \wedge Q) \rightarrow Q)$. The reason for this not being a wff is that one of the parenthesis in the beginning is missing. So

$((P \wedge Q) \rightarrow Q)$ is a wff.

Tautologies :- A Tautology is a universally true formula whose truth value is T for all possible assignments of truth values to the propositional variables.

Ex: Show that $(p \vee \sim p)$ is a Tautology.

sol: The truth table for the proposition as follows

p	$\sim p$	$(p \vee \sim p)$
T	F	T
F	T	T

Now since the final truth values of the statement is always true. Hence the given proposition is Tautology.

Ex: Show that $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ is a Tautology.

Sol: The truth table for the given proposition is as follows

p	q	r	$q \to r$	$p \to (q \to r)$	$p \to q$	$p \to r$	$(p \to q) \to (p \to r)$	statement
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Ex: Show that $(p \wedge q) \vee (\sim p \vee \sim q)$ is a Tautology.

Sol:

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$p \wedge q$	$(p \wedge q) \vee (\sim p \vee \sim q)$
T	T	F	F	F	T	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

Ex: show that $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ is a Tautology.

Sol:- Let $a = ((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r)))$

$b = (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$.

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \vee q$	$\sim q \vee \sim r$	$(\sim p \wedge (\sim q \vee \sim r))$	$\sim(\sim p \wedge (\sim q \vee \sim r))$	
T	T	T	F	F	F	T	F	F	T	
T	T	F	F	F	T	T	T	F	T	
T	F	T	F	T	F	T	T	F	T	
T	F	F	F	T	T	T	T	F	T	
F	T	T	T	F	F	T	F	F	T	
F	T	F	T	F	T	T	T	T	F	
F	F	T	T	T	F	F	T	T	F	
F	F	F	T	T	T	F	T	T	F	

a	$\sim p \wedge \sim q$	$\sim p \wedge \sim r$	b	$a \vee b$
T	F	F	F	T
T	F	F	F	T
T	F	F	F	T
T	F	F	F	T
T	F	F	F	T
F	F	T	T	T
F	T	F	T	T
F	T	T	T	T.

Now since the final truth values of the statement are always true

Hence the given proposition is Tautology.

Contradictions :- A proposition is said to be a contradiction if its truth value is F for any assignment of truth values to its components.

Ex: proposition $p \wedge \sim p$ is a contradiction

p	$\sim p$	$p \wedge \sim p$
T	F	F
T	F	F
F	T	F
F	T	F

Ex: Determine whether the compound proposition $(\sim(q \to r) \wedge r \wedge (p \to q))$ is a contradiction or not ?

Sol: The truth table of the expression is made by considering the preliminary proposition and then at least all of the small propositions are combined to form the request proposition.

p	q	r	$p \to q$	$q \to r$	$\sim(q \to r)$	$\sim(q \to r) \wedge r$	$(\sim(q \to r) \wedge r \wedge (p \to q))$
T	T	T	T	T	F	F	F
T	T	F	T	F	T	F	F
T	F	T	F	T	F	F	F
T	F	F	F	T	F	F	F
F	T	T	T	T	F	F	F
F	T	F	T	F	T	F	F
F	F	T	T	T	F	F	F
F	F	F	T	T	F	F	F

The last column contains only 'F' as the truth values of the given statement. Hence it is a contradiction.

Ex: show that $[(p \to q) \wedge p] \to q$ is a Tautology

Ex: show that $(p \wedge q) \to (p \vee q)$ is a Tautology.

Ex: show that $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a Contradiction.

Equivalence of formulas :- The compound propositions have the same truth values no matter what truth value their constituent propositions have, they are called logically Equivalent. It is denoted by the symbol '$\equiv$'.

Ex: $p \to q$ and $\sim p \to q$ are logically Equivalent and it is denoted by $p \to q \equiv \sim p \to q$.

p	q	$\sim p$	$p \to q$	$\sim p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Ex: show that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.

Sol:

same

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Ex: show that $(p \vee q)$ and $(q \vee p)$ are Equivalent.

Sol:

p	q	$p \vee q$	$q \vee p$	$(p \vee q) \leftrightarrow (q \vee p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

EX: show that the following are Equivalence.

i, $p \to (q \to r) \equiv p \to (\sim q \vee r) \equiv (p \wedge q) \to r$.

ii, $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$

sol: i,

p	q	r	$\sim q$	$q \to r$	$\sim q \vee r$	$p \wedge q$	$p \to (q \to r)$	$p \to (\sim q \vee r)$	$(p \wedge q) \to r$
T	T	T	F	T	T	T	T	T	T
T	T	F	F	F	F	T	F	F	F
T	F	T	T	T	T	F	T	T	T
T	F	F	T	T	T	F	T	T	T
F	T	T	F	T	T	F	T	T	T
F	T	F	F	F	F	F	T	T	T
F	F	T	T	T	T	F	T	T	T
F	F	F	T	T	T	F	T	T	T

ii)

p	q	r	$q \vee r$	$p \to q$	$p \to r$	$p \to (q \vee r)$	$(p \to q) \vee (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

EX: show that $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$

EX: show that $p \to q \equiv \sim p \vee q$

EX: show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically Equivalent.

EX: show that $\sim (p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim q$.

principle of duality :- Two formulas A and A^* are said to be duals of each other if either one can be obtained from the other by replacing '$\wedge$' by '$\vee$' and '$\vee$' by '$\wedge$'. The connectives '$\vee$' and '$\wedge$' are called duals of each other. If the formula 'A' contains the special variable T (or) F, then A^* its dual is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

Ex: Write the duals of the following formulas.

i) $(p \vee q) \wedge r$ Ans: $(p \wedge q) \vee r$

ii) $(p \wedge q) \vee r$ Ans: $(p \vee q) \wedge r$

iii) $\sim(p \vee q) \wedge (p \vee \sim(q \wedge \sim s))$ Ans: $\sim(p \wedge q) \vee (p \wedge \sim(q \vee \sim s))$.

iv) $(p \wedge q) \vee T$ Ans: $(p \vee q) \wedge F$.

Tautological Implications :- A statement formula A is said to Tautologically imply a statement B if and only if $A \rightarrow B$ is a Tautology. This is denoted by $A \Rightarrow B$ and read as "A implies B"

Note: $\Rightarrow$ is not a connective, $A \Rightarrow B$ is not a statement.
$A \Rightarrow B$ states that $A \rightarrow B$ is tautology.

Ex: $(p \rightarrow q) \Rightarrow (\sim q \rightarrow \sim p)$

Sol:

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$(p\rightarrow q)\rightarrow(\sim q\rightarrow \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

$\therefore (p \rightarrow q) \rightarrow (\sim q \rightarrow \sim p)$ is Tautology.

Normal forms :- The problem of determining whether a given statement formula is a Tautology or a Contradiction or at least satisable is called a decision problem.

It will be convenient to use the word "product" in place of 'conjuction' and sum in place of 'disjunction' in our current discussion.

A product of the variables and their negations in a formula is called an elementary product. similarly a sum of the variables and their negations is called an elementary sum.

Let 'p' and 'q' be any atomic variables. Then P, $\sim p \wedge q$, $\sim q \wedge P$, $P \wedge \sim q$, ... are some examples of elementary product. on the other hand P, $\sim p \vee q$, $\sim q \vee P$, $P \vee \sim q$ .... are some examples of elementary sum.

There are two Normal forms

i) Disjunctive Normal forms (DNF)

ii) Conjunctive Normal forms (CNF).

i) **Disjunctive Normal Forms :-** A formula which is equivalent to a given formula and which consist of a sum of elementary products is called a disjunctive Normal form of the given formula.

ii) **Conjunctive Normal forms :-** A Formula which is equivalent to a given formula and which consist of a product of elementary sum is called a Conjunctive Normal form of the given formula.

procedure to obtain DNF & CNF of given formula:

i) If the connectives $\rightarrow$ and $\leftrightarrow$ are present in the given formula they are replaced by $\wedge, \vee, \sim \ldots$ $p\rightarrow q$ is replaced by $\sim p \vee q$ and $p \leftrightarrow q$ is replaced by either $(p\wedge q) \vee (\sim p \wedge \sim q)$ (or) $(\sim p \vee q) \wedge (\sim q \vee p)$.

ii) If the negation is present before the given formula or a part of the given formula (not variable), De Morgan's Laws are applied so that the negation is brought before the variables only.

iii) If necessary the distributive Law and Idempotent Law are applied.

iv) If there is an elementary product which is equivalent to the truth value F in the DNF, it is ommitted, similarly If there is an elementary sum which is equivalent to the truth value T in the CNF, it is omitted.

EX: Find the Disjunctive Normal Forms of the following statements.

i) $\sim(\sim(p \leftrightarrow q) \wedge r)$

ii) $p \vee (\sim p \rightarrow (q \vee (q \rightarrow \sim r)))$

iii) $p \wedge \sim (q \wedge r) \vee (p \rightarrow q)$

iv) $p \wedge \sim (q \wedge r) \vee (((p\wedge q) \vee \sim r) \wedge p)$.

Sol: i) $\sim(\sim(p \leftrightarrow q) \wedge r) \equiv \sim(\sim((p\wedge q) \vee (\sim p \wedge \sim q)) \wedge r)$

$\equiv \sim[(\sim(p\wedge q) \wedge \sim(\sim p \wedge \sim q)) \wedge r]$

$\equiv \sim[((\sim p \vee \sim q) \wedge (p \vee q)) \wedge r]$

$\equiv \sim[((\sim p \wedge p) \vee (\sim p \wedge q) \vee (\sim q \wedge q) \vee (\sim q \wedge p)) \wedge r]$

$\equiv \sim\left[((\sim p \vee \sim q) \wedge (\sim p \vee p) \wedge (q \vee \sim q) \wedge (q \vee p)) \wedge r\right]$

$\equiv \sim\left[((p \vee q) \wedge (\sim p \vee \sim q)) \wedge r\right]$

$\equiv \sim(p \vee q) \vee \sim(\sim p \vee \sim q) \vee \sim r$

$\equiv \sim(p \vee q) \vee (p \wedge q) \vee \sim r$

$\equiv (\sim p \wedge \sim q) \vee (p \wedge q) \vee \sim r.$ //.

ii) $p \vee (\sim p \rightarrow (q \vee (q \rightarrow \sim r)))$

$\equiv p \vee (\sim p \rightarrow (q \vee (\sim q \vee \sim r)))$

$\equiv p \vee (p \vee (q \vee (\sim q \vee \sim r)))$

$\equiv p \vee p \vee q \vee \sim q \vee \sim r$

$\equiv p \vee q \vee \sim q \vee \sim r.$

iii) $p \wedge \sim(q \wedge r) \vee (p \rightarrow q)$

$\equiv p \wedge \sim(q \wedge r) \vee (\sim p \vee q)$

$\equiv p \wedge (\sim q \vee \sim r) \vee (\sim p \vee q)$

$\equiv (p \wedge \sim q) \vee (p \wedge \sim r) \vee (\sim p \vee q)$

$\equiv (p \wedge \sim q) \vee (p \wedge \sim r) \vee \sim p \vee q.$

iv) $(p \wedge \sim(q \vee r)) \vee (((p \wedge q) \vee \sim r) \wedge p)$

$\equiv p \wedge (\sim q \wedge \sim r) \vee ((p \wedge q) \wedge p) \vee (\sim r \wedge p)$

$\equiv (p \wedge \sim q \wedge \sim r) \vee (p \wedge q) \vee (p \wedge \sim r).$

Ex: Find the conjunctive Normal forms of the following statements.

i) $(p \wedge \sim(q \wedge r)) \vee (p \rightarrow q)$

ii) $(q \vee (p \wedge q)) \wedge \sim((p \vee r) \wedge q)$

iii) $(p \wedge \sim(q \vee r)) \vee (((p \wedge q) \vee \sim r) \wedge p).$

Sol:- i) $(p \wedge \sim(q \wedge r)) \vee (p \rightarrow q)$

$\equiv (p \wedge (\sim q \vee \sim r)) \vee (\sim p \vee q)$

$\equiv (p \wedge \sim q) \vee (p \wedge \sim r) \vee (\sim p \vee q)$

$\ldots (p \vee \sim r) \wedge (p \vee \sim q) \wedge (\sim q \vee \sim r) \vee (\sim p \vee q)$

$\equiv (p \vee p) \wedge (p \vee \sim r) \wedge (p \vee \sim q) \wedge (\sim p \vee q \vee \sim q \vee \sim r)$

$\equiv (p \vee p) \wedge (p \vee \sim r) \wedge (p \vee \sim q) \wedge (\sim p \vee \sim r)$

$\equiv p \wedge (p \vee \sim r) \wedge (p \vee \sim q) \wedge (\sim p \vee \sim r)$ //

ii) $(q \vee (p \wedge q)) \wedge \sim((p \vee r) \wedge q)$

$\equiv q \wedge \sim((p \vee r) \wedge q)$ ∵ by absorption Law.

$\equiv q \wedge (\sim(p \vee r) \vee \sim q)$

$\equiv q \wedge (\sim p \wedge \sim r) \vee \sim q$

$\equiv q \wedge (\sim p \vee \sim q) \wedge (\sim q \vee \sim r)$.

iii) $(p \wedge \sim(q \vee r)) \vee (((p \wedge q) \vee \sim r) \wedge p)$

$\equiv (p \wedge (\sim q \wedge \sim r)) \vee ((p \vee \sim r) \wedge (q \vee \sim r)) \wedge p)$

$\equiv (p \wedge (\sim q \wedge \sim r)) \vee [(p \wedge (p \vee \sim r) \wedge (q \vee \sim r))]$

$\equiv [(p \wedge ((\sim q \wedge \sim r)) \vee p] \wedge [(p \wedge (\sim q \wedge \sim r)) \vee (q \vee \sim r)]$

$p \wedge [(p \wedge \sim q \wedge \sim r) \vee \sim r] \vee q$

$\equiv \cancel{p \wedge [(p \wedge (\sim q \wedge \sim r)) \vee (\sim q \wedge \sim r)]}$ ∵ by absorption Laws

$p \wedge [\sim r] \vee q$

$\equiv \cancel{p \wedge} \; \cancel{(\sim q \wedge \sim r)} \equiv p \wedge (q \vee \sim r)$ //

Principal disjunctive Normal Form (PDNF) :

Minterm :- For a given number of variables the minterm consists of conjunctions in which each statement variable or its negation but not both appears only once.

Let p and q be the two statement variables Then there are 2^2 minterms given by $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$, $\sim p \wedge \sim q$.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$\sim p \wedge q$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F	F
T	F	F	T	F	T	F	F
F	T	T	F	F	F	T	F
F	F	T	T	F	F	F	T.

i) No two minterms are equivalent.

ii) Each minterm has the truth value T for exactly one combination of the truth values of the variables 'p' and 'q'

Def: For given formula an equivalent formula consisting of disjunctions of minterms only is called the principal disjunctive normal form of the formula.

The principal disjunctive normal form is also called the sum of products canonical form.

Ex: obtain the PDNF of $p \to q$.

Sol: From the truth table of $p \to q$

p	q	Minterm	$p \to q$
T	T	$p \wedge q$	T
T	F	$p \wedge \sim q$	$\boxed{F}$
F	T	$\sim p \wedge q$	T
F	F	$\sim p \wedge \sim q$	T

The PDNF of $p \to q$ is $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

Ex: obtain the PDNF of $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

Sol:

p	q	r	Minterm	$p \wedge q$	$\sim p \wedge r$	$q \wedge r$	$(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$
T	T	T	$p \wedge q \wedge r$	T	F	T	T
T	T	F	$p \wedge q \wedge \sim r$	T	F	F	T
T	F	T	$p \wedge \sim q \wedge r$	F	F	F	F
T	F	F	$p \wedge \sim q \wedge \sim r$	F	F	F	F
F	T	T	$\sim p \wedge q \wedge r$	F	T	T	T
F	T	F	$\sim p \wedge q \wedge \sim r$	F	F	F	F
F	F	T	$\sim p \wedge \sim q \wedge r$	F	T	F	T
F	F	F	$\sim p \wedge \sim q \wedge \sim r$	F	F	F	F

$\therefore$ The PDNF of the given formula is

$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$ //

with out truth table :

$(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

$\equiv (p \wedge q \wedge T) \vee (\sim p \wedge r \wedge T) \vee (q \wedge r \wedge T)$

$\equiv [p \wedge q \wedge (r \vee \sim r)] \vee (\sim p \wedge r \wedge (q \vee \sim q)) \vee (q \wedge r \wedge (p \vee \sim p))$

$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge r \wedge q) \vee (\sim p \wedge r \wedge \sim q)$

$\vee (q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p)$

$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$ //.

Ex: Find the PDNF of the following statements :

i) $(\sim p \rightarrow q) \wedge (q \leftrightarrow p)$

Sol: $(\sim p \rightarrow q) \wedge (q \Leftrightarrow p) \equiv (p \vee q) \wedge [(q \wedge p) \vee (\sim q \wedge \sim p)]$

$\equiv (p \vee q) \wedge [(q \wedge p) \vee \sim (p \vee q)]$

$\equiv ((p \vee q) \wedge (p \wedge q)) \vee ((p \vee q) \wedge \sim (p \vee q))$

$\equiv ((p \vee q) \wedge (p \wedge q)) \vee F$

$\equiv (p \wedge (p \wedge q) \vee q \wedge (p \wedge q))$

$\equiv (p \wedge q) \vee (p \wedge q) \equiv p \wedge q.$

ii) $(q \vee (p \wedge r)) \wedge \sim ((p \vee r) \wedge q)$

Sol: $(q \vee (p \wedge r)) \wedge \sim ((p \vee r) \wedge q)$

$\equiv (q \vee (p \wedge r)) \wedge (\sim (p \vee r) \vee \sim q)$

$\equiv (q \vee (p \wedge r)) \wedge ((\sim p \wedge \sim r) \vee \sim q)$

$\equiv (q \wedge \sim p \wedge \sim r) \vee (q \wedge \sim q) \vee (p \wedge r \wedge \sim p \wedge \sim r) \vee (p \wedge r \wedge \sim q)$

$\equiv (\sim p \wedge q \wedge \sim r) \vee F \vee F \vee (p \wedge \sim q \wedge r)$

$\equiv (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$ //

Ex: $(p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge r)$

Ex: $p \wedge \sim (q \wedge r) \vee (p \rightarrow q)$

Principal Conjunctive Normal Form (PCNF)

Max term :- For a given number of variables the Maxterm consists of disjunctions in which each variable or its negation but not both appears only once.

Let p and q be the two statements variables Then there are 2^2 maxterms given by $p \vee q$, $p \vee \sim q$, $\sim p \vee q$, $\sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$p \vee q$	$p \vee \sim q$	$\sim p \vee q$	$\sim p \vee \sim q$
T	T	F	F	T	T	T	F
T	F	F	T	T	T	F	T
F	T	T	F	T	F	T	T
F	F	T	T	F	T	T	T

i) No two maxterms are equivalent

ii) Each maxterm has the truth value F for exactly one combination of the truth values of the variables 'p' and 'q'

Def: For a given formula an equivalent formula consisting of conjunctions of maxterms only is called the principal conjunctive Normal form of the formula This normal form is also called the product of sums canonical form.

Ex: obtain the PCNF of the formula $(\sim p \to r) \wedge (q \leftrightarrow p)$

sol: $(\sim p \to r) \wedge (q \leftrightarrow p)$

$$\equiv (p \vee r) \wedge [(q \to p) \wedge (p \to q)]$$

$$\equiv (p \vee r) \wedge [(\sim q \vee p) \wedge (\sim p \vee q)]$$

$$\equiv (p \vee r \vee F) \wedge (\sim q \vee p \vee F) \wedge (\sim p \vee q \vee F)$$

$$\equiv [p \vee r \vee (q \wedge \sim q)] \wedge [\sim q \vee p \vee (r \wedge \sim r)] \wedge [\sim p \vee q \vee (r \wedge \sim r)]$$

$$\equiv (p \vee r \vee q) \wedge (p \vee r \vee \sim q) \wedge (\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r)$$

$$\wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$$

$$\equiv (p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r).$$

which is required PCNF.

Ex :- Find the principal Conjunctive Normal forms of the following statements :

i) $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

Sol: $S = (p \wedge q) \vee (\sim p \wedge q \wedge r) \equiv (p \wedge q \wedge F) \vee (\sim p \wedge q \wedge r)$

$$\equiv (p \wedge q \wedge (r \vee \sim r)) \vee (\sim p \wedge q \wedge r)$$

$$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r)$$

$$\sim S \equiv (p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r)$$

$$\sim(\sim S) \equiv \sim(p \wedge \sim q \wedge r) \wedge \sim(\sim p \wedge \sim q \wedge r) \wedge \sim(\sim p \wedge q \wedge \sim r) \wedge \sim(p \wedge \sim q \wedge \sim r) \wedge \sim(\sim p \wedge \sim q \wedge \sim r)$$

$$\equiv (\sim p \vee q \vee \sim r) \wedge (p \vee q \vee \sim r) \wedge (p \vee \sim q \vee r) \wedge (\sim p \vee q \vee r) \wedge (p \vee q \vee r)$$

which is required PCNF.

ii) $(p \to (q \wedge r)) \wedge (\sim p \to (\sim q \wedge \sim r))$

Sol: $(\sim p \vee (q \wedge r)) \wedge (p \vee (\sim q \wedge \sim r))$

$$\equiv (\sim p \vee q) \wedge (\sim p \vee r) \wedge (p \vee \sim q) \wedge (p \vee \sim r)$$

$$\equiv (\sim p \vee q \vee F) \wedge (\sim p \vee r \vee F) \wedge (p \vee \sim q \vee F) \wedge (p \vee \sim r \vee F)$$

$$\equiv ((\sim p \vee q) \vee (r \wedge \sim r)) \wedge ((\sim p \vee r) \vee (q \wedge \sim q)) \wedge ((p \vee \sim q) \vee (r \wedge \sim r)) \wedge ((p \vee \sim r) \vee (q \wedge \sim q))$$

$$\equiv (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (\sim p \vee r \vee q) \wedge (\sim p \vee r \vee \sim q) \wedge (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (p \vee \sim r \vee q) \wedge (p \vee \sim r \vee \sim q)$$

$$\equiv (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$$

Ordering and uniqueness of Normal Forms :-

In principal disjunctive Normal form it is summation of min terms. Minterm is product of all the variables given in the formula with their negations and minterm must be Unique.

Ex: set A : $p \wedge q$, $\sim p \wedge q$, $p \wedge \sim q$, $\sim p \wedge \sim q$

set B : $q \wedge p$, $\sim q \wedge p$, $q \wedge \sim p$, $\sim q \wedge \sim p$.

The possible minterms of three variables p, q, r

$p \wedge q \wedge r$, $\sim p \wedge q \wedge r$, $p \wedge \sim q \wedge r$, $p \wedge q \wedge \sim r$

$\sim p \wedge \sim q \wedge r$, $\sim p \wedge q \wedge \sim r$, $p \wedge \sim q \wedge \sim r$, $\sim p \wedge \sim q \wedge \sim r$.

If no. of variables in the given formula go on increasing from '3' to 'n'. Let us 'm' for denoting the minterm. The total No. of minterms will be from m_0 to m_{2^n-1}.

If $n=2$ then the total No. of minterms are '4' ie

m_0, m_1, m_2, m_3.

If $n=3$ then the total No. of minterms are '8' ie

$m_0, m_1, m_2, m_3, m_4, m_5, m_6, m_7$.

Ex: What are the possible minterms for Three variables ?

sol: Let p, q, r are the three variables.

The Total No. of minterms are

$m_0 = m_{000} = \phi$

$m_1 = m_{001} = \sim p \wedge \sim q \wedge r$

$m_2 = m_{010} = \sim p \wedge q \wedge \sim r$

$m_3 = m_{011} = \sim p \wedge q \wedge r$

$m_4 = m_{100} = p \wedge \sim q \wedge \sim r$

$m_5 = m_{101} = p \wedge \sim q \wedge r$

$m_6 = m_{110} = p \wedge q \wedge \sim r$

$m_7 = m_{111} = p \wedge q \wedge r$.

PDNF will be above sum of minterms

Theory of Inference for statement calculus:-

The main aim of logic is to provide rules of inference to infer a conclusion from certain premises. The theory associated with rules of inference is known as Inference theory.

Rules of inference :- The following are two important rules of Inferences.

i) **Rule P :-** A premise may be introduced at any point in the derivation.

ii) **Rule T :-** A formula 's' may be introduced in a derivation if 's' is tautologically implied by one or more of the preceding formulas in the derivation.

Implication formulas :-

I_1 : $(p \wedge q) \Rightarrow p$ (simplication)

I_2 : $(p \wedge q) \Rightarrow q$

I_3 : $p \Rightarrow (p \vee q)$ (Addition)

I_4 : $q \Rightarrow (p \vee q)$

I_5 : $\sim p \Rightarrow p \rightarrow q$

I_6 : $q \Rightarrow p \rightarrow q$

I_7 : $\sim (p \rightarrow q) \Rightarrow p$

I_8 : $\sim (p \rightarrow q) \Rightarrow \sim q$

I_9 : $p, q \Rightarrow p \wedge q$ (conjunction)

I_{10} : $\sim p \wedge (p \vee q) \Rightarrow q$ (Disjunctive syllogism)

I_{11} : $p \wedge (p \rightarrow q) \Rightarrow q$ (Modus ponens)

I_{12} : $\sim q \wedge (p \rightarrow q) \Rightarrow \sim p$ (Modus tollens)

I_{13} : $(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$ (Hypothetical Syllogism)

I_{14} : $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \Rightarrow r$ (dilemma)

I_{15} : $(p \vee q) \wedge (\sim p \vee r) \Rightarrow q \vee r$ (Resolution)

EX: Demonstrate that 'r' is a valid inference from the premises $p \to q$, $q \to r$ and p.

Sol:

1. $p \to q$	Rule P	
2. p	Rule P	
3. q	Rule T	(1), (2) and I_{11}
4. $q \to r$	Rule P	
5. r	Rule T	(3), (4) and I_{11}

Hence the result.

EX: Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$.

Sol:

1. $P \vee Q$	Rule P	
2. $\sim P \to Q$	Rule T	(1) $P \to Q \Leftrightarrow \sim P \vee Q$
3. $Q \to S$	Rule P	
4. $\sim P \to S$	Rule T	(2), (3) and I_{13}
5. $\sim S \to P$	Rule T	(4), $P \to Q \Leftrightarrow \sim Q \to \sim P$
6. $P \to R$	Rule P	
7. $\sim S \to R$	Rule T	(5), (6) and I_{13}
8. $S \vee R$	Rule T	(7) $P \to Q \Leftrightarrow \sim P \vee Q$

Hence the result.

EX: Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$, and $\sim M$

1. $P \to M$	Rule P

3. $\sim P$ Rule T (1),(2) and I_{12}

4. $P \vee Q$ Rule P

5. Q Rule T (3),(4) and I_{10}

6. $Q \rightarrow R$ Rule P

7. R Rule T (5),(6) and I_{11}

8. $R \wedge (P \vee Q)$ Rule T (4),(7) and I_9

Hence the result.

Ex: show that $\sim q,\ p \rightarrow q \Rightarrow \sim p$.

Sol:

1. $p \rightarrow q$ Rule P

2. $\sim q \rightarrow \sim p$ Rule T $p \rightarrow q \Leftrightarrow \sim q \rightarrow \sim p$

3. $\sim q$ Rule P

4. $\sim p$ Rule T (2),(3) and I_{11}

Hence the result.

Ex: show that $p \rightarrow q,\ \sim p \rightarrow r,\ r \rightarrow s \Rightarrow \sim q \rightarrow s$

Sol.

1. $p \rightarrow q$ Rule P

2. $\sim q \rightarrow \sim p$ Rule T $p \rightarrow q \Leftrightarrow \sim q \rightarrow \sim p$.

3. $\sim p \rightarrow r$ Rule P

4. $\sim q \rightarrow r$ Rule T (2),(3) and I_{13}

5. $r \rightarrow s$ Rule P

6. $\sim q \rightarrow s$ Rule T (4),(5) and I_{13}

Hence the result.

Ex: show that $R \vee S$ follows logically from the premises $C \vee D,\ (C \vee D) \rightarrow \sim H,\ \sim H \rightarrow (A \wedge \sim B),\ (A \wedge \sim B) \rightarrow R \vee S$.

Sol:

1. $(C \vee D) \rightarrow \sim H$ Rule P

2. $\sim H \rightarrow (A \wedge \sim B)$ Rule P

3. $(C \vee D) \rightarrow (A \wedge \sim B)$ Rule T (1),(2) and I_{13}

4. $(A \wedge \sim B) \rightarrow R \vee S$ Rule P

5. $(C \vee D) \rightarrow R \vee S$ Rule T (3),(4) and I_{13}

6. $C \vee D$ Rule P

7. $R \vee S$ Rule T (5),(6), I_{11}

Hence the result.

Consistency of premises :-

A set of formulas $H_1, H_2, \ldots H_m$ is said to be consistent if their conjunction has the truth value 'T' for some assignment of the truth values to the atomic variables appearing in $H_1, H_2, \ldots H_m$

If for every assignment of the truth values to the atomic variables atleast one of the formulas $H_1, H_2, \ldots H_m$ is false, so that their conjunction is identically false then the formulas $H_1, H_2, \ldots H_m$ are called inconsistent.

ie $H_1 \wedge H_2 \wedge \cdots \wedge H_m \Rightarrow R \wedge \sim R$ (F)

where 'R' is any formula.

Ex: show that the following set of premises is inconsistent " If the contract is valid, then john is liable for penalty. If john is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go brankrupt. As a matter of fact, the contract is valid and the bank will loan him money.".

Sol: Let us indicate the statements as follows.

V = The contract is valid.

L = John is liable for penalty

M = Bank will loan him money

B = John will go Bankrupt.

1.	$V \to L$	Rule P	
2.	$L \to B$	Rule P	
3.	$V \to B$	Rule T	(1),(2) and I_{13}
4.	$M \to \sim B$	Rule P	
5.	$B \to \sim M$	Rule T	(4), $P \to Q \Leftrightarrow \sim Q \to \sim P$
6.	$V \to \sim M$	Rule T	(3),(5) and I_{13}

7. $(\sim V) \vee (\sim M)$ Rule T (6), $P \to q \Leftrightarrow \sim p \vee q$

8. $\sim(V \wedge M)$ Rule T (7) $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

9. $V \wedge M$ Rule P

10. $\sim(V \wedge M) \wedge (V \wedge M)$ Rule T (8), (9) and I_9.

11. F (contradiction)

Hence it is inconsistent.

Indirect method of proof :-

The method of using the rule of conditional proof and the notion of an inconsistent set of premises is called the indirect method of proof (or) proof by contradiction.

In order to show that a conclusion 'c' follows logically from the premises $H_1, H_2, \ldots H_m$. we assume that 'c' is false and consider '$\sim c$' as an additional premise. If the new set of premises is inconsistent, so that they imply a contradiction. Hence 'c' is true whenever $H_1, H_2, \cdots H_m$ are true.

EX: show that $\sim(P \wedge Q)$ follows from $\sim P \wedge \sim Q$.

Sol: we introduce $\sim(\sim(P \wedge Q))$ as additional premise. Now we show that this additional premise leads to a contradiction.

1. $\sim(\sim(P \wedge Q))$ Rule P (assumed)
2. $P \wedge Q$ Rule T, (1) and $\sim(\sim P) \Leftrightarrow P$
3. P Rule T, (2) and I_1
4. $\sim P \wedge \sim Q$ Rule P
5. $\sim P$ Rule T, (4) and I_1
6. $P \wedge \sim P$ Rule T, (3), (5) and I_9

... our assumption is wrong

EX: using the indirect method of proof show that
$P \to Q,\ Q \to R,\ \sim(P \wedge R),\ P \vee R \Rightarrow R$.

Sol: we include $\sim R$ as an additional premise.

1.	$P \to Q$	Rule P
2.	$Q \to R$	Rule P
3.	$P \to R$	Rule T, (1), (2) and I_{13}
4.	$\sim R$	Rule P (assumed)
5.	$\sim P$	Rule T, (3), (4) and I_{12}
6.	$P \vee R$	Rule P
7.	R	Rule T, (5), (6) and I_{10}
8.	$R \wedge \sim R$	Rule T (4), (7) and I_9.
9.	F	(contrdiction)

Hence our assumption is wrong.

EX: show that the following set of premises are incon-sistent using proof by contradiction.
$P \to (Q \vee R),\ Q \to \sim P,\ S \to \sim R,\ P \Rightarrow P \to \sim S$.

Sol: we include $\sim(P \to \sim S)$ as an additional premise.

P. $\therefore \sim(P \to \sim S) \Leftrightarrow \sim(\sim P \vee \sim S) \Leftrightarrow P \wedge S$

1.	$P \to (Q \vee R)$	Rule P
2.	P	Rule P
3.	$Q \vee R$	Rule T (1), (2) and Modus ponens
4.	$P \wedge S$	Rule P assumed
5.	S	Rule T (4) and $P \wedge Q \Rightarrow P$
6.	$S \to \sim R$	Rule P
7.	$\sim R$	Rule T, (5), (6) and Modus ponens
8.	Q	Rule T (3), (7) and I_{10}.
9.	$Q \to \sim P$	Rule P
10.	$\sim P$	Rule T, (8), (9) and I_{11}
11.	$P \wedge \sim P$	Rule T (2), (10) and I_9
	F	(contradiction) Hence given premises

The predicate calculus :-

predicate :- A part of a declarative sentence describing the properties of an object is called a predicate.

We consider the statement " x is greater than 3" has two parts. The first part is the variable 'x' is the subject of the statement. The second part " is greater than 3" is the object of the statement and it is called predicate.

ie $p(x)$: " x is greater than 3"

where 'p' denotes the predicate and 'x' is the variable.

Note : $p(x)$ is not a statement, but just an Expression.

Quantifiers :- Quantifiers are words that are refer to quantities such as "some" (or) "all". We will focus on two types of quantifiers.

1. **Universal quantifier :-** The universal quantification of $p(x)$ is the statement " $p(x)$ for all values of 'x' in the domain".

The notation $\forall x\ p(x)$ denotes the universal quantification of $p(x)$. Here " $\forall$ " is called the universal quantifier. We read as "for all x $p(x)$" (or) "for every x $p(x)$".

2. **Existential quantifier :-** The existential quantification of $p(x)$ is the statement " $p(x)$ there an element 'x' in the domain" such that $p(x)$"

The notation $\exists x\, p(x)$ denotes the Existential quantification of $p(x)$. Here '$\exists$' is called the Existential quantifier. we read as "there exist x $p(x)$ ".

Ex: write the following statements in symbolic form.

i, something is good

ii, Every thing is good

iii, Nothing is good

iv, something is not good.

sol: i, There is atleast one x such that, x is good.

ie $(\exists x)\, p(x)$.

ii, For all x, x is good ie $(\forall x)\, p(x)$

iii, For all x, x is not good, ie $(\forall x) \sim p(x)$

iv, There is atleast one x such that 'x' is not good

ie $(\exists x) \sim p(x)$.

Negations of Quantified statements :-

$$\boxed{\begin{array}{l} \sim(\forall x)\, p(x) \Longleftrightarrow \exists x\,(\sim p(x)) \\ \hline \sim(\exists x)\, p(x) \Longleftrightarrow \forall x\,(\sim p(x)) \end{array}}$$

Ex: Let $p(x)$ denote the statement " x is a professional athlete " and Let $Q(x)$ denote the statement " x plays soccer ". write the each of the following statement in English.

i, $\forall x\,[p(x) \to Q(x)]$

ii, $\exists x\,[p(x) \wedge Q(x)]$

iii, $\forall x\,[p(x) \vee Q(x)]$

sol: i, For all 'x' if 'x' is an professional athlete then x plays soccer.

" All professional athletes plays soccer ".

ii, There Exist an 'x' such that 'x' is a professional athlete and 'x' plays soccer".

" Some professional athlets play soccer ".

iii, For all 'x', x is a professional athlete (or) x plays soccer.

" Every person is either professional athlete or plays soccer ".

EX: Express the statement " Every student in this class has studied calculus " using predicates and Quantifiers.

Sol: We introduce the variable 'x'.

$s(x)$ = The person 'x' is in this class

$c(x)$ = x has studied calculus.

Now our statement can be expressed as

$$\forall x \{s(x) \rightarrow c(x)\}.$$

Inference Theory of the predicate calculus :-

To understand the inference theory of the predicate calculus, it is important to be following rules.

1. Rule US :- Universal specification

$$\forall x \; p(x) \Rightarrow p(c) \quad \text{for all } 'c'.$$

2. Rule ES :- Existential specification

$$\exists x \; p(x) \Rightarrow p(c) \quad \text{for some } 'c'.$$

3. Rule UG :- Universal generalization

$$p(c), \text{ for all } c \Rightarrow \forall x \; p(x).$$

4. Rule EG :- Existential generalization

$$p(c), \text{ for some } 'c' \Rightarrow \exists x\, p(x).$$

Equivalence formulas :-

E_1 : $\exists x\,[A(x) \vee B(x)] \Leftrightarrow \exists x\, A(x) \vee \exists x\, B(x).$

E_2 : $\forall x\,[A(x) \wedge B(x)] \Leftrightarrow \forall x\, A(x) \wedge \forall x\, B(x)$

E_3 : $\sim(\exists(x))\, A(x) \Leftrightarrow \forall x \cdot (\sim A(x))$

E_4 : $\sim(\forall x)\, A(x) \Leftrightarrow \exists x\, (\sim A(x))$

E_5 : $\forall x\,[A \vee B(x)] \Leftrightarrow A \vee \forall x\, B(x)$

E_6 : $\exists x\,[A \wedge B(x)] \Leftrightarrow A \wedge \exists x\, B(x).$

E_7 : $\forall x\, A(x) \rightarrow B \Leftrightarrow \forall x\,(A(x) \rightarrow B)$

E_8 : $\exists x\, A(x) \rightarrow B \Leftrightarrow \exists x\,(A(x) \rightarrow B)$

E_9 : $A \rightarrow \forall x\, B(x) \Leftrightarrow \forall x\,(A \rightarrow B(x))$

E_{10} : $A \rightarrow \exists x\, B(x) \Leftrightarrow \exists x\,(A \rightarrow B(x))$

E_{11} : $\exists x\,[A(x) \rightarrow B(x)] \Leftrightarrow \forall x\, A(x) \rightarrow \exists x\, B(x)$

E_{12} : $\forall x\,[A(x) \rightarrow B(x)] \Leftrightarrow \exists x\, A(x) \rightarrow \forall x\, B(x).$

Ex: Show that $\forall x\,[H(x) \rightarrow M(x)] \wedge H(S) \Rightarrow M(S).$

Sol:

1.	$\forall x\,[H(x) \rightarrow M(x)]$	Rule P
2.	$H(S) \rightarrow M(S)$	Rule US, (1)
3.	$H(S)$	Rule P
4.	$M(S)$	Rule T, (2), (3) and I_{11}.

Ex: Establish the validity of the following argument " All integers are rational numbers. Some integers are powers of '2'. There fore some rational numbers are powers of 2 "

Sol: Let $R(x) =$ x is a rational number

$P(x) =$ x is an integer

$S(x) =$ x is a power of '2'.

Hence the given statements becomes

$$\forall x\,[P(x) \rightarrow R(x)],\ \exists x\,[P(x) \wedge S(x)] \Rightarrow \exists x\,[R(x) \wedge S(x)]$$

1.	$\exists x\,[P(x) \wedge S(x)]$	Rule P
2.	$P(c) \wedge S(c)$	Rule ES, (1)
3.	$P(c)$	Rule T, (2), $P \wedge q \Rightarrow P$
4.	$S(c)$	Rule T, (2) $P \wedge q \Rightarrow q$
5.	$\forall x\,[P(x) \rightarrow R(x)]$	Rule P
6.	$P(c) \rightarrow R(c)$	Rule US, (5)
7.	$R(c)$	Rule T (3), (6), $P \wedge (P \rightarrow q) \Rightarrow q$
8.	$R(c) \wedge S(c)$	Rule T (4), (7), $P, q \Rightarrow P \wedge q$
9.	$\exists x\,[R(x) \wedge S(x)]$	Rule EG, (8)

Hence given statement is valid.

Ex: Show that $\forall x\,[P(x) \rightarrow Q(x)] \wedge \forall x\,[Q(x) \rightarrow R(x)] \Rightarrow$
$\forall x\, P(x) \rightarrow R(x)$.

Sol:

1.	$\forall x\,[P(x) \rightarrow Q(x)]$	Rule P
2.	$P(c) \rightarrow Q(c)$	Rule US, (1)
3.	$\forall x\,[Q(x) \rightarrow R(x)]$	Rule P
4.	$Q(c) \rightarrow R(c)$	Rule US, (3)
5.	$P(c) \rightarrow R(c)$	Rule T, (2), (4) and I_{13}
6.	$\forall x\,[P(x) \rightarrow R(x)]$	Rule UG, (5)

Ex: Show that $\exists x\, M(x)$ follows logically from the premises $\forall x\,[H(x) \rightarrow M(x)]$ and $\exists x\, H(x)$.

Sol:

1. $\exists x\ H(x)$ Rule P
2. $H(c)$ Rule ES, (1)
3. $\forall x[H(x) \rightarrow M(x)]$ Rule P
4. $H(c) \rightarrow M(c)$ Rule US, (3)
5. $M(c)$ Rule T (2), (4) and I_{11}
6. $\exists x\ M(x)$ Rule EG, (5)

Hence the result.

Ex: Show that $\exists x\,[P(x) \wedge Q(x)] \Rightarrow \exists x\, P(x) \wedge \exists x\, Q(x)$

Sol:

1. $\exists x\ [P(x) \wedge Q(x)]$ Rule P
2. $P(c) \wedge Q(c)$ Rule ES, (1)
3. $P(c)$ Rule T, (2) and I_1
4. $\exists x\ P(x)$ Rule EG, (3)
5. $Q(c)$ Rule T, (2) and I_2
6. $\exists x\ Q(x)$ Rule EG, (5)
7. $\exists x\ P(x) \wedge \exists x\, Q(x)$, Rule T, (4), (5) and I_9.

Hence the result.

Ex: Show that $\forall x\,[P(x) \vee Q(x)] \Rightarrow \forall x\, P(x) \vee \exists x\, Q(x)$.

Sol: We shall use the indirect proof of Method by assuming $\sim[\forall x\, P(x) \vee \exists x\, Q(x)]$ as an additional premise.

1. $\sim[\forall x\ P(x) \vee \exists x\ Q(x)]$ Rule P (assumed)
2. $\sim\forall x\ P(x) \wedge \sim\exists x\ Q(x)$ Rule T, $\sim(P \vee Q) \Rightarrow \sim P \wedge \sim Q$
3. $\sim\forall x\ P(x)$ Rule T, (2) and I_1
4. $\exists x(\sim P(x))$ Rule T, (3), Negation rule.
5. $\sim\exists x\ Q(x)$ Rule T, (2) and I_2
6. $\forall x\,(\sim Q(x))$ Rule T, (5) Negation rule.

7. $\sim P(C)$ Rule ES, (4)

8. $\sim Q(C)$ Rule US, (6)

9. $\sim P(C) \wedge \sim Q(C)$ Rule T, (7), (8) and I_9

10. $\sim(P(C) \vee Q(C))$ Rule T, (9) and $\sim(P \vee Q) = \sim P \wedge \sim Q$

11. $\forall x\,(P(x) \vee Q(x))$ Rule P

12. $P(C) \vee Q(C)$ Rule T, US, (11)

13. $\sim(P(C) \vee Q(C)) \wedge (P(C) \vee Q(C))$, Rule T, (10), (12), I_9

14. F Rule T, and (13)

which is a Contradiction, the statement is valid.

Ex: Using predicate logic prove that validity of the following argument " Every husband argues with his wife, x is a husband There fore 'x' argues with his wife ".

Sol: Let $P(x)$: x is a husband

$Q(x)$: x argues with his wife.

Thus we have to show that

$$\forall x\,(P(x) \rightarrow Q(x)) \wedge P(C) \Rightarrow Q(C).$$

1. $\forall x\,(P(x) \rightarrow Q(x))$ Rule P

2. $P(C) \rightarrow Q(C)$ Rule US, (1)

3. $P(C)$ Rule P,

4. $Q(C)$ Rule T, (2), (3) and I_{11}

UNIT - II

SET THEORY.

Basic Concepts of set theory :- A set is any well-defined collection of objects called the elements (or) members of the set. A collection of well-defined objects is called a set. Each of the objects in the set is called a member of an element of the set.

Ex: Books, cities, numbers, animals etc.

Elements of a set are usually denoted by Lower-case letters while sets are denoted by capital letters

Ex: Natural Numbers $N = \{1, 2, 3, 4, \ldots\ldots\}$

The owels $O = \{a, e, i, o, u\}$

Formalization of sets :- There are two ways by which the sets can be described.

1. **Roster Notation :-** Roster Notation is a complete list of all the elements of the set.

   Ex: $B = \{2, 4, 6, 8, \ldots 40\}$

2. **Set - Builder Notation :-** It is used the roster method is impossible.

   Ex: $B = \{ x / 2 \leq x \leq 40, \text{ and } 'x' \text{ is Even} \}$.

The vertical bar '/' is read as such that.

Symbol	Meaning
$\in$	Belongs to
$\forall$	for all

Sub set :- If Every element in a set A is also element of a set B then A is called Sub set of B. It is denoted by $A \subseteq B$.

Note : The null set is a subset of Every set and Every set is a subset of itself.

ie $\phi \subseteq A$ and $A \subseteq A$.

EX: If $A = \{1, 3, 4, 5, 8, 9\}$, $B = \{1, 2, 3, 5, 7\}$, $C = \{1, 5\}$

Then $C \subseteq A$ and $C \subseteq B$.

proper subset :- If A is a subset of B and 'B' is not subset of A then A is called a proper subset of B it is denoted by $A \subset B$.

If A is subset of B then 'B' is called a super set of A and it is denoted by $B \supset A$.

Note : Two different sets are Equal ie $A = B$ if and only if $A \subset B$ and $B \subset A$.

Types of set :-

1. **Null set :-** The set with no elements is called an Empty set (or) Null set. A null set is denoted by ϕ. The null set is a sub set of Every set ie $\phi \subset A$

2. **countable set :-** A countable set is a set with the same cardinality as some subset of the set of natural Numbers.

EX: $R = \{ x / x$ is a natural number $\}$

$\Rightarrow$ 'R' is an infinite set

$A = \{ x / x$ is a Even number between '2' and 40 $\}$

3. uncountable set :- The uncountability of a set is closely related to its cardinal Number, a set is uncountable if its cardinal number is larger than that of the natural numbers.

4. Universal set :- In many discussions all the sets are considered to be subsets of one particular set. This set is called the Universal set for that discussion. The universal set is denoted by 'μ' (or) 'U'

5. power set :- The set of all subsets of a set 'A' is called the power set of A. and it is denoted by P(A). If A has 'n' elements in it then P(A) has 2^n elements.

EX: If $A = \{1, 2, 3\}$ then

$$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\} \{1,3\} \{2,3\}, \{1,2,3\}\}$$

6. Equivalent sets :- Two sets are said to be Equivalent if there is a one-one correspondence between the elements of the two sets.

EX: If $A = \{a, b, c\}$, $B = \{\alpha, \beta, \gamma\}$ then 'A' and 'B' are Equivalent sets.

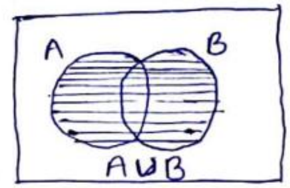
7. Disjoint sets :- Two sets are said to be disjoint if they have no element in common.

EX: If $A = \{1, 2, 3\}$, $B = \{6, 7, 8, 9\}$ then 'A' and 'B' are disjoint sets. Here $A \cap B = \phi$.

Union of two sets :- The union of two sets 'A' and 'B' is the set whose elements are all of the elements "in A" or "in B" or "in both". The union of sets 'A' and 'B' is denoted by $A \cup B$ is read as "A union 'B'".

EX: Let $A = \{1, 2, 3, 4\}$ $B = \{3, 4, 5, 6, 7\}$

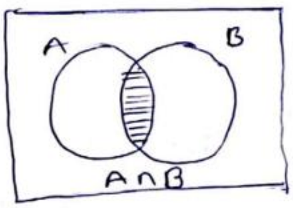
$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$.



Intersection of two sets :- The intersection of two sets 'A' and 'B' is the set whose elements are all of the elements common to both 'A' and 'B'. The intersection of the sets of 'A' and 'B' is denoted by $A \cap B$ is read as "A intersection B".

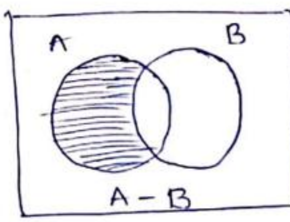
Ex: Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$

$A \cap B = \{3, 4\}$.



Difference of sets :- If A and B are subsets of the universal set μ then the complement of B in A is the set of all elements in A which are not in A. and it is denoted by $A - B$

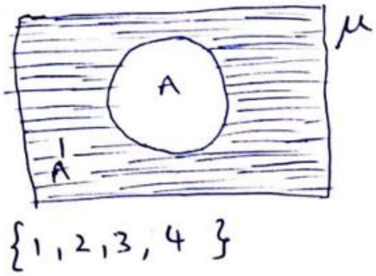
$$A - B = \{ x / x \in A \text{ and } x \notin B \}.$$



EX: If $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 4\}$

$A - B = \{1, \cancel{2}, \cancel{3}, \cancel{4}, 5\} - \{\cancel{2}, \cancel{3}, \cancel{4}\} = \{1, 5\}$

Complement of a set :- If 'μ' is a universal set containing the set A then $\mu - A$ is called the complement of A and it is denoted by A'.



$$\mu - A = A' = \{ x / x \notin A \}.$$

EX: Let $\mu = \{1, 2, 3, 4, 5, \text{-----}\}$, $A = \{1, 2, 3, 4\}$

$B = \{2, 4, 6, 8\}$

$A' = \mu - A = \{5, 6, 7, 8, \dots\}$.

$B' = \mu - B = \{1, 3, 5, 7, 9, 10 \dots\}$.

Inclusion - Exclusion principle :- The inclusion-Exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets. Symbolically Expressed as

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

where $n(A)$ = number of elements in A

$n(B)$ = number of elements in B.

$n(A \cup B)$ = no. of elements both 'A' and 'B'

$n(A \cap B)$ = no. of elements in A and B with common.

For any finite sets A, B, C we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C).$$

Ex: Find the no. of mathematics students at a college taking at least one of the languages Telugu, Hindi and English given the following data.

65 study Telugu , 20 study Telugu and Hindi

45 study Hindi 25 study Telugu and English

42 study English, 15 study Hindi and English.

8 study all three languages.

sol: Given $n(T) = 65$, $n(H) = 45$, $n(E) = 42$

$n(T \cap H) = 20$, $n(T \cap E) = 25$, $n(H \cap E) = 15$,

$n(T \cap H \cap E) = 8$

By inclusion - Exclusion principle

$$n(T \cup H \cup E) = n(T) + n(H) + n(E) - n(T \cap H) - n(T \cap E) - n(H \cap E) + n(T \cap H \cap E)$$

$$= 65 + 45 + 42 - 20 - 25 - 15 + 8$$

Cartesian product :-

Cartesian product is a direct product of sets. Let us consider two sets A and B the set of all order pair (a, b) where $a \in A$ and $b \in B$ is called the product of A and B and is denoted by $A \times B$.

Ex: Let $A = \{1, 2, 3\}$ $B = \{4, 5\}$ then

$A \times B = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$

$B \times A = \{(4,1)\ (5,1), (4,2), (5,2), (4,3), (5,3)\}$

$\therefore\ A \times B \neq B \times A$.

Laws of set theory :-

1. Identity Laws : $A \cup \phi = A$ and $A \cap \mu = A$
2. Domination Laws: $A \cup \mu = \mu$ and $A \cap \phi = \phi$
3. Idempotent Laws : $A \cup A = A$ and $A \cap A = A$
4. Complementation Law : $(A')' = A$
5. Commutative Laws : $A \cup B = B \cup A$ and $A \cap B = B \cap A$
6. Associative Laws : $A \cup (B \cup C) = (A \cup B) \cup C$

   $A \cap (B \cap C) = (A \cap B) \cap C$
7. Distributive Laws : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
8. DeMorgan's Laws : $(A \cup B)' = A' \cap B'$

   $(A \cap B)' = A' \cup B'$.
9. Absorption Laws : $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$
10. Complement Laws : $A \cup A' = \mu$ and $A \cap A' = \phi$.

Ex: prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof: Let 'x' is an arbitrary element of $A \cup (B \cap C)$

Then $x \in A \cup (B \cap C) \Leftrightarrow x \in A$ (or) $x \in (B \cap C)$

$\Leftrightarrow x \in A$ or ($x \in B$ and $x \in C$)

$\Leftrightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)

$\Leftrightarrow x \in (A \cup B)$ and $x \in (A \cup C)$

$\Leftrightarrow x \in (A \cup B) \cap (\cancel{x}A \cup C)$

Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Ex: prove that $(A \cap B)' = A' \cup B'$

Proof: Let $x \in (A \cap B)'$

$\Leftrightarrow x \notin A \cap B$

$\Leftrightarrow \sim (x \in A \cap B) \Leftrightarrow \sim (x \in A$ and $x \in B)$

$\Leftrightarrow \sim (x \in A)$ or $\sim (x \in B)$

$\Leftrightarrow x \notin A$ or $x \notin B \Leftrightarrow x \in A'$ or $x \in B'$

$\Leftrightarrow x \in A' \cup B'$ Hence $(\cancel{x}A \cap B)' = A' \cup B'$.

Ex: prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by using membership table.

Proof:

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
Y	Y	Y	Y	Y	Y	Y	Y
Y	Y	N	Y	Y	Y	N	Y
Y	N	Y	Y	Y	N	Y	Y
Y	N	N	N	N	N	N	N
N	Y	Y	Y	N	N	N	N
N	Y	N	Y	N	N	N	N
N	N	Y	Y	N	N	N	N
[illegible]	N	N	N	N	N	N	N

Relations :- Any set of order pairs defines a binary relation. and it is arbitrary association of elements with in a set or with elements of another set.

Mathematically Let A and B are any two sets. A relation from A to B is a subset of $A \times B$.

Ex: $A = \{a, b\}$ and $B = \{a\}$

$A \times B = \{(a, a), (b, a)\}$

Any subset of $A \times B$ is called relation, ie $(a, a) \in A \times B$

(a, a) is a relation.

Note: 1. $(a, b) \in R$ then we say "a is related to b" and it can be written as "aRb" (or) $R(a, b)$

2. $(a, b) \notin R$ then we say "a is not related to b" and it can be written as "$a \not R b$".

ordered pair :- If $a \in B$ and $b \in B$ then the ordered pair is the set $\{\{a\}, \{a, b\}\}$ consisting of the pair $\{a, b\}$ and the singleton $\{a\}$. It is represented by (a, b).

In the order pair (a, b), the element 'a' is called the first element and 'b' is called the second element.

If (a, b) and (a', b') are two ordered pairs then.

$$(a, b) = (a', b') \Rightarrow a = a' \text{ and } b = b'.$$

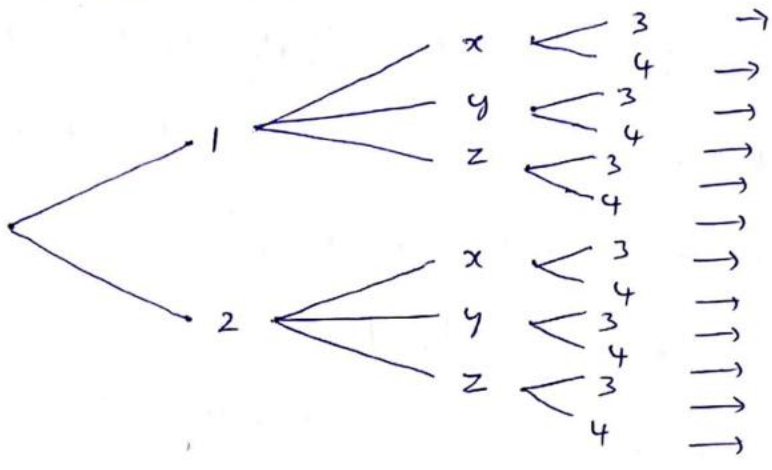
Note: 1. (a, b) and (b, a) are two different ordered pairs

$$(a, b) \neq (b, a).$$

2. The two objects in an ordered pairs do not have to be distinct. ie (a, a) is a well-defined ordered

Ex: Let $A = \{1, 2\}$, $B = \{x, y, z\}$, $C = \{3, 4\}$ then find $A \times B \times C$.

Sol: $A = \{1, 2\}$, $B = \{x, y, z\}$, $C = \{3, 4\}$



$(1, x, 3)$
$(1, x, 4)$
$(1, y, 3)$
$(1, y, 4)$
$(1, z, 3)$
$(1, z, 4)$
$(2, x, 3)$
$(2, x, 4)$
$(2, y, 3)$
$(2, y, 4)$
$(2, z, 3)$
$(2, z, 4)$

Domain and Range of relations :-

The domain of a relation R is the set of all first elements of the ordered pair which belongs to R and the Range of 'R' is the set of second element.

1. Domain $(R) = \{ a \in A / aRb$ for some $b \in B \}$
2. Range $(R) = \{ b \in B / aRb$ for some $a \in A \}$.

Ex: If $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ then find domain and Range. If $R = \{(1, a), (1, c), (2, b), (3, a)\}$.

Sol: Given $A = \{1, 2, 3\}$, $B = \{a, b, c\}$

$A \times B = \{(1, a), (1, b) (1, c), (2, a) (2, b) (2, c), (3, a) (3, b) (3, c)\}$

Given Relation $R = \{(1, a), (1, c), (2, b), (3, a)\}$

Domain R = The set of first elements in R

$= \{1, 2, 3\}$

Range R = The set of second elements in R

$= \{a, c, b\}$

properties of Binary Relations :-

1. **Universal relation** :- A relation R on a set A is Universal if $R = A \times A$.

Ex: If $A = \{1,2,3\}$ then $A \times A = \{(1,1), (1,2), (1,3)\ (2,1)\ (2,2)\ (2,3), (3,1)\ (3,2), (3,3)\}$ is the universal relation on A.

2. **Void relation** :- A relation R on a set A is void relation if R is the null set. ϕ.

Ex: If $A = \{3,4,5\}$ then $R = \{a+b > 10\}$ is a null set

3. **Identity relation** :- A relation R on a set A is Identity relation If $R = \{(a,a) / a \in A\}$ and it is denoted by I_A. Ex: If $A = \{1,2,3\}$ then $R = \{(1,1)\ (2,2)\ (3,3)\}$.

4. **Inverse relation** :- If 'R' is any relation from a set 'A' to set 'B' the inverse of R is denoted by R^{-1}.

$$R^{-1} = \{(y,x) / y \in B,\ x \in A,\ (x,y) \in R\}$$

Ex: If $A = \{2,3,5\}$, $B = \{6,8,10\}$ then $R = \{(2,6), (2,8), (2,10)\ (3,6), (5,10)\}$

Now $R^{-1} = \{(6,2), (8,2), (10,2), (6,3), (10,5)\}$

5. **Reflexive relation** :- A relation on a set A is reflexive if aRa for every $a \in A$ ie If $(a,a) \in R,\ \forall x \in a$

Ex: If $A = \{1,2,3,4\}$ then $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$\therefore$ 'R' is called reflexive relation. otherwise it is called Irreflexive. Ex: $R = \{(1,1), (2,2), (2,3), (4,4)\}$.

6. Symmetric relation :- A relation R on a set A is called symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ for all $a, b \in A$.

7. Anti symmetric relation :- R is an anti symmetric if there exists (a,b) and $(b,a) \in R$ then $a = b$.

8. Transitive relation :- A relation R on a set A is transitive if whenever aRb and bRc then aRc.
ie $(a,b), (b,c) \in R$ then $(a,c) \in R$.

EX: If $A = \{1,2,3,4\}$ and $R = \{(1,1)\ (1,3), (2,4), (3,1)\ (3,3)\ (4,3)\}$ then find if relation is reflexive, symmetric and transitive.

Sol: $R = \{(1,1), (1,3), (2,4), (3,1), (3,3), (4,3)\}$

i) Reflexive : $R \cup \{(2,2), (4,4)\}$

$\therefore R = \{(1,1), (1,3), (2,4), (3,1), (3,3), (4,3), (2,2), (4,4)\}$

ii) Symmetric :- $R \cup \{(4,2), (3,4)\}$

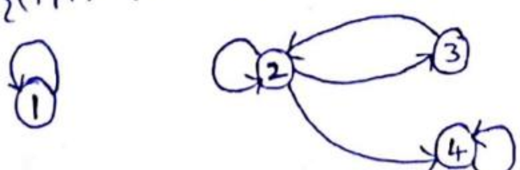
$\therefore R = \{(1,1), (1,3), (2,4), (3,1), (3,3), (4,3), (4,2), (3,4)\}$

iii) Transitive :- $R \cup \{(2,3), (4,1)\}$

$\therefore R = (1,1), (1,3), (2,4), (3,1), (3,3), (4,3), (2,3), (4,1)\}$

EX: If $A = \{1,2,3,4\}$ and $R = (1,1), (2,2), (2,3), (3,2), (4,2)$ $(4,4)\}$ then draw its directed graph.

Sol: $R = \{(1,1)\ (2,2)\ (2,3)\ (3,2), (4,2), (4,4)\}$



Matrix representation of relations :-

Let A and B are both finite sets and 'R' is a relation from A to B. Then R may be represented as a matrix called the relation matrix of R. It is denoted by M_R.

Let $A = \{a_1, a_2, \ldots a_m\}$ and $B = \{b_1, b_2 \ldots b_n\}$ are two finite sets containing 'm' and 'n' elements respectively.

$M_R = [m_{ij}]_{m \times n}$ is defined by

$$m_{ij} = \begin{cases} 0 & \text{if } (a_i, b_j) \in R \\ 1 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

Ex: Let $A = \{1, 2, 3, 4\}$ and $R = \{(x, y) / x > y\}$ find M_R.

Sol: Given $A = \{1, 2, 3, 4\}$

$R = \{(4,1)\ (4,3)\ (4,2),\ (3,1)\ (3,2),\ (2,1)\}$

The corresponding relation matrix for the R are given

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{4 \times 4}. \end{array}$$

Ex: If $A = \{1, 2, 3, 4\}$ defined by $R = (1,1)\ (2,2)\ (3,3),\ (4,4),\ (4,3),\ (4,2),\ (4,1),\ (3,2),\ (3,1)\}$. Find the matrix and directed graph of relation R.

Sol: The matrix of relation R is a 4×4 matrix is

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Directed graph is

Equivalence relations :- A relation R in a set A is called an Equivalence relation if it is reflexive, symmetric and transitive.

Let a, b, c are arbitrary elements of some set A. Then "$a \sim b$" (or) "$a \equiv b$" denotes 'a' is Equivalent to 'b'.

i) Reflexive : $a \sim a$

ii) Symmetric : If $a \sim b$ then $b \sim a$

iii) Transitive : If $a \sim b$ and $b \sim c$ then $a \sim c$.

Ex: Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,4), (4,1) (4,4), (2,2), (2,3), (3,2), (3,3)\}$. prove that 'R' is an Equivalence.

Sol: Given $R = \{(1,1)\ (1,4)\ (4,1)\ (4,4)\ (2,2)\ (2,3)\ (3,2)\ (3,3)\}$

i) Reflexive :- For any $x \in A \Rightarrow (x, x) \in R$

$\therefore xRx \Rightarrow$ 'R' is reflexive. $\because \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$

ii) Symmetric :- For any $x, y \in A \Rightarrow$ If $(x, y) \in R$ then $(y, x) \in R$

$\because (1,4) \to (4,1)$ and $(2,3) \to (3,2)$

$\therefore xRy \Rightarrow yRx \Rightarrow$ 'R' is symmetric.

iii) Transitive :- For any $x, y, z \in A$

If (x, y) and $(y, z) \in R$ then $(x, z) \in R$

$\therefore xRy$ and $yRz \Rightarrow xRz$. 'R' is transitive.

Hence the relation 'R' is an Equivalence Relation.

$\because (1,1), (1,4) \to (1,4)$
$(2,3), (3,2) \to (2,2)$
$(3,2)\ (2,3) \to (3,3)$
etc...

Congruence Relation :- Congruence relation is an example of equivalence relation. Let 'a' and 'b' are any two integers and 'r' be a fixed positive integer then 'a' and 'b' are said to be congruent modulo 'r' ie Mathematically $a \equiv b \pmod{r}$ if 'r' devides $(a-b)$.

Ex: $83 \equiv 13 \pmod{5}$

since $83 - 13 = 70$ is divisible by '5'

Ex: Prove that the relation " Congruence modulo m" over the set of positive integers is an Equivalence relation

Sol: Let $x, y \in N$, $x \equiv y \pmod{m}$ if and only if $x-y$ is divisible by 'm'.

i) Let $x, y, z \in N$ then

$x - x = 0 \cdot m \Rightarrow x \equiv x \pmod{m}$ for all $x \in N$

ii) Let $x \equiv y \pmod{m}$. Then $x-y$ is divisible by 'm'

$\Rightarrow -(x-y) = y-x$ is also divisible by 'm'

ie $y \equiv x \pmod{m}$

$\therefore$ The relation $\equiv$ is symmetric.

iii) Let $x \equiv y \pmod{m}$ and $y \equiv z \pmod{m}$

ie $(x-y)$ and $(y-z)$ are divisible by 'm'

Now $(x + \cancel{y}) + (\cancel{y} - z)$ is divisible by 'm'.

ie $x-z$ is divisible by 'm'

$\Rightarrow x \equiv z \pmod{m}$ $\therefore$ The relation $\equiv$ is transitive

Hence '$\equiv$' is Equivalence relation.

Ex: Let 'R' denote a relation on the set of order pairs of positive integers such that $(x,y) R (u,v)$ iff $xv = yu$ show that R is an Equivalence relation.

Sol: Let x, y, u and v be positive integers

Given $(x,y) R (u,v) \Leftrightarrow xv = yu$

i) since $xy = yx$ is true for all positive integers

$\Rightarrow (x,y) R (x,y)$

$\therefore$ The reation R is reflexive.

ii) Let $(x,y) R (u,v) \Rightarrow xv = yu$

$\Rightarrow yu = xv \Rightarrow uy = vx$

$\Rightarrow (u,v) R (x,y)$

$\therefore$ The relation 'R' is symmetric.

iii) Let x, y, u, v, m and 'n' are positive integers

Let $(x,y) R (u,v)$ and $(u,v) R (m,n)$

$\Rightarrow xv = yu$ and $un = vm$

$\Rightarrow xvun = yuvm \Rightarrow xn = ym$ $\because$ by cancelling uv

$\Rightarrow (x,y) R (m,n)$.

$\therefore$ The relation 'R' is transitive.

Hence The relation 'R' is Equivalence relation.

Ex: N is the set of Natural numbers. The relation 'R' is defined by $(a,b) R (c,d) \Leftrightarrow a+d = b+c$. plove that R is an Equivalence relation.

Sol: Let a, b, 'c' and 'd' be natural numbers.

Given $(a,b) R (c,d) \Leftrightarrow a+d = b+c$.

i) $(a,b) R (a,b) \Rightarrow a+b = b+a$

R is reflexive.

ii) $(a,b)\,R\,(c,d) \Rightarrow a+d = b+c$

$\Rightarrow b+c = a+d$

$\Rightarrow c+b = d+a \Rightarrow (c,d)\,R\,(a,b)$

$\therefore$ 'R' is symmetric

iii) $(a,b)\,R\,(c,d)$ and $(c,d)\,R\,(m,n)$

$\Rightarrow a+d = b+c$ and $c+n = d+m$

$\Rightarrow a+\cancel{d}+\cancel{c}+n = b+\cancel{c}+\cancel{d}+m$

$\Rightarrow a+n = b+m \Rightarrow (a,b)\,R\,(m,n)$

$\therefore$ 'R' is transitive

Hence 'R' is an Equivalence relation.

Compatibility relation :- A Relation 'R' in X is said to be a compatibility relation if it is reflexive and symmetric. So all equivalence relations are compatibility relations and it is sometimes denoted by "$\approx$".

Composition of relations :- Let 'R' be a relation from X to Y and 'S' be a relation from Y to Z. Then a relation written as RoS is called a composite relation of 'R' and 'S' where $RoS = \{(x,z) / x \in X, z \in Z$ there exist $y \in Y$ with $(x,y) \in R$ and $(y,z) \in S\}$.

Ex: Let $R = \{(1,2)\ (3,4)\ (2,2)\}$ and $S = \{(4,2), (2,5), (3,1), (1,3)\}$. Find RoS and SoR.

Sol: Given $R = \{(1,2), (3,4), (2,2)\}$

$S = \{(4,2), (2,5), (3,1), (1,3)\}$

$\therefore RoS = \{(1,5), (3,2), (2,5)\}$

$SoR = \{(4,2), (3,2), (1,4)\}$ clearly $RoS \neq SoR$.

partial ordering :- A binary relation 'R' in P is called a partial order relation (or) a partial ordering in P iff R is reflexive, antisymmetric and transitive. ie

1. aRa for all $a \in P$
2. aRb and $bRa \Rightarrow a=b$
3. aRb and $bRc \Rightarrow aRc$

Ex: show that the relation "greater than or equal to" is a partial ordering on the set of integers..

sol: Let 'Z' be the set of all integers and the relation $R = $ "$\geq$".

i) since $a \geq a$ for every integer a,
the relation '$\geq$' is reflexive

ii) Let 'a' and 'b' be any two integers
Let aRb and $bRa \Rightarrow a \geq b$ and $b \geq a \Rightarrow a=b$
∴ The relation "$\geq$" is antisymmetric.

iii) Let a, b and 'c' be any three integers.
Let aRb and $bRc \Rightarrow a \geq b$ and $b \geq c. \Rightarrow a \geq c$
∴ The relation '$\geq$' is transitive.

Hence the relation '$\geq$' is partial ordering on the set of integers ∴ $(Z, \geq)$ is a poset.

poset (partially ordered set) :- A set 'P' together with a partial ordering 'R' is called a partial ordered set or poset. The relation R is often denoted by the symbol '$\leq$' then the ordered pair $(P, \leq)$ is called poset.

Hasse diagrams :- A partial order $\leq$ on a set P can be represented by means of a diagram known as Hasse diagram of $(P, \leq)$.

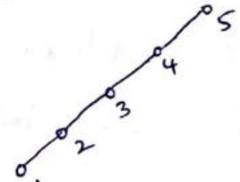
i) Each element is represented by a small circle or dot

ii) The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$ and a line is drawn between x and y if y covers 'x'.

iii) If $x < y$ but y does not cover 'x' then x and y are not connected directly by a single line.

EX: Let $P = \{1, 2, 3, 4, 5\}$ and $\leq$ be the relation "Less than or Equal to" then the Hasse diagram is drawn below.

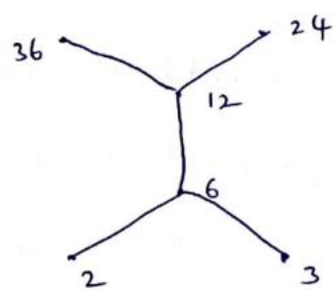
Sol:



It is totally order set.

EX: Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$ if x divides y. Draw the Hasse diagram of $(X, \leq)$.

Sol:



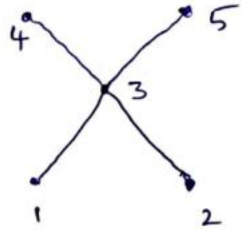
it is not a total order set.

EX: Draw the Hasse diagram for the relation R on $A = \{1, 2, 3, 4, 5\}$ whose relation matrix given below.

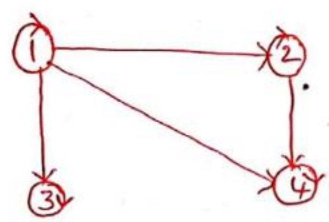
$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Sol: $R = \{ (1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (5,5) \}$.

Hasse diagram for M_R is



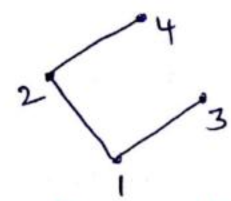
Ex: A partial order 'R' on the set $A = \{1,2,3,4\}$ is represented by the following digraph. Draw the Hasse diagram for R.



Sol: By Examining the given digraph we find that

$R = \{ (1,1) (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4) \}$.

∴ Hasse diagram is

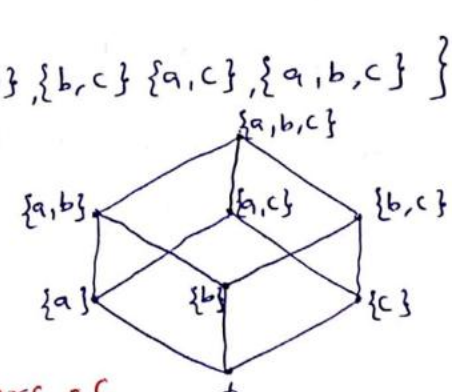


Ex: Draw the Hasse diagram for the partial ordering $\subseteq$ on the power set $P(S)$ where $S = \{a,b,c\}$

Sol: Given $S = \{a,b,c\}$

$P(S) = \{ \phi, \{a\}, \{b\} \{c\}, \{a,b\}, \{b,c\} \{a,c\}, \{a,b,c\} \}$.

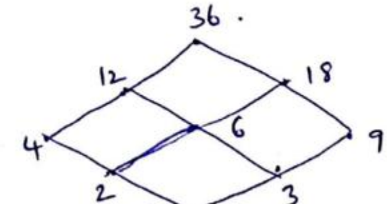
∴ Hasse diagram is



Ex: Draw the Hasse diagram representing the positive divisors of 36.

Sol: $D_{36} = \{ 1,2,3,4,6,9,12,18,36 \}$

∴ Hasse diagram is

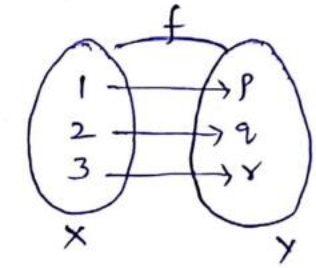


Functions :- Let X and Y be any two sets. A relation 'f' from 'X' to 'Y' is called a function if for every $x \in X$ there is a unique element $y \in Y$ such that $(x, y) \in f$.

Ex: Let $X = \{1, 2, 3\}$, $Y = \{p, q, r\}$ and $f = \{(1, p), (2, q), (3, r)\}$

Then $f(1) = p$, $f(2) = q$, $f(3) = r$

clearly 'f' is a function from $X \to Y$



Domain and range of a function :-

If $f : X \to Y$ is a function then 'X' is called the Domain of 'f' and the set 'Y' is called the codomain of f.

The range of 'f' is defined as the set of all images under 'f'. It is denoted by $f(x) = \{ y \mid \text{for some } x \text{ in } X, f(x) = y \}$

Ex: If the function 'f' is defined by $f(x) = x^2 + 1$ on the set $\{-2, -1, 0, 1, 2\}$ find the range of 'f'.

sol: Given $f(x) = x^2 + 1$

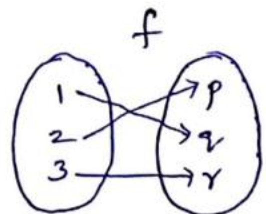
$f(-2) = (-2)^2 + 1 = 5$, $f(0) = 0 + 1 = 1$ $f(2) = 4 + 1 = 5$

$f(-1) = (-1)^2 + 1 = 2$ $f(1) = 1 + 1 = 2$,

$\therefore$ The range of 'f' $= \{1, 2, 5\}$.

Types of functions :-

1. **one-to-one function (Injection) :-** A mapping $f : X \to Y$ is called one-to-one if distinct elements of 'X' are mapped into distinct elements of Y



i.e f is one-to-one if

Ex: $f : R \to R$ defined by $f(x) = 3x$, $\forall x \in R$ is one-one, since

$f(x_1) = f(x_2) \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2$, $\forall x_1, x_2 \in R$.

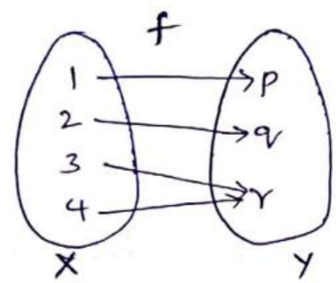
2. **onto function (Surjection) :-** A mapping $f : X \to Y$ is called onto if the range set $R_f = Y$.

If $f : X \to Y$ is onto, then each element of Y is f-image of atleast one element of X.

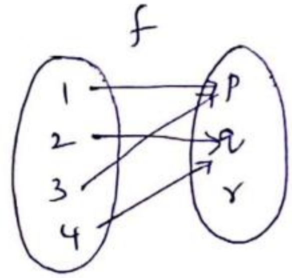
Mathematically $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.

Note: If 'f' is not onto then it is said to be into.

Ex:



Surjective



not surjective.

3. **Bijection function :-** A mapping $f : X \to Y$ is called Bijective function if it is one-to-one and onto.

Such a mapping is also called a one-to-one correspondence between 'X' and 'Y'.

Ex: Show that a mapping $f : R \to R$ defined by $f(x) = 2x+1$ for $x \in R$ is a bijective map from R to R.

Sol: Let $f : R \to R$ defined by $f(x) = 2x+1$, for $x \in R$.

i) one-to-one : Let 'x' and 'y' be any two elements in R such that $f(x) = f(y)$

$\Rightarrow 2x+1 = 2y+1$

$\therefore f(x) = f(y) \Rightarrow x = y$

$\therefore f$ is one-to-one

ii) **onto** :- Let 'y' be any element in the domain R

$\Rightarrow f(x) = y \quad \Rightarrow 2x+1 = y \quad \Rightarrow x = \frac{(y-1)}{2}$

clearly $x = \frac{(y-1)}{2} \in R$. $\therefore f$ is onto

Hence 'f' is a bijective map.

Composition of Functions :- Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then the composition of 'f' and 'g' denoted by gof is the function from 'x' to 'z' defined as $(gof)(x) = g(f(x))$. for all $x \in X$.

Ex :- Let $f(x) = x+2$, $g(x) = x-2$, and $h(x) = 3x$ for $x \in R$ where R is the set of real numbers. Find gof, fog, fof, gog, foh, hog, hof and $fogoh$.

Sol: Given $f(x) = x+2$, $g(x) = x-2$, $h(x) = 3x$

i) $(gof)(x) = g(f(x)) = g(x+2) = x+\cancel{2}-\cancel{2} = x$

ii) $(fog)(x) = f(g(x)) = f(x-2) = x-\cancel{2}+\cancel{2} = x$

iii) $(fof)(x) = f(f(x)) = f(x+2) = x+2+2 = x+4$

iv) $(gog)(x) = g(g(x)) = g(x-2) = x-2-2 = x-4$

v) $(foh)(x) = f(h(x)) = f(3x) = 3x+2 = 3x+2$

vi) $(hog)(x) = h(g(x)) = h(x-2) = 3(x-2) = 3x-6$

vii) $(hof)(x) = h(f(x)) = h(x+2) = 3(x+2) = 3x+6$.

viii) $(fogoh)(x) = f((goh)(x)) = f(g(h(x))) = f(g(3x))$

$= f(3x-2) = 3x-\cancel{2}+\cancel{2} = 3x$

Ex: If $f(x) = x^2+3x+1$ and $g(x) = 2x-3$ then find the composition of i) fof ii) fog iii) gof.

Sol: . . . .

Inverse functions :- A function $f: X \to Y$ is said to be invertible of its inverse function f^{-1} is also function from the range of 'f' into 'x'.

If $f(x) = y$ then $x = f^{-1}(y)$.

Ex: Let $f: R \to R$ be given by $f(x) = x^3-2$. Find f^{-1}.

Sol: i) Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow x_1^3 - 2 = x_2^3 - 2 \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

$\therefore f$ is one-to-one.

ii) If $f(x) = y$ then $y = x^3 - 2 \Rightarrow x^3 = y+2$

$\Rightarrow x = \sqrt[3]{y+2}$ $\therefore$ 'f' is onto.

$\Rightarrow f^{-1}$ is invertible and $f^{-1}(x) = \sqrt[3]{x+2}$.

Ex: $f: R \to R$ is defined by $f(x) = ax+b$ for $a, b \in R$. Show that 'f' is invertible and find the inverse of 'f'.

Sol: 'f' is invertible and $f^{-1}(x) = \frac{x-b}{a}$

Floor and ceiling functions :- Let 'x' be a real number, then the least integer that is not less than 'x' is called the ceiling of x. The ceiling of x is denoted by $\lceil x \rceil$.

Ex: $\lceil 2.15 \rceil = 3$, $\lceil \sqrt{5} \rceil = 3$, $\lceil -7.4 \rceil = -7$, $\lceil -2 \rceil = -2$

Let 'x' be a real number, then the greatest integer that does not exceed 'x' is called the 'Floor' of 'x' is denoted by $\lfloor x \rfloor$.

Ex: $\lfloor 5.14 \rfloor = 5$ $\lfloor \sqrt{5} \rfloor = 2$, $\lfloor -7.6 \rfloor = 8$

Recursive function :- A recursive function is a function which is defined in terms of itself. We can also define functions recursively.

A recursive definition has two parts.

1. Definition of the smallest argument ie $f(0)$ or $f(1)$
2. Definition of $f(n)$, given $f(n-1)$, $f(n-2)$ etc...

Ex: An example of a recursively defined function.

$f(0) = 3$, $f(n+1) = 2f(n) + 3$. Then

$f(1) = 2f(0) + 3 = 2\cdot(3) + 3 = 9$

$f(2) = 2f(1) + 3 = 2\cdot(9) + 3 = 21$

$f(3) = 2f(2) + 3 = 2(21) + 3 = 45$

$f(4) = 2f(3) + 3 = 2(4.5) + 3 = 93$ .....

Ex: calculate the values of the following recursively defined function $f(0) = 1$ and $f(n) = n\,f(n-1)$.

Sol: Given $f(0) = 1$ and $f(n) = n\,f(n-1)$

$f(1) = 1\cdot f(0) = 1\cdot 1 = 1$

$f(2) = 2\cdot f(1) = 2\cdot 1 = 2$

$f(3) = 3\cdot f(2) = 3\cdot 2 = 6$

$f(4) = 4\cdot f(3) = 4\cdot 6 = 24$ ....

This is the recursive definition of the factorial function, $f(n) = n!$.

Ex: $f(1) = 1$, $f(2) = 1$ and $f(n) = f(n-2) + f(n-1)$

Ex: $f(0) = 0$, $f(n) = f(n-1) + 2n - 1$

Ex: $f(0) = 5$, $f(n) = f(n-1) + 2$.

Ex: Consider the following recursive function

If $x < y$ then $f(x,y) = 0$, if $y \leq x$ then $f(x,y) = f(x-y, y)+1$

Find the value of $f(4,7)$ and $f(19,6)$.

Sol: Given $f(x,y) = \begin{cases} 0, & x < y \\ f(x-y, y)+1, & y \leq x. \end{cases}$

i) $f(4,7) = 0$.

ii) $f(19,6) = f(19-6, 6)+1 = f(13,6)+1$

$f(13,6) = f(13-6, 6)+1 = f(7,6)+1$

$f(7,6) = f(7-6, 6)+1 = f(1,6)+1 = 0+1 = 1$

$f(13,6) = f(7,6)+1 = 1+1 = 2$.

Now $f(19,6) = f(13,6)+1 = 2+1 = 3$ //.

LATTICES :- In this section, we introduce Lattices which have important applications in the theory and design of computers.

Def :- A Lattice is a partially ordered set $(L, \leq)$ in which every subset consisting of pair of elements has a lowest upper bound and greatest lower bound.

A Lattice 'L' is an algebraic system with binary operations '$\vee$' and '$\wedge$' It is denoted by $(L, \wedge, \vee)$.

Thus 'L' is called lattice if the following axioms hold where $a, b, c \in L$.

i) Commutative Law : $(a \wedge b) = (b \wedge a)$, $(a \vee b) = (b \vee a)$

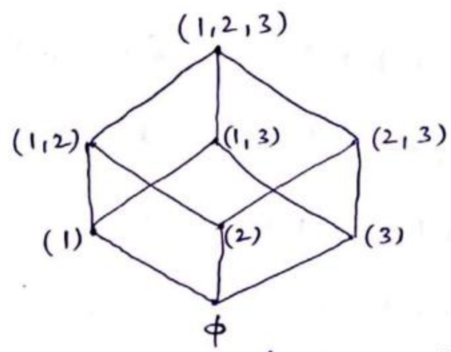
ii) Associative Law : $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,
$a \vee (b \vee c) = (a \vee b) \vee c$

iii) Absorption Law : $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$.

The LUB is denoted by $(a \vee b)$ and is called "join" of 'a' and 'b'. The 'GLB' is denoted by $(a \wedge b)$ and is called "meet" of 'a' and 'b'.

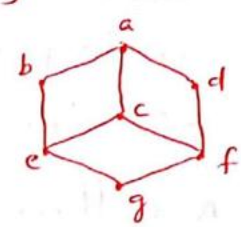
<u>EX:</u> Let $X = \{1,2,3\}$ then the poset $(P(X), \subseteq)$ of all Sub sets of X is a Lattice 'or' not

<u>Sol:</u> $P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}\ \{1,3\}\ \{2,3\}, \{1,2,3\}\}$



Each subset has a lub and glb. So it is Lattice.

<u>EX:</u> Determine whether the following Hasse diagram represent Lattice or not.



<u>Sol:</u>

<u>LUB:</u>

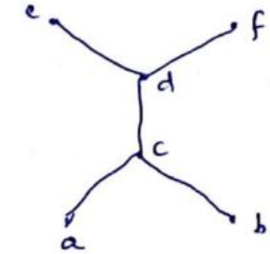
$\vee$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	a	b	a	b
c	a	a	c	a	c	c	c
d	a	a	a	d	a	d	d
e	a	b	c	a	e	c	e
f	a	a	c	d	c	f	f
g	a	b	c	d	e	f	g

Each subset of two elements has a least upper bound and a greatest lower bound so it is a lattice.

<u>GLB:</u>

$\wedge$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	e	g	e	g	g
c	e	e	c	f	e	f	g
d	d	g	f	d	g	f	g
e	e	e	e	g	e	g	g
f	f	g	f	f	g	f	g
g	g	g	g	g	g	g	g

EX: The following diagram is a not a lattice because GLB (a,b) and LUB (e,f) does not exist.



Some properties of Lattices :-

If 'L' is any Lattice then $\forall$ a, b, c $\in$ L

i) $a \wedge a = a$, $a \vee a = a$ Idempotent

ii) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$ Commutative.

iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
$a \vee (b \vee c) = (a \vee b) \vee c$ Associative.

iv) $a \wedge (a \vee b) = a$
$a \vee (a \wedge b) = a$ Absorption.

v) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ Distributive.

Duality Law in Lattice :- The dual of any statement in a Lattice $(L, \wedge, \vee)$ is defined to be the statement that is obtained by interchanging '$\wedge$' and '$\vee$'

EX: The dual of $a \wedge (b \vee c) = a \vee (b \wedge c)$

vi) $a \leq b$, $c \leq d \Leftrightarrow a \wedge c \leq b \wedge d$, $a \vee c \leq b \vee d$.

vii) $a \wedge b \leq a$, $b \leq a \vee b$

EX: Let $(L, \leq)$ be a Lattice in which '$\wedge$' and '$\vee$' denote the operations of meet and join respectively. For any $a \in L$, $a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$.

Sol: i) We shall first prove that $a \leq b \Leftrightarrow a \wedge b = a$

Let us assume that $a \leq b$, Also we know that $a \leq a$

$\Rightarrow a \leq a \wedge b$ {∵ From $a \wedge b \leq a$

Hence $a \leq b \Rightarrow a \wedge b = a$

Next assume that $a \wedge b = a$, But it is only possible if $a \leq b$, that is $a \wedge b = a \Rightarrow a \leq b$.

Finally $a \leq b \Rightarrow a \wedge b = a$ and

$a \wedge b = a \Rightarrow a \leq b$

Hence $a \leq b \Leftrightarrow a \wedge b = a$.

Similarly we can prove that $a \leq b \Leftrightarrow a \vee b = b$.

Ex: Let $(L, \leq)$ be a lattice Then $b \leq c \Rightarrow \begin{cases} a \wedge b \leq a \wedge c \\ a \vee b \leq a \vee c \end{cases}$

Sol: By above Theorem $a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$.

Now we have to show that $b \leq c \Rightarrow a \wedge b \leq a \wedge c$

We shall show that $(a \wedge b) \wedge (a \wedge c) = a \wedge b$

$$\begin{aligned}
\Rightarrow (a \wedge b) \wedge (a \wedge c) &= a \wedge (b \wedge a) \wedge c \\
&= a \wedge (a \wedge b) \wedge c \\
&= \cancel{(a \wedge a) \wedge (b \wedge c)} \\
&= (a \wedge a) \wedge (b \wedge c) \\
&= a \wedge (b \wedge c) \\
&= a \wedge b
\end{aligned}$$

∴ If $b \leq c$ then $a \wedge b \leq a \wedge c$.

Similarly we can prove that

If $b \leq c$ then $a \vee b \leq a \vee c$.

Some special Lattices :-

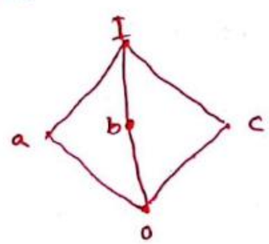
1. **Distributive Lattices :-** A Lattice $(L, \wedge, \vee)$ is distributive if the following additional identity holds for all x, y and z in L.

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{and}$$

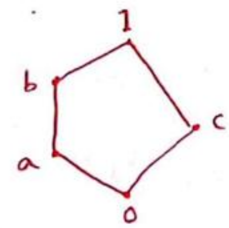
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Ex: Show that the following simple But ~~no~~ significant Lattices are not distributive.

i,



ii,



Sol: i, The diamond Lattice is not distributive.

Now $a \wedge (b \vee c) = a \wedge I = a$ but

$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

$\therefore a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$.

ii, The pentagon Lattice is also not distributive.

for these three Elements ..

$a \wedge (b \vee c) = a \wedge (I) = a \qquad a \neq 0$.

$(a \wedge b) \vee (a \wedge c) = a \vee 0 = 0$

Ex: Consider the poset $(L, \leq)$ where $L = \{1, 2, 3, 5, 30\}$ and partial ordered relation $\leq$ is defined as if x and $y \in L$ then $x \leq y$ means 'x divides y' Then show that poset is a Lattice.

Sol: since $GLB(x,y) = x \wedge y = lcm(x,y)$

$LUB(x,y) = x \vee y = gcd(x,y)$

Now we can construct the operation table I and II for GLB and LUB respectively and the Hasse diagram is shown in Fig.

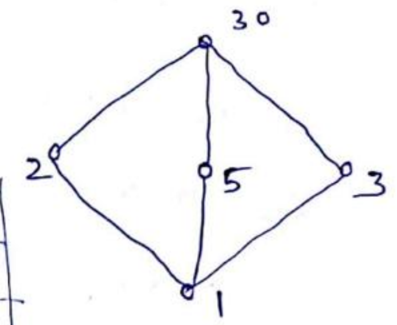


Table I :

LUB	1	2	3	5	30
1	1	2	3	5	30
2	2	2	30	30	30
3	3	30	3	30	30
5	5	30	30	5	30
30	30	30	30	30	30

Table II

GLB	1	2	3	5	30
1	1	1	1	1	1
2	1	2	1	1	2
3	1	1	3	1	3
5	1	1	1	5	5
30	1	2	3	5	30

Now Test for distributive Lattice.

Assume $x = 2$, $y = 3$, $z = 5$ then

$x \wedge (y \vee z) = 2 \wedge (3 \vee 5) = 2 \wedge 30 = 2$

$(x \wedge y) \vee (x \wedge z) = (2 \wedge 3) \vee (3 \wedge 5) = 1 \vee 1 = 1$

since LHS $\neq$ RHS

Hence Lattice is not a distributive Lattice.

2. **Bounded Lattice :-** A bounded lattice is an algebraic system $(L, \wedge, \vee)$ if there exist a greatest element '1' and a least element '0' in the Lattice i.e

1) for all $x \in L$, $x \wedge 1 = x$ and $x \vee 1 = 1$

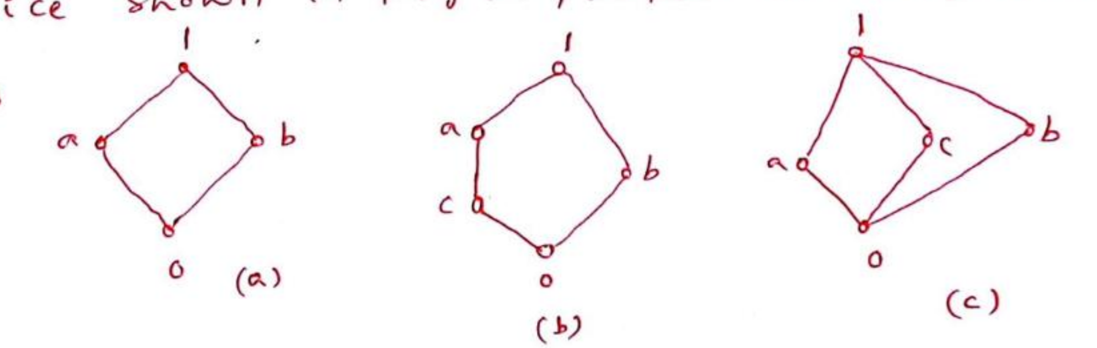
2) for all $x \in L$, $x \wedge 0 = 0$ and $x \vee 0 = x$

The element '1' is called the upper bound (or) top of L and the element '0' is called the Lower bound (or) bottom of L.

3 Complemented Lattice :- A Lattice $(L, \wedge, \vee)$ is said to complemented Lattice if, Every element in the Lattice has a Complement. (or)

A bounded Lattice with greatest element '1' and Least element '0' then for the elements $a, b \in L$ element 'b' is complement of a if $a \vee b = 1$ and $a \wedge b = 0$.

Ex: Lattice shown in Fig (a), (b), (c) are complemented Lattices



Sol:

a) $GLB(a,b) = a \wedge b = 0$

$LUB(a,b) = a \vee b = 1$

Hence it is complemented Lattice

b) $GLB(a,b) = a \wedge b = 0$, $GLB(c,b) = c \wedge d = 0$

$LUB(a,b) = a \vee b = 1$ $\quad$ $LUB(c,b) = c \vee b = 1$

So both 'a' and 'c' are Complements of 'b'.

Hence it is Complemented Lattice.

c) $GLB(a,c) = a \wedge c = 0$, $GLB(a,b) = a \wedge b = 0$

$LUB(a,b) = a \vee b = 1$, $LUB(a,c) = a \vee c = 1$.

So both 'b' and 'c' are complements of 'a'.

Hence it is complemented Lattice :

4. Modular Lattice :- A modular Lattice is a Lattice that satisfies the following self-dual condition.

$x \leq b \Rightarrow x \vee (a \wedge b) = (x \vee a) \wedge b$.

Ex: In a distributive Lattice $(L, \leq)$, $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$ imply that $b = c$

Sol: Given $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$

Now
$$
\begin{aligned}
b &= b \wedge (a \vee b) && \because \text{absorption Law} \\
&= b \wedge (a \vee c) && \because a \vee b = a \vee c \\
&= (b \wedge a) \vee (b \wedge c) \\
&= (a \wedge b) \vee (b \wedge c) \\
&= (a \wedge c) \vee (b \wedge c) && \because a \wedge b = a \wedge c \\
&= (c \wedge a) \vee (c \wedge b) \\
&= c \wedge (a \vee b) \\
&= c \wedge (a \vee c) \\
b &= c && \because \text{by absorption}
\end{aligned}
$$
//

Ex: In a distributive Lattice If an element has a complement then this complement is unique.

Sol: Suppose that an element a has two complements 'b' and 'c'. Then
$$a \vee b = 1 \qquad a \wedge b = 0$$
$$a \vee c = 1 \qquad a \wedge c = 0$$

We have
$$
\begin{aligned}
b &= b \wedge 1 \\
&= b \wedge (a \vee c) && \because a \vee c = 1 \\
&= (b \wedge a) \vee (b \wedge c) \\
&= 0 \vee (b \wedge c) \\
&= (a \wedge c) \vee (b \wedge c) \\
&= (a \vee b) \wedge c \\
&= 1 \wedge c \\
b &= c
\end{aligned}
$$
//

Ex: Every distributive Lattice is Modular.

Sol: Let $(L, \leq)$ be a distributive Lattice if $a \leq c \Rightarrow a \vee c = c$

Now $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c$

... distributive Lattice is modular.

Algebraic structures

Binary operation :- Let 'S' be a non empty set. If $f: S\times S \to S$ is a mapping, then 'f' is called a binary operation or binary composition in 'S'. The symbols $+, \cdot, *, \circ, \oplus$ etc are used to denote binary operations

Ex: For $a,b \in S \Rightarrow a+b \in S \Rightarrow$ '+' is a binary operation

For $a,b \in S \Rightarrow a*b \in S \Rightarrow$ '*' is a binary operation.

This is said to be the closure property of the binary operation and the set 'S' is said to be closed with respect to the binary operation.

Algebraic structures :- A non empty set 'G' equipped with one or more binary operations is called an algebraic structure or an algebraic system. If '*' is a binary operation on G then the algebraic structure is written as $(G, *)$.

Ex: $(N,+)$, $(Q,-)$, $(R,+)$ etc are algebraic structures

Semi group :- An algebraic structure $(G, *)$ is called a semi group if the binary operation '*' is a binary operation Then $(G,*)$ is said to be a semi group if

i) $a,b \in G \Rightarrow a*b \in G$ for all $a,b \in G$

ii) $a*(b*c) = (a*b)*c$ for all $a,b,c \in G$

i) property is called closure property

ii) property is called Associative property.

Ex: The operation $*$ is defined by $a*b = a$ for all $a,b \in G$ show that $(G, *)$ is a semi group.

Sol: Let $a,b \in G \Rightarrow a*b = a \in G$

$\therefore$ '$*$' is a binary operation in G

i, closure property : Let $a,b \in G$ then $a*b = a \in G$

ii, Associative property :- Let $a,b,c \in G$

$a*(b*c) = a*b = a$

$(a*b)*c = a*c = a$

$\therefore a*(b*c) = (a*b)*c$

'$*$' is associative in G

Hence $(G, *)$ is a semi group.

Ex: The operation $*$ is defined by $a*b = a+b-ab$ for all $a,b \in Z$ show that $(Z, *)$ is a semi group.

Sol: Let $a,b \in Z \Rightarrow a*b = a+b-ab \in Z$

$\therefore$ '$*$' is a binary operation in G

i, closure property :- Let $a,b \in G$ then $a*b = a+b-ab \in G$

ii, Associative property :- Let $a,b,c \in G$

$$\begin{aligned} a*(b*c) &= a*(b+c-bc) \\ &= a+(b+c-bc) - a(b+c-bc) \\ &= a+b+c-bc-ab-ac+abc \end{aligned}$$

$$\begin{aligned} (a*b)*c &= (a+b-ab)*c \\ &= a+b-ab+c-(a+b-ab).c \\ &= a+b+c-ab-ac-bc-abc \end{aligned}$$

$\therefore a*(b*c) = (a*b)*c$

Hence $(Z, *)$ is a semi group.

Ex: 'Q' is the set of rational numbers, $*$ is a binary operation defined on Q such that $a*b = a-b+ab$ for $a,b \in Q$ Then $(Q,*)$ is not a semi group.

Sol: Let $a,b \in Q \Rightarrow a*b = a-b+ab \in Q$

'$*$' is a binary operation in Q.

i, closure property : Let $a,b \in Q \Rightarrow a*b = a-b+ab \in Q$

ii, Associative property : Let $a,b,c \in G$

$$a*(b*c) = a*(b-c+bc)$$
$$= a-(b-c+bc)+a(b-c+bc)$$
$$= a-b+c-bc+ab-ac+abc$$
$$(a*b)*c = (a-b+ab)*c$$
$$= (a-b+ab)-c+(a-b+ab)\cdot c$$
$$= a-b-c+ab+ac-bc+abc$$

$\therefore a*(b*c) \neq (a*b)*c$

Hence $(Q,*)$ is not a semi group.

Ex: Let $(A,*)$ be a semi group. show that for a,b,c in A if $a*c = c*a$ and $b*c = c*b$ then $(a*b)*c = c*(a*b)$.

Sol: Given $(A,*)$ is semi group

$a*c = c*a$ and $b*c = c*b$ for $a,b,c \in A$

Now $(a*b)*c = a*(b*c)$ $\langle \because$ Associative

$= a*(c*b)$ $\langle \because c*b = b*c$

$= (a*c)*b$ $\langle$ Associative

$= (c*a)*b$ $\langle \because a*c = c*a$

$= c*(a*b)$ $\langle \because$ Associative

$\therefore a*(b*c) = c*(a*b)$

Ex: show that the binary operation $*$ defined on $(R, *)$ where $x * y = \max(x, y)$ is a semi group.

Sol: Let $x, y \in R \Rightarrow x * y = \max(x, y) \in R$

'$*$' is a binary operation in R

i) closure property : Let $x, y \in R \Rightarrow x * y = \max(x, y) \in R$

ii) Associative property :- Let $x, y, z \in R$

$$(x * y) * z = \max(x, y) * z$$
$$= \max(\max(x, y), z) = \max(x, y, z)$$

$$x * (y * z) = x * \max(y, z)$$
$$= \max(x, \max(y, z)) = \max(x, y, z)$$

$$\therefore x * (y * z) = (x * y) * z$$

Hence $(R, *)$ is a semi group.

Ex :- show that the binary operation $*$ defined on $(R, *)$ where $x * y = x^y$ is not a semi group.

Sol: Let $x, y \in R \Rightarrow x * y = x^y \in R$

'$*$' is a binary operation in R

i) closure property :- Let $x, y \in R \Rightarrow x * y = x^y \in R$

ii) Associative property : Let $a, b, c \in R$

$$(x * y) * z = x^y * z$$
$$= (x^y)^z = x^{yz}$$

$$x * (y * z) = x * y^z$$
$$= x^{y^z}$$

$$\therefore (x * y) * z \neq x * (y * z)$$

Hence $(R, *)$ is not a semi group.

Homomorphism of semi-group :-

Let $(S, *)$ and (T, o) be any two semi-groups. A mapping $f : S \to T$ such that for any two elements $a, b \in S$ $f(a * b) = f(a) \, o \, f(b)$ is called a semi group homomorphism.

A homomorphism of a semi group into itself is called a semi group endomorphism.

Identity element :- Let G be a non empty set and '$*$' be a binary operation on G. If there exists an element $e \in G$ such that $a * e = e * a = a$ for $a \in G$. Then 'e' is called an identity element on G.

Monoid :- A semi group $(G, *)$ with an identity element with respect to the binary operation '$*$' is known as a Monoid

EX: $(Z, +)$ is a Monoid and the identity is '0'

$(Z, \cdot)$ is a Monoid and the identity is '1'.

Inverse element :- Let $(G, *)$ be an algebraic structure with the identity element 'e' in G w.r.t '$*$'. An element $a \in G$ is said to be invertible if there exist an element $x \in G$ such that $a * x = x * a = e$.

Group :- If G is a nonempty set and $*$ is a binary operation defined on G such that the following Laws are satisfied then $(G, *)$ is a group.

i) Closure Law : Let $a, b \in G \Rightarrow a * b \in G$

ii) Associative Law : For $a, b, c \in G \Rightarrow a*(b*c) = (a*b)*c$

iii) Identity Law : For $a \in G$ there exist $e \in G$ such that

iv) Inverse Law :- For $a \in G$ there exist an element $b \in G$ such that $a * b = b * a = e$, b is called an inverse of 'a'

Ex:- prove that $\{1, \omega, \omega^2\}$ is a group with respect to multiplication where $1, \omega, \omega^2$ are cube roots of unity.

Sol: we construct the composition table as follows.

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	$\omega^3=1$	ω

$\omega^3 = 1$, $\omega^4 = \omega^3 \cdot \omega = 1 \cdot \omega = \omega$

i) closure Law :- From the composition table it is clear that $(G, \cdot)$ is closed w.r.to multiplication.

ii) Associative Law :- From the composition table it is clear that $(G, \cdot)$ is Associative w.r.to multiplication.

iii) Identity Law :- '1' is identity element in G such that $a \cdot 1 = 1 \cdot a = 1$, $\forall a \in G$.

iv) Inverse Law :- Each element of G is invertible

$1 \cdot 1 = 1 \cdot 1 = 1 \Rightarrow$ '1' is its own inverse

$\omega \cdot \omega^2 = \omega^3 = 1 \Rightarrow \omega^2$ is the inverse of ω.

Hence $(G, \cdot)$ is a group.

Note :- $(G, \cdot)$ is a group and $a \cdot b = b \cdot a$, $\forall a, b \in G$ that is commutative Law holds in G with respect to the binary operation. then $(G, \cdot)$ is an abelian group.

EX:- show that the set $G = \{1, -1, i, -i\}$ where $i^2 = 1$ is an abelian group with respect multiplication as a binary operation.

Sol: we construct the composition table as follows

·	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

i) Closure Law :- From the table $(G, \cdot)$ is closed w.r.to multiplication

ii) Associative Law :- For any three elements $a, b, c \in G \Rightarrow$

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

since $1 \cdot (-1 \cdot i) = 1 \cdot -i = -i$

$(1 \cdot -1)\, i = -1 \cdot i = -i$

$\therefore$ Associative Law is satisfied.

iii) Identity Law :- '1' is the identity element in $(G, \cdot)$ such that $a \cdot 1 = 1 \cdot a = a, \ \forall\, a \in G$.

iv) Inverse Law :- $1 \cdot 1 = 1 \cdot 1 = 1 \Rightarrow$ '1' is inverse of '1'

$(-1) \cdot (-1) = (-1) \cdot (-1) = 1 \Rightarrow$ '-1' is the inverse of '-1'

$i \cdot -i = -i \cdot i = 1 \Rightarrow$ 'i' is the inverse of '-i'

Hence Inverse of every element in G exists.

v) Commutative Law :- For $a, b \in G \Rightarrow a \cdot b = b \cdot a$

$1 \cdot 1 = 1 \cdot 1 = 1, \quad -1 \cdot 1 = 1 \cdot -1 = -1$

$i \cdot 1 = 1 \cdot i = i, \quad -i \cdot i = i(-i) = 1$ etc...

Commutative Law is satisfied.

Hence $(G, \cdot)$ is an abelian group.

Ex: Prove that the set 'Z' of all integers with binary operation '$*$' defined by $a * b = a + b + 1, \ \forall\, a, b \in Z$ is an abelian group.

Sol: Given $a * b = a + b + 1, \ \forall\, a, b \in Z$

i) Closure Law :- Let $a, b \in Z \Rightarrow a * b = a + b + 1 \in Z$

$\therefore$ 'Z' is closed under '$*$'

ii) Associative Law :- Let $a, b, c \in Z$

$(a * b) * c = (a+b+1) * c$

$= a+b+1+c+1 = a+b+c+2$

$a * (b * c) = a * (b+c+1)$

$= a+b+c+1+1 = a+b+c+2$

$\therefore a * (b * c) = (a * b) * c.$

iii, Identity Law :- Let $a \in Z$, there exist $e \in Z$ such that $a * e = e * a = a$

$a+e+1 = a \Rightarrow e+1 = 0 \Rightarrow e = -1$

$\therefore e = -1$ is the identity element in Z.

iv) Inverse Law :- Let $a \in Z$, there exist $b \in Z$ such that $a * b = e$

$a+b+1 = -1 \Rightarrow a+b = -2 \Rightarrow b = -2-a$

$\therefore b = -2-a$ is an Inverse element in Z.

v) Commutative Law :- For $a, b \in G \Rightarrow a * b = b * a$

$a * b = a+b+1 = b+a+1 = b * a$

$\therefore$ Commutative Law is satisfied.

Hence $(Z, *)$ is an abelian group.

Ex: Show that the set Q_+ of all positive rational numbers forms an abelian group under the composition defined by $*$ such that $a * b = \frac{ab}{3}$ for $a, b \in Q_+$

sol: Let $a, b \in Q_+ \Rightarrow a * b = \frac{ab}{3}$.

i, closure Law :- For $a, b \in Q_+ \Rightarrow a * b = \frac{ab}{3} \in Q_+$

ii, Associative Law : For $a, b, c \in Q_+$

$a * (b * c) = a * \frac{bc}{3} = \frac{\frac{abc}{3}}{3} = \frac{abc}{9}.$

$(a * b) * c = \frac{ab}{3} * c = \frac{\frac{ab}{3} \cdot c}{3} = \frac{abc}{9}$

$a * (b * c) = (a * b) * c,$ Associative Law

iii) Identity Law :- Let $a \in Q_+$ there exist $e \in Q_+$ such that $a \cdot e = e \cdot a = a$.

$$\frac{ae}{3} = a \Rightarrow ae - 3a = 0 \Rightarrow a(e-3) = 0 \quad \{\because a \neq 0$$

$$\therefore e - 3 = 0 \Rightarrow \boxed{e = 3}$$

$\therefore$ $e = 3$ is the identity element in Q_+

iv) Inverse Law :- Let $a \in Q_+$ there exist $b \in Q_+$ such that $a * b = e = b * a$

$$\frac{ab}{3} = e \Rightarrow \frac{ab}{3} = 3 \Rightarrow ab = 9 \Rightarrow \boxed{b = \frac{9}{a}}$$

$\therefore$ $b = \frac{9}{a}$ is an inverse element in Q_+.

v) Commutative Law :- For $a, b \in Q_+ \Rightarrow a * b = b * a$

$$a * b = \frac{ab}{3} = \frac{ba}{3} = b * a.$$

$\therefore$ Commutative Law is satisfied.

Hence $(Q_+, *)$ is an abelian group.

EX: Prove that the set G of rational numbers other than '1' with operation $\oplus$ such that $a \oplus b = a + b - ab$ for $a, b \in G$ is abelian group. : practice problem

EX! Consider the algebraic system $(G, *)$ where 'G' is the set of all non-zero real numbers and '$*$' is a binary operation defined by $a * b = \frac{ab}{4}$ $\forall a, b \in G$. Show that $(G, *)$ is an abelian group. : practice problem

Addition modulo 'm' :- If 'a' and 'b' are any two integers and 'r' is the least non-negative reminder obtaining by dividing the ordinary sum of 'a' and 'b' by 'm' then the addition modulo 'm' of 'a' and 'b' is 'r'

symbolically $a +_m b = r$, $0 \leq r < m$

Ex: $20 +_6 5 = 1$ and $-15 +_5 3 = 3$

Multiplication modulo p : If 'a' and 'b' are any two integers and 'r' is the least non-negative reminder obtained by dividing the ordinary product of 'a' and 'b' by 'p' then the multiplication modulo 'p' of 'a' and 'b' is 'r' symbolically $a \times_p b = r$, $0 \le r < p$

Ex: $5 \times_3 4 = 2$,

Ex: show that the set $G = \{0,1,2,3,4\}$ is an abelian group with respect to addition modulo '5'.

Sol: We form the composition table as follows.

$+_5$	0	1	2	3	4
0	$\boxed{0}$	1	2	3	4
1	1	2	3	4	$\boxed{0}$
2	2	3	4	$\boxed{0}$	1
3	3	4	$\boxed{0}$	1	2
4	4	$\boxed{0}$	1	2	3

i, Closure Law :- since all the entries in the composition table are elements in G.

$\therefore$ closure Law satisfied.

ii, Associative Law :- For $a,b,c \in G$

$(a +_5 b) +_5 c = a +_5 (b +_5 c)$

$(1 +_5 3) +_5 4 = 1 +_5 (3 +_5 4) = 3$ etc.

iii, Identity Law :- clearly $0 \in G$ is the identity element

$0 +_5 a = a = a +_5 0$, $\forall a \in G$

iv) Inverse Law :- From the table

'0' is its own inverse ; '1' is the inverse of '4'

'2' is the inverse of 3 ; '3' is the inverse of '2'

'4' is the inverse of '1'.

v) Commutative Law :- From the table

$a +_5 b = b +_5 a$, $\forall a,b \in G$

$(G, +_5)$ is an abelian group.

Ex: show that the set $G = \{1, 2, 3, 4\}$ is an abelian group with respect to multiplication modulo 5.

Sol: The composition table for multiplication modulo '5' is

$\times_5$	1	2	3	4.
1	$\boxed{1}$	2.	3	4
2	2	4	$\boxed{1}$	3
3	3	$\boxed{1}$	4	2
4	4	3	2	$\boxed{1}$

i, closure Law :- since all the entries in the composition table are elements in G.

∴ closure Law is satisfied.

ii, Associative Law :- For $a, b, c \in G$

$(a \times_5 b) \times_5 c = a \times_5 (b \times_5 c)$

$(1 \times_5 3) \times_5 4 = 1 \times_5 (3 \times_5 4) = 2$ etc.

iii, Identity Law :- clearly $1 \in G$ is the identity element

$1 \times_5 a = 4 = a \times_5 1 \quad \forall a \in G$

iv, Inverse Law :- From the table

'1' is the inverse of '1' ; '2' is the inverse of '3'

'3' is the inverse of '2' ; '4' is the inverse of '4'

v) Commutative Law :- From the table

$a \times_5 b = b \times_5 a, \quad \forall a, b \in G$

Hence $(G, \times_5)$ is an abelian group.

Ex: Show that $G = \{x / x = 2^a 3^b \text{ for } a, b \in Z\}$ is a group under multiplication.

Sol: Let $x, y, z \in G$, we can take

$x = 2^p 3^q, \quad y = 2^r 3^s, \quad z = 2^l 3^m, \quad p, q, r, s, l, m \in Z$

i, closure Law :- For $x, y \in G \Rightarrow x \cdot y \in G$

since $x \cdot y = (2^p \cdot 3^q) \cdot (2^r \cdot 3^s)$

$= 2^{p+r} \cdot 3^{q+s} \in G.$

ii) Associative Law :- For $x, y, z \in G$

$$x\cdot(y\cdot z) = x\cdot(yz)$$
$$= 2^p 3^q (2^r \cdot 3^s \cdot 2^l 3^m)$$
$$= 2^p 3^q (2^{r+l} 3^{s+m})$$
$$= 2^{p+r+l} \cdot 3^{q+s+m}$$
$$= (2^{p+r} \cdot 3^{q+s})\, 2^l \cdot 3^m$$
$$= (2^p \cdot 3^q \cdot 2^r \cdot 3^s)\, 2^l \cdot 3^m = (x\cdot y)\cdot z$$

$\therefore$ Associative is satisfied.

iii) Identity Law : Let $x \in G$

We know that $e = 2^0 3^0 \in G$, since $0 \in \mathbb{Z}$

$$\therefore x\cdot e = 2^p 3^q \cdot 2^0 \cdot 3^0 = 2^{p+0} 3^{q+0} = 2^p 3^q = x$$

$$\therefore x\cdot e = e\cdot x = x$$

iv) Inverse Law :- Let $x \in G$

Now $y = 2^{-p} 3^{-q} \in G$, since $-p, -q \in \mathbb{Z}$

$$\therefore x\cdot y = 2^p \cdot 3^q \cdot 2^{-p} \cdot 3^{-q} = 2^{p-p} 3^{q-q} = 2^0 \cdot 3^0 = e$$

$$\therefore x\cdot y = y\cdot x = e$$

Hence $(G, \cdot)$ is a group.

Ex: If $(G, *)$ is a group then $(a*b)^{-1} = b^{-1} * a^{-1}$, $\forall a, b \in G$

Sol: Let $a, b \in G$ and 'e' is the identity element in G

Let $a \in G \Rightarrow a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$

$b \in G \Rightarrow b^{-1} \in G$ such that $b * b^{-1} = b^{-1} * b = e$

Now $a, b \in G \Rightarrow a*b \in G$ and $(a*b)^{-1} \in G$

Consider

$$(a*b)*(b^{-1}*a^{-1}) = a*[b*(b^{-1}*a^{-1})]$$
$$= a*[(b*b^{-1})*a^{-1}]$$

$$= a * (e * a^{-1})$$

$$= a * a^{-1} = e$$

$$(b^{-1} * a^{-1}) * (a * b) = b^{-1} * [a^{-1} * (a * b)]$$

$$= b^{-1} * [(a^{-1} * a) * b]$$

$$= b^{-1} * (e * b)$$

$$= b^{-1} * b = e$$

$$\therefore (a * b) * (b^{-1} * a^{-1}) = (b^{-1} * a^{-1}) * (a * b) = e$$

Hence $(a * b)^{-1} = b^{-1} * a^{-1}$

Ex: Cancelation Laws holds in G

For all $a, b, c \in G$ then $a * b = a * c \Rightarrow b = c$ (L.C.L)

$b * a = c * a \Rightarrow b = c$ (R.C.L)

Sol: G is a group

Let 'e' be the identity element in G.

$a \in G \Rightarrow a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$

Consider $a * b = a * c$

$$\Rightarrow a^{-1} * (a * b) = a^{-1} * (a * c)$$

$$\Rightarrow (a^{-1} * a) * b = (a^{-1} * a) * c$$

$$\Rightarrow e * b = e * c \Rightarrow b = c$$

Now $b * a = c * a$

$$\Rightarrow (b * a) * a^{-1} = (c * a) * a^{-1}$$

$$\Rightarrow b * (a * a^{-1}) = c * (a * a^{-1})$$

$$\Rightarrow b * e = c * e \Rightarrow b = c$$

//.

Ex: If Every element of a group G is its own inverse show that G is an abelian group.

Sol: Let $a, b \in G$ By Hypothesis $a^{-1} = a$, $b^{-1} = b$

Then $ab \in G \Rightarrow (ab)^{-1} = ab$

$(ab)^{-1} = ab \Rightarrow b^{-1}a^{-1} = ab \Rightarrow ba = ab$

Ex: prove that If $a^2 = a$ then $a = e$, 'a' being an element of a group G.

Sol: Let 'a' be an element of a group G such that

$a^2 = a$

Now $a^2 = a \Rightarrow aa = a$

$\Rightarrow (aa)a^{-1} = aa^{-1}$

$\Rightarrow a(aa^{-1}) = aa^{-1}$ $\quad \{\because aa^{-1} = e$

$\Rightarrow ae = e \Rightarrow \boxed{a = e}$

Ex: Show that in a group G, for $a, b \in G$, $(ab)^2 = a^2b^2$ $\Leftrightarrow$ G is an abelian.

Sol: Let $a, b \in G$ and $(ab)^2 = a^2b^2$ Then

$(ab)^2 = a^2b^2 \Rightarrow (ab)(ab) = a^2b^2 = (aa)(bb)$

$\Rightarrow a(ba)b = a(ab)b$

$\Rightarrow ba = ab$ (by cancellation Law)

$\Rightarrow$ 'G' is abelian

Conversely, Let G is an abelian Then

$(ab)^2 = (ab)(ab) = a(ba)b = a(ab)b = (aa)(bb) = a^2b^2$ //.

Ex: If a, b are any two elements of a group $(G, \cdot)$ which commute. Show that 1. a^{-1} and 'b' commute 2. b^{-1} and 'a' commute 3. 'a^{-1}' and 'b^{-1}' commute

Sol: $(G, \cdot)$ is a group such that $ab = ba$

1. $ab = ba \Rightarrow a^{-1}(ab) = a^{-1}(ba)$

$\Rightarrow (a^{-1}a)b = a^{-1}(ba)$

$\Rightarrow eb = (a^{-1}b)a \Rightarrow b = (a^{-1}b)a$

$\Rightarrow ba^{-1} = (a^{-1}b)aa^{-1}$

$\Rightarrow ba^{-1} = (a^{-1}b)e$

$\Rightarrow ba^{-1} = a^{-1}b \Rightarrow$ 'a^{-1}' and 'b' commute.

2. $ab = ba \Rightarrow (ab)b^{-1} = (ba)b^{-1}$

$\Rightarrow a(bb^{-1}) = (ba)b^{-1}$

$\Rightarrow ae = b(ab^{-1})$

$\Rightarrow a = b(ab^{-1})$

$\Rightarrow b^{-1}a = b^{-1}(b(ab^{-1}))$

$\Rightarrow b^{-1}a = b^{-1}b(ab^{-1}) \Rightarrow b^{-1}a = e(ab^{-1})$

$\Rightarrow b^{-1}a = ab^{-1}$

$\therefore$ b^{-1} and 'a' commute

3. $ab = ba \Rightarrow (ab)^{-1} = (ba)^{-1} \Rightarrow b^{-1}a^{-1} = a^{-1}b^{-1}$

$\therefore$ 'a^{-1}' and 'b^{-1}' are commute.

Order of an element :- Let $(G, *)$ be a group and $a \in G$ then the least positive integer 'n' if it exist such that $a^n = e$ is called the order of $a \in G$. The order of an element $a \in G$ is be denoted by $O(a)$.

Ex: If $G = \{1, -1, i, -i\}$

$1^1 = 1^2 = 1^3 = \cdots = 1 \Rightarrow O(1) = 1$

$(-1)^2 = (-1)^4 = (-1)^6 = \cdots = 1 \Rightarrow O(-1) = 2$

$i^4 = i^8 = \cdots = 1 \Rightarrow O(i) = 4$

$(-i)^4 = (-i)^8 = \cdots = 1 \Rightarrow O(-i) = 4.$

Sub groups :- Let $(G, *)$ be a group and 'H' be a non empty subset of G. If $(H, *)$ is itself is a group then $(H, *)$ is called sub-group of $(G, *)$

Ex: i) $(Z, +)$ is a sub group of $(Q, +)$

ii) $(N, +)$ is not a sub group of the group $(Z, +)$.

Ex: If $(G, *)$ is a group and $H \subseteq G$ then $(H, *)$ is a sub group of $(G, *)$ if and only if i) $a, b \in H \Rightarrow a*b \in H$
ii) $a \in H \Rightarrow a^{-1} \in H$.

Sol: Let $(H, *)$ be a sub group of $(G, *)$

since $(H, *)$ is a group

By closure property we have $a, b \in H \Rightarrow ab \in H$

By inverse property $a \in H \Rightarrow a^{-1} \in H$.

Conversely Let $a, b \in H \Rightarrow ab \in H$ and $a \in H \Rightarrow a^{-1} \in H$.

since H is non empty

Let $a \in H \Rightarrow a^{-1} \in H$

$\therefore a \in H, a^{-1} \in H \Rightarrow aa^{-1} \in H \Rightarrow e \in H$, e is identity in G

$\Rightarrow$ 'e' is the identity in H.

'$*$' is associative in H follows from the fact that '$*$' is associative in G.

Hence H Itself is a group, $\therefore$ H is a Subgroup of 'G'.

Ex: If H_1 and H_2 are two subgroups of a group G then $H_1 \cap H_2$ is also a Sub group of G.

Sol: Let H_1 and H_2 be two subgroups of a group G.

Let 'e' be the identity element in G.

$\therefore e \in H_1$ and $e \in H_2 \Rightarrow e \in H_1 \cap H_2 \Rightarrow H_1 \cap H_2 \neq \phi$

Let $a \in H_1 \cap H_2$, and $b \in H_1 \cap H_2$

$a \in H_1, a \in H_2$ and $b \in H_1, b \in H_2$

since H_1 is a subgroup, $a \in H_1$ and $b \in H_1 \Rightarrow ab^{-1} \in H_1$

similarly $ab^{-1} \in H_2$ $\quad \therefore ab^{-1} \in H_1 \cap H_2$

we have $a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

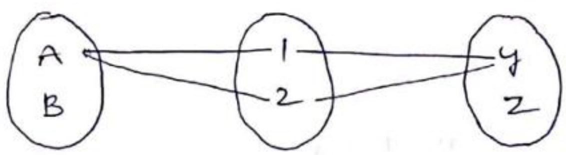
$\therefore H_1 \cap H_2$ is a subgroup of G.

UNIT – III

ELEMENTARY COMBINATORICS

Basics of counting :– Basic rules of counting are two types. They are i) sum Rule ii) product rule.

i) sum Rule :– If an event can be performed in 'm' ways and another event can be performed in 'n' ways and if these two events cannot be performed simultaneously, then one of the two events can be performed in m+n ways.



2+2 = 4 ways.

Ex: If there are 14 boys and 12 girls in a class, find the number of ways of selecting one student as C.R

Sol: For seclection of one student either of the following two tasks is to be performed

i) selecting a boy among 14 boys or

ii) selecting a girl among 12 girls

The first of these can be performed in 14 ways and second in 12 ways. By sum Rule

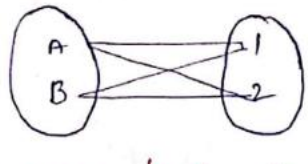
The no. of ways of selecting a student is

$14+12 = 26$ ways.

Ex: If a Library has 12 books of mathematics, 10 books of physics, 16 books of chemistry, 11 books of English. find the no. of books of choosing one of the above books.

Sol: No. of ways $= 12+10+16+11 = 49$ ways.

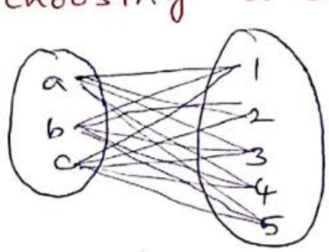
ii) product Rule :- If an event can be performed in 'm' ways and a second event can be performed in 'n' ways. and if the no. of ways the second event perform can not be depend upon the perform of the first event, then the two events can be performed simultaneously in 'mn' ways



$2 \times 2 = 4$ ways

Ex: A person has 3 shirts and 5 ties then find the no. of ways of choosing a shirt and tie.

Sol:



No. of ways $= 3 \times 5 = 15$ ways.

Ex: Three persons enter into car, where there are 5 seats. In How many ways can they take up their seats?

Sol: The first person has a choice of 5 seats. He can sit in any one of those 5 seats, so there are 5 ways of occupying the first seat.

The second person has a choice of 4 seats.

Similarly the third person has a choice of '3' seats.

$\therefore$ The total No. of ways $= 5 \times 4 \times 3 = 60$ ways.

Ex: There are 20 married couples in a party. Find no. of ways of choosing one women and one man from the party such that two are not married to each other.

Sol: Women can be choosen in 20 ways. One man is her husband. So one man can be choosen in 19 ways.

$20 \times 19 = 380$ ways

permutations and Combinations :

Factorial Notation :- The product of first 'n' natural numbers $1, 2, \ldots, n$ is denoted by $n!$ and it is read as factorial 'n' (or) n factorial. Thus

$$n! = n(n-1)(n-2)\ldots\ldots 3\cdot 2\cdot 1$$

$$4! = 4\times 3\times 2\times 1 = 24, \quad 5! = 5\times 4\times 3\times 2\times 1 = 120.$$

Ex: Find 'n' if $(n+1)! = 12\times(n-1)!$

Sol:

$$(n+1)! = 12\times(n-1)!$$

$$(n+1)\,n\,\cancel{(n-1)!} = 12\times\cancel{(n-1)!}$$

$$(n+1)n = 12 \Rightarrow n^2+n-12=0 \Rightarrow (n+4)(n-3)=0$$

$\therefore n=3$ (or) $n=-4$. Here $n=-4$ is not defined

Ex. compute $\dfrac{20!}{18!}$

Sol.

$$\frac{20!}{18!} = \frac{20\times 19\times \cancel{18!}}{\cancel{18!}} = 380.$$

Note: If $n=0$ then $n!=1$ ie $0!=1$

If $n>0$ then $n! = n(n-1)!$.

permutations :- An ordered selection (or) arrangements of a given set of objects taken some or all of them at a time is called a permutation of the objects.

Any arrangements of 'r' objects from a set of 'n' objects is called r- permutation of 'n' objects taken 'r' at a time and it is denoted by $p(n,r)$ (or) $^{n}p_{r}$.

$$\boxed{^{n}p_{r} = \frac{n!}{}}$$

Note: The number of permutations of 'n' distinct objects taken 'r' at a time without repitition is

$$p(n,r) = n(n-1)(n-2)\ldots\ldots(n-r+1).$$

proof: The no. of permutations of 'n' distinct objects 'r' at a time is equivalent to filling 'r' positions with 'n' objects.

The first place can be filled in 'n' ways as any one of the given objects can be put in it.

After filling the first place in any one of the 'n' ways the second place can be filled up by any one of the remaining $(n-1)$ objects.

there are $(n-1)$ ways of filling the second place Thus, the first two places can be felled up in $n\times(n-1)$ ways similarly there remains $(n-2)$ ways to fill up the third place. Thus, the first three places can be filled in $n(n-1)(n-2)$ ways.

proceeding in the same way there remains $n-(r-1) = n-r+1$ ways to fill up the r^{th} place. Thus, the no. of ways in which 'r' place can be filled up is

$$^{n}p_{r} = p(n,r) = n(n-1)(n-2)\ldots\ldots(n-r+1).$$

important results :-

1. $p(n,r) = {}^{n}p_{r} = \dfrac{n!}{(n-r)!}$

2. $p(n,n) = {}^{n}p_{n} = n!$

3. $p(n, n-1) = p(n,n)$ i.e. ${}^{n}p_{n-1} = {}^{n}p_{n}$

4. $0! = 1$.

Combinations :- An ordered selection of a given set of objects taken some or all of them at a time is called a combination. Thus any ordered selection of 'r' objects from a set of 'n' objects is called an r- combinations of 'n' objects and it is denoted by $C(n,r)$ (or) $^{n}C_{r}$.

Note: The no. of all combinations of 'n' distinct objects taken 'r' at a time is given by $^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$.

proof: Let $c(n,r)$ be the required no. of combination of 'n' different objects taken 'r' at a time.

Then each of these combinations is a group of 'r' different objects which can be arranged among themselves in $r!$ ways.

So all the $c(n,r)$ combinations will produced $r!\,c(n,r)$ permutation of 'n' objects taken 'r' at a time

But this number also equal to $p(n,r)$

$$\therefore r!\,c(n,r) = p(n,r) = \frac{n!}{(n-r)!}$$

$$\boxed{c(n,r) = {}^{n}c_{r} = \frac{n!}{(n-r)!\,r!}}$$

Important results :-

1. $c(n,0) = {}^{n}c_{0} = 1$

2. $c(n,n) = {}^{n}c_{n} = 1$

3. $c(n,r) = c(n,n-r)$ i.e $^{n}c_{r} = {}^{n}c_{n-r}$

4. $c(n+1,r) = c(n,r-1) + c(n,r)$

Ex: How many diff. strings of length 4 can be formed using the letters of the word "FLOWER".

Sol: Given word is "FLOWER"

No. of Letters = 6. All are distinct.

Req no. of strings is

$$p(6,4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6\times5\times4\times3\times2!}{2!} = 360.$$

Ex: Find the no. of permutations of letters of the word "COMPUTER" be arranged ?

Sol: Given word is "COMPUTER"

No. of letters = 8, All are distinct.

No. of permutations = 8! = 40320 ways.

Ex: How many four digit numbers are there with distinct digits ?

Sol: The total No. of arrangements of Ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Taking '4' at a time is $p(10,4)$

But these numbers also include those numbers which has '0' at the entreme Left (thousand's place)

Such numbers are not four digit numbers.

Keeping '0' at the extreme left, 3 places can be filled up with the remaining '9' digits in $p(9,3)$ ways.

Hence the total No. of four digit number

$$= p(10,4) - p(9,3)$$

$$= \frac{10!}{(10-4)!} - \frac{9!}{(9-3)!}$$

Ex: Suppose there are 5 boys and 3 girls

i) In how many ways can they sit in a row ?

ii) How many sitting arrangements are there with no two girls sitting together ?

iii) How many arrangements when all the girls never together

iv) How many ways all the girls sit together and all the Boys sit together

v) In how many than can sit in a row if the boys are to sit together and the girls do not sit together ?

Sol: i) Total No. of students = 8 (5+3).

No. of ways = $8! = 40320$ ways.

ii) The five boys can be seated in a row = $p(5,5) = 5!$

In each of these arrangements place are created for the girls as given below

GBGBGBGBGBG

So '3' girls can sit in 6 places in $p(6,3)$ ways so that no two girls are to sit together.

Hence the total No. of sitting arrangements

$$= 5! \times p(6,3) = 5! \times 6 \times 5 \times 4 = 14400$$

iv) 5 boys can be arranged among themselves in $5!$ ways.

3 girls can be arranged among themselves in $3!$ ways.

They can be considered as 2-Unit and can be arranged in $2!$ ways

The total No. of seating arrangements $= 2! \times 5! \times 3!$

$$= 1440.$$

iii) The total no. of ways when all the girls are never together = Total No. of arrangements without any restriction

$= 8! - 6! \times 3! = 36000.$

v) The boys can 5! ways and girls can sit in 3 ways since boys have to sit together they can be considered as one unit.

Then there '0' or '1' or '2' or '3' girls have to sit to the left of boys unit

But '0' and '3' cases are to be ommitted as the girls do not sit together.

Hence the required no. of arrangements

$= 2! \times 5! \times 3! = 1440.$

permutation with repition :-

If repetition is allowed then the no. of permutations of r objects from a set of 'n' objects is n^r.

There are n ways to select the first object. After we have selected the first object. There are n ways to select the second object. We continue selecting objects untill we have selected all 'r' 'objects.

By the product rule these selections can be made in

$n \times n \times n \times \cdots \times n = n^r$ ways.

EX: Consider the 6 digits number 2, 3, 4, 5, 6 and 8 and repititions of digits are allowed.

a) How many 3 digit numbers can be formed ?

b) How many 3 digit numbers must contain the digit '5'

Sol: a) For a 3- digit numbers we have to fill up three places. Since repititions of the digits is allowed, each of the places can be filled up in 6 ways.

Hence the required 3- digit number

$= 6 \times 6 \times 6 = 6^3 = 216.$

b) Excluding the digit 5 the no. of 3 digit numbers that can be formed from the remaining 5 digits 2, 3, 4, 6 and 8 is $5 \times 5 \times 5 = 5^3 = 125$

$\therefore$ Hence the number must contain the digit 5

= Total 3 digit number − the no. of 3 digit number that do not contain 5

$= 216 - 125 = 91.$

Note: The no. of permutations of n objects in which 'p' objects of one type, q objects are of second type, r objects are of third type and rest are all distinct

$$= \frac{n!}{p!\,q!\,r!}.$$

Ex: How many different words can be formed with the Letters of the word "MISSISSIPPI"

Sol: The no. of Letters in the word = 11

S = 4 Letters , I = 4 letters , P = 2 letters.

Hence, the total no. of words $= \frac{11!}{4!\,4!\,2!} = 34650.$

Ex: How many seven letter words can be formed using the Letters of the word "BENZENE"

Sol: No. of Letters in the word = 7

E = 3 letters , N = 2 letters

Hence the total No. of words $= \frac{7!}{3!\,2!} = \frac{7.6.5.4.3.2.1}{3.2.1.2.1} = 420$

Ex: How many 4-digit numbers can be formed by using the digits 2, 4, 6, 8 when repititions of digits is allowed?

Sol: No. of digits = 4

[illegible] of filling Units place = 4.

No. of ways of filling ten's place = 4

No. of ways of filling Hundred's place = 4

No. of ways of filling thousand's place = 4

The total No. of 4-digits numbers $= 4 \times 4 \times 4 \times 4 = 256$.

Ex: How many 2-digits even numbers can be formed by using the digits 1, 3, 4, 6, 8 when repititions of digits is allowed.

Sol: We have three even numbers and two odd number

No. of ways of filling units place = 3

No. of ways of filling Ten's place = 5

Total No. of two digits even numbers $= 3 \times 5 = 15$.

Ex: Find the no. of words that can be formed by using the letters of the word "MATHEMATICS" that starts as well as end with 'T'

Sol: No. of letters in the word = 11

M = 2 letters, A = 2 letters, T = 2 letters

The no. of words that begin with T and end with T

is $= \dfrac{9!}{2!\,2!} = 90720$ ∵ Here 'T' is ommitted

Ex: How many Four digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5 if i) repititions of digits is not allowed ii) repititions of digits is allowed?

Sol: i) In a four digit number '0' cannot appear in the thousand's place. So thousand's place can be filled in 5 ways.

Since repitition of digits is not allowed

'0' can be used at hundred's place, so hundred's place can be filled in 5 ways.

Now any one of the remaining four digits can be used to fill up ten's place. So, ten's place can be filled 4 ways

one's place can be filled from the remaining three digits in 3 ways.

Hence the required four digit numbers

$$= 5 \times 5 \times 4 \times 3 = 300.$$

ii) For a four digit number we have to fill up four places and '0' cannot appear in the thousand's place

So thousand's place can be filled in 5 ways.

since repititions of digits is allowed So each of the remaining three places, hundreds, ten's and one's can be filled in 6 ways.

Hence the required four digit numbers

$$= 5 \times 6 \times 6 \times 6 = 1080.$$

problems on combinations :-

Ex: compute the no. of sub committies of three members each that can be formed from a committee of 25 members ?

Sol: The no. of Sub committees of three people each is equal to the number of selection of three members from 25 members.

Hence the required number of sub committees

$$= c(25,3) = {}^{25}c_3 = \frac{25!}{(25-3)!\,3!}$$

$$= \frac{25.24.23.\cancel{22!}}{} = 2300.$$

Ex: In how many ways a committee of 5 members can be selected from 6 men and 5 women consisting of 3 men and 2 women.

Sol: Three men out of 6 men can be selected in $C(6,3)$ ways.

Two women out of 5 women can be selected in $C(5,2)$ ways

Hence the required number is

$$C(6,3) \times C(5,2) = \frac{6 \times 5 \times 4}{3!} \times \frac{5 \times 4}{2!} = 200.$$

Ex: The question paper of mathematics contains two questions divided into two groups of five questions each. In how many ways can an examine answer six questions taking atleast two questions from each group

Sol: The examine can answer questions from two groups in following ways.

i) 2 from first group and 4 from the second group

ii) 3 from first group and 3 from the second group

iii) 4 from first group and 2 from the second group

Now the no. of ways of selecting the question in (i)

$$= C(5,2) \cdot C(5,4) = {}^5C_2 \cdot {}^5C_4 = 10 \times 5 = 50$$

No. of ways of selecting the questions in (ii)

$$= C(5,3) \cdot C(5,3) = 10 \times 10 = 100$$

No. of ways of selecting the questions in (iii)

$$= C(5,4) \cdot C(5,2) = 5 \times 10 = 50$$

Therefore, the required no. of ways

$$= 50 + 100 + 50$$

$$= 200 \text{ ways.}$$

EX: A man has 7 relatives, 4 of them Ladies and '3' gentlemen, his wife has 7 relatives, 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives?

Sol: They can invite '4' possible ways.

i) 3 ladies from husband's side and '3' gents from wife's side

No. of ways = $C(4,3) \times C(4,3) = {}^4C_3 \times {}^4C_3 = 16$.

ii) 3 gents from husband's side and '3' ladies from wife's side, No. of ways = $C(3,3) \times C(3,3) = 1$

iii) 2 ladies and 1 gent from husband's side and one lady and 2 gents from wife's side

No. of ways = $\{C(4,2) \times C(3,1)\} \times \{C(3,1) \times C(4,2)\}$

$= 324$.

iv) one lady and two gents from husband's side and '2' ladies and one gent from wife's side

No. of ways = $\{C(4,1) \times C(3,2)\} \times \{C(3,2) \times C(4,1)\}$

$= 144$

$\therefore$ The total No. of ways $= 16 + 1 + 324 + 144 = 485$.

Combinations with repitition :-

The number of unordered choices of 'r' from 'n' with repititions is allowed is

$${}^{n+r-1}C_r = C(n+r-1, r)$$

EX: Consider the set $\{a, b, c, d\}$. In how many ways we can selected two of these letters when repitition is allowed.

Sol: In order matters and repitition is allowed there are $2^4 = 16$ possible selections and they are

aa	ba	ca	da
ab	bb	cb	db
ac	bc	cc	dc
ad	bd	cd	dd.

If order does not matter but repititions are allowed there are 10 possibilities and they are

aa bb cc dd

ab bc cd ad ac bd

EX: In how many ways can 12 ballons be distributed at a Birthday party among 10 children ?

Sol: This is unordered selection with repitition

Here $n = 10$ and $r = 12$

Hence the number of selection is

$$C(12+10-1, 12) = C(21, 12) = C(21, 9)$$

If we want to ensure that every child gets atleast one ballon, we must give a ballon to each child, then distribute the remaining two ballons which can be done in $C(10+2-1, 2) = C(11, 2) = 55$ ways.

EX: A farmer buys 3 hens, 2 pigs, and 4 cows from a man who has 6 hens, 5 pigs and 8 cows. How many choices does the farmer have ?

Sol: The farmer can choose the hens in $C(6, 3)$ ways the pigs in $C(5, 2)$ ways, ~~hens~~ cows in $C(8, 4)$ ways

Hence the required number of ways are

$= C(6,3) \times C(5,2) \times C(8,4)$

$= {}^{6}C_3 \times {}^{5}C_2 \times {}^{8}C_4 = \dfrac{(6\cdot5\cdot4)(5\cdot4)(8\cdot7\cdot6\cdot5)}{3\cdot2\cdot1\cdot2\cdot1\cdot4\cdot3\cdot2\cdot1} = 20 \times 10 \times 70$

$= 14{,}000$ ways.

Ex: If there are 12 persons in a party and if each two of them shake hands with each other, how many handshakes happen in the party ?

Sol: When two persons shake hands. It is counted as one hand shack, not two.

$\therefore$ The total No. of handshakes is the same as no. of ways of selecting 2 persons from among 12 persons

$= C(12,2) = {}^{12}C_2 = \dfrac{12 \times 11}{2 \times 1} = 66.$

Ex: consider a pack of cards

i) How many 5-card hands consist only of Hearts ?

ii) How many 5-card hands consist of cards from a single suit ?

Sol: i) out of total 52 cards we have 13 hearts.

since there are 13 hearts to choose from, each such hand is a 5-combination of 13 objects.

$\therefore C(13,5) = \dfrac{13!}{8!\,5!} = \dfrac{13 \times 12 \times 11 \times 10 \times 9}{5 \times 4 \times 3 \times 2 \times 1} = 13 \times 11 \times 9 = 1287$

ii) out of each 4 suits, spades, hearts, diamonds or clubs there are $5 \cdot C(13,5)$ cards in hand.

Hence there are a total of $4 \cdot C(13,5)$ such hands.

Ex: Find the number of unordered samples of size five (repetitions allowed) from the set $\{a,b,c,d,e,f\}$

a) No further restrictions b) a occurs at least twice

Sol: a) Here $n = 6$, $r = 5$

So, the required member

$$= C(6+5-1, 5) = C(10,5) = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = 252$$

b) Since 'a' occurs at least twice, we have to find the number of unordered samples of size 3 from 6-element set. So the required number is

$$C(6+3-1, 3) = C(8,3) = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56.$$

c) Since 'a' occurs exactly twice we have to find the number of unordered samples of size 3 from 5-element set. So the required number is

$$C(5+3-1, 3) = C(7,3) = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35.$$

EX: How many ways are there to place 20 identical balls into 6 different boxes in which exactly 2 boxes are empty

Sol: The number of ways of choosing 2 boxes to be empty out of '6' boxes is $C(6,2)$.

Now first place one ball in each of the remaining 4 boxes. Then we count the number of ways of distributing the 16 identical balls into 4 boxes with repetitions.

This can be done in $C(16+4-1, 16)$

$$= C(19,16) = C(19,3)$$

Hence the required number of ways

$$= C(6,2) \times C(19,3)$$

$$= 15 \times 369$$

$$= 14535.$$

Mixed problems on permutations and combinations

Ex: Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

Sol: Three consonants out of 7 and 2 vowels out of 4 can be choosen in $C(7,3) \times C(4,2)$ ways.

Hence there are $C(7,3) \times C(4,2)$ groups each containing 3 consonants and 2 vowels.

Since each group contains 5 letters, which can be arranged among themselves in 5! ways.

So the required no. of words $= C(7,3) \times C(4,2) \times 5!$

$$= {}^7C_3 \times {}^4C_2 \times 5!$$

$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1} \times 5 \times 4 \times 3 \times 2 \times 1 = 25200.$$

Ex: Find the number of ways to seat 5 boys in a row of 12 chairs?

Sol: Using permutation :-

The problem is to arrange 12 objects that are of '6' kinds. The 6 different objects are 5 boys and 7 unoccupied chairs. Thus the number of arrangements is

$$= \frac{12!}{1!\,1!\,1!\,1!\,1!\,7!} = \frac{12!}{7!} = \frac{\overset{6}{\cancel{12}} \times 11 \times \overset{2}{\cancel{10}} \times \overset{3}{\cancel{9}} \times \overset{2}{\cancel{8}} \times \cancel{7} \times \cancel{6}}{\cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1} = 792$$

Using combination :-

Five boys can be arranged in a row in 5! ways.

Distribute the 7 unoccupied chairs arbitrarily in 6 places

Then Total Number of ways

$$= 5! \times C(6+7-1, 7) = 5! \times C(12,7)$$

$$= 5!\ \frac{12!}{5! \times 7!} = \frac{12!}{7!} = 792 \ //$$

Ex: Show that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ where $n \geq r \geq 1$ and n, r are natural numbers.

Sol: $${}^nC_r + {}^nC_{r-1} = \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$= \frac{n!}{r(r-1)!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)(n-r)!}$$

$$= \frac{n! \times (n-r+1) + n! \times r}{r(r-1)! \times (n-r)! \times (n-r+1)} = \frac{n! \times (n-r+1+r)}{r!\,(n-r+1)!}$$

$$= \frac{n! \times (n+1)}{r!\,(n-r+1)!} = \frac{(n+1)!}{(n+1-r)!\,r!} = {}^{n+1}C_r.$$

Ex: Show that $c(m+n,2) - c(m,2) - c(n,2) = mn$

Sol: $c(m+n,2) - c(m,2) - c(n,2)$

$$= \frac{(m+n)!}{(m+n-2)!\,2!} - \frac{m!}{(m-2)!\,2!} - \frac{n!}{(n-2)!\,2!}$$

$$= \frac{(m+n)(m+n-1)\cancel{(m+n-2)!}}{\cancel{(m+n-2)!}\,2!} - \frac{m(m-1)\cancel{(m-2)!}}{\cancel{(m-2)!}\,2!} - \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\,2!}$$

$$= \frac{(m+n)(m+n-1) - m(m-1) - n(n-1)}{2}$$

$$= \frac{\cancel{m^2} + mn - \cancel{m} + mn + \cancel{n^2} - \cancel{n} - \cancel{m^2} + \cancel{m} - \cancel{n^2} + \cancel{n}}{2} = \frac{\cancel{2}mn}{\cancel{2}} = mn$$

Constrained representations :-

1. **permutations:** The number of permutations of 'n' different objects takes 'r' at a time in which 'm' particular objects are

i) Never included be $p(n-m, r)$

ii) Always included will be $p(n-m, r-m) \cdot p(r, m)$

where $n \geq m$, $r \geq m$.

Ex: How many words of four letters can be formed with the letters a, b, c, d, e, f, g, h ~~and~~ when

i) 'e' and 'f' are not to be included

ii) 'e' and 'f' are to be included.

Sol: i) We are given with Eight letters

We are to take four letters after keeping aside '2'

Here $n = 8$, $m = 2$, $r = 4$, so 'e' and 'f' are not to be included then

the required no. of ways $= p(n-m, r)$

$= p(8-2, 4) = p(6,4) = {}^6P_4 = 360$ ways.

ii) Now we are interested in included 2 letters 'e' and 'f' for '2' places out of 4 places of 4 letters words.

$\therefore$ The required no. of ways $= p(n-m, r-m) \cdot p(r, m)$

$= p(8-2, 4-2) \cdot p(4,2)$

$= p(6,2) \cdot p(4,2) = {}^6P_2 \cdot {}^4P_2 = 360$ ways.

Ex: How many 6-digit numbers can be formed by using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 if every number is to start with '30' with no digit repeated?

Sol: All the numbers begin with 30.

So we have to choose 4-digits from the remaining 7-digits.

$\therefore$ The total Number of numbers that begin with '30' is ${}^7P_4 = \dfrac{7!}{(7-4)!} = \dfrac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 840$

2. combinations:- The number of combinations of 'n' object taken 'r' at a time in which 'm' particular objects are

i) Always included is $c(n-m, r-m)$ ways

ii) Never included is $c(n-m, r)$ ways where $n \geq m$.

Ex: In how many ways a football eleven can be chosen out of 17 players when i) Five particular players are to be always included ii) Two particular players are to be always excluded.

Sol: i) First we are interested in including five particular players, Thus after selecting five we are left with 12 players out of which we are select remaining '6' players. Here $n=17$, $m=5$, $r=11$

$\therefore$ The required No. of ways $= c(n-m, r-m)$

$$= c(17-5, 11-5) = c(12,6)$$

$$= \frac{12!}{(12-6)!\,6!} = \frac{12\times 11\times 10\times 9\times 8\times 7\times \cancel{6!}}{\cancel{6!}\,6!} = 924 \text{ ways.}$$

ii) We wish to exclude two particular players then we are left with 15 players out of which we are to select eleven which can be done. Here $n=17$, $m=2$, $r=11$

$\therefore$ The required No. of ways $= c(n-m, r)$

$$= c(17-2, 11) = c(15,11)$$

Ex: How many different selection of 6 books can be made from 11 different books, if

i) Two particular book are ~~a~~always selected

ii) Two particular book are never selected.

Sol: i) since Two particular books are Always Selected

Here $n = 11$, $m = 2$, $r = 6$

$\therefore$ No. of Required ways $= c(n-m, r-m)$

$= c(11-2, 6-2) = c(9,4)$

$= {}^9c_4 = \dfrac{9\times8\times7\times6}{4\times3\times2\times1} = 126.$

ii) since two particular books are never selected.

Here $n = 11$, $m = 2$, $r = 6$

$\therefore$ No. of Required Ways $= c(n-m, r)$

$= c(9,6) = {}^9c_6 = \dfrac{9\times8\times7\times6\times5\times4}{6\times5\times4\times3\times2\times1} = 84.$

EX: From 10 programmers in how many ways can 5 be selected when i) A particular programmer is included every time ii) A particular programmer is not included at all.

Sol: i) We have to select 5 programmers from the 10.

So the no. of ways selected them in ${}^{10}c_5 = 252.$

i) When a particular programmer is included every time then the required No. of ways   Here $n = 10$, $m = 1$, $r = 5$

$= c(n-m, r-m)$

$= c(10-1, 5-1) = c(9,4) = {}^9c_4 = \dfrac{9\times8\times7\times6}{4\times3\times2\times1} = 126$

ii) When a particular programmer is not included at all then the required No. of ways

$= c(n-m, r)$

$= c(10-1, 5) = c(9,5) = {}^9c_5 = {}^9c_4 = 126$

Binomial Coeffecients :-

The quantity $\frac{n!}{(n-r)!\,r!}$ written as $c(n,r)$ (or) $\binom{n}{r}$ is known as Binomial coeffecient. The symbol $c(n,r)$ has two meanings i) combinatorial meaning ii) algebraic meaning : In the first case it represents the no. of ways of choosing 'r' objects from 'n' distinct objects and in the second case $c(n,r) = \frac{n!}{(n-r)!\,r!}$.

Ex: $\binom{8}{2} = \frac{8 \times 7}{2 \times 1} = 28$, $\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 126 = \binom{9}{4}$

Note: $\binom{n}{a} = \binom{n}{b} \Rightarrow \boxed{n = a+b}$.

Binomial Theorem :-

For any real number x, y and any integer $n \geq 0$ then $(x+y)^n = nc_0\, x^n + nc_1\, x^{n-1} y + nc_2\, x^{n-2} y^2 + \cdots + nc_r\, x^{n-r} y^r + \cdots + nc_n\, x^0 y^n$.

Proof: Multiplying out the left hand side we get

$(x+y)^n = (x+y)(x+y) \cdots \cdots (x+y)$ ($\because$ n times)

This gives a sum of terms, each of which is obtained by multiplying together one choice of 'x' or 'y' from each bracket. If y is chosen from exactly 'r' brackets ~~x~~ then 'x' must be chosen from the remaining $(n-r)$ brackets. So the resulting term will be $x^{n-r} y^r$.

But this can be done in $c(n,r)$ ways.

Since $c(n,r)$ counts the number of ways of selecting

Thus $x^{n-r} y^r$ appears $c(n,r)$ times.

It follows that

$$(x+y)^n = c(n,0)\, x^n y^0 + c(n,1)\, x^{n-1} y^1 + c(n,2)\, x^{n-2} y^2 + \cdots + c(n,n)\, x^0 y^n$$

$$= {}^nc_0\, x^n + {}^nc_1\, x^{n-1} y + {}^nc_2\, x^{n-2} y^2 + \cdots + {}^nc_n\, y^n.$$

$$(x+y)^n = \sum_{r=0}^{n} {}^nc_r\, x^{n-r} y^r.$$

Note: 1. ${}^nc_0 + {}^nc_1 + \cdots + {}^nc_n = 2^n$.

2. ${}^nc_1 + {}^nc_3 + \cdots = {}^nc_0 + {}^nc_2 + \cdots = 2^{n-1}$

3. ${}^nc_r = {}^nc_{n-r}$

4. ${}^nc_r \cdot {}^rc_K = {}^nc_K \cdot {}^{n-K}c_{r-K}$ (∵ Newton's Identity

5. ${}^{n+1}c_r = {}^nc_r + {}^nc_{r-1}$ (∵ pascal Identity)

6. ${}^{n+m}c_r = \sum_{K=0}^{r} {}^mc_{r-K} \cdot {}^nc_K$ (∵ Vandermonde's Identity

Multinomial coeffecients :-

The expression in the form $x_1 + x_2$ is a binomial, a multinomial is an expression of the form $x_1 + x_2 + \cdots + x_n$ with $n \geq 3$

If $n_1, n_2, \cdots n_K$ are non negative integers such that $n_1 + n_2 + \cdots n_K = n$ then

$$c(n_1, n_2, \cdots, n_K) = \frac{n!}{n_1!\, n_2! \cdots n_K!}$$ is called a

Multinomial coeffecient. This coeffecient denotes the number of distinguishable arrangements of n objects.

Multinomial Theorem :-

For any positive integer 'n' and 'k' then

$$(x_1 + x_2 + \cdots + x_k)^n = \sum \binom{n}{n_1, n_2, \ldots n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

the summation is taken over all k-tuples of non negative integers $(n_1, n_2 \ldots n_k)$ such that $n_1 + n_2 + \cdots + n_k = n$.

proof: When the 'n' factors $x_1 + x_2 + \cdots + x_k$ are multiplied a typical product term $x_1^{n_1} \cdot x_2^{n_2} \cdots x_k^{n_k}$ arises by choosing ~~them~~ x_1 term from n_1 of the factors, the x_2 term from n_2 of the factors and so on.

In other words the typical term corresponds to a function from the set of n factor to the set $\{x_1, x_2 \ldots x_k\}$, with the property that n_1 of the factors go to x_1, n_2 of them go to x_2 and so on.

By the definition multinomial numbers there are $\binom{n}{n_1, n_2 \ldots n_k}$ functions of this kind and so this number of terms $x_1^{n_1} \cdot x_2^{n_2} \cdots x_k^{n_k}$ in the product.

$$\therefore (x_1 + x_2 + x_3 \cdots + x_k)^n = \sum \binom{n}{n_1, n_2, \ldots n_k} x_1^{n_1} \cdot x_2^{n_2} \cdots x_k^{n_k}.$$

Ex: $(x_1 + x_2 + x_3 + x_4 + x_5)^{10}$

The coeffecient of $x_1^2 x_3 x_4^3 x_5^4$ is

$$p(10; 2, 0, 1, 3, 4) = \frac{10!}{2!\,0!\,1!\,3!\,4!} = 12{,}600.$$

Ex: What is the coefficient of x^2y^4 in $(x+y)^6$,

Sol: Here $(x+y)^6 = (x+y)(x+y)\cdots(x+y)$ (6 times)

We have to choose an 'x' from two factors and a 'y' from the remaining four factors.

$$c(6,2) = {}^6c_2 = \frac{6\times5}{2\times1} = 15$$

$\therefore$ The coefficient of $x^2y^4 = 15$.

Ex: What is the coefficient of $x^3y^2z^2$ in $(x+y+z)^9$?

Sol: The coefficient is the same as how many ways one can choose 'x' from three brackets, a 'y' from '2' brackets and a 'z' from '2' brackets in the expansion.

$(x+y+z)(x+y+z)\cdots\cdots(x+y+z)$ (9 times)

$\therefore$ The coefficient of $x^3y^2z^2$

$$= \frac{9!}{3!\,2!\,2!} = 15120.$$

Ex: Find the coefficient of x^{16} in the expansion of $\left(2x^2-\frac{x}{2}\right)^{12}$

Sol: By Binomial Theorem

$$\binom{12}{r}(2x^2)^{12-r}\left(-\frac{x}{2}\right)^r = \binom{12}{r}2^{12-r}\left(-\frac{1}{2}\right)^r x^{24-r}$$

We want $24-r = 16 \Rightarrow r = 8$

$$\therefore \text{The coefficient} = \binom{12}{8}2^4\left(-\frac{1}{2}\right)^8 = \frac{1}{16}\binom{12}{8} = \frac{495}{16}.$$

Ex: Find the coefficient of $a^4b^3c^2d$ in the expansion of $(a-b+c-d)^9$.

Sol: The coefficient of $a^4b^3c^2d$ is

$$= (-1)^4\frac{9!}{4!\,3!\,2!\,1!} = 12600$$

powers of 'b' and 'd' are '3' and '1'

$\therefore (-1)^3(-1)^1 = (-1)^4$

Ex: Find the coeffecient of b^4c^2 in $(a+b+c)^6$?

Sol: The coeffecient of b^4c^2 is

$$= \frac{6!}{4!\,2!} = \frac{6\times5\times\cancel{4!}}{\cancel{4!}\,2!} = \frac{30}{2} = 15$$

Ex: Find the coeffecient of x^9y^3 in the expansion of $(2x-3y)^{12}$.

Sol: By the binomial theorem

$$(2x-3y)^{12} = \sum_{r=0}^{12} \binom{12}{r} (2x)^r (-3y)^{12-r}$$

$$= \sum_{r=0}^{12} \binom{12}{r} 2^r (-3)^{12-r} \cdot x^r \cdot y^{12-r}$$

In this expansion coeffecient of x^9y^3 is

$$= \binom{12}{9} 2^9 (-3)^3 = (-2)^9 \cdot 3^3 \times \frac{12!}{9!\,3!} \quad \leftarrow r=9$$

$$= -2^9 \times 3^3 \times \frac{12\times11\times10\times\cancel{9!}}{\cancel{9!}\,3\times2\times1} = -2^{10}\times3^3\times11\times10 = 1946.$$

Ex: Find the coeffecient of xyz^2 in the expansion of $(2x-y-z)^4$

Sol: By the multinomial theorem

$$(2x-y-z)^4 = \binom{4}{n_1\ n_2\ n_3} \cdot (2x)^{n_1} (-y)^{n_2} (-z)^{n_3}$$

Here $n_1 = 1$, $n_2 = 1$, $n_3 = 2$

$$\therefore \binom{4}{1\ 1\ 2} \cdot 2^1 (-1)^1 \cdot (-1)^2 \cdot xyz^2$$

$\therefore$ The coeffecient of xyz^2 is

$$= 2\cdot(-1)(1)\cdot\frac{4!}{1!\,1!\,2!} = -24.$$

Ex: Find the coeffecient of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$.

Sol: Given

$$(a+2b-3c+2d+5)^{16} = \binom{16}{n_1\ n_2\ n_3\ n_4\ n_5} a^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}$$

Here $n_1 = 2,\ n_2 = 3,\ n_3 = 2,\ n_4 = 5,\ n_5 = 16-(2+3+2+5)$
$= 16-12 = 4$

$$\therefore \binom{16}{2\ 3\ 2\ 5\ 4} \cdot a^2 (2b)^3 (-3c)^2 (2d)^5\ 5^4.$$

$\therefore$ The coeffecient of $a^2b^3c^2d^5$ is

$$= \frac{16!}{2!\,3!\,2!\,5!\,4!} \times 2^3 \times (-3)^2\, 2^5 \cdot 5^4$$

$$= \frac{16!\ \ 2^3 \cdot 3^2 \cdot 2^5 \cdot 5^4}{2\times1\times3\times2\times1\times2\times1\times5\times4\times3\times2\times1\times4\times3\times2\times1}$$

$$= \frac{16!\ \ 2^3\times5^4}{3\times5\times4^2}$$

Ex: What is the coeffecient of x^6y^3 in $(x+y)^9$.

Sol: 84

Ex: what is the coeffecient of x^5y^7 in $(x-y)^{12}$

Sol: -101376.

Ex: what is the coeffecient of x^3y^2 in $(3x-y)^5$

Sol. 2700

Ex: what is the coeffecient of $x^2y^3z^2$ in $(x+y+z)^7$

Sol: 210

Ex: what is the coeffecient of $x^3y^2z^5$ in $(x+y+z)^{10}$.

Sol: 2520.

Ex: what is the coeffecient of x^2yz in $(2x-y+z+1)^7$.

Sol: -280.

principle of Inclusion - Exclusion :-

Let $A_1, A_2, \ldots A_n$ be finite sets. To calculate $|A_1 \cup A_2 \cup A_3 \cdots \cup A_n|$ the sizes of all possible intersections of sets from $\{A_1, A_2, \cdots A_n\}$, add the result obtained by intersecting an odd number of sets and then subtract the results obtained by intersecting an even number of sets. In general

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Ex: How many integers in $S = \{1, 2, \ldots, 1000\}$ are divisible by either '3' 'or' '5' or both ?

Sol: Let $D_3 = \{ n \in S ; n \text{ is divisible by } 3 \}$

$D_5 = \{ n \in S ; n \text{ is divisible by } 5 \}$

Now $|D_3| = \lfloor 1000/3 \rfloor = 333$

$|D_5| = \lfloor 1000/5 \rfloor = 200$

$|D_3 \cap D_5| = \lfloor 1000/15 \rfloor = 66$

$\therefore |D_3 \cup D_5| = |D_3| + |D_5| - |D_3 \cap D_5|$

$= 333 + 200 - 66 = 467.$

Ex: of 32 people who save paper or bottles for recycling 30 save paper and 14 save bottles Find the no. of people who i) save both ii) save only paper iii) save only bottles.

Sol: Let P and B denote the sets of people saving paper and bottles respectively

i. m = $n(P \cap B) = n(P) + n(B) - n(P \cup B)$

$= 30 + 14 - 32 = 12$

ii) $m = n(P \setminus B) = n(P) - n(P \cap B)$

$= 30 - 12 = 18$

iii) $m = n(B \setminus P) = n(B) - n(P \cap B)$

$= 14 - 12 = 2.$

Ex: In a sample of 100 logic chips, 23 have a defect D_1, 26 have a defect D_2, 30 have defect D_3, 7 have defect D_1 and D_2, 8 has D_1 & D_3, 10 have D_2 & D_3, 3 have all '3'. Find the no. of chips having i) at least one defect ii) no defect

Sol:

$|U| = 100$ $\quad |A \cap B| = 7$

$|A| = 23$ $\quad |B \cap C| = 8$

$|B| = 26$ $\quad |A \cap C| = 10$

$|C| = 30$ $\quad |A \cap B \cap C| = 3.$

i) At least one defect is $A \cup B \cup C$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 23 + 26 + 30 - 7 - 8 - 10 + 3.$

$= 57$

ii) No defect is $|\overline{(A \cup B \cup C)}|$

$|\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C| = 100 - 57 = 43$ //

UNIT - IV

RECURRENCE RELATIONS

Generating functions of sequences :-

Let $a_0, a_1, a_2, \ldots a_K \ldots$ be a sequence of real numbers then $G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_K x^K + \cdots$ is called a Generating function for the sequence $a_0, a_1, a_2 \ldots a_K, \ldots$.

$$G(x) = \sum_{K=0}^{\infty} a_K x^K. \quad \text{(or)} \quad \sum_{r=0}^{\infty} a_r x^r.$$

Note: If 'n' is positive integer then

1. $(1+x)^n = \sum_{r=0}^{\infty} {}^nC_r \, x^r.$

2. $(1-x)^n = \sum_{r=0}^{\infty} {}^nC_r \, (-x)^r.$

Ex: Find the sequence generated by the following functions.

1. $(3+x)^3$.

Sol. $(3+x)^3 = \left[3\left(1+\frac{x}{3}\right)\right]^3$

$= 3^3\left[1+\frac{x}{3}\right]^3 = 27\left[1 + 3\cdot(1)^2\left(\frac{x}{3}\right) + 3(1)\left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3\right]$

$= 27\left(1 + x + \frac{x^2}{3} + \frac{x^3}{27}\right)$

$= 27 + 27x + 9x^2 + x^3 = 27 + 27x + 9x^2 + x^3 + 0\cdot x^4 + 0\cdot x^5 + 0\cdot x^6 + \cdots$

$\therefore$ The sequence generated by the given function is

$27, 27, 9, 1, 0, 0, 0, \ldots$

2. $2x^2(1-x)^{-1}$

Sol: Let $f(x) = 2x^2(1-x)^{-1}$

$= 2x^2(1+x+x^2+x^3+\cdots)$

$= 2x^2+2x^3+2x^4+2x^5+\cdots$

$= 0+0\cdot x+2x^2+2x^3+2x^4+2x^5+\cdots$

$\therefore$ sequence of $f(x) = 0, 0, 2, 2, 2, \cdots$

3. $\frac{1}{1-x}+2x^3$

Sol: Let $f(x) = \frac{1}{1-x}+2x^3$

$f(x) = (1-x)^{-1}+2x^3 = 1+x+x^2+x^3+\cdots+2x^3$

$= 1+x+x^2+3x^3+x^4+x^5+x^6+\cdots$

$\therefore$ Sequence of $f(x) = 1, 1, 1, 3, 1, 1, 1, \cdots$

4. $3x^3+e^{2x}$.

Sol. Let $f(x) = 3x^3+e^{2x}$

$f(x) = 3x^3+1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots$

$= 1+\frac{2x}{1!}+\frac{2^2x^2}{2}+\left(3+\frac{2^3}{3!}\right)x^3+\frac{(2x)^4}{4!}+\cdots$

$= 1+2x+2x^2+\left(3+\frac{2^3}{3!}\right).x^3+\frac{2^4}{4!}x^4+\cdots$

$\therefore$ Sequence of $f(x) = 1, 2, 2, 3+\frac{2^3}{3!}, \frac{2^4}{4!}, \cdots$

5. $(1-x)^{-1}-x^2$

Sol: Let $f(x) = (1-x)^{-1}-x^2$

$f(x) = 1+x+x^2+x^3+\cdots - x^2$

$= 1+x+0\cdot x^2+x^3+x^4+\cdots$

$\therefore$ sequence of $f(x) = 1, 1, 0, 1, 1, 1, \cdots$

Ex: Find the sequence of $\frac{x^5}{x^2-5x+6}$.

Sol: Let $f(x) = \frac{x^5}{x^2-5x+6}$ then $f(x) = x^5 \cdot \frac{1}{x^2-5x+6}$.

$$f(x) = x^5\left[\frac{1}{(x-2)(x-3)}\right] = x^5\left[\frac{1}{x-3} - \frac{1}{x-2}\right]$$

$$= x^5\left[\frac{1}{-3(1-\frac{x}{3})} - \frac{1}{-2(1-\frac{x}{2})}\right]$$

$$= x^5\left[-\frac{1}{3}(1-\frac{x}{3})^{-1} + \frac{1}{2}(1-\frac{x}{2})^{-1}\right]$$

$$= x^5\left[-\frac{1}{3}(1+\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\cdots) + \frac{1}{2}(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\cdots)\right]$$

$$= x^5\left[(\frac{1}{2}-\frac{1}{3}) + (\frac{1}{2^2}-\frac{1}{3^2})x + (\frac{1}{2^3}-\frac{1}{3^3})x^2 + (\frac{1}{2^4}-\frac{1}{3^4})x^3 + \cdots\right]$$

$$= (\frac{1}{2}-\frac{1}{3})x^5 + (\frac{1}{2^2}-\frac{1}{3^2})x^6 + (\frac{1}{2^3}-\frac{1}{3^3})x^7 + \cdots$$

$\therefore$ sequence of $f(x) = 0, 0, 0, 0, (\frac{1}{2}-\frac{1}{3}), (\frac{1}{2^2}-\frac{1}{3^2}), \cdots$

Note: If 'n' is a real number and 'x' is a non negative integer then

$$\sum_{r=0}^{\infty} {}^nC_r\, x^r = 1 + \sum_{r=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!} \cdot x^r$$

Ex: Find the sequence generated by the following functions

i) $(1+3x)^{-1/3}$ ii) $(1+5x)^{-1/5}$ iii) $(1-4x)^{-1/2}$.

Sol: (i) Let $f(x) = (1+3x)^{-1/3}$. Then

$$f(x) = 1 + \sum_{r=1}^{\infty} \frac{(-\frac{1}{3})(-\frac{1}{3}-1)(-\frac{1}{3}-2)\cdots(-\frac{1}{3}-(r-1))}{r!}(3x)^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})\cdots(-\frac{3r+2}{3})}{r!}\, 3^r \cdot x^r.$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)(-4)(-7)\cdots(-3r+2)}{(3\cdot3\cdot3\cdots3)\, r!}\, 3^r\cdot x^r.$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)(-4)(-7)\cdots(-3r+2)}{\cancel{3^r}\cdot r!}\, \cancel{3^r}\cdot x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)(-4)(-7)\cdots(-3r+2)}{r!}\, x^r.$$

$$f(x) = 1 + \frac{(-1)}{1!}x + \frac{(-1)(-4)}{2!}x^2 + \frac{(-1)(-4)(-7)}{3!}x^3 + \cdots$$

$\therefore$ Sequence of $f(x) = 1, \frac{-1}{1!}, \frac{(-1)(-4)}{2!}, \frac{(-1)(-4)(-7)}{3!}, \ldots$

iii) Let $f(x) = (1-4x)^{-1/2}$ then

$$f(x) = 1 + \sum_{r=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!}\, x^r.$$

$$= 1 + \sum_{r=1}^{\infty} \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)\cdots\left(\frac{-1}{2}-(r-1)\right)}{r!}\,(-4x)^r.$$

$$= 1 + \sum_{r=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(\frac{-2r+1}{2}\right)}{r!}\,(-1)^r(2^2)^r\cdot x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)(-3)(-5)\cdots(-2r+1)}{(2\cdot2\cdot2\cdots2)\, r!}\,(-1)^r(2^r)^2\cdot x^r.$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)(-1)(3)(-1)5\cdots(-1)(2r-1)}{\cancel{2^r}\cdot r!}\cdot(-1)^r(\cancel{2^r})^2\cdot x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)^r\cdot(1\cdot3\cdot5\cdots(2r-1))}{r!}\,(-1)^r\, 2^r\cdot x^r.$$

$$= 1 + \sum_{r=1}^{\infty} \frac{[(-1)^2]^2\; 1\cdot3\cdot5\cdots(2r-1)\; r!}{r!\, r!}\, 2^r\cdot x^r.$$

$$= 1 + \sum_{r=1}^{\infty} \frac{1\cdot(2r)!}{r!\,r!}\, x^r. = 1 + \sum_{r=1}^{\infty} {}^{2r}C_r\cdot x^r$$

$$= 1 + {}^{2}C_1\, x + {}^{4}C_2\, x^2 + {}^{6}C_3\, x^3 + \cdots$$

$1, {}^{2}C_1, {}^{4}C_2, {}^{6}C_3, \ldots$

Ex: Find the generating functions for the following sequences.

1. $1, 2, 3, 4, \ldots$

Sol: Given sequence $1, 2, 3, 4, \ldots$

Let $a_0 = 1,\ a_1 = 2,\ a_2 = 3,\ a_3 = 4, \ldots$

$\therefore$ The generating function is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= 1 + 2x + 3x^2 + 4x^3 + \cdots = (1-x)^{-2}$$

2. $1, -2, 3, -4, 5, \ldots$

Sol: Given sequence $1, -2, 3, -4, 5, \ldots$

Let $a_0 = 1,\ a_1 = -2,\ a_2 = 3,\ a_3 = -4,\ a_4 = 5, \ldots$

$\therefore$ The generating function is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$
$$= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots = (1+x)^{-2}.$$

3. $0, 1, 2, 3, \ldots$

Sol: $f(x) = 0 + x + 2x^2 + 3x^3 + \cdots$
$$= x(1 + 2x + 3x^2 + \cdots) = x(1-x)^{-2}.$$

4. $0, 1, -2, 3, -4, \ldots$

Sol: $f(x) = 0 + x - 2x^2 + 3x^3 - 4x^4 \cdots$
$$= x(1 - 2x + 3x^2 - 4x^3 + \cdots) = x(1+x)^{-2}.$$

5. $1^2, 2^2, 3^2, 4^2, \ldots$

Sol: $f(x) = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots \longrightarrow (1)$

W.K.T $\quad x(1-x)^{-2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots$

differentiating w.r.to 'x' we get

$$\frac{d}{dx}\left(x(1-x)^{-2}\right) = \frac{d}{dx}\left(x + 2x^2 + 3x^3 + 4x^4 + \cdots\right)$$

$$x \frac{d}{dx}(1-x)^{-2} + (1-x)^{-2}\frac{d}{dx}(x) = 1 + 2(2x) + 3(3x^2) + 4(4x^3) + \dots$$

$$\Rightarrow x(-2)(1-x)^{-3}(0-1) + (1-x)^{-2} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots$$

$$\Rightarrow \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2} = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \dots$$

$$\therefore f(x) = \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2} = \frac{2x+1-x}{(1-x)^3} = \frac{1+x}{(1-x)^3} \text{ //.}$$

6. $0^2, 1^2, 2^2, 3^2, \dots$

Sol: $f(x) = 0 + 1^2x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \dots$

$$= x(1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots)$$

$$= x\left(\frac{1+x}{(1-x)^3}\right).$$

7. $0, 2, 6, 12, 20, 30, 42, \dots$

Sol: $f(x) = 0 + 2x + 6x^2 + 12x^3 + 20x^4 + 30x^5 + 42x^6 + \dots$

$$= 0 + (1+1)x + (2+4)x^2 + (3+9)x^3 + (4+16)x^4 + \dots$$

$$= 0 + (1+1)x + (2+2^2)x^2 + (3+3^2)x^3 + (4+4^2)x^4 + \dots$$

$$= (0 + x + 2x^2 + 3x^3 + 4x^4 + \dots) + (0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \dots)$$

$$= x(1 + 2x + 3x^2 + 4x^3 + \dots) + x(1 + 2^2x + 3^2x^2 + 4^2x^3) \dots$$

$$= x(1-x)^{-2} + x\left(\frac{1+x}{(1-x)^3}\right) = \frac{x(1-x) + x(1+x)}{(1-x)^3}$$

$$= \frac{x - x^2 + x + x^2}{(1-x)^3} = \frac{2x}{(1-x)^3} \text{ //}$$

8. $8, 26, 54, 92, \dots$

Sol. $f(x) = 8 + 26x + 54x^2 + 92x^3 + \dots$

$$= (3(1) + 5(1)) + (3(2) + 5(2)^2)x + (3(3) + 5(3)^2)x^2$$

$$+ (3(4) + 5(3)^3)x^3 + \dots$$

$$= \left(3(1) + 3(2)x + 3(3)x^2 + 3(4)x^3 + \cdots\right) + \left(5(1) + 5(2)^2 x + 5(3^2)x^2 + 5(4^2)x^3 + \cdots\right)$$

$$= 3\left(1 + 2x + 3x^2 + 4x^3 + \cdots\right) + 5\left(1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots\right)$$

$$= 3\left(\frac{1}{(1-x)^2}\right) + 5\left(\frac{1+x}{(1-x)^3}\right) = \frac{3(1-x) + 5(1+x)}{(1-x)^3}$$

$$= \frac{3 - 3x + 5 + 5x}{(1-x)^3} = \frac{8 + 2x}{(1-x)^3}. \quad //$$

Ex: Find the generating functions for the sequence $\langle a_r \rangle$ for each of the following

1) a^r.

Sol: Let $a_r = a^r$

W.K.T $\quad f(x) = \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} a^r x^r$

$$= \sum_{r=0}^{\infty} (ax)^r = (ax)^0 + (ax)^1 + (ax)^2 + (ax)^3 + \cdots$$

$$= 1 + (ax) + (ax)^2 + (ax)^3 + \cdots$$

$$= (1 - ax)^{-1} = \frac{1}{1 - ax} \quad //$$

2. $(r+1)\, a^r$.

Sol: Let $a_r = (r+1)\, a^r$

$$f(x) = \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} a^r (r+1) x^r$$

$$= \sum_{r=0}^{\infty} (r+1)(ax)^r$$

$$= (0+1)(ax)^0 + (1+1)(ax)^1 + (2+1)(ax)^2 + \cdots$$

$$= 1 + 2(ax) + 3(ax)^2 + 4(ax)^3 + \cdots$$

$$= (1 - ax)^{-2} = \frac{1}{(1-ax)^2} \quad //$$

3. $a_r = r^2 a^r$.

Sol: Let $a_r = r^2 a^r$.

$$f(x) = \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} r^2 a^r \cdot x^r$$

$$= \sum_{r=0}^{\infty} r^2 (ax)^r = 0(ax)^0 + 1(ax)^1 + 2^2(ax)^2 + \cdots$$

$$= 0 + (ax) + 2^2(ax)^2 + 3^2(ax)^3 + \cdots$$

$$= ax\left[1 + 2^2(ax) + 9(ax)^2 + 16(ax)^3 + \cdots\right]$$

$$= ax\left[(1 + 3ax + 6(ax)^2 + 10(ax)^3 + \cdots) + (ax + 3(ax)^2 + 6(ax)^3 + 10(ax)^4 + \cdots)\right]$$

$$= ax\left[1(1 + 3ax + 6(ax)^2 + 10(ax)^3 + \cdots) + ax(1 + 3(ax) + 6(ax)^2 + 10(ax)^3 + \cdots)\right]$$

$$= ax\left[(1 + 3ax + 6(ax)^2 + 10(ax)^3 + \cdots)(1 + ax)\right]$$

$$= ax(1-ax)^{-3} \cdot (1+ax) = ax(1+ax)(1-ax)^{-3} \;//.$$

4. $(r+1)(r+2)(r+3)$

Sol: Let $a_r = (r+1)(r+2)(r+3)$

$$f(x) = \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} (r+1)(r+2)(r+3) \cdot x^r$$

$$= (0+1)(0+2)(0+3)x^0 + (1+1)(1+2)(1+3)x^1 + (2+1)(2+2)(2+3)x^2 + (3+1)(3+2)(3+3)x^3 + \cdots$$

$$= (1)(2)(3)(1) + (2)(3)(4)x + (3)(4)(5)x^2 + (4)(5)(6)x^3 + \cdots$$

$$= 6 + 24x + 60x^2 + 120x^3 + 210x^4 + \cdots$$

$$= 6(1 + 4x + 10x^2 + 20x^3 + 35x^4 + \cdots)$$

$$= 6(1-x)^{-4} = \frac{6}{(1-x)^4} \;//.$$

Calculating coeffecients of generating functions :-

Ex: Find the coeffecient of i) x^{12} in $x^3(1-2x)^{10}$.

Sol: W.K.T $(1-x)^n = \sum_{r=0}^{\infty} {}^{n}C_{r} (-x)^r$.

$$\therefore x^3(1-2x)^{10} = x^3 \sum_{r=0}^{\infty} {}^{10}C_{r} (-2x)^r = \sum_{r=0}^{\infty} {}^{10}C_{r} (-2)^r x^r \cdot x^3.$$

$$= \sum_{r=0}^{\infty} {}^{10}C_{r} (-2)^r x^{3+r}.$$

We find x^{12} coeffecient of $x^3(1-2x)^{10}$.

$$\therefore 3+r = 12 \Rightarrow r = 12-3 = 9 \qquad \therefore r = 9$$

$$\therefore \text{The coeffecient of } x^{12} = {}^{10}C_{9} (-2)^9 = -5120.$$

2) x^0 in $\left(3x^2 - \frac{2}{x}\right)^{15}$.

Sol: given $\left(3x^2 - \frac{2}{x}\right)^{15} = \left[3x^2\left(1 - \frac{2}{3x^3}\right)\right]^{15}$

$$= (3x^2)^{15} \cdot \left(1 - \frac{2}{3x^3}\right)^{15}$$

$$= 3^{15} (x^2)^{15} \sum_{r=0}^{\infty} {}^{15}C_{r} \left(-\frac{2}{3x^3}\right)^r.$$

$$= 3^{15} x^{30} \sum_{r=0}^{\infty} {}^{15}C_{r} \left(-\frac{2}{3}\right)^r \cdot \left(\frac{1}{x^3}\right)^r.$$

$$= 3^{15} x^{30} \sum_{r=0}^{\infty} {}^{15}C_{r} \left(\frac{-2}{3}\right)^r \cdot (x^{-3})^r = \sum_{r=0}^{\infty} {}^{15}C_{r} \frac{(-2)^r \cdot 3^{15}}{3^r} \cdot x^{30} \cdot x^{-3r}$$

$$= \sum_{r=0}^{\infty} {}^{15}C_{r} (-2)^r \cdot 3^{15-r} \cdot x^{(30-3r)}.$$

We find x^0 coeffecient of $\left(3x^2 - \frac{2}{x}\right)^{15}$

$$\therefore 30 - 3r = 0 \Rightarrow 3r = 30 \Rightarrow \boxed{r = 10}$$

$$\therefore \text{The coeffecient of } x^0 = {}^{15}C_{10} (-2)^{10} 3^{15-10}$$

$$= {}^{15}C_{10} (2)^{10} 3^5 = (1024)(243)(273)$$

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3. The coeffecient of x^5 in $(1-2x)^{-7}$.

Sol: Note: $(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$

Here $n=7$, $x = 2x$

$\therefore (1-2x)^{-7} = \sum_{r=0}^{\infty} \binom{7+r-1}{r} (2x)^r = \sum_{r=0}^{\infty} \binom{6+r}{r} (2x)^r$.

The coeffecient of x^5 in $(1-2x)^{-7}$

$\therefore r = 5$

$\therefore$ The coeffecient of $x^5 = \binom{6+5}{5} 2^5$

$= {}^{11}C_5 \, 2^5 = \frac{11!}{6!\,5!} 32 = 14784$.

4. The coeffecient of x^{10} in $\frac{x^3-5x}{(1-x)^3}$.

Sol: $\frac{x^3-5x}{(1-x)^3} = (x^3-5x)(1-x)^{-3}$

$= (x^3-5x) \sum_{r=0}^{\infty} \binom{3+r-1}{r} x^r$.

$= (x^3-5x) \sum_{r=0}^{\infty} \binom{2+r}{r} x^r$.

$= \sum_{r=0}^{\infty} \binom{2+r}{r} x^{r+3} - 5 \sum_{r=0}^{\infty} \binom{2+r}{r} x^{r+1} \longrightarrow (1)$

we find ~~the~~ coeffecient of x^{10}, so $r=7$ & $r=9$

in I^{st} and 2^{nd} terms respectively.

coeffecient of $x^{10} = \sum \binom{2+7}{7} x^{7+10} - 5 \sum \binom{9+2}{9} x^{9+1}$

$= \binom{9}{7} - 5\binom{11}{9} = {}^{9}C_7 - 5\,{}^{11}C_9 = 36 - 275 = -239$.

5. The coeffecient of x^8 in $\frac{1}{(x-3)(x-2)^2}$.

Sol: $\frac{1}{(x-3)(x-2)^2} = \frac{1}{(-3)(1-\frac{x}{3})(-2)^2(1-\frac{x}{2})^2} = \frac{1}{(-3)(4)(1-\frac{x}{3})(1-\frac{x}{2})^2}$

$$= \frac{-1}{12}\left(1-\frac{x}{3}\right)^{-1}\left(1-\frac{x}{2}\right)^{-2}$$

$$= \frac{-1}{12}\left[\sum_{r=0}^{\infty}\left(\frac{x}{3}\right)^{r}\sum_{s=0}^{\infty}\binom{2+s-1}{s}\left(\frac{x}{2}\right)^{s}\right]$$

$$= \frac{-1}{2}\left[\sum_{r=0}^{\infty}\frac{1}{3^{r}}\cdot x^{r}\cdot\sum_{s=0}^{\infty}\frac{1}{2^{s}}\binom{1+s}{s}x^{s}\right]$$

The coeffecient of x^{8} in above series is

$$c_8 = \frac{-1}{12}\left[\frac{1}{3^0}\frac{1}{2^8}\binom{8+1}{8}+\frac{1}{3^1}\frac{1}{2^7}\binom{7+1}{7}+\frac{1}{3^2}\cdot\frac{1}{2^6}\binom{6+1}{6}+\cdots\right.$$

$$\left.\cdots+\frac{1}{3^8}\frac{1}{2^0}\binom{0+1}{0}\right]$$

$$= \frac{-1}{12}\left[\binom{9}{8}\left(\frac{1}{3}\right)^{0}\left(\frac{1}{2}\right)^{8}+\binom{8}{7}\left(\frac{1}{3}\right)^{1}\left(\frac{1}{2}\right)^{7}+\cdots+\binom{1}{0}\left(\frac{1}{3}\right)^{8}\left(\frac{1}{2}\right)^{0}\right]$$

$$= \frac{-1}{12}\left[\binom{9}{1}1\cdot\frac{1}{256}+\binom{8}{1}\frac{1}{3}\frac{1}{128}+\binom{7}{1}\frac{1}{9}\cdot\frac{1}{64}+\binom{6}{1}\frac{1}{27}\cdot\frac{1}{32}\right.$$

$$+\binom{5}{1}\frac{1}{81}\cdot\frac{1}{16}+\binom{4}{1}\frac{1}{243}\cdot\frac{1}{8}+\binom{3}{1}\frac{1}{729}\cdot\frac{1}{4}+\binom{2}{1}\frac{1}{2187}\cdot\frac{1}{2}$$

$$\left.+\binom{1}{0}\frac{1}{6561}\cdot\frac{1}{1}\right]$$

(or) $\displaystyle\sum_{k=0}^{8}\binom{9-k}{8-k}\left(\frac{1}{3}\right)^{k}\left(\frac{1}{2}\right)^{8-k}.$

6. The coeffecient of x^{20} in $(x^2+x^3+x^4+x^5+x^6)^5$.

Sol: $(x^2+x^3+x^4+x^5+x^6)^5 = [x^2(1+x+x^2+x^3+x^4)]^5$

$$= (x^2)^5(1+x+x^2+x^3+x^4)^5 = x^{10}\left[\frac{(1-x^5)}{1-x}\right]^5$$

$$= x^{10}(1-x^5)^5(1-x)^{-5}$$

$$= x^{10}(1-x^5)^5(1-x)^{-5} = x^{10}\sum_{r=0}^{\infty}\binom{5}{r}(-x^5)^{r}\cdot\sum_{s=0}^{\infty}\binom{5+s-1}{s}x^{s}$$

$$= x^{10}\sum_{r=0}^{\infty}\binom{5}{r}(-1)^{r}x^{5r}\cdot\sum_{s=0}^{\infty}\binom{4+s}{s}x^{s}.$$

The coeffecient of x^{20} in above series is

$$c_{20} = \binom{5}{0}(-1)^0 \binom{4+10}{10} + \binom{5}{1}(-1)^1 \binom{4+5}{5} + \binom{5}{2}(-1)^2 \binom{4+0}{0}$$

$$= 1\cdot 1\cdot \binom{14}{10} + 5(-1)\binom{9}{5} + \binom{5}{2}(1)\cdot(1)$$

$$= \binom{14}{4} - 5\binom{9}{4} + \binom{5}{2} = 1001 - 630 + 10 = 1011 - 630 = 381.$$

7. The coeffecient of x^{27} in $(x^4 + x^5 + x^6 + \cdots)^5$

Sol: $(x^4 + x^5 + x^6 + \cdots)^5 = [x^4(1 + x + x^2 + \cdots)]^5$

$$= (x^4)^5 [(1-x)^{-1}]^5 = x^{20}(1-x)^{-5}$$

$$= x^{20} \sum_{r=0}^{\infty} \binom{5+r-1}{r} x^r = \sum_{r=0}^{\infty} \binom{4+r}{r} x^{20+r}$$

$\therefore r = 7$

$\therefore$ The coeffecient of x^{27} in above series is

$$c_{27} = \binom{4+7}{7} = \binom{11}{7} = \binom{11}{4} = \frac{11\times 10\times 9\times 8}{4\times 3\times 2\times 1} = 330.$$

8. the coeffecient of x^n in $(1 + x^2 + x^4 + \cdots)^7$

Sol: $(1 + x^2 + x^4 + \cdots)^7 = (1 + x^2 + (x^2)^2 + \cdots)^7$

$$= [(1-x^2)^{-1}]^7 = (1-x^2)^{-7} = \sum_{r=0}^{\infty} \binom{6+r}{r} (x^2)^r$$

$$= \sum_{r=0}^{\infty} \binom{6+r}{r} x^{2r}$$

$\therefore$ the coeffecient of x^n is $\binom{6+\frac{n}{2}}{\frac{n}{2}} = \binom{6+\frac{n}{2}}{6}$.

9. The coeffecient of x^{18} in $(x + x^2 + x^3 + x^4 + x^5)(x^2 + x^3 + \cdots)^5$

Sol: $(x + x^2 + x^3 + x^4 + x^5)(x^2 + x^3 + x^4 + \cdots)^5$

$$= x(1 + x + x^2 + x^3 + x^4)(x^2(1 + x + x^2 + x^3 + \cdots))^5$$

$$= x(1 + x + x^2 + x^3 + x^4)(x^{10})(1 + x + x^2 + x^3 \cdots)^5$$

$$= x^{11}(1 + x + x^2 + x^3 + x^4)[(1-x)^{-1}]^5$$

$=$

$$= x^{11}(1+x+x^2+x^3+x^4)\sum_{r=0}^{\infty}\binom{4+r}{r}x^r.$$

$$= \sum_{r=0}^{\infty}\binom{4+r}{r}x^{11+r} + \sum_{r=0}^{\infty}\binom{4+r}{r}x^{12+r} + \sum_{r=0}^{\infty}\binom{4+r}{r}x^{13+r}$$

$$+ \sum_{r=0}^{\infty}\binom{4+r}{r}x^{14+r} + \sum_{r=0}^{\infty}\binom{4+r}{r}x^{15+r}.$$

$\therefore$ The coeffecient of x^{18} is

$$c_{18} = \binom{4+7}{7} + \binom{4+6}{6} + \binom{4+5}{5} + \binom{4+4}{4} + \binom{4+3}{3}$$

$$= \binom{11}{7} + \binom{10}{6} + \binom{9}{5} + \binom{8}{4} + \binom{7}{3} = 771.$$

Recurrence relations :-

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence namely $a_0, a_1, a_2, \ldots\ldots a_{n-1}$, for all integers $n \geq 1$

Linear recurrence relation :-

Suppose 'n' and 'K' are non-negative integers A sequence relation of the form $c_0(n)\,a_n + c_1(n)\,a_{n-1} + c_2(n)\,a_{n-2} + \cdots + c_K(n)\,a_{n-K} = f(n) \longrightarrow ①$ is said to be Linear recurrence relation. where $c_0(n), c_1(n), c_2(n), \ldots\ldots c_K(n)$ are functions of 'n'.

If $f(n)=0$ then the eq ① is called Homogeneous linear recurrence relation.

If $f(n) \neq 0$ then the eq ① is called inhomogeneous Linear recurrence relation.

Ex:

1. $a_n = n + a_{n-1}$
2. $a_n - 3a_{n-1} + 2a_{n-2} = 0$
3. $a_n - 5a_{n-1} + 6a_{n-2} = n^2+1$. (Not a linear)

Solving recurrence relation by substitution and generating functions :-

we shall consider three methods of solving recurrence relations in this method.

1. substitution method (iteration method)
2. Generating functions
3. characteristic roots method.

1. Substitution Method :-

In this method the recurrence relation for an is used repeated to solve for a general expression for an interms of n.

Ex: solve the recurrence relation $a_n = a_{n-1} + n^2$ where $a_0 = 7$ by substitution method.

Sol: Given $a_n = a_{n-1} + n^2 \rightarrow ①$ and $a_0 = 7 \rightarrow ②$

put $n=1$ in Eq ① we get

$$a_1 = a_0 + (1)^2 = 7 + 1^2$$

put $n=2 \Rightarrow a_2 = a_1 + 2^2 = 7 + 1^2 + 2^2$

put $n=3 \rightarrow a_3 = a_2 + 3^2 = 7 + 1^2 + 2^2 + 3^2$

$\vdots$

$$\Rightarrow a_n = 7 + (1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$= 7 + \frac{n(n+1)(2n+1)}{6} = 7 + \frac{(n^2+n)(2n+1)}{6}$$

$$= 7 + \frac{(2n^3 + n^2 + 2n^2 + n)}{6}$$

Ex: Solve the recurrence relation $a_n = a_{n-1} + \frac{1}{n(n+1)}$ where $a_0 = 1$

Sol: Given $a_n = a_{n-1} + \frac{1}{n(n+1)}$

$n=1 \Rightarrow a_1 = a_0 + \frac{1}{1(1+1)} = 1 + \frac{1}{1(2)}$

$n=2 \Rightarrow a_2 = a_1 + \frac{1}{2(2+1)} = 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3}$

$n=3 \Rightarrow a_3 = a_2 + \frac{1}{3(3+1)} = 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4}$

$\vdots$

$a_n = 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$

$a_n = 1 + \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$= 1 + 1 - \frac{1}{n+1} = 2 - \frac{1}{n+1} = \frac{2n+2-1}{n+1} = \frac{2n+1}{n+1}$ //.

Ex: Solve the recurrence relation $a_n = a_{n-1} + f(n)$ for $n \geq 1$

Sol: Given $a_n = a_{n-1} + f(n)$

$n=1 \Rightarrow a_1 = a_0 + f(1)$

$n=2 \Rightarrow a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$

$n=3 \Rightarrow a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3)$

$\vdots$

$a_n = a_0 + f(1) + f(2) + f(3) + \cdots + f(n)$.

$a_n = a_0 + \sum_{K=1}^{n} f(K)$.

Ex: $a_n = a_{n-1} + n$, where $a_0 = 2$

Sol: Given $a_n = a_{n-1} + n$

$n=1 \Rightarrow a_1 = a_0 + 1 = 2 + 1$

$n=2 \Rightarrow a_2 = a_1 + 2 = 2 + 1 + 2$

$n=3 \Rightarrow a_3 = a_2 + 3 = 2 + 1 + 2 + 3$

$\vdots$

$a_n = 2 + (1 + 2 + 3 + \cdots + n) = 2 + \frac{n(n+1)}{2} = \frac{4 + n^2 + n}{2}$

Ex: $a_n = a_{n-1} + n^3$ where $a_0 = 5$

Sol: Given $a_n = a_{n-1} + n^3$ where $a_0 = 5$

$n=1 \Rightarrow a_1 = a_0 + 1^3 = 5 + 1^3$

$n=2 \Rightarrow a_2 = a_1 + 2^3 = 5 + 1^3 + 2^3$

$n=3 \Rightarrow a_3 = a_2 + 3^3 = 5 + 1^3 + 2^3 + 3^3.$

$\vdots$

$a_n = 5 + (1^3 + 2^3 + 3^3 + \cdots + n^3) = 5 + \frac{n^2(n+1)^2}{4}$

$\therefore a_n = \frac{20 + n^2(n^2 + 1 + 2n)}{4} = \frac{20 + n^4 + n^2 + 2n^3}{4}$ //

Ex: $a_n = a_{n-1} + n(n-1)$ where $a_0 = 1$

Sol: Given $a_n = a_{n-1} + n(n-1)$ where $a_0 = 1$

$n=1 \Rightarrow a_1 = a_0 + 1(1-1) = 1 + 0.$

$n=2 \Rightarrow a_2 = a_1 + 2(2-1) = 1 + 2(1)$

$n=3 \Rightarrow a_3 = a_2 + 3(3-1) = 1 + 2(1) + 3(2).$

$\vdots$

$a_n = 1 + 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + (n-1)n.$

$a_n = 1 + \sum_{K=1}^{n} K(K-1).$

Ex: $a_n = a_{n-1} + 3n^2 + 3n + 1$ where $a_0 = 1$

Sol. Given $a_n = a_{n-1} + 3n^2 + 3n + 1.$

$n=1 \Rightarrow a_1 = a_0 + 3(1)^2 + 3(1) + 1 = 1 + 3(1)^2 + 3(1) + 1$

$n=2 \Rightarrow a_2 = a_1 + 3(2)^2 + 3(2) + 2 = 1 + 3(1)^2 + 3(1) + 1 + 3(2)^2 + 3(2) + 2.$

$\vdots$

$a_n = 1 + 3(1)^2 + 3(1) + (1) + 3(2)^2 + 3(2) + 2 + \cdots + 3(n^2) + 3(n) + n.$

$= 1 + 3(1^2 + 2^2 + \cdots + n^2) + 3(1 + 2 + 3 + \cdots + n) + 1 + 2 + \cdots + n$

$= 1 + \cancel{3}\left(\frac{n+1)(2n+1)}{}\right) + 3\frac{n(n+1)}{} + \frac{n(n+1)}{}$

$$a_n = 1 + \frac{(n+1)(2n+1)}{2} + \frac{4n(n+1)}{2}$$

$$= \frac{2 + (n+1)(2n+1) + 4n(n+1)}{2} \quad //.$$

Ex: $a_n = a_{n-1} + n \cdot 3^n$ where $a_0 = 1$

Ex: $a_n = a_{n-1} + \frac{n(n+1)}{2}$, where $n \geq 1$

Ex: $a_n = 2a_{n-1} + 1$, where $a_1 = 7$ and $n \geq 1$

II. Method of generating function :-

Recurrence relations can also be solved by using generating functions. Some equivalent expressions used are given below.

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ then

$$\sum_{n=k}^{\infty} a_n x^n = A(x) - a_0 - a_1 x - \cdots a_{k-1} x^{k-1}$$

$$\sum_{n=k}^{\infty} a_{n-1} x^n = x\left[A(x) - a_0 - a_1 x - \cdots a_{k-2} x^{k-2}\right]$$

$$\sum_{n=k}^{\infty} a_{n-2} x^n = x^2\left[A(x) - a_0 - a_1 x - \cdots a_{k-3} x^{k-3}\right]$$

$$\vdots$$

$$\sum_{n=k}^{\infty} a_{n-k} x^n = x^k A(x).$$

where $A(x)$ is called a generating function for a given recurrence relation and 'a_n' is the solution of given recurrence relation.

Ex: Use the generating function and solve the following 1) $y_{n+2} + 3y_{n+1} + 9y_n = 0$, $y_1 = 0$, $y_0 = 2$

Sl: let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

Given relation is

$$y_{n+2} + 3y_{n+1} + 9y_n = 0, \quad y_1 = 0, \quad y_0 = 2 \longrightarrow ①$$

put $y = a$ in eq ① we get

$$a_{n+2} + 3a_{n+1} + 9a_n = 0 \qquad a_1 = 0 \quad a_0 = 2$$

Now multiply with x^{n+2} and sum from '0' to ∞

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} + 3\sum_{n=0}^{\infty} a_{n+1} x^{n+2} + 9\sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

$$\Rightarrow x^2\left[\sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x\right] + 3x\left[\sum_{n=0}^{\infty} a_n x^n - a_0\right] + 9\cdot\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow x^2\left(A(x) - a_0 - a_1 x\right) + 3x\left(A(x) - a_0\right) + 9A(x) = 0$$

$$\Rightarrow x^2\left(A(x) - 2 - (0)x\right) + 3x\left(A(x) - 2\right) + 9A(x) = 0$$

$$\Rightarrow A(x)\left[x^2 + 3x + 9\right) - 2x^2 - 6x = 0$$

$$A(x)\left(x^2 + 3x + 9\right) = 2x^2 + 6x$$

$$A(x) = \frac{2x^2 + 6x}{x^2 + 3x + 9}$$

Ex: Solve $a_n = 3a_{n-1} + 2$, where $a_0 = 1$.

sol: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function.

Given $a_n = 3a_{n-1} + 2 \qquad a_0 = 1$

Multiply each term by x^n and taking summation from '1' to ∞ we get $\qquad \Leftarrow \because$ order '1'.

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 3a_{n-1} x^n + 2\cdot\sum_{n=1}^{\infty} x^n \longrightarrow ①$$

$$\sum_{n=1}^{\infty} a_{n-1} x^n = a_0 x + a_1 x^2 + a_2 x^3 + \cdots$$
$$= x(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$\sum_{n=1}^{\infty} a_n x^n = a_1 x + a_2 x^2 + \cdots + a_0 - a_0$$

$$\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots$$
$$= x(1 + x + x^2 + x^3 + \cdots) = x(1-x)^{-1} = \frac{x}{1-x}$$

$$① \Rightarrow A(x) - a_0 = 3x A(x) + 2 \cdot \left(\frac{x}{1-x}\right)$$

$$A(x) - 3x A(x) = a_0 + \frac{2x}{1-x} \qquad a_0 = 1$$

$$A(x)(1-3x) = 1 + \frac{2x}{1-x} = \frac{1-x+2x}{1-x} = \frac{1+x}{1-x}$$

$$\therefore A(x) = \frac{1+x}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{2}{1-3x} - \frac{1}{1-x} \qquad \langle \because \text{partial fractions}$$
$$= 2(1-3x)^{-1} - (1-x)^{-1}$$
$$= 2 \sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (2 \cdot 3^n - 1) x^n.$$

$\therefore a_n = 2 \cdot 3^n - 1$ is the required solution.

Ex: Solve, $a_n - 9a_{n-1} + 20a_{n-2} = 0$, $a_0 = -3$, $a_1 = -10$

Sol: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function

Given $a_n - 9a_{n-1} + 20a_{n-2} = 0 \longrightarrow ①$, $a_0 = -3$, $a_1 = -10$

Multiply by x^n & taking the summation from '2' to ∞ in Eq ① we get $\qquad \langle \because$ order '2'.

$$\sum_{n=2}^{\infty} a_n x^n - 9 \sum_{n=2}^{\infty} a_{n-1} x^n + 20 \sum_{n=2}^{\infty} a_{n-2} x^n = 0 \longrightarrow ②$$

$$\sum_{n=2}^{\infty} a_n x^n = a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_0 + a_1 x - a_0 - a_1 x$$
$$= A(x) - a_0 - a_1 x$$

$$\sum_{n=2}^{\infty} a_{n-1} x^n = a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots$$
$$= x[a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_0 - a_0]$$

$$= x\left[A(x) - a_0\right]$$

$$\sum_{n=2}^{\infty} a_{n-2}\, x^n = a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots$$

$$= x^2 (a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$= x^2 A(x).$$

$$② \Rightarrow (A(x) - a_0 - a_1 x) - 9x(A(x) - a_0) + 20x^2 A(x) = 0$$

$$(A(x) + 3 + 10x) - 9x(A(x) + 3) + 20x^2 A(x) = 0$$

$$A(x) - 9x A(x) + 20x^2 A(x) = \cancel{a_0} - 3 - 10x + 27x$$

$$A(x)\left[1 - 9x + 20x^2\right] = -3 + 17x$$

$$A(x) = \frac{-3+17x}{1-9x+20x^2} = \frac{-3+17x}{(1-5x)(1-4x)}$$

$$= \frac{A}{1-5x} + \frac{B}{1-4x} = \frac{2}{1-5x} - \frac{5}{1-4x}$$

$A = 2,\ B = -5$
$\because$ partial fraction

$$\sum_{n=0}^{\infty} a_n x^n = 2(1-5x)^{-1} - 5(1-4x)^{-1}$$

$$= 2\sum_{n=0}^{\infty} 5^n x^n - 5\sum_{n=0}^{\infty} 4^n x^n$$

$$= \sum_{n=0}^{\infty} (2\cdot 5^n - 5\cdot 4^n)\, x^n$$

$\therefore\ a_n = 2\cdot 5^n - 5\cdot 4^n$ is the required solution.

Ex: Solve $a_{n+2} - 2a_{n+1} + a_n = 0$, $a_0 = 2$, $a_1 = 1$

Sol: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function

Given $a_{n+2} - 2a_{n+1} + a_n = 0 \longrightarrow ①$

Replace 'n' by 'n-2' in Eq ①

$\therefore\ a_n - 2a_{n-1} + a_{n-2} = 0 \longrightarrow ②$

Multiply by x^n and taking summation from '2' to '∞'

$$\sum a_n x^n - 2\sum a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0 \longrightarrow ③ \qquad a_0 = 2,\ a_1 = 1$$

$$\sum_{n=2}^{\infty} a_n x^n = A(x) - a_0 - a_1 x$$

$$\sum_{n=2}^{\infty} a_{n-1} x^n = x(A(x) - a_0)$$

$$\sum_{n=2}^{\infty} a_{n-2} x^n = x^2 A(x).$$

$$③ \Rightarrow (A(x) - a_0 - a_1 x) - 2(x(A(x) - a_0)) + x^2 A(x) = 0$$

$$A(x)[1 - 2x + x^2] - a_0 - a_1 x + 2x a_0 = 0$$

$$A(x)[1 - 2x + x^2] - 2 - x + 2x(2) = 0$$

$$A(x)[1 - 2x + x^2] = 2 - 3x$$

$$A(x) = \frac{2-3x}{1-2x+x^2} = \frac{2-3x}{(1-x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2}$$

$$\therefore A(x) = \frac{3}{1-x} - \frac{1}{(1-x)^2}$$

$A = 3, \ B = -1$ $\because$ partial fractions

$$\sum_{n=0}^{\infty} a_n x^n = 3(1-x)^{-1} - 1(1-x)^{-2}$$

$$= 3 \cdot \sum_{n=0}^{\infty} (1)^n x^n - 1 \sum_{n=0}^{\infty} (n+1) x^n$$

$$= 3 \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} (n+1) x^n$$

$$= \sum_{n=0}^{\infty} [3 - (n+1)] x^n$$

$\therefore a_n = 3 - n - 1 = 2 - n$ is the required solution.

Ex: solve $y_{n+2} + 4y_{n+1} + 4y_n = 0$, $y_1 = 0$ and $y_0 = 2$

sol: Given $y_{n+2} + 4y_{n+1} + 4y_n = 0 \rightarrow ①$, $y_1 = 0$, $y_0 = 2$

put $y = a$ and $n = n-2$ in Eq ① we get

$$a_n + 4a_{n-1} + 4a_{n-2} = 0, \quad a_0 = 2, \quad a_1 = 0$$

Now multiply with x^n and sum from '2' to '∞'

$$\sum_{n=2}^{\infty} a_n x^n + 4 \sum_{n=2}^{\infty} a_{n-1} x^n + 4 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow (A(x) - a_0 - a_1 x) + 4\big(x(A(x) - a_0)\big) + 4x^2 A(x) = 0$$

$$(A(x) - 2 - 0) + 4x(A(x) - 2) + 4x^2 A(x) = 0$$

$$A(x)[1 + 4x + 4x^2] - 2 - 8x = 0$$

$$A(x)[1 + 4x + 4x^2] = 2 + 8x$$

$$A(x) = \frac{2+8x}{1+4x+4x^2} = \frac{2+8x}{(1+2x)^2} = \frac{A}{1+2x} + \frac{B}{(1+2x)^2}$$

$$= \frac{4}{1+2x} - \frac{2}{(1+2x)^2}$$

$A = 4, \; B = -2$

$\because$ partial fractions

$$\therefore A(x) = 4(1+2x)^{-1} - 2(1+2x)^{-2}$$

$$\sum_{n=0}^{\infty} a_n x^n = 4 \sum_{n=0}^{\infty} (-1)^n (2x)^n - 2 \cdot \sum_{n=0}^{\infty} (-1)^n (n+1)(2x)^n$$

$$= 4 \sum_{n=0}^{\infty} (-1)^n 2^n x^n - 2 \sum_{n=0}^{\infty} (-1)^n (n+1) 2^n x^n.$$

$$= \sum_{n=0}^{\infty} \left[4(-1)^n 2^n - 2(-1)^n (n+1) 2^n\right] x^n$$

$$= \sum_{n=0}^{\infty} \left[(-1)^n \left(2^{n+2} - (n+1) 2^{n+1}\right)\right] x^n$$

$$\therefore a_n = (-1)^n 2^{n+1} [2 - n - 1] = (-1)^n 2^{n+1} (1-n) \quad //.$$

Ex: solve $a_n - 7a_{n-1} + 12a_{n-2} = 0, \; n \geq 2$

Sol: Given $a_n - 7a_{n-1} + 12a_{n-2} = 0, \; n \geq 2$

$$\sum_{n=2}^{\infty} a_n x^n - 7 \sum_{n=2}^{\infty} a_{n-1} x^n + 12 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow (A(x) - a_0 - a_1 x) - 7x(A(x) - a_0) + 12x^2 A(x) = 0$$

$$\Rightarrow A(x)\left[1-7x+12x^2\right] - a_0 - a_1 x + 7a_0 x = 0$$

$$\Rightarrow A(x)\left[1-7x+12x^2\right] = a_0 + a_1 x - 7a_0 x$$

$$A(x) = \frac{a_0 + (a_1 - 7a_0)x}{1-7x+12x^2} = \frac{a_0 + (a_1-7a_0)x}{(1-3x)(1-4x)}$$

$$= \frac{c_1}{1-3x} + \frac{c_2}{1-4x} = c_1(1-3x)^{-1} + c_2(1-4x)^{-1}$$

$$\sum_{n=0}^{\infty} a_n x^n = c_1 \sum_{n=0}^{\infty} 3^n x^n + c_2 \sum_{n=0}^{\infty} 4^n x^n$$

$$= \sum_{n=0}^{\infty} (c_1 3^n + c_2 4^n)\, x^n$$

$\therefore$ $a_n = c_1 3^n + c_2 4^n$ is the required solution.

Ex: Solve $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$, for $n \geq 3$

Sol: Given $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$, $n \geq 3$

$$\sum_{n=3}^{\infty} a_n x^n - 8\sum_{n=3}^{\infty} a_{n-1} x^n + 21\sum_{n=3}^{\infty} a_{n-2} x^n - 18\sum_{n=3}^{\infty} a_{n-3} x^n = 0$$

$$\left[A(x) - a_0 - a_1 x - a_2 x^2\right] - 8x\left[A(x) - a_0 - a_1 x\right] + 21x^2\left[A(x) - a_0\right]$$
$$-18x^3\left(A(x)\right) = 0$$

$$A(x) - a_0 - a_1 x - a_2 x^2 - 8x A(x) + 8x a_0 + 8a_1 x^2 + 21x^2 A(x)$$
$$-21x^2 a_0 - 18x^3 A(x) = 0$$

$$\Rightarrow A(x)\left(1 - 8x + 21x^2 - 18x^3\right) - a_0 - a_1 x - a_2 x^2 + 8x a_0 + 8a_1 x^2$$
$$-21x^2 a_0 = 0$$

$$A(x)\left[1 - 8x + 21x^2 - 18x^3\right] = a_0 + (a_1 - 8a_0)x + (a_2 - 8a_1 + 21a_0)x^2$$

$$A(x) = \frac{a_0 + (a_1 - 8a_0)x + (a_2 - 8a_1 + 21a_0)x^2}{(1 - 8x + 21x^2 - 18x^3)}$$

$$= \frac{a_0 + (a_1 - 8a_0)x + (a_2 - 8a_1 + 21a_0)x^2}{(1-2x)(1-3x)^2}$$

$$= \frac{c_1}{1-2x} + \frac{c_2}{1-3x} + \frac{c_3}{(1-3x)^2}$$ $\angle$ ∴ partial fractions

$$A(x) = c_1 \sum_{n=0}^{\infty} (1-2x)^{-1} + c_2 \sum_{n=0}^{\infty} (1-3x)^{-1} + c_3 \sum_{n=0}^{\infty} (1-3x)^{-2}$$

$$\sum_{n=0}^{\infty} a_n x^n = c_1 \sum_{n=0}^{\infty} 2^n \cdot x^n + c_2 \sum_{n=0}^{\infty} 3^n x^n + c_3 \sum_{n=0}^{\infty} (n+1)\, 3^n \cdot x^n$$

$$= \sum_{n=0}^{\infty} \left[c_1 2^n + c_2 3^n + c_3 (n+1) 3^n \right] x^n$$

$\therefore a_n = c_1 2^n + c_2 3^n + c_3 (n+1) 3^n$ is the required solution

<u>Ex:</u> Solve $a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0$, $a_0 = 0$, $a_1 = 1$, $a_2 = 10$.

<u>Sol:</u> Given $a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0$ for $n \geq 3$

$a_0 = 0,\ a_1 = 1,\ a_2 = 10$

$$\sum_{n=3}^{\infty} a_n x^n - 9 \sum_{n=3}^{\infty} a_{n-1} x^n + 26 \sum_{n=3}^{\infty} a_{n-2} x^n - 24 \sum_{n=3}^{\infty} a_{n-3} x^n = 0$$

$$(A(x) - a_0 - a_1 x - a_2 x^2) - 9x(A(x) - a_0 - a_1 x) + 26x^2(A(x) - a_0) - 24x^3(A(x)) = 0$$

$$(A(x) - 0 - x - 10x^2) - 9x(A(x) - 0 - x) + 26x^2(A(x) - 0) - 24x^3 A(x) = 0$$

$$A(x)\left[1 - 9x + 26x^2 - 24x^3\right] - x + 9x^2 - 10x^2 = 0$$

$$A(x)\left[1 - 9x + 26x^2 - 24x^3\right] = x + x^2$$

$$A(x) = \frac{x+x^2}{(1-9x+26x^2-24x^3)}$$

$$= \frac{x+x^2}{(1-2x)(1-3x)(1-4x)} = \frac{A}{1-2x} + \frac{B}{1-3x} + \frac{C}{1-4x}.$$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{3}{2}\cdot(1-2x)^{-1} - 4(1-3x)^{-1} + \frac{5}{2}(1-4x)^{-1}$$

$$= \frac{3}{2}\sum_{n=0}^{\infty} 2^n x^n - 4\sum_{n=0}^{\infty} 3^n x^n + \frac{5}{2}\sum_{n=0}^{\infty} 4^n x^n$$

$\because A = \frac{3}{2}$, $B = -4$, $C = \frac{5}{2}$

$$= \sum_{n=0}^{\infty}\left[\frac{3}{2}2^n - 4(3)^n + \frac{5}{2}\cdot 4^n\right]x^n$$

$\therefore a_n = \frac{3}{2}\cdot 2^n - 4\cdot 3^n + \frac{5}{2}\cdot 4^n$ is the required Solution

Ex: solve $y_{n+2} - y_{n+1} - 6y_n = 0$, $y_1 = 1$, $y_0 = 2$

Ex: solve $s(n) = s(n-1) + 2(n-1)$ with $s(0) = 3$, $s(1) = 1$.

Hint: put $s = a$, $a_n = a_{n-1} + 2(n-1)$, $a_0 = 3$, $a_1 = 1$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1}x^n + 2\sum_{n=1}^{\infty}(n-1)x^n$$

$$(A(x) - a_0) = x\,A(x) + 2\cancel{x^2}[0 + x^2 + 2x^3 + 3x^4 + \cdots]$$

$$A(x) - a_0 - x\,A(x) = 2x^2(1 + 2x + 3x^2 + 4x^3 + \cdots)$$

$$A(x)(1-x) - 3 = 2x^2(1-x)^{-2}$$

$$A(x)(1-x) = 3 + \frac{2x^2}{(1-x)^2} \Rightarrow A(x) = \frac{3(1-x)^2 + 2x^2}{(1-x)^3}$$

$$A(x) = \frac{3(1+x^2-2x) + 2x^2}{(1-x)^3} = \frac{3-6x+5x^2}{(1-x)^3}$$

$$\frac{3-6x+5x^2}{(1-x)^3} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3}$$

$$A(x) = \frac{-5}{1-x} + \frac{6}{(1-x)^2} + \frac{2}{(1-x)^3} \qquad \therefore \begin{cases} A = -5 \\ B = 6 \\ C = 2 \end{cases}$$

$$\sum_{n=0}^{\infty} a_n x^n = -5(1-x)^{-1} + 6(1-x)^{-2} + 2(1-x)^{-3}$$

$$= -5\sum_{n=0}^{\infty} 1^n . x^n + 6\sum_{n=0}^{\infty}(n+1)x^n + 2\sum_{n=0}^{\infty}(n+1)(n+2)x^n$$

$$= \sum_{n=0}^{\infty} [-5 + 6(n+1) + 2(n^2+3n+2)] x^n$$

$$= \sum_{n=0}^{\infty} (-5 + 6n + 6 + 2n^2 + 6n + 4) x^n$$

$$= \sum_{n=0}^{\infty} (2n^2 + 12n + 5) x^n$$

$\therefore a_n = 2n^2 + 12n + 5$ is the required Solution.

EX: Solve $y_{n+2} + 3y_{n+1} + 9y_n = 0$, $y_1 = 0$ and $y_0 = 2$

3. characteristic roots method :-

EX: Solve $a_n - 3a_{n-1} + 2a_{n-2} = 0$ for $n \geq 2$

Sol: given $a_n - 3a_{n-1} + 2a_{n-2} = 0 \longrightarrow (1)$

order $= n-(n-2) = \cancel{n} - \cancel{n} + 2 = 2$

$\therefore$ The characteristic equ is $t^2 - 3t + 2 = 0$

$\Rightarrow (t-1)(t-2) = 0 \Rightarrow t-1=0 \qquad t-2=0$

$\boxed{t=1} \qquad t=2$

$\therefore$ The General solution of eq (1) is

$a_n = c_1 \alpha_1^n + c_2 \alpha_2^n$

$= c_1(1)^n + c_2(2)^n = c_1 + c_2 \cdot 2^n$

Ex: solve $a_n - 3a_{n-1} - 4a_{n-2} = 0$ for $n \ge 2$

Sol. given $a_n - 3a_{n-1} - 4a_{n-2} = 0 \longrightarrow ①$

order $= n-(n-2) = \cancel{n}-\cancel{n}+2 = 2$

$\therefore$ The characteristic eq is $t^2 - 3t - 4 = 0$

$\Rightarrow (t+1)(t-4) = 0 \quad \Rightarrow t = -1$ or $t = 4$

$\therefore$ The general solution of eq ① is

$$a_n = c_1 \alpha_1^n + c_2 \alpha_2^n$$
$$= c_1(-1)^n + c_2(4)^n.$$

Ex: Solve $a_n - 6a_{n-1} + 9a_{n-2} = 0$ for $n \ge 2$

Sol: Given $a_n - 6a_{n-1} + 9a_{n-2} = 0 \longrightarrow ①$

order $= 2$

The characteristic eq is

$\Rightarrow t^2 - 6t + 9 = 0$

$\Rightarrow (t-3)(t-3) = 0 \quad \therefore t = 3, 3$

$\therefore$ The General solution of eq ① is

$$a_n = (c_1 + c_2 n)\alpha_1^n \quad \left\{ \because \text{roots are equal} \right.$$
$$= (c_1 + c_2 n) 3^n.$$

Ex: solve $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \ge 2$ with $a_0 = 10$ and $a_1 = 41$

Sol: Given $a_n - 7a_{n-1} + 10a_{n-2} = 0 \longrightarrow ①$

order $= 2$

The characteristic eq is

$\Rightarrow t^2 - 7t + 10 = 0 \Rightarrow (t-2)(t-5) = 0$

$\therefore t = 2, 5$

The general solution is $a_n = c_1 \alpha_1^n + c_2 \alpha_2^n$

$c_1(2)^n + c_2(5)^n \longrightarrow ②$

Given intial conditions $a_0 = 10$, $a_1 = 41$

put $n=0$ in eq ② weget

$a_0 = c_1 2^0 + c_2 (5)^0 \Rightarrow c_1 + c_2 = 10 \longrightarrow$ ③

put $n=1$ in eq ② we get

$a_1 = c_1 2^1 + c_2 5^1 \Rightarrow 2c_1 + 5c_2 = 41 \longrightarrow$ ④

from ③ & ④ $\Rightarrow$

$$\begin{aligned} \cancel{2c_1} + 2c_2 &= 20 \\ \cancel{2c_1} + 5c_2 &= 41 \\ \hline -3c_2 &= -21 \\ \boxed{c_2 = 7} \end{aligned}$$

$$\begin{aligned} c_1 + c_2 &= 10 \\ c_1 &= 10 - c_2 \\ &= 10 - 7 \\ \boxed{c_1 = 3} \end{aligned}$$

$\therefore$ The general solution $a_n = 3 \cdot (2)^n + 7 \cdot (5)^n$

Ex: Solve $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$ for $n \geq 3$. with $a_0 = 1$, $a_1 = 4$ and $a_2 = 8$.

Sol: Given $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0 \longrightarrow$ ①

order = 3

The characteristic eqn is

$\Rightarrow t^3 - 7t^2 + 16t - 12 = 0$

$\Rightarrow (t-2)(t-2)(t-3) = 0$

$\therefore t = 2, 2, 3.$

2	1	-7	16	-12
	0	2	-10	12
2	1	-5	6	0
	0	2	-6	
3	1	-3	0	
	0	3		
	1	0		

$\therefore$ General solution of eq ① is

$a_n = (c_1 + c_2 n) \alpha_1^n + c_3 \alpha_2^n$

$= (c_1 + c_2 n) 2^n + c_3 (3)^n. \longrightarrow$ ②

given intial conditions $a_0 = 1$, $a_1 = 4$, $a_2 = 8$

put $n=0$ in Eq ① we get

$a_0 = [c_1 + c_2(0)] 2^0 + c_3 (3)^0$

$1 = c_1 + c_3 \Rightarrow c_1 + c_3 = 1 \longrightarrow$ ③

put $n=1$ in Eq ② we get

$$a_1 = [c_1 + c_2(1)]\, 2^1 + c_3(3)^1$$

$$4 = (c_1+c_2)\,2 + 3c_3 \Rightarrow 2c_1 + 2c_2 + 3c_3 = 4 \longrightarrow ④$$

put $n=2$ in eq ② we get

$$a_2 = [c_1 + c_2(2)]\, 2^2 + c_3\, 3^2$$

$$8 = (c_1 + 2c_2)\,4 + c_3(9) \Rightarrow 4c_1 + 8c_2 + 9c_3 = 8 \longrightarrow ⑤$$

solve the eq (3), (4) & (5) we get

$$c_1 = 5,\quad c_2 = 3,\quad c_3 = -4$$

$\therefore$ the general solution of eq (1) is

$$a_n = (5+3n)\,2^n - 4(3)^n. \quad //$$

<u>Ex:</u> Find the characteristic polynomial for the homogeneous recurrence relations whose the general solution has the following forms.

1. $a_n = c_1 + n c_2.$

<u>Sol.</u> $a_n = c_1 + nc_2 = c_1(1)^n + c_2(n)(1)^n$

$= (c_1 + c_2 n)(1)^n.$ Here '1' is a root

$\therefore$ The characteristic eq is $\Rightarrow (t-1)^2 = 0$

$$t^2 - 2t + 1 = 0$$

2. $a_n = c_1 2^n + c_2 3^n$

<u>Sol:</u> $a_n = c_1 2^n + c_2 3^n.$ Here 2, 3 are roots

$\therefore$ The characteristic ~~roo~~ eq is

$(t-2)(t-3) = 0 \Rightarrow t^2 - 5t + 6 = 0.$

3. $a_n = c_1 2^n + n c_2 2^n$

5. $a_n = c_1 3^n + c_2 n.3^n + c_3 2^n$

6. $a_n = c_1 2^n + c_2 n 2^n + c_3 n^2 2^n$

Solution of Inhomogeneous linear recurrence relations:-

$c_0 a_0 + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_K a_{n-K} = f(n)$ is called Inhomogeneous linear recurrence relation where $c_K \neq 0$

For solving inhomogeneous linear recurrence relation, we use two methods.

1. characteristic roots method :-

Ex: solve the recurrence relation $a_n - 9a_{n-1} + 20a_{n-2} = 1$

sol: Given $a_n - 9a_{n-1} + 20a_{n-2} = 1 \longrightarrow (1)$

we consider homogeneous relation

$$a_n - 9a_{n-1} + 20a_{n-2} = 0 \longrightarrow (2)$$

order = 2

The characteristic eq of (2) is

$$t^2 - 9t + 20 = 0 \Rightarrow (t-4)(t-5) = 0$$

$$\therefore t = 4, 5$$

The general solution is $a_n = c_1 4^n + c_2 5^n \longrightarrow (3)$

where c_1 & c_2 are constants. Here '1' is not a characteristic root

since $f(n) = 1$ is a constant from eq (1) so that particular solution will also a constant. say 'q'

$$q - 9q + 20q = 1 \Rightarrow 12q = 1 \Rightarrow \boxed{q = \frac{1}{12}}$$

Hence the general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = c_1 4^n + c_2 5^n + \frac{1}{12} \text{ //.}$$

Ex: solve $a_n - a_{n-1} - 6a_{n-2} = -30$ with $a_0 = 20$, $a_1 = -5$

sol: Given $a_n - a_{n-1} - 6a_{n-2} = -30 \longrightarrow (1)$

consider homogeneous relation

The characteristic eq of (2) is order = 2

$$t^2 - t - 6 = 0 \Rightarrow (t-3)(t+2) = 0$$

$\therefore t = 3, -2$

The homogeneous solution is $a_n^{(h)} = c_1 3^n + c_2(-2)^n \longrightarrow (3)$

Here '1' is not a characteristic root and

$f(n) = -30$ is a constant.

$\therefore \quad q - q - 6q = -30 \Rightarrow -6q = -30 \Rightarrow \boxed{q = 5}$

$\therefore$ particular solution of eq (1) is $a_n^{(p)} = 5$

Hence the general solution is $a_n = a_n^h + a_n^p$

$$a_n = c_1 3^n + c_2(-2)^n + 5 \longrightarrow (4)$$

Ex: given intial conditions are $a_0 = 20$, $a_1 = -5$

put $n=0$ in eq (4) we get

$$a_0 = c_1 3^0 + c_2(-2)^0 + 5$$

$$20 = c_1 + c_2 + 5 \Rightarrow c_1 + c_2 = 15 \longrightarrow (5)$$

put $n=1$ in eq (4) we get

$$a_1 = c_1 3^1 + c_2(-2)^1 + 5$$

$$-5 = c_1(3) + c_2(-2) + 5 \Rightarrow 3c_1 - 2c_2 = -10 \longrightarrow (6)$$

from (5) & (6) $\rightarrow$

$$\begin{array}{r} 3c_1 + 3c_2 = 45 \\ 3c_1 - 2c_2 = -10 \\ \hline 5c_2 = 55 \end{array}$$

$$\boxed{c_2 = 11}$$

$c_1 + c_2 = 15$

$c_1 = 15 - c_2$

$= 15 - 11$

$\boxed{c_1 = 4}$

The complete general solution eq (1) is

$$a_n = 4 \cdot 3^n + 11 \cdot (-2)^n + 5$$

Ex: solve $a_n - 7a_{n-1} + 10a_{n-2} = 4^n$.

Sol: Given relation $a_n - 7a_{n-1} + 10a_{n-2} = 4^n \longrightarrow (1)$

consider homogeneous relation.

$a_n - 7a_{n-1} + 10a_{n-2} = 0 \longrightarrow (2)$

The characteristic eq is $\qquad$ order = 2

$t^2 - 7t + 10 = 0 \Rightarrow (t-2)(t-5) = 0$

$\therefore t = 2, 5$

The homogeneous solution is $a_n^{(h)} = c_1 2^n + c_2 5^n \longrightarrow (3)$

Here '4' is not a characteristic root

the particular solution of eq (1) is $q \cdot 4^n$ so

$(1) \Rightarrow \quad q4^n - 7q4^{n-1} + 10q4^{n-2} = 4^n$

$q4^n - 7q\frac{4^n}{4} + 10q\frac{4^n}{4^2} = 4^n$

$q\cancel{4^n}\left[1 - \frac{7}{4} + \frac{10}{4^2}\right] = \cancel{4^n} \Rightarrow q\left[1 - \frac{7}{4} + \frac{10}{16}\right] = 1$

$\Rightarrow q\left[\frac{16-28+10}{16}\right] = 1 \quad \Rightarrow q\left(-\frac{2}{16}\right) = 1$

$\Rightarrow q\left[-\frac{1}{8}\right] = 1 \quad \Rightarrow \boxed{q = -8}$

$\therefore$ particular solution of eq(1) is $a_n^{(p)} = (-8)4^n \longrightarrow (4)$

Hence the general solution of eq (1) is

$a_n = a_n^{(h)} + a_n^{(p)}$

$= c_1(2)^n + c_2(5)^n - 8(4)^n$ //

Ex: solve $a_n - 3a_{n-1} - 4a_{n-2} = 4^n$

Sol: Given relation $a_n - 3a_{n-1} - 4a_{n-2} = 4^n \longrightarrow (1)$

we consider homogeneous relation

$a_n - 3a_{n-1} - 4a_{n-2} = 0 \longrightarrow (2)$

order = 2

The characteristic eq is

$t^2 - 3t - 4 = 0 \Rightarrow (t+1)(t-4) = 0$

$\therefore t = -1, 4$

The homogeneous solution is $a_n^{(h)} = c_1(-1)^n + c_2 4^n \longrightarrow (3)$

Here '4' is a characteristic root then the particular solution of eq (1) is $a_n^{(P)} = nq4^n$

Now we find q

$(1) \Rightarrow \quad qn4^n - 3q(n-1)4^{n-1} - 4q(n-2)4^{n-2} = 4^n$

$\Rightarrow qn4^n - 3qn4^{n-1} + 3q4^{n-1} - 4qn4^{n-2} + 8q4^{n-2} = 4^n$

$\Rightarrow qn4^n - \frac{3qn4^n}{4} + \frac{3q4^n}{4} - \frac{4qn4^n}{4^2} + \frac{8q4^n}{4^2} = 4^n$

$\Rightarrow qn4^n\left[1 - \frac{3}{4} - \frac{1}{4}\right] + q4^n\left[\frac{3}{4} + \frac{2}{4}\right] = 4^n$

$\Rightarrow qn4^n[1-1] + q4^n\left[\frac{5}{4}\right] = 4^n$

$\Rightarrow \quad 0 + q4^n\left(\frac{5}{4}\right) = 4^n \quad \Rightarrow \frac{5}{4}q\cancel{4^n} = \cancel{4^n} \Rightarrow \boxed{q = \frac{4}{5}}$

The particular solution of eq (1) is

$$a_n^{(P)} = n\left(\frac{4}{5}\right)4^n = \frac{n}{5}4^{n+1}$$

$\therefore$ Hence the general solution of eq(1) is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = c_1(-1)^n + c_2 4^n + \frac{n}{5}4^{n+1} \quad //$$

Ex: Solve $a_n - 6a_{n-1} + 8a_{n-2} = n4^n$ where $a_0 = 8, a_1 = 22$

Sol: Given relation $a_n - 6a_{n-1} + 8a_{n-2} = n4^n \longrightarrow (1)$

We consider homogeneous ~~eq~~ relation

$$a_n - 6a_{n-1} + 8a_{n-2} = 0 \longrightarrow (2)$$

order = 2

The characteristic eq is

$t^2 - 6t + 8 = 0 \quad \Rightarrow (t-2)(t-4) = 0$

$t = 2, 4$

$\therefore$ The homogeneous solution is $a_n^{(h)} = c_1(2)^n + c_2(4)^n \longrightarrow (3)$

Here '4' is the characteristic root then the particular solution is of the form $n(q_0 + q_1 n)4^n$

$$(1) \Rightarrow n(q_0+q_1 n)4^n - 6(n-1)(q_0 + q_1(n-1))4^{n-1} + 8(n-2)(q_0+q_1(n-2))4^{n-2} = n4^n. \longrightarrow (3)$$

put $n=0$ in eq (3) we get

$$0 - 6(-1)(q_0 - q_1)4^{-1} + 8(-2)(q_0 - 2q_1)4^{-2} = 0$$

$$\frac{\cancel{6}^{3}(q_0 - q_1)}{\cancel{4}_{2}} - \frac{\cancel{16}(q - 2q_1)}{\cancel{16}} = 0$$

$$\Rightarrow \frac{3}{2}(q_0 - q_1) - (q - 2q_1) = 0 \quad \Rightarrow 3(q_0 - q_1) - 2(q_0 - 2q_1) = 0$$

$$\Rightarrow 3q_0 - 3q_1 - 2q_0 + 4q_1 = 0 \quad \Rightarrow q_0 + q_1 = 0 \longrightarrow (4)$$

put $n=1$ in eq (3) we get

$$1(q_0+q_1)4^1 - 6(0) + 8(-1)(q_0 - q_1)4^{-1} = 4$$

$$1(q_0 - q_0)4^1 - 0 + \frac{8(-1)(q_0+q_0)}{4} = 4 \quad \angle \because q_0 = -q_1$$

$$0 - 0 - \frac{\cancel{8}^{2}}{\cancel{4}}(q_0+q_0) = 4 \quad \Rightarrow -4q_0 = 4 \quad \Rightarrow \boxed{q_0 = -1}$$

$$(4) \Rightarrow q_0 + q_1 = 0 \quad \Rightarrow q_1 = -q_0 \quad \Rightarrow \boxed{q_1 = 1}$$

$\therefore$ The particular solution is

$$a_n^{(P)} = n(-1+(1)n)4^n = n(n-1)4^n$$

Hence the general solution is $a_n = a_n^h + a_n^p$

$$a_n = c_1 2^n + c_2 4^n + n(n-1)4^n \longrightarrow (5)$$

put $n=0 \Rightarrow a_0 = c_1 + c_2 \Rightarrow c_1 + c_2 = 8 \longrightarrow (6)$

put $n=1 \Rightarrow a_1 = c_1(2) + c_2(4) \Rightarrow 2c_1 + 4c_2 = 22 \longrightarrow (7)$

Solve eq (6) & (7) we get $c_1 = 5$ and $c_2 = 3$

$\therefore$ Hence the complete solution is

$$= 5(2)^n + 3(4)^n + n(n-1)4^n$$

Ex: solve $a_n + a_{n-1} = 3n\,2^n$.

Sol: Given $a_n + a_{n-1} = 3n\,2^n \rightarrow (1)$

We consider homogeneous relation

$a_n + a_{n-1} = 0$ order $= 1$

$\Rightarrow t+1 = 0 \Rightarrow t = -1$

Homogeneous solution is $a_n^{(h)} = c_1 \alpha^n = c_1(-1)^n \longrightarrow (2)$

Here '2' is not a characteristic root

The particular solution of eq (1) is $a_n^{(p)} = (q_0 + q_1 n)2^n$

Now we find q_0 and q_1.

$(2) \Rightarrow (q_0 + q_1 n)\,2^n + [q_0 + q_1(n-1)]\,2^{n-1} = 3n\,2^n \longrightarrow (3)$

put $n=0$ in eq (3) we get

$(q_0 + 0)\,2^0 + (q_0 + q_1(0-1))\,2^{0-1} = 0$

$q_0 + \frac{q_0 - q_1}{2} = 0 \Rightarrow \frac{q_0 - q_1}{2} = -q_0$

$\Rightarrow q_0 - q_1 = -2q_0 \Rightarrow +q_1 = +3q_0 \Rightarrow q_1 = 3q_0 \longrightarrow (4)$

put $n=1$ in eq (3) we get

$[q_0 + q_1(1)]\,2^1 + (q_0 + q_1(1-1)]\,2^0 = 3(2^1)$

$(q_0 + q_1)\,2 + (q_0 + 0)(1) = 6$

$\Rightarrow 2q_0 + 2q_1 + q_0 = 6 \Rightarrow 3q_0 + 2q_1 = 6$ $\{\because q_1 = 3q_0$

$\Rightarrow 3q_0 + 2(3q_0) = 6$

$\Rightarrow 9q_0 = 6 \Rightarrow q_0 = \frac{6}{9} = \frac{2}{3}$

$\boxed{\therefore q_0 = \frac{2}{3}}$

$(4) \Rightarrow q_1 = 3(q_0) = 3 \cdot \frac{2}{3} = 2$ $\boxed{\therefore q_1 = 2}$

$\therefore$ The particular solution is $a_n^{(p)} = (\frac{2}{3} + 2n)\,2^n$.

Hence the general solution of eq (1) is

$\ldots(-1)^n + (\frac{2}{3} + 2n)\,2^n$

Ex: Solve $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$

Sol: Given $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1 \longrightarrow (1)$

order = 2

We consider homogeneous relation is

$a_n + 5a_{n-1} + 6a_{n-2} = 0$

The characteristic eq is $t^2 + 5t + 6 = 0$

$\Rightarrow (t+2)(t+3) = 0 \Rightarrow t = -2, -3$

$\therefore$ The homogeneous solution is $a_n^{(h)} = c_1(-2)^n + c_2(-3)^n \longrightarrow (2)$

Here '1' is not a characteristic root

since $f(n)$ is a polynomial then particular

Solution is of the form $(q_0 + q_1 n + q_2 n^2)$

Here $a_n = q_0 + q_1 n + q_2 n^2$

$a_{n-1} = q_0 + (n-1)q_1 + (n-1)^2 q_2$

$a_{n-2} = q_0 + (n-2)q_1 + (n-2)^2 q_2$

$(1) \Rightarrow (q_0 + q_1 n + q_2 n^2) + 5(q_0 + (n-1)q_1 + (n-1)^2 q_2) + 6(q_0 + (n-2)q_1 + (n-2)^2 q_2) = 3n^2 - 2n + 1 \longrightarrow (3)$

put $n=0$ in eq (3) we get

$(q_0 + 0 + 0) + 5(q_0 - q_1 + q_2) + 6(q_0 - 2q_1 + 4q_2) = 1$

$\Rightarrow q_0 + 5q_0 - 5q_1 + 5q_2 + 6q_0 - 12q_1 + 24q_2 = 1$

$\Rightarrow 12q_0 - 17q_1 + 29q_2 = 1 \longrightarrow (4)$

put $n=1$ in eq (3) we get

$(q_0 + q_1 + q_2) + 5(q_0 + 0 + 0) + 6(q_0 - q_1 - q_2) = 3-2+1$

$\Rightarrow q_0 + q_1 + q_2 + 5q_0 + 6q_0 - 6q_1 - 6q_2 = 2$

$12q_0 - 5q_1 + 7q_2 = 2 \longrightarrow (5)$

put $n=2$ in eq (3) we get

$$(q_0+2q_1+4q_2)+5(q_0+q_1+q_2)+6(q_0+0+0)=3(2)^2-2(2)+1$$

$$\Rightarrow q_0+2q_1+4q_2+5q_0+5q_1+5q_2+6q_0=12-4+1$$

$$\Rightarrow 12q_0+7q_1+9q_2=9 \longrightarrow (6)$$

Solve (4), (5) & 6 we get $q_0=\frac{71}{288}$, $q_1=\frac{13}{24}$, $q_2=\frac{1}{4}$

$\therefore$ particular solution of eq (1) is

$$a_n^{(p)}=q_0+q_1 n+q_2 n^2=\frac{71}{288}+\frac{13}{24}n+\frac{1}{4}n^2$$

$\therefore$ Hence the general solution of eq (1)

$$a_n=a_n^{(h)}+a_n^{(p)}$$
$$=c_1(-2)^n+c_2(-3)^n+\frac{71}{288}+\frac{13}{24}n+\frac{1}{4}n^2$$

Ex: Solve $a_{n+2}-6a_{n+1}+9a_n=3\cdot(2)^n+7\cdot(3)^n$. for $n\geq 0$ given $a_0=1$, $a_1=4$.

Sol: Given $a_{n+2}-6a_{n+1}+9a_n=3\cdot(2)^n+7\cdot(3)^n \longrightarrow (1)$

$a_0=1$, $a_1=4$

The characteristic eq of (1) is

order = 2

$$t^2-6t+9=0 \Rightarrow (t-3)^2=0$$
$$\Rightarrow t=3,3$$

$\therefore$ The homogeneous solution of eq (1) is

$$a_n^{(h)}=(c_1+c_2 n)3^n \longrightarrow (2)$$

Here '3' is the characteristic root with multiply '2'

The particular solution is the form

$$C(2)^n+Dn^2 3^n \longrightarrow (3)$$

$$(1)\Rightarrow \left(C\,2^{n+2}+D(n+2)^2 3^{n+2}\right)-6\left(C\cdot 2^{n+1}+D(n+1)^2 3^{n+1}\right)$$

$$\ldots = 3\cdot 2^n+7\cdot(3)^n.$$

$$\Rightarrow \left(C\,2^{n+2} - 6C\,2^{n+1} + 9C\,2^{n}\right) + \left(D(n+2)^2 3^{n+2} - 6D(n+1)^2 3^{n+1} + 9Dn^2 3^n\right) = 3(2^n) + 7(3^n)$$

Equating the corresponding terms on b.s we get

$$C\,2^{n+2} - 6C\,2^{n+1} + 9C\,2^{n} = 3(2)^n$$

$$\Rightarrow \cancel{2^n}\left(C\,2^2 - 6C(2) + 9C\right) = 3\,\cancel{2^n}$$

$$\Rightarrow 4C - 12C + 9C = 3 \quad \Rightarrow \boxed{C = 3}$$

$$D(n+2)^2 3^{n+2} - 6D(n+1)^2 3^{n+1} + 9Dn^2 3^n = 7(3^n)$$

$$\left(D(n+2)^2 - 6D(n+1)^2 + 9Dn^2\right)\cancel{3^n} = 7\cdot\cancel{3^n}$$

$$\Rightarrow D\left(9(n^2+4n+4)\right) - 18\left(n^2+2n+1\right) + 9n^2) = 7$$

$$\Rightarrow D\left(\cancel{9n^2} + \cancel{36n} + 36 - \cancel{18n^2} - \cancel{36n} - 18 + \cancel{9n^2}\right) = 7$$

$$\Rightarrow 18D = 7 \quad \Rightarrow \boxed{D = \frac{7}{18}}$$

$\therefore$ particular solution of eq (1) is $a_n^{(P)} = C\,2^n + Dn^2 3^n$

$$\Rightarrow a_n^{(P)} = 3(2^n) + \frac{7}{18}n^2 3^n$$

Hence the general solution of eq (1) is $a_n = a_n^{(h)} + a_n^{(P)}$

$$\Rightarrow a_n = (c_1 + c_2 n)\,3^n + 3(2^n) + \frac{7}{18}n^2 3^n \longrightarrow (3)$$

put $n=0$ in eq (3) we get

$$a_0 = (c_1+0)\,3^0 + 3(2^0) + \frac{7}{18}(0)$$

$$1 = c_1 + 3 + 0 \quad \Rightarrow c_1 = 1-3 \quad \Rightarrow \boxed{c_1 = -2}$$

put $n=1$ in eq (3) we get

$$a_1 = (c_1 + c_2(1))\,3^1 + 3(2)^1 + \frac{7}{18}(1)^2 3^1$$

$$4 = (c_1+c_2)(3) + 6 + \frac{7}{6}$$

$$\Rightarrow 3c_1 + 3c_2 = 4 - 6 - \frac{7}{6} \quad \Rightarrow 3(-2) + 3c_2 = -2 - \frac{7}{6}$$

$$3c_2 = -\frac{19}{6} + 6$$

$$3c_2 = \frac{17}{6} \Rightarrow \boxed{c_2 = \frac{17}{18}}$$

∴ Hence the complete solution of eq (1) is

$$a_n = \left(-2 + \frac{17}{18}n\right)3^n + 3(2^n) + \frac{7}{18}n^2(3^n).$$

Ex: Solve the following recurrence relations.

i) $a_n - 2a_{n-1} + a_{n-2} = 5^n$ for $n \geq 2$

ii) $a_n - 5a_{n-1} + 6a_{n-2} = 42(4^n)$, for $n \geq 2$

iii) $a_n - 5a_{n-1} + 6a_{n-2} = 1$

iv) $a_n + 5a_{n-1} = 9$ with intial conditions $a_0 = 6$

v) $a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n$, $a_0 = 1$, $a_1 = 2$.

II. Generating function method :-

Ex: Find the solution to $a_n - a_{n-1} = 3(n-1)$ for $n \geq 1$ and $a_0 = 2$

Sol: Given $a_n - a_{n-1} = 3(n-1)$, $a_0 = 2 \longrightarrow ①$

multiply each term x^n and sum from '1' to ∞

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} = \sum_{n=1}^{\infty} 3(n-1)x^n$$

$$\Rightarrow (A(x) - a_0) - xA(x) = 3(0 + 1.x^2 + 2.x^3 + 3.x^4 + \cdots)$$

$$A(x) - 2 - xA(x) = 3x^2(1 + 2x + 3x^2 + \cdots)$$

$$A(x)(1-x) - 2 = 3x^2(1-x)^{-2}$$

$$A(x)(1-x) = \frac{3x^2}{(1-x)^2} + 2$$

$$A(x) = \frac{3x^2}{(1-x)^3} + \frac{2}{(1-x)} = 3x^2(1-x)^{-3} + 2(1-x)^{-1}$$

$$= \frac{3}{} \sum n(n-1)x^n + 2\sum_{n=0}^{\infty} x^n.$$

$$\Rightarrow A(x) = \sum_{n=0}^{\infty} \left[\frac{3}{2}(n)(n-1) + 2\right] x^n$$

$\therefore \quad a_n = \frac{3}{2}(n^2 - n) + 2$ is a required solution.

Ex: Find the solution of $a_n - 5a_{n-1} + 6a_{n-2} = n(n-1)$ for $n \geq 1$, $a_0 = 1$, $a_1 = 5$.

Sol: Given $a_n - 5a_{n-1} + 6a_{n-2} = n(n-1) \longrightarrow (1)$

$$(1) \Rightarrow \sum_{n=2}^{\infty} a_n x^n - 5\sum_{n=2}^{\infty} a_{n-1} x^n + 6\sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=2}^{\infty} n(n-1) x^n$$

$$\Rightarrow (A(x) - a_0 - a_1 x) - 5x(A(x) - a_0) + 6x^2(A(x)) = \frac{2x^2}{(1-x)^3}$$

$$\Rightarrow (A(x) - 1 - 5x) - 5x(A(x) - 1) + 6x^2 A(x) = \frac{2x^2}{(1-x)^3}$$

$$\Rightarrow A(x)(1 - 5x + 6x^2) - \cancel{5x} + \cancel{5x} - 1 = \frac{2x^2}{(1-x)^3}$$

$$\Rightarrow A(x)(1-2x)(1-3x) = \frac{2x^2}{(1-x)^3} + 1$$

$$A(x) = \frac{2x^2 + (1-x)^3}{(1-x)^3(1-2x)(1-3x)} = \frac{2x^2 + 1 - 3x + 3x^2 - x^3}{(1-2x)(1-3x)(1-x)^3}$$

$$= \frac{(1 - 3x + 5x^2 - x^3)}{(1-x)^3(1-2x)(1-3x)}$$

Let $\frac{1 - 3x + 5x^2 - x^3}{(1-x)^3(1-2x)(1-3x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{(1-2x)} + \frac{E}{(1-3x)} \longrightarrow (2)$

By using partial fraction solve the eq (2) we get

$A = \frac{13}{4}$, $B = \frac{3}{2}$, $C = 1$, $D = -10$, $E = \frac{21}{4}$,

$$(2) \Rightarrow A(x) = \frac{13}{4}(1-x)^{-1} + \frac{3}{2}(1-x)^{-2} + (1-x)^{-3} - 10(1-2x)^{-1} + \frac{21}{4}(1-3x)^{-1}$$

$$= \frac{13}{4}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty}(n+1)x^n + \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n - 10\sum_{n=0}^{\infty} 2^n x^n + \frac{21}{4}\sum_{n=0}^{\infty} 3^n x^n$$

$$13 + 2(n+1)(n+5) - 10\, 2^n + 21\, 3^n$$

procedure of Solving Inhomogeneous recurrence relation :-

A solution, which satisfies the recurrence relation when the right hand side of the eq is set to '0' is called homogeneous solution and is denoted by $a_n^{(h)}$.

A solution which satisfies the recurrence with f(n) on the right hand side is called particular solution and is denoted by $a_n^{(p)}$.

$\therefore$ The General solution $a_n = a_n^{(h)} + a_n^{(p)}$.

Rule (1) :- If f(n) is of the form of a polynomial of degree 'm' in 'n' ie $b_0 + b_1 n + b_2 n^2 + \cdots + b_n n^m$. Then the particular solution will be of the form $q_0 + q_1 n + q_2 n^2 + \cdots + q_m n^m$. provided one is not a characteristic root of recurrence relation.

Rule 2 :- If f(n) is of the form $(b_0 + b_1 n + b_2 n^2 + \cdots + b_m n^m)\, a^m$ Then particular solution is of the form $(q_0 + q_1 n + q_2 n^2 + \cdots + q_m n^m)\, a^n$ where 'a' is not a characteristic root of the recurrence relation

Rule 3 :- If 'a' is the characteristic root of the multiplicity $(r-1)$ when f(n) is of the form $(b_0 + b_1 n + b_2 n^2 + \cdots + b_m n^m)\, a^n$ then particular solution is of the form $n^{r-1}(q_0 + q_1 n + q_2 n^2 + \cdots + q_m n^m)\, a^n$

→ If no initial conditions are given then you have finished the problem.

→ If 'm' initial conditions are given we can get 'm' linear equations in unknowns and the solve system to get a complete solution.

— . . —

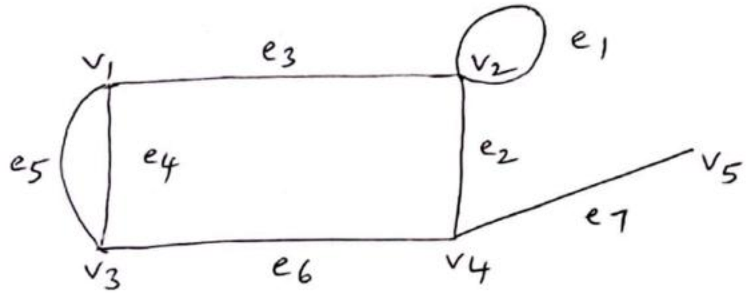
UNIT – V

Graphs

Basic concepts :– Graph :–

A Graph $G = \langle V, E, \phi \rangle$ consists of a non-empty set of vertices $V = \{v_1, v_2, \cdots\}$, a set of edges $E = \{e_1, e_2, \cdots\}$ and ϕ is a mapping from 'V' to 'E'

Ex:



In the above di-graph vertices are 5 and edges are '7'

$\therefore G = \{V, E\}$, $V = \{v_1, v_2, v_3, v_4, v_5, \cancel{v_6}\ \cancel{v_7}\}$

$E = \{e_1\ e_2, e_3, e_4, e_5, e_6, e_7\}$

Self loop is '1' $\Rightarrow e_1 = (v_2, v_2)$

parallel edger are '2' $\Rightarrow e_4, e_5$

$e_1 = (v_2, v_2)$, $e_2 = (v_2, v_4)$, $e_3 = (v_1, v_2)$,

$e_4 = (v_1, v_3)$, $e_5 = (v_3, v_1)$, $e_6 = (v_3, v_4)$, $e_7 = (v_4, v_5)$

Note: 1. vertex of a graph is also called 'node' (or) 'junction' (or) 'a point'.

2. Edge of a graph is also called a 'branch' (or) 'line' (or) 'a element'.

3. An edge associated with a vertex pair is called a loop (or) 'self loop' . $\{v_i, v_i\}$

4. If there is more than one edge associated with a given pair of vertices then these edges are called

Simple graph: A graph that has neither self-loops nor parallel edges is called simple graph.

General graph :- A graph containing either parallel edges or loops is called a general graph

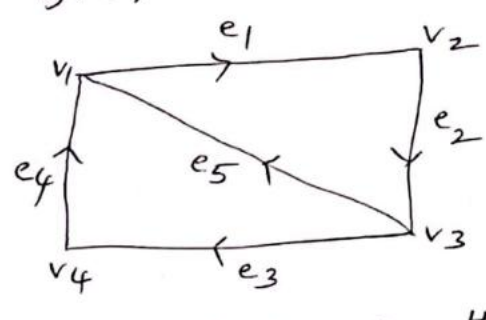
finite and infinite graphs : A graph 'G' with finite numbers of vertices and finite number of edges is called a finite graph.

A graph 'G' that is not a finite graph is said to be 'infinite graph'.

Types of Graphs :-

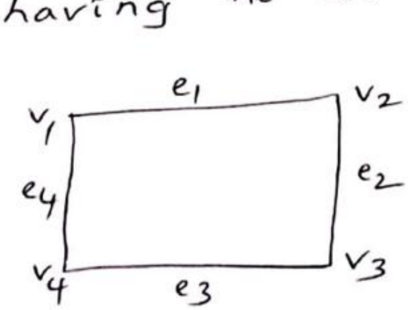
1. Directed graphs : In the graph 'G' all the edges having directions then the graph 'G' is called 'Directed graph'.

Ex:



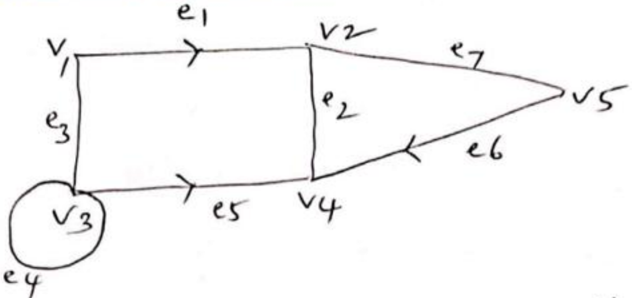
2. General graph :- In the graph 'G' all the edges doesnot having the directions is known as general graph.

Ex:



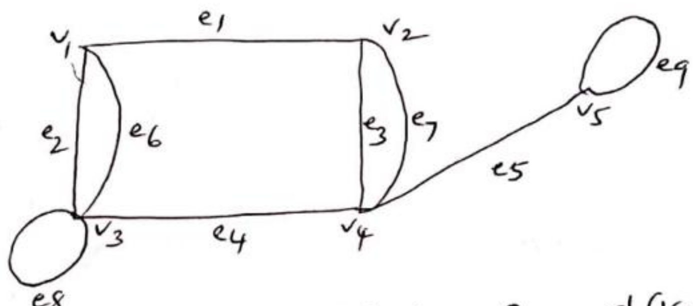
3. Mixed graph :- In the graph 'G' some edges having directions and some are not having the

Ex:



Degree of a vertex :- In the graph 'G' the no. of edges incident on a vertex 'v' is called "the degree of a vertex" and it is denoted by $d(v)$.

Ex:



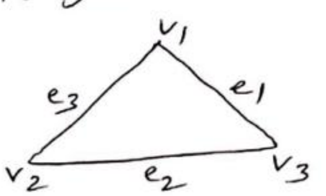
In the graph G, $d(v_1) = 3$ $d(v_3) = 5$ $d(v_5) = 3$

$d(v_2) = 3$, $d(v_4) = 4$,

Note: loop is treated as two edges.

Regular Graph :- In a graph 'G' all the vertices of G are of equal degree then the graph 'G' is called 'Regular graph'.

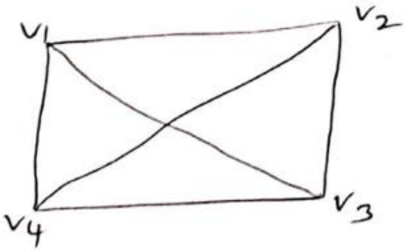
Ex:



$d(v_1) = 2$

$d(v_2) = 2$

$d(v_3) = 2$



$d(v_1) = 3$ $d(v_3) = 3$

$d(v_2) = 3$ $d(v_4) = 3$.

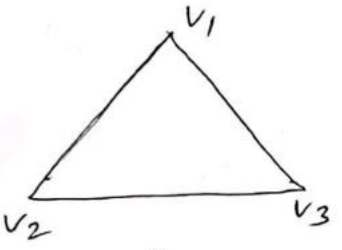
Null graph :- In a graph 'G' all the vertices of degree '0' i.e all the vertices are not having any edge (Isolated) then the graph is called "null graph"

Ex: v_1 v_2 $d(v_1) = 0$

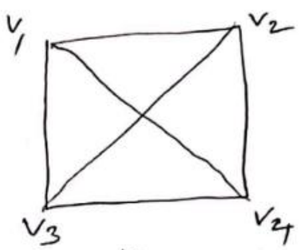
$d(v_2) = 0$.

Complete graph :- A simple graph G in which all the pair of vertices are connected by the means of a edge. It is denoted by K_n. where 'n' is ~~nof~~ no. of ~~ed~~ vertices.

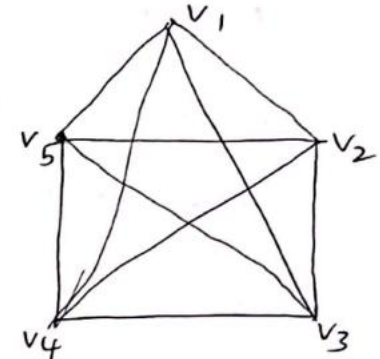
Ex:



K_3



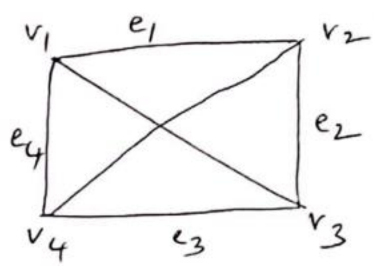
K_4



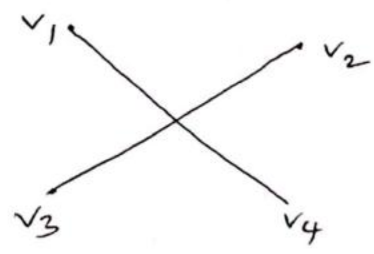
K_5

Bi-partite graph :- A Graph 'G' in which the vertex set can be divided into two subsets, where $v_1 \cup v_2 = V$ and $v_1 \cap v_2 = \phi$ and in the edge set one end edge is in v_1 and another end edge is in v_2 then the graph is "Bipartite graph".

Ex:



K_4



Bipartite graph

$v_1 = \{v_1, v_4\}$, $v_2 = \{v_2, v_3\}$

Matrix representation of Graphs :-

There are Two types of representations.

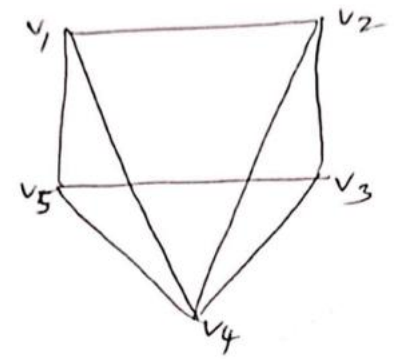
1. adjacency matrix representation
2. Incident matrix representation.

1. **Adjacency matrix :-** It is represented between vertices it is denoted by $M(G)$ and

is defined as

$$M(G) = \begin{cases} 1 & v_i \text{ join to } v_j \\ 0 & v_i \text{ not join to } v_j \end{cases}$$

Ex:

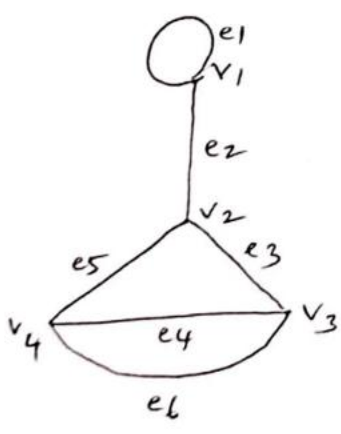


$$M(G) = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 0 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 1 & 1 \\ v_4 & 1 & 1 & 1 & 0 & 1 \\ v_5 & 1 & 0 & 1 & 1 & 0 \end{array}$$

2. **Incident matrix** :- It is represented between the connectivity of the vertices and the edges of the graph G. It is denoted by $M(G)_{V\times E}$ and is defined as

$$M(G) = \begin{cases} 0 & \text{if the vertex not ends with the edge} \\ 1 & \text{if the vertex ends with edge not loop} \\ 2 & \text{if the vertex ends with loop edge.} \end{cases}$$

Ex:

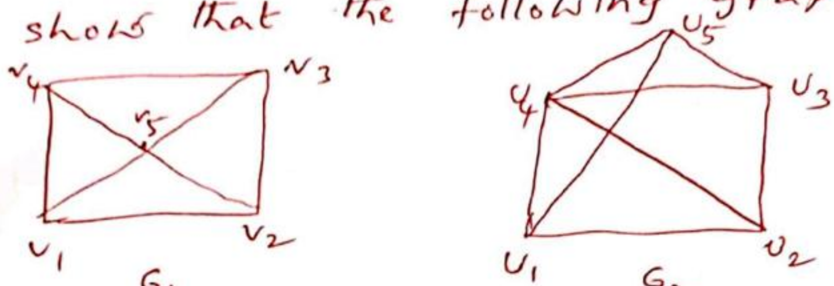


$$\therefore M(G) = \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_1 & 2 & 1 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 1 & 1 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

Isomorphism of graphs :- Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there is a one-to-one and onto mapping $f: V_1 \to V_2$ with the property that v_1 and v_2 are adjacent in G_2 for all $v_1, v_2 \in G_1$ Then the two graphs G_1 and G_2

If G_1 and G_2 are isomorphic then the no. of vertices and the no. of edges in both graphs are equal. The degree of vertices in G_1 is same as the vertices in G_2.

EX: shows that the following graphs are isomorphic



G_1 G_2

sol: $v(G_1) = \{v_1, v_2, v_3, v_4, v_5\} = 5$

$v(G_2) = \{u_1, u_2, u_3, u_4, u_5\} = 5$

$E(G_1) = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\ (v_4, v_5)\ (v_2, v_5), (v_1, v_5)\ (v_3, v_5)\} = 7$

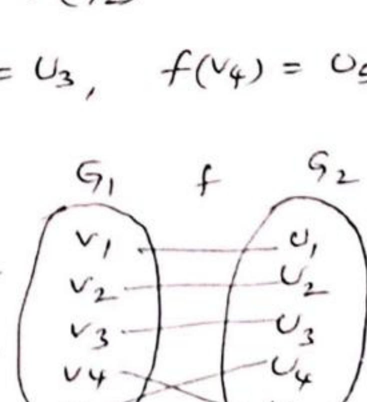
$E(G_2) = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_1)\ (u_1, u_5), (u_4, u_5)\ (u_3, u_5)\} = 7$

Both graphs are have equal no. of vertices and edges

In graph (G_1) and (G_2) Four vertices have degree '3' and one vertex having degree '4'.

Define a function $f : v(G_1) \rightarrow v(G_2)$

$f(v_1) = u_1$, $f(v_2) = u_2$, $f(v_3) = u_3$, $f(v_4) = u_5$, $f(v_5) = u_4$.



$\therefore$ There is a one to one correspondence between the edges and the vertices of G and G_1.

$(v_1, v_2) \longleftrightarrow (u_1, u_2)$ $(v_3, v_4) \longleftrightarrow (u_3, u_5)$

$(v_1, v_4) \longleftrightarrow (u_1, u_5)$ $(v_3, v_5) \longleftrightarrow (u_3, u_4)$

$(v_1, v_5) \longleftrightarrow (u_1, u_4)$ $(v_4, v_5) \longleftrightarrow (u_5, u_4)$

$(v_2, v_3) \longleftrightarrow (u_2, u_3)$

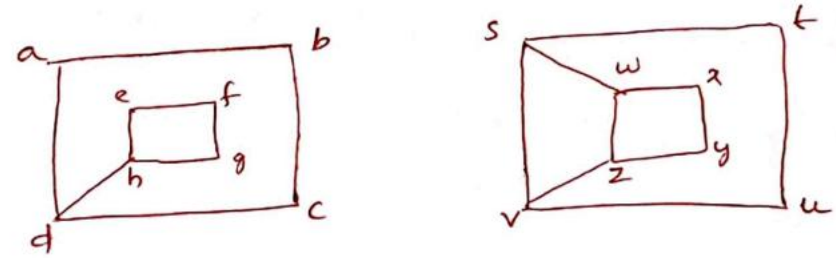
$(v_2, v_5) \longleftrightarrow (u_2, u_4)$,

since Every element of G_2 is having pre-image in G_1 and 'f' is on-to

Hence $f: V(G_1) \to V(G_2)$ is one-to-one and onto

$\therefore$ G_1 and G_2 are isomorphic.

Ex: Determine whether the graphs G and H are isomorphic



Sol: The given two graphs contains same no. of vertices $(V = 8)$ and same no. of edges $(E = 10)$ in both graphs four vertices having degree '2' and four vertices having degree '3'.

correspondence between the edges

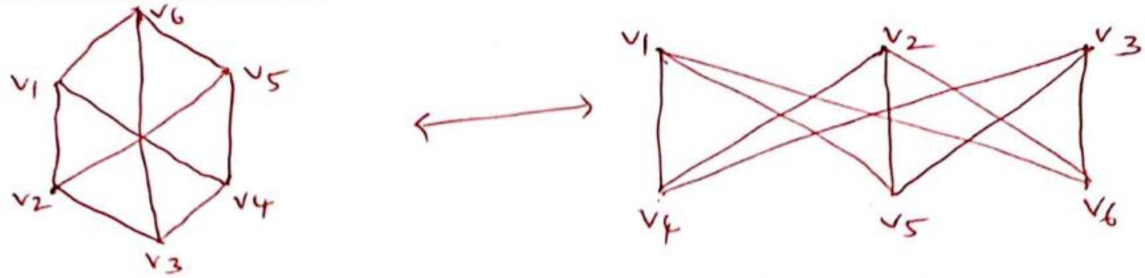
$(a, b) \longleftrightarrow (t, s)$

$(a$, not exist $) \longleftrightarrow (s, w)$

$\therefore$ There is no one-to-one correspondence because the edges of G and H.

Hence the two graphs G and H are not isomorphic

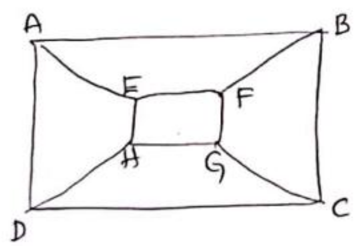
Ex: i)

ii) 

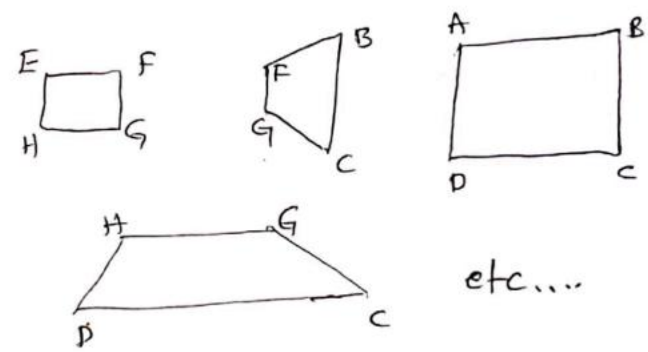
iii) 

Sol: The above (i), (ii), (iii) graphs are isomorphic.

Sub graphs :— A Graph 'H' is said to be a Subgraph of a graph 'G' if all the vertices and all the edges of H are in G and each edge of H has the same end vertices as in G. It is denoted by $H \subseteq G$.

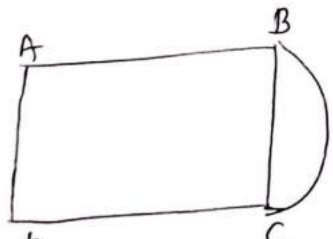
EX: 1) 

Graph G

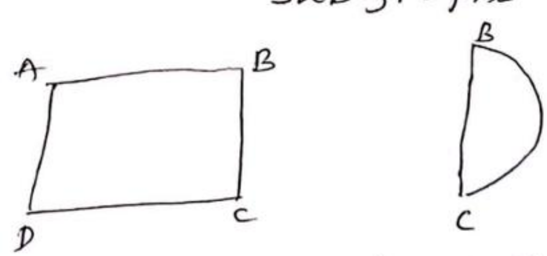


etc....

Subgraphs of G.

2) 

Graph G



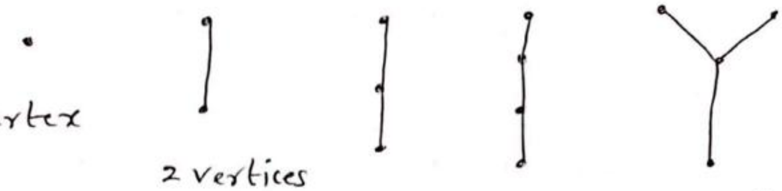
etc...

Subgraphs of G.

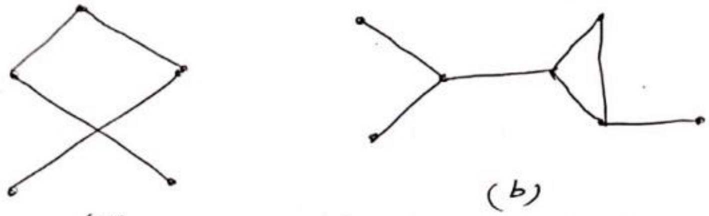
Trees and their properties :—

A Tree is a connected acyclic graph ie a connected graph having no cycle. its edges are called

Ex:



1 vertex 2 vertices 3 Vertices 4 Vertices 4 vertices



(a) (b)

(a), (b) are not trees since they have cycles.

Note: A tree with only one vertex is called a "trivial tree" otherwise it is a non trivial tree.

Theorem :- 1. There is one and only one path between every pair of vertices in a tree T.

proof: since 'T' is a connected graph.

There exist at least one path between every pair of vertices in T.

Let there are two distinct paths between two vertices u and v of T.

But union of these two paths will contain a cycle

Then 'T' can not be a tree.

Hence only one path between every pair of vertices in a tree T.

Theorem :- A Tree 'T' with n vertices has n-1 edges.

proof: The Theorem is proved by induction on 'n'

If n = 1 then 'T' has only one vertex

T can not have any edge it has $e = 0$

ie $e = n - 1$

Suppose the theorem is true for $n = k \geq 2$ where 'k' is some positive integer

Now we have to show that the result is true for $n = k+1$.

Let 'T' be a tree with $k+1$ vertices and let uv be the edge of T. Then if we remove the edge uv from T we obtain the graph $T - uv$.

Thus $T - uv$ is disconnected. So $T - uv$ has two components say T_1 and T_2.

Thus T_1 and T_2 are trees and each has fewer than 'n' vertices, to give

$$e(T_1) = v(T_1) - 1$$
$$e(T_2) = v(T_2) - 1$$

But the ~~contra~~ construction of T_1 and T_2 by removal of a single edge from T gives that

$$e(T) = e(T_1) + e(T_2) + 1$$
$$v(T) = v(T_1) + v(T_2)$$

It follows that $e(T) = v(T_1) - 1 + v(T_2) - 1 + 1$

$$= v(T) - 1 = k + 1 - 1 = k$$

Thus 'T' has 'k' edges as required

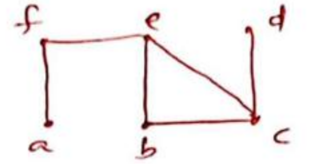
Hence the theorem is proved.

Spanning Tree :- A subgraph H of a graph G is called a spanning tree of G if

a) H is a tree

b) H contains all the vertices of G.

Ex: Find all spanning trees for the graph G shown below.



Sol: The graph is connected

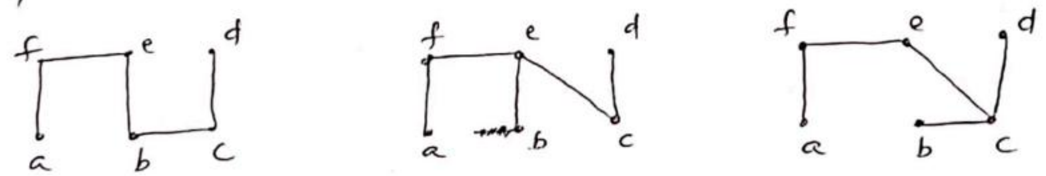
It has 6 edges and 6 vertices

Hence each spanning tree must have $6-1=5$ edges

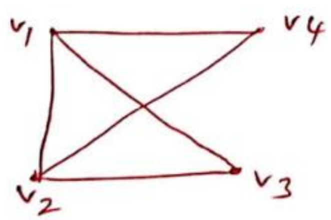
So $6-5=1$ edge has to be deleted from G.

The graph G has one cycle cbec and removal of any edge of the cycle gives a tree.

There are three trees which contains all the vertices of G and Hence the spanning trees are



Ex: How many spanning trees has the graph follows and find all the spanning trees.



Sol: The matrix form is

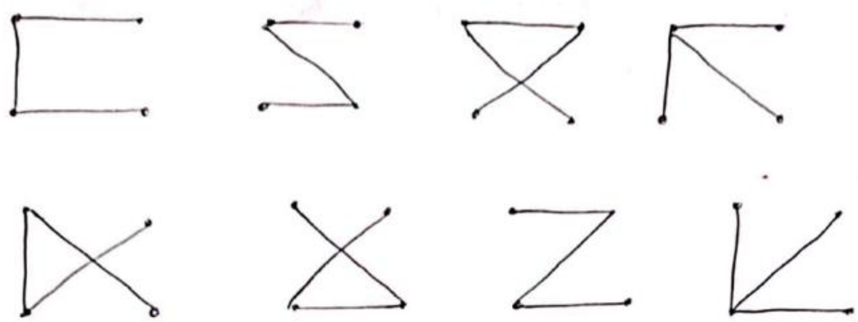
$$M = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 3 & -1 & -1 & -1 \\ v_2 & -1 & 3 & -1 & -1 \\ v_3 & -1 & -1 & 2 & 0 \\ v_4 & -1 & -1 & 0 & 2 \end{array}$$

The cofactor of the element '3' (m_{11}) is the determinant

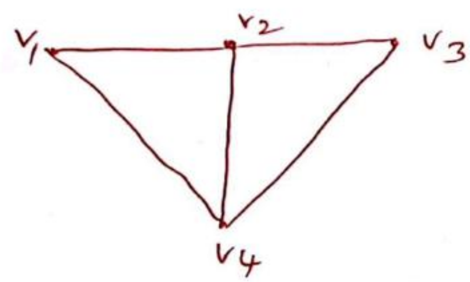
$$\begin{vmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 3(4-0) + (-2-0) - 1(0+2)$$

$$= 8.$$

Th... H... are eight spanning trees and they are



Ex: Find all spanning trees of the graph G shown.



Sol: The matrix form is

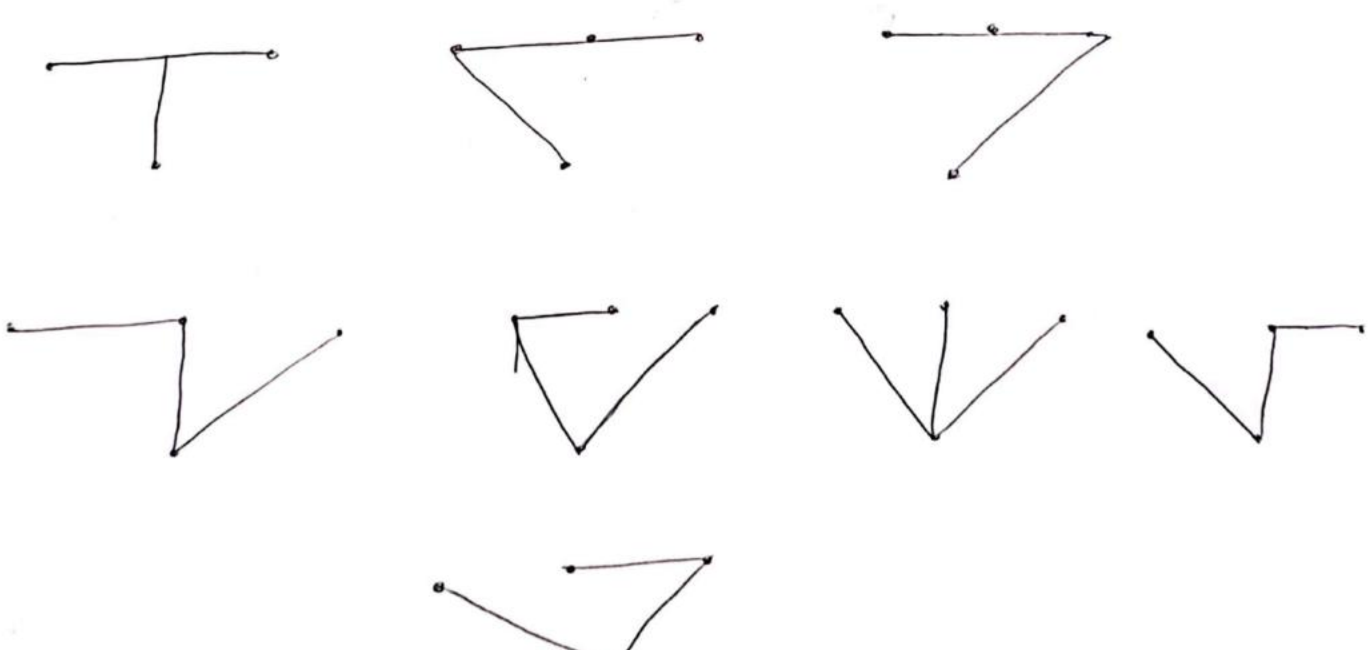
$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \end{array}$$

The cofactor of the element 2 (m_{11}) is the determinant

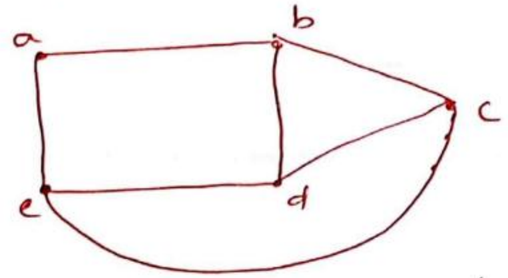
$$D = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 3(6-1) + 1(-3-1) - 1(1+2)$$

$$= 8$$

Thus there are eight spanning trees and they are



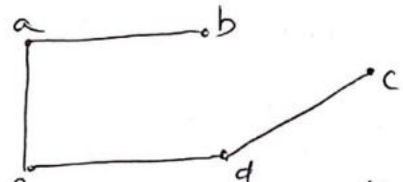
Ex: Find the six spanning trees of the graph given below



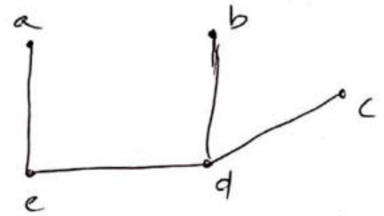
Sol: There are 7 edges and 5 vertices of the given graph G.

Each spanning tree must have $5-1=4$ edges.

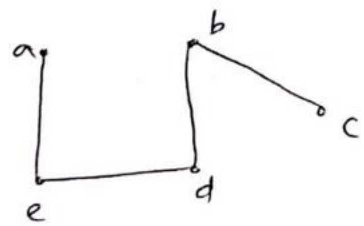
i) Deleting edges (b, c), (b, d) and (c, e) we get



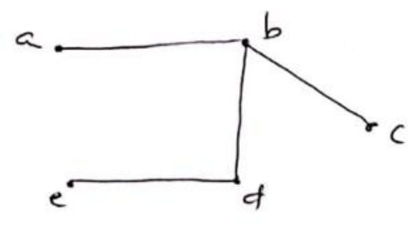
ii) Deleting edges (a, b), (b, c), and (c, e) we get



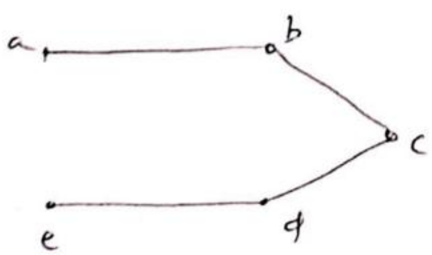
iii) deleting edges (a, b), (d, c), and (c, e) we get



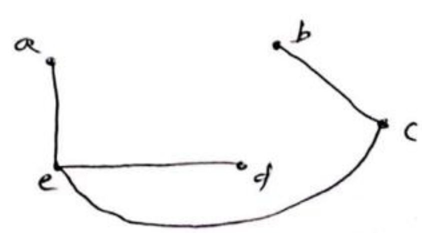
iv) Deleting edges (a, e), (d, c) and (c, e) we get



v) deleting (a, e), (b, d) and (c, e) we get



vi) Deleting edges (a, b) (b, d) and (c, d) we get



Algorithms for constructing spanning trees :-

Two algorithms based on this principle for finding a spanning tree are.

1. BFS (Breath - First - Search) algorithm
2. DFS (Depth - First - Search) algorithm.

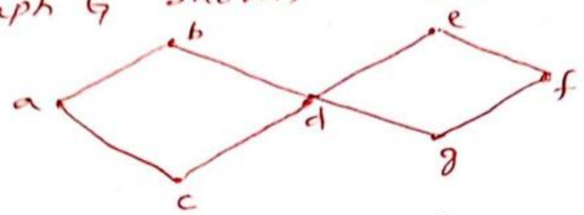
1. BFS algorithm :- (Always harizontal)

procedure :-

1. Arbitrarily choose a vertex and designate it as the root. Then add all edges incident to this vertex such that the addition of edges does not produce any clycle.
2. The new vertices added at this stage become the vertices at level 1 in the spanning tree.
3. Next for each vertex at level 1 visited in order add each edge incident to this vertex to the tree as long as it does not produce any cycle.
4. Arbitrarily order the children of each vertex at level 1. This produces the vertices at level '2' in the tree.
5. continue the same procedure until all the vertices in the tree have been added.
6. The procedure ends there are only a finite number of edges in the graph.
7. A spanning tree is produced since we have produced a tree with out cycle containing every vertex of the graph.

Ex: Use BFS algorithm to find a spanning tree

of graph G shown below



Sol: i) choose the vertex 'a' to be the root.

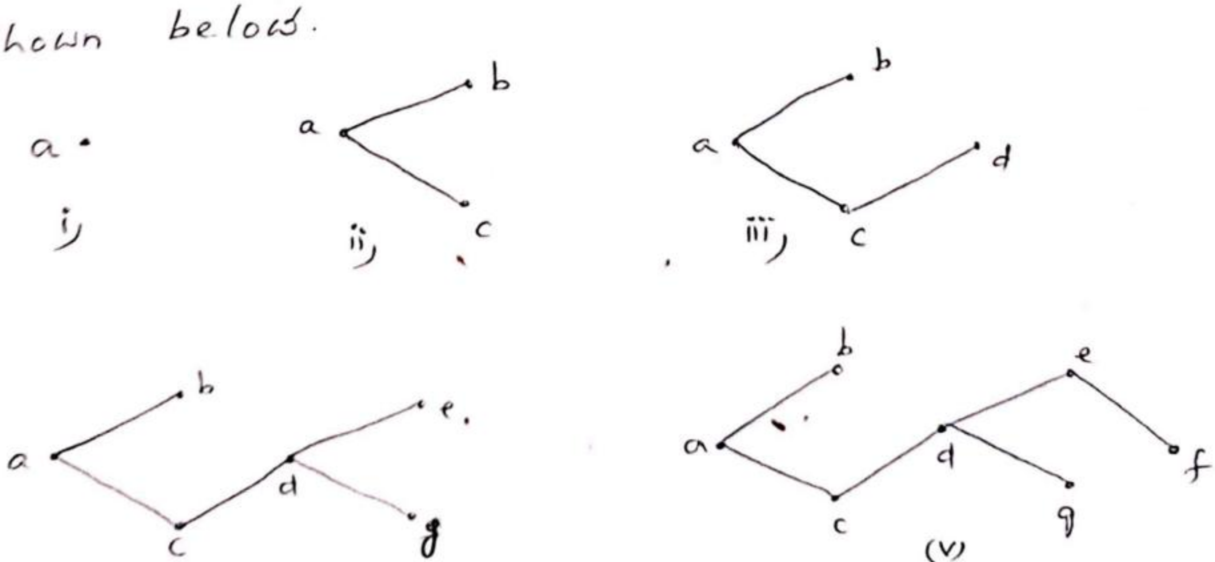
ii) Add edges incident with all vertices adjacent to a, so that (a,b), (a,c) are added. The two vertices 'b' and 'c' are in level 1 in the tree.

iii) Add edges from these vertices at level 1 to adjacent vertices not already in the tree. Hence the edge (c,d) is added. The vertex d is in level 2.

iv) Add edge from 'd' in level 2 to adjacent vertices not already in the tree. The edge (d,e) and (d,g) are added. Hence 'e' and 'g' are in level '3'

v) Add edge from 'e' at level 3 to adjacent vertices not already in the tree. Hence (e,f) is added.

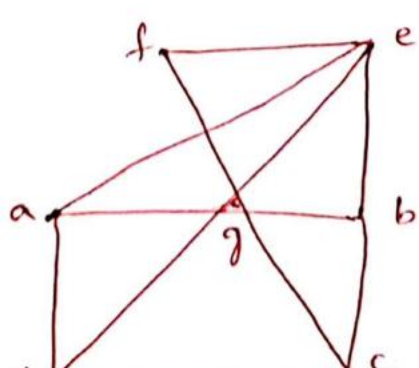
The steps of Breath first search procedure are shown below.



2. DFS algorithm :— (Always vertical)

1. Arbitrarily choose a vertex from the vertices of the graph and designate it as the root.
2. Form a path starting at this vertex by successively adding edges as long as possible where each new edge is incident with the last vertex in the path without producing any cycle.
3. If the path goes through all vertices of the graph the tree consisting of this path is a spanning tree.
4. If this can not be done move back another vertex in the path that is two vertices back in the path and repeat.
5. Repeat this procedure beginning at the last vertex visited moving back up the path one vertex at a time forming new paths that are as long as possible untill no more edges can be added.
6. This process ends since the graph has a finite number of edges and is connected. A spanning tree is produced.

Ex: Using DFS method find a spanning tree for the graph



Sol: i) start with the vertex 'a' as a root.

ii) we cannot add 'b' and 'c' to 'a' because there is a vertex 'g' between 'a' and 'b'

iii) So add 'd' to 'a' ⇒ level 1 ⇒

a
|
d

iv) Now start with 'd' we cannot add 'b' so add 'c' to 'd' ∴ level - 2 ⇒

a
|
d
|
c

v) Now start with 'c' we can ~~not~~ add 'b' to 'c'

level - 3 ⇒

a
|
d
|
c
|
b

vi) start at the vertex 'b' add 'e' to 'b'

level - 4 ⇒

a
|
d
|
c
|
b
|
e

vii) start the vertex 'e' add 'f' to 'e'.

Level - 5 ⇒

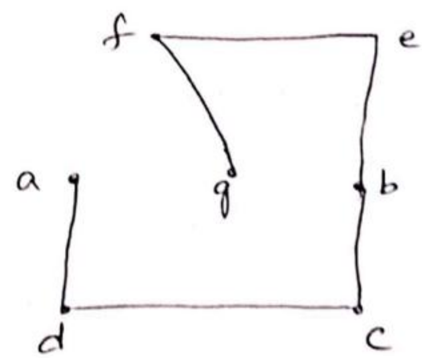
a
|
d
|
c
|
b
|
e
|
f

viii) start at the vertex 'f' add 'g' to f

Level - 6 ⇒

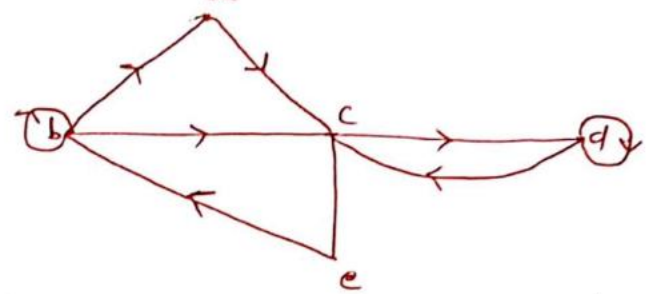
a
|
d
|
c
|
b
|
e
|
f
|

Now spanning tree is



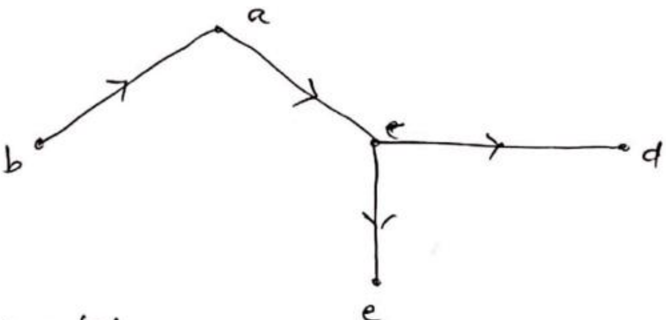
Directed tree :- A spanning tree that is a directed graph then it is called a "directed spanning tree" of G.

Ex: Derive the directed spanning tree from the graph shown below

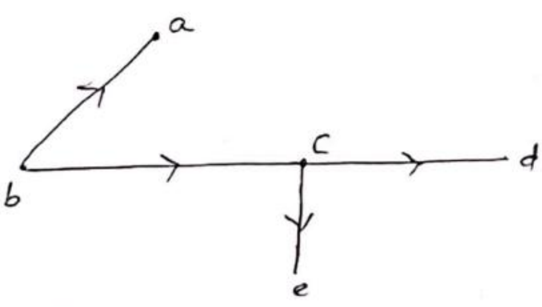


Sol: There are many directed spanning trees of G.

i) one of them can be obtained by deleting the loops at 'b' and 'd' ii) deleting the edges (d, c), (e, b), (b, e).



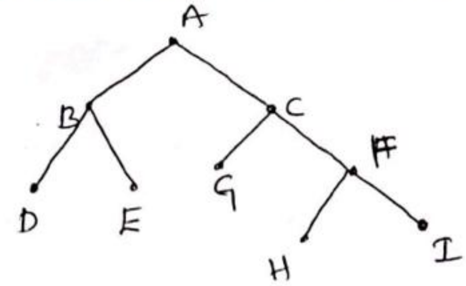
Another directed spanning tree also obtained by deleting the loops and (d, c), (e, b), (a, c)



etc....

Binary tree :- A rooted tree is an m-ary if every internal vertex has at most 'm' children. A m-ary tree is a full m-ary tree if every internal vertex has exactly m children. In particular the 2-ary

Ex:



Ex: Suppose a tree has n_1 vertices of degree 1, 2 vertices of degree 2, 4 vertices of degree 3, and 3 vertices of degree 4, find n_1.

Sol: W.K.T. Sum of degrees $= 2 \cdot e$

Now the sum of degrees $= n_1 \times 1 + 2 \times 2 + 4 \times 3 + 3 \times 4$

$= 28 + n_1$

The no. of edges in a tree is $n-1$ where 'n' is the no. of vertices.

$\therefore$ The no. of edges equal to $(n_1 + 2 + 4 + 3) - 1$

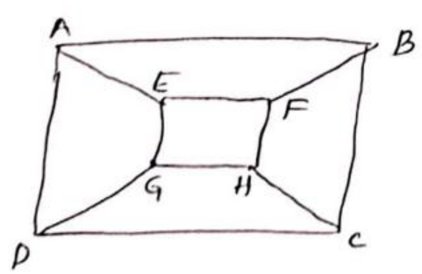
$= n_1 + 8$

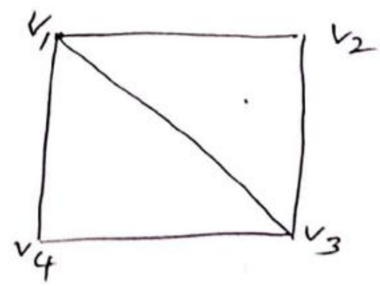
$\therefore 28 + n_1 = 2(n_1 + 8) \Rightarrow n_1 = 28 - 16 = 12$ //

planar Graph :-

A Graph is said to be ~~planar~~ planar graph if there exist some geometric representation of G which can be drawn on a plane such that no two of its edges intersects.

Ex:

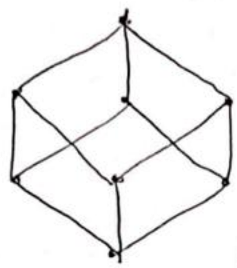




Non planar graph :-

A graph that cannot be drawn on a plane without a cross over between its edges. is called "Non planar graph"

Ex:



Walk :- It is a sequence of vertices and edges. Here repititions are allowed.

Ex: $v_1 e_1 v_2 e_2 v_3 \cdots e_n v_{n+1}$ is a walk.

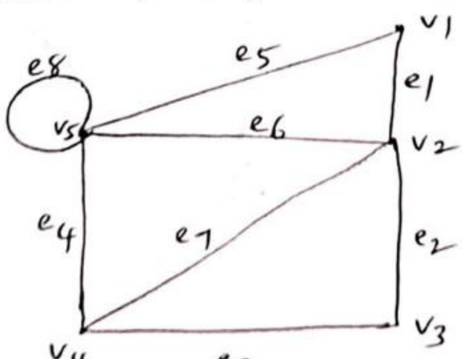
Trail :- It is a walk in which all the edges are different but vertices are coinside

path :- It is a walk in which all the vertices and all the edges are different.

cycle / circuit :- It is a closed trail.

Length of a path :- The no. of edges appearing in the sequence of a path is called "length of the path"

Ex:



In this graph. we have.

1. $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_7 v_2 e_6 v_5$ is a open walk.

2. $v_2 e_6 v_5 e_8 v_5 e_4 v_4 e_3 v_3 e_2 v_2$ is closed walk.

3. $v_2 e_7 v_4 e_3 v_3 e_2 v_2 e_6 v_5 e_8 v_5 e_5 v_1$ is a trail

4. $v_1 e_5 v_5 e_4 v_4 e_3 v_3 e_2 v_2$ is a path

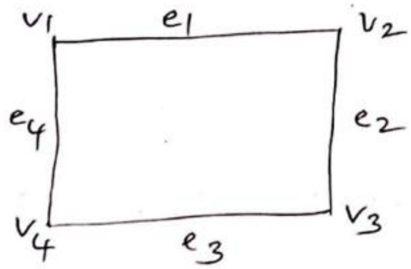
5. $v_1 e_1 v_2 e_6 v_5 e_8 v_5 e_5 v_1$ is a cycle.

→ Here Length of the path = 4. (no. of edges).

Connected graph :—

A Graph G is said to be connected if there is atleast one path between every pair of vertices in G

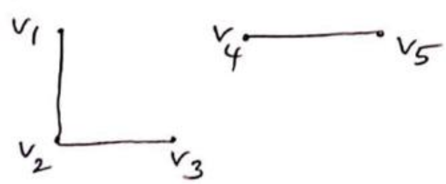
Ex:



it is connected graph.

Disconnected graph :— which is not connected

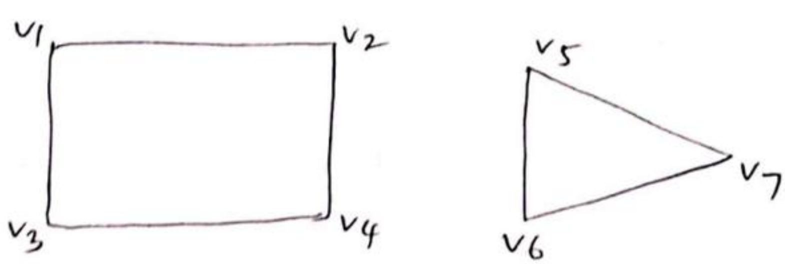
Ex:



it is Disconnected graph

Component :— A disconnected graph contains two or more connected subgraphs each of their connected subgraphs is called a "components".

Ex:



It is a disconnected graph and it contains two connected sub graphs which are called Components

* Euler Trail :-

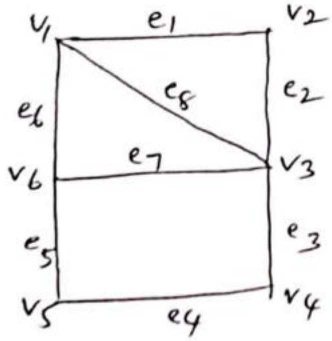
A Trail in the graph 'G' if it include every edge of G exactly one's is called "Euler Trail" Here all the edges are distinct.

Tour :- A closed walk of the graph, which includes all the edges atleast once.

Euler's Tour :- It is a closed Trail of a graph G which includes all the edges exactly once.

Euler graph :- A graph in which the Euler's Tour exist in G.

Ex:



In the graph

$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_6 e_7 v_3$ $e_8 v_1 e_6$ in Euler Tour

Hence it is Euler graph.

Euler's Formula :- Let 'G' be a graph then

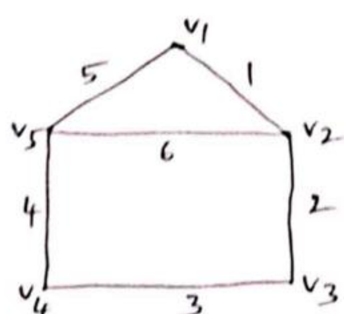
$|V| - |E| + |R| = 2$ where

$|V|$ = Set of vertices , $|E|$ = Set of edges

$|R|$ = set of Regions.

If any graph which sastisfy Euler's formula then it is planar graph.

Ex:



In the given graph

$|V| = 5$, $|E| = 6$, $|R| = 3$

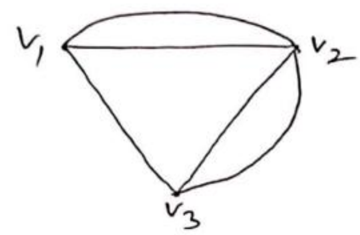
$\therefore |V| - |E| + |R| = 5-6+3$

$= 8-6 = 2$

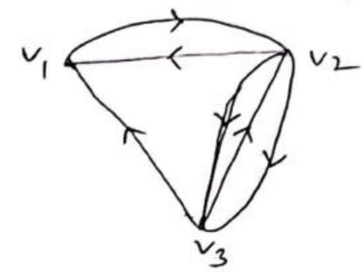
Hence it is planar graph.

Multi graphs :- Any graph which contains some multiple edges is called a multigraph. In a multi graph no loops are allowed.

Ex:



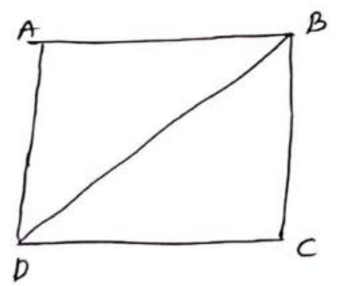
a) Undirected multigraph



b) Directed multigraph

Euler's circuits :- Let $G = (V, E)$ be an undirected graph (or) multi graph with no isolated vertices Then G is said to be an Euler's circuit if there is a circuit in G that traverses (visits) every edge of the graph exactly once.

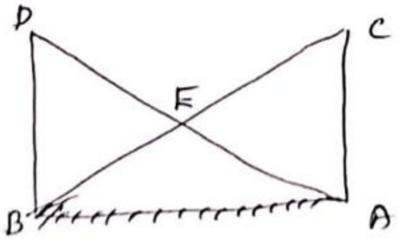
Ex:



B – D – C – B – A – D.

Theorem :- Graph G is an Euler's circuit if and only if G is connected and every vertex in G has even degree.

proof:



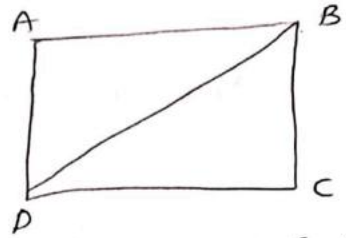
Deg (A) = 2, Deg (B) = 2, Deg(C) = 2, Deg (D) = 2, Deg (E) = 4

$\therefore$ In the graph each degree as an even number

$\therefore$ Hence the theorem is proved.

Theorem :- We can construct an Euler's circuit in G if and only if G is connected and exactly two vertices of odd degree.

proof:



$D(A) = 2$, $D(B) = 3$, $D(C) = 2$, $D(D) = 3$.

In the graph two vertices are odd degree

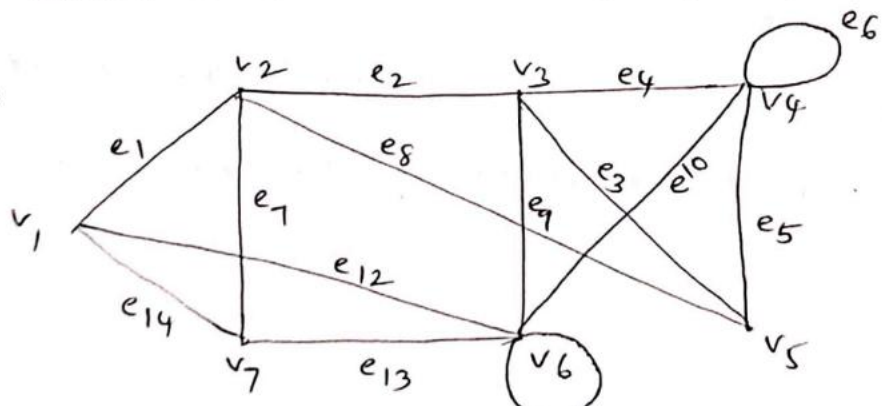
Hence the theorem is proved.

Hamiltonian graphs :-

In a connected graph a closed walk running through every vertex of G exactly once (except the starting vertex) at which the walk terminates is called "Hamiltonian circuit (or) graph"

A graph containing a hamiltonian circuit is called a Hamiltonian graph.

EX:



In this graph

$v_1 e_1 v_2 e_2 v_3 e_3 v_5 e_5 v_4 e_{10} v_6 e_{13} e_7 e_{14} v_1$

We have to see that all vertices must be present for one time (except starting) should be there.

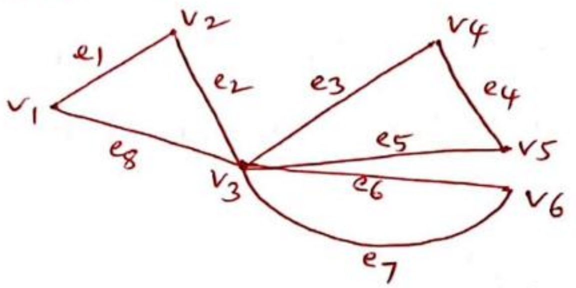
since vertices are not repeated

and edges are also not repeated

Hence it is a Hamiltonian cycle.

It is also not a Eulerian circuit as some of the vertices like $v_1, v_7, \ldots$ have odd degree But it is a connected graph..

Ex: consider the graph. Is this graph a Eulerian circuit? Is this graph a Hamiltonian circuit?

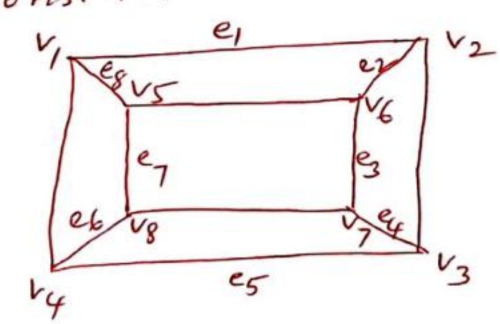


Sol: From the above graph

$v_1 e_1 v_2 e_2 \underline{v_3} e_3 v_4 e_4 v_5 e_5 \underline{v_3} e_6 v_6 e_7 \underline{v_3} e_8 v_1$

This is not a hamiltonian graph since the vertices are repeated But it is a Eulerian graph since the degree of all vertices is even.

Ex: Consider the graph. Is this graph a Hamiltonian circuit.



Sol: From the above graph

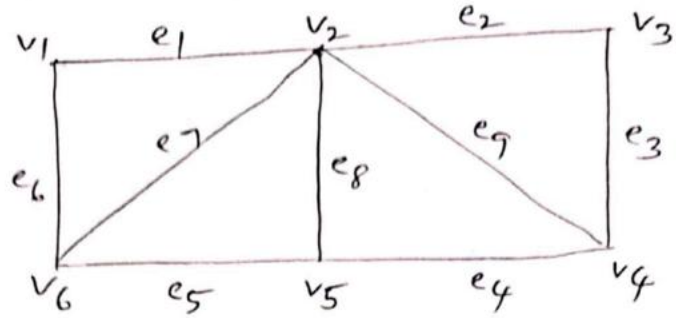
$v_1 v_4 v_8 v_7 v_3 v_2 v_6 v_5 v_1$ (or)

$v_1 v_5 v_6 v_2 v_3 v_7 v_8 v_4 v_1$

which is a closed walk and also a Hamiltonian circuit

Ex: Give an example of graph which is hamiltonian

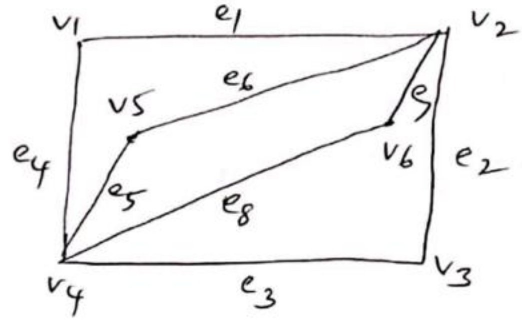
Sol: i) The following graph



From the graph ($v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_6 e_6 v_1$)

This is Hamiltonian graph But not a Eulerian circuit as the degree of some vertices is odd.

ii) The following graph.



From the graph

($v_1 e_1 \underline{v_2} e_2 v_3 e_3 v_4 e_8 v_6 v_7 \underline{v_2} e_6 v_5 e_5 v_4 e_4 v_1$)

This graph is a Eulerian graph since the vertices are repeated and the degree of all vertices are even

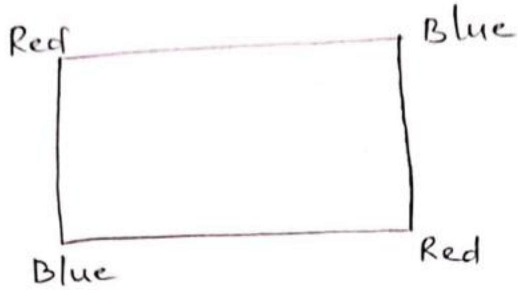
But this is not a Hamiltonian graph as the vertices are repeated.

Chromatics number :-

A graph G is said to be K- colourable if we can colour properly with K- colour. A K- colourable graph is a graph which can be coloured properly with K- colour But not less.

The Chromatic number of a graph is the minimum number of colours required to colour a graph properly. It is denoted by $\chi(G)$.

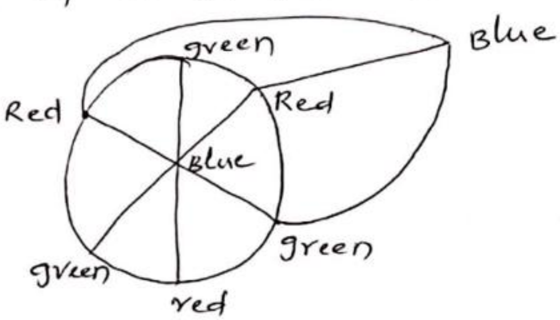
Ex:



chromatic number of graph = 2

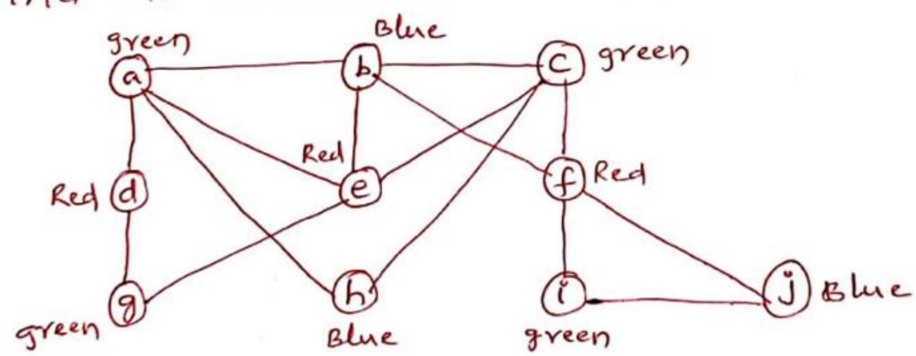
since the minimum No. of colours required for colouring of a graph = 2.

Ex:



∴ chromatic Number of graph = 3, $\chi(G) = 3$.

Ex: Find the chromatic Number of the graph



Sol:

Deg(e) = 5 Deg(h) = 3

Deg(a) = 4 Deg(i) = 3

Deg(b) = 4 Deg(d) = 2

Deg(c) = 4 Deg(j) = 2

Deg(f) = 4 Deg(g) = 2

Decreasing order : e, a, b, c, f, h, i, d, j, g

i) e, a, b c f h i d g j

e → Red, f → Red, d → Red

ii) e a b c d h i d g j

↓ ↓ ↓ ↓ ↓ ↓ ↓

Red green green Red green Red green

iii) e a b c d h i d g j

↓ ↓ | ↓ ↓ ↓ ↓ ↓ ↓ |

Red green Blue green Red Blue green Red green Blue.

∴ The chromatic no. of graph = 3

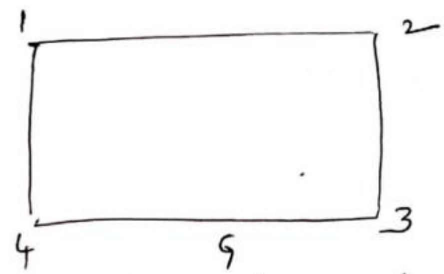
Red colour = e, f, d

Green colour = a, c, i, g

Blue colour = b, h, j.

The Four - color problem :-

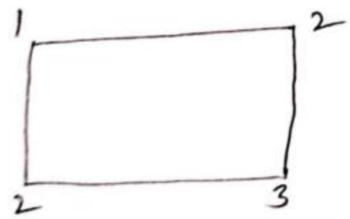
Each vertex of G can be assigned with different colours



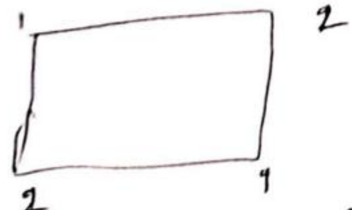
G

So it is 4 - Colourable graph

we can use colours 1, 2, 3 for G as



so it is 3- colourable. we can also use only two colours '1' and '2' for G as



So it is 2- colourable. But G is not one - colourable. since '1' colour is not sufficient to assign colours to G. so that adjacent colours are different.

— . —

some practice problems

1. write the adjacency matrix for the graph

Sol: Adjacency matrix

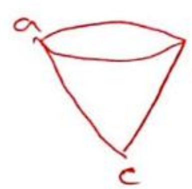
$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \rightarrow 3 \\ \rightarrow 5 \\ \rightarrow 3 \\ \rightarrow 3 \end{array}$$

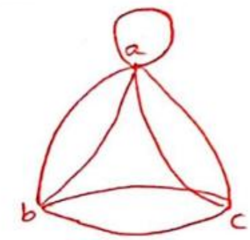
Deg (A) = 3

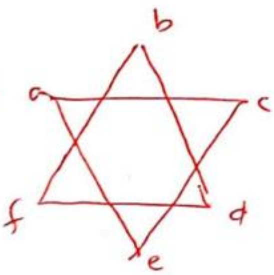
Deg (B) = 5

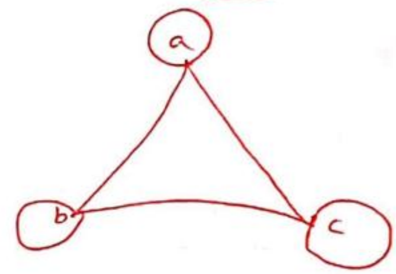
Deg (C) = 3, Deg (D) = 3

2. Find the degrees of the vertices of the graphs shown below

a) 

b) 

c) 

d) 

Sol: a) Deg (a) = 3, Deg (b) = 3, Deg (c) = 2

b) Deg (a) = 6, Deg (b) = 4, Deg (c) = 4

c) Deg (a) = 2, Deg (b) = 2, Deg (c) = 2

Deg (d) = 2, Deg (e) = 2, Deg (f) = 2

d) Deg (a) = 4, Deg (b) = 4 Deg (c) = 4.

3. Find the adjacency matrices of the above graphs.

Sol: a)

$$M = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \end{array}$$

b)

$$M = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \end{array}$$

d)

$$M = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

c)

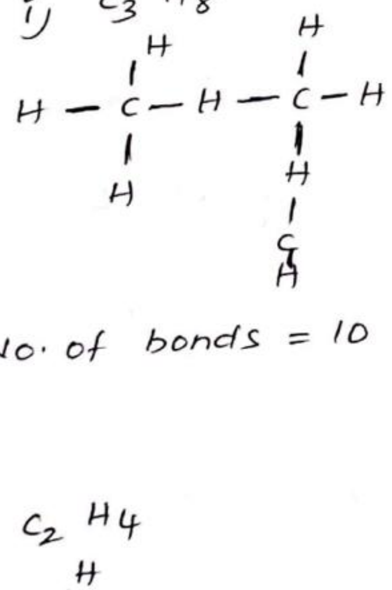
$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

4. Find the no. of bonds in each hydrocarbon molecule (Assume each carbon atom is bonded to four atoms)

i) C_3H_8 2) C_4H_{10} 3) C_2H_4 4) C_4H_8 v) C_2H_6

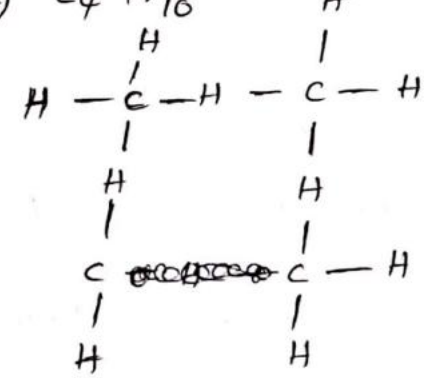
Sol:

i) C_3H_8



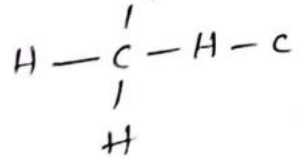
No. of bonds = 10

2) C_4H_{10}



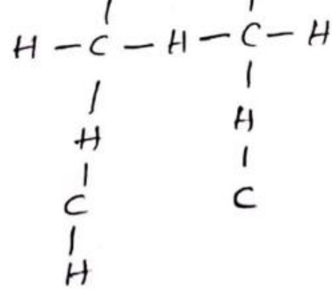
No. of bonds = 13.

3) C_2H_4



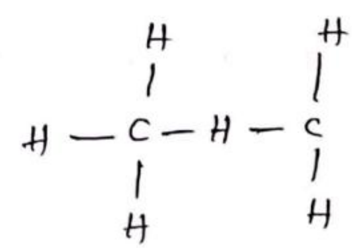
No. of bonds = 5

4) C_4H_8



No. of bonds = 11

5) C_2H_6

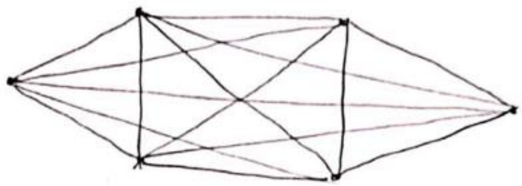


No. of bonds = 7.

5. Compute the number of edges in each complete graph.

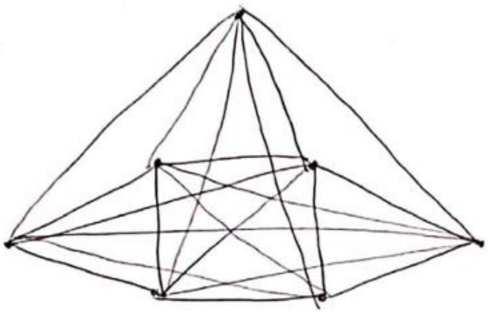
i) K_6 ii) K_7 iii) K_{10} iv) K_{11}.

Sol: 1) K_6



No. of edges $= 15$

2) K_7.



$\therefore$ No. of edges $= 21$

3) K_{10}. This process is critical for large numbers. So we have a formula to find out the no. of edges for a complete graph

ie $\frac{n(n-1)}{2}$.

$$1)\ K_6 = \frac{6(6-1)}{2} = \frac{6(5)}{2} = 15$$

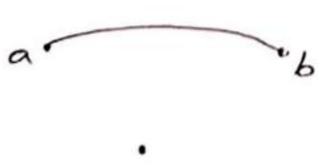
$$K_7 = \frac{7(7-1)}{2} = \frac{7(6)}{2} = 21$$

$$K_{10} = \frac{10(10-1)}{2} = \frac{10(9)}{2} = 45$$

$$K_{11} = \frac{11(11-1)}{2} = \frac{11(10)}{2} = 55.$$

6. How are the adjacency matrices of G and G' are related?

Sol: For Example let us take a graph G



Now its complement is G'

Adjacency matrix of G is

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{array}$$

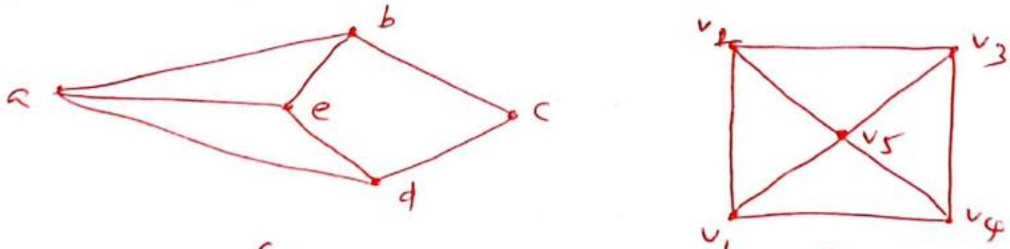
Adjacency matrix of G' is

$$A' = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\end{array}$$

Now from A and A' we observe that only diagonal matrices are equal and the others are complements to each other.

So Adjacency matrices of G and G' are complementlly related to each other.

7. The following graphs are isomorphic or not



G_1 $\qquad\qquad$ G_2

Sol: Here $V(G_1) = \{a, b, c, d, e\} = 5$

$V(G_2) = \{v_1, v_2, v_3, v_4, v_5\} = 5$

$E(G_1) = \{(a,b), (a,e), (a,d), (e,b), (e,d), (b,c), (d,c)\} = 7$

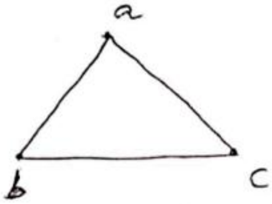
$E(G_2) = \{(v_1,v_2), (v_2,v_3), (v_3,v_4), (v_4,v_1), (v_1,v_5), (v_5,v_3)$
$(v_4,v_5), (v_5,v_2)\} = 8$

$\therefore E(G_1) \neq E(G_2)$

Two graphs are not isomorphic.

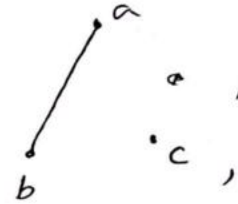
8. Find the no. of sub graphs of the complete graph K_3.

Sol: $K_3 \rightarrow$

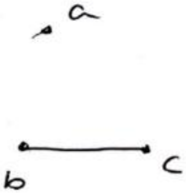


The sub graphs are i)

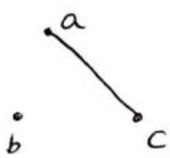
ii)



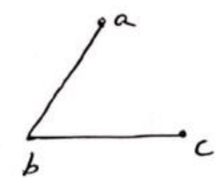
iii)



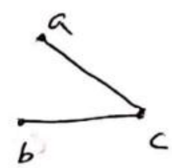
iv)



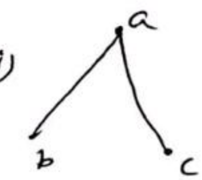
v)



vi)



vii)

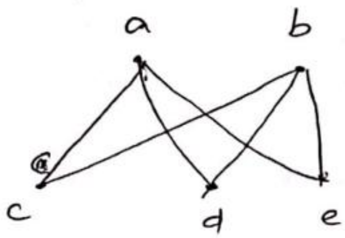


$\therefore$ No. of sub graphs $= 7$.

9. Is $K_{m,n}$ is a complete graph?

Sol: $K_{m,n}$ is not a complete graph it is bipartite

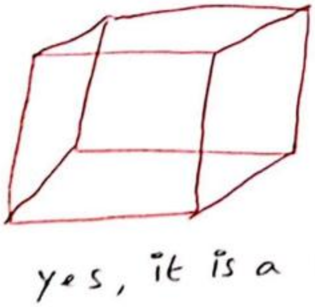
For Example $K_{2,3}$



Here 'a' is not connected to 'b' since in a complete graph every vertex should be connected to each and every other vertex.

10. Are the following graphs are connected or not?

i)



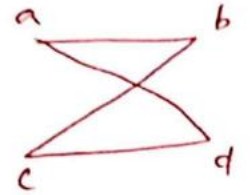
ii) $K_{3,4}$

Sol: i) yes, it is a connected graph

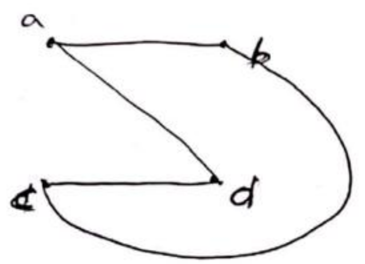
ii) $K_{3,4}$

yes it is also a connected graph

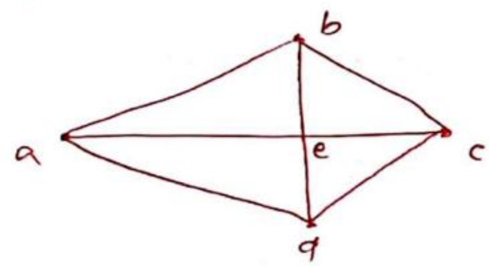
11. Is the graph planar? If so, draw its plane graph



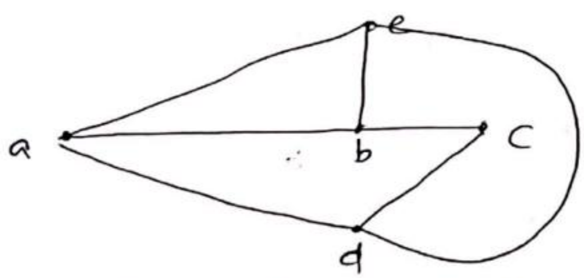
Sol: yes it is a planar graph, since we can remove crossovers. its plane graph is



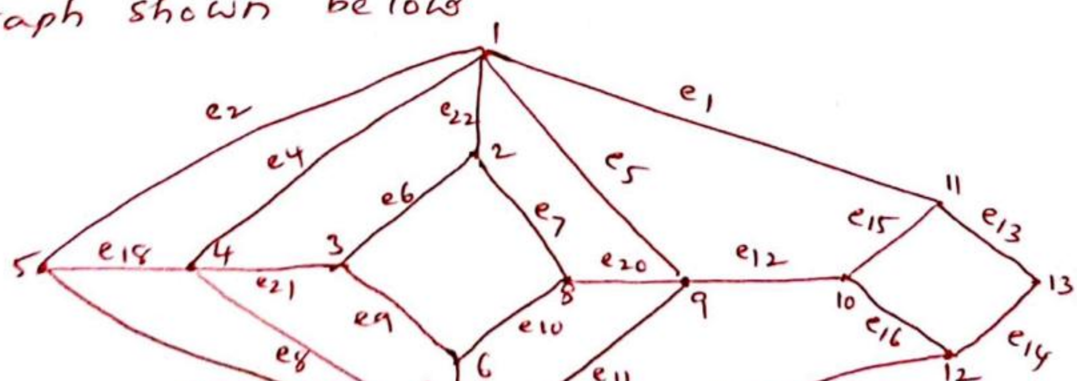
12. Is the graph planar? If so, draw its plane graph



Sol: yes, it is a planar graph since we can remove the crossovers from it, So, its plane graph is



13. verify Euler's formula for the connected planar graph shown below



Sol: By Euler's formula is

$$|V| - |E| + |R| = 2$$

In the above graph we have

Edges $E = 22$

vertices $V = 13$

regions $R = 11$ (including the outer region of graph)

$\therefore\ 13 - 22 + 11 = 2$

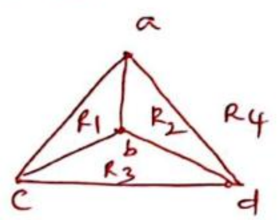
$24 - 22 = 2$

$2 = 2$

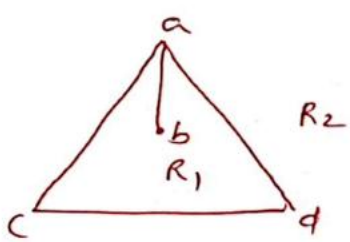
Hence verified.

14. Find the degree of each region formed by the graph shown below.

i)



ii)



Sol: i) Degree of region represents to closed path incident edges.

$Deg(R_1) = 3$, $Deg(R_2) = 3$, $Deg(R_3) = 4$

$Deg(R_4) = 3$

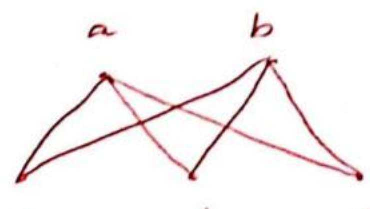
ii) $Deg(R_1) = 5$, since the closed path is

a - c - d - a - b - a

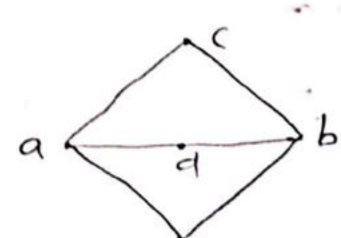
$Deg(R_2) = 3$

15. Draw a planar representation of each graph

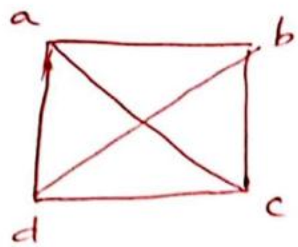
i)



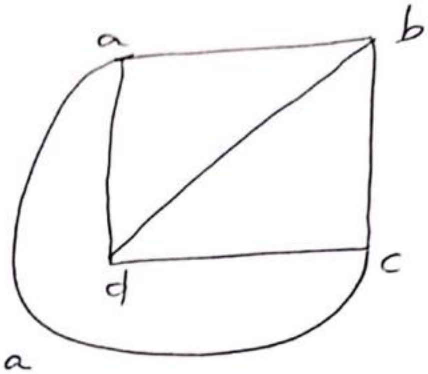
Ans:



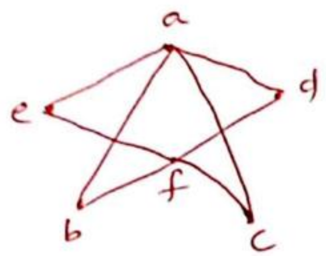
ii)



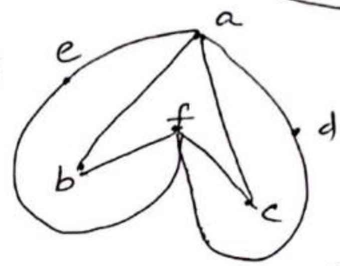
Ans:



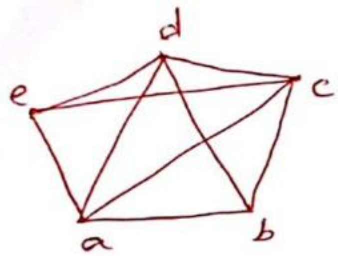
iii)



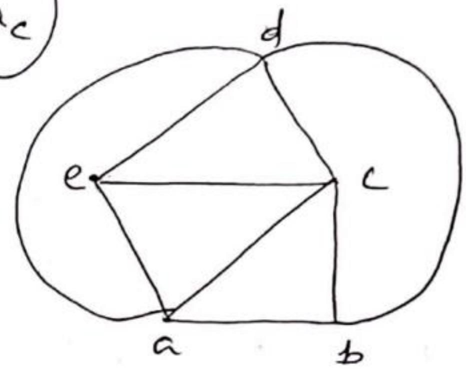
Ans:



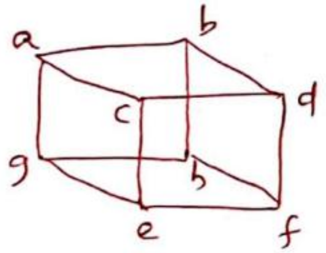
iv)



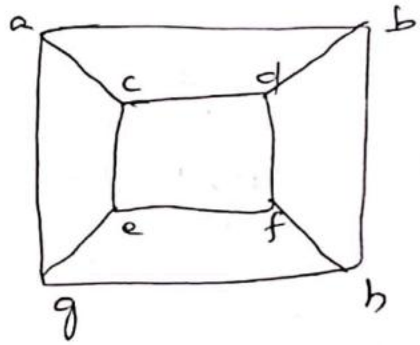
Ans:



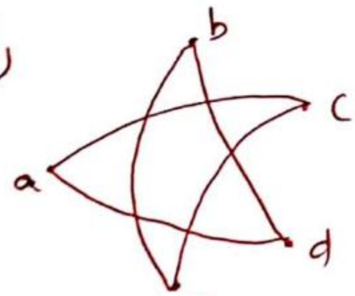
v)



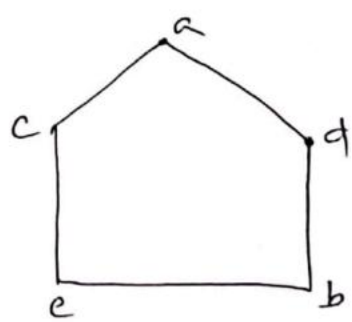
Ans:



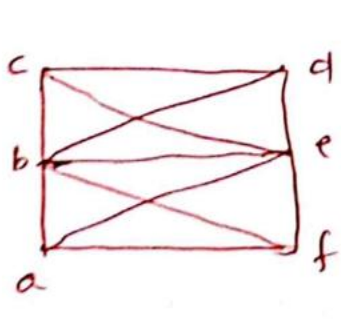
vi)



Ans:



vii)



Ans: