ADVANCED STRUCTURAL ANALYSIS
(23HPE012a)

LECTURER NOTES

III-B.TECH & II-SEM

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CIVIL ENGINEERING					
III B.Tech – II Semester					
Course Code	**ADVANCED STRUCTURAL ANALYSIS**	**L**	**T**	**P**	**C**
23HPE012a	**Professional Elective-II**	3	0	0	3

Course Objectives:

The objectives of this course are to make the student to:

1. **Understand** the fundamental concepts of arches, including three-hinged and two-hinged arches, and analyze the effects of horizontal thrust, bending moment, normal thrust, and radial shear.
2. **Apply** the moment distribution method to analyze single-bay, single-story portal frames with and without side sway.
3. **Analyze** continuous beams and portal frames using Kani's Method, including cases with and without settlement of supports.
4. **Solve** structural problems using the flexibility method for continuous beams and single-bay portal frames, considering support settlements and side sway effects.
5. **Evaluate** the stiffness method for analyzing continuous beams and single-bay portal frames with and without side sway, ensuring structural stability and performance.

Course Outcomes (COs):

After successful completion of this course, students will be able to:

1. **Explain** the behavior of three-hinged and two-hinged arches and **analyze** the effects of horizontal thrust, bending moment, normal thrust, and radial shear. (Bloom's Level: Understand - L2, Analyze - L4)
2. **Apply** the moment distribution method to **analyze** single-bay, single-story portal frames with and without side sway. (Bloom's Level: Apply - L3, Analyze - L4)
3. **Analyze** continuous beams and portal frames using Kani's Method, including cases with and without settlement of supports.
4. **Solve** structural problems using the flexibility method for continuous beams and single-bay portal frames, considering support settlements and side sway effects.
5. **Evaluate** the stiffness method for analyzing continuous beams and single-bay portal frames with and without side sway, ensuring structural stability and performance.

CO – PO Articulation Matrix

Course Outcomes	**PO 1**	**PO 2**	**PO 3**	**PO 4**	**PO 5**	**PO 6**	**PO 7**	**PO 8**	**PO 9**	**PO 10**	**PO 11**	**PO 12**	**PSO1**	**PSO2**
CO -1	3	2	2	2	1	-	-	-	-	2	-	1	2	2
CO -2	3	3	2	2	2	-	-	-	-	2	-	2	2	2
CO -3	3	3	3	2	2	-	-	-	-	2	-	2	3	2
CO -4	3	2	3	3	3	-	-	-	-	2	1	2	3	3
CO -5	3	3	3	3	3	-	-	-	-	2	1	2	3	3

UNIT – I

ARCHES: Three Hinged and Two Hinged Arches, Elastic Theory of Arches– Eddy_S Theorem –Determination of Horizontal Thrust, Bending Moment, Normal Thrust and Radial Shear–Effect of Temperature-Determination of Horizontal Thrust Bending Moment, Normal Thrust and Radial Shear–Rib Shortening and Temperature Stresses.

UNIT –II

MOMENT DISTRIBUTION METHOD for FRAMES:

Analysis of Single Bay Single Storey Portal Frame Including Sides Way–Substitute Frame Analysis By Two Cycle Method

UNIT – III

KANI`S METHOD: -

Analysis of Continuous Beams with and Without Settlement of Supports-Single Bay Single Storey Portal Frames with and Without Side Sway.

UNIT – IV

FLEXIBILITY METHODS: -

Flexibility Methods- Introduction-Application to Continuous Beams Including Support Settlements—Analysis of Single Bay Single Storey Portal Frames Without and With Side Sway.

UNIT – V

Stiffness Methods:

Stiffness Methods – Introduction – Application to Continuous Beams Including Support Settlements – Analysis of Single Bay Single Storey Portal Frames Without and With Side Sway.

TEXT BOOKS:

1. Analysis of structures by Vazrani&Ratwani– Khanna Publications.
2. Theory of structures by Ramamuratam, jain book depot , New Delhi.

REFRENCE BOOKS:

1. Structural analysis by R.S.Khurmi, S.Chand Publications, NewDelhi.
2. Basic Structural Analysis by K.U.Muthuetal .,I.K.International Publishing House Pvt. Ltd
3. Theoryof Structures by Gupta SP, GSPundit and R Gupta,Vol II, Tata McGraw Hill Publications Company td.
4. D. S. PrakashRao,―Structural Analysis: A Unified Approach‖, UniversitiesPress

Online Learning Resources:

https://archive.nptel.ac.in/courses/105/106/105106050/

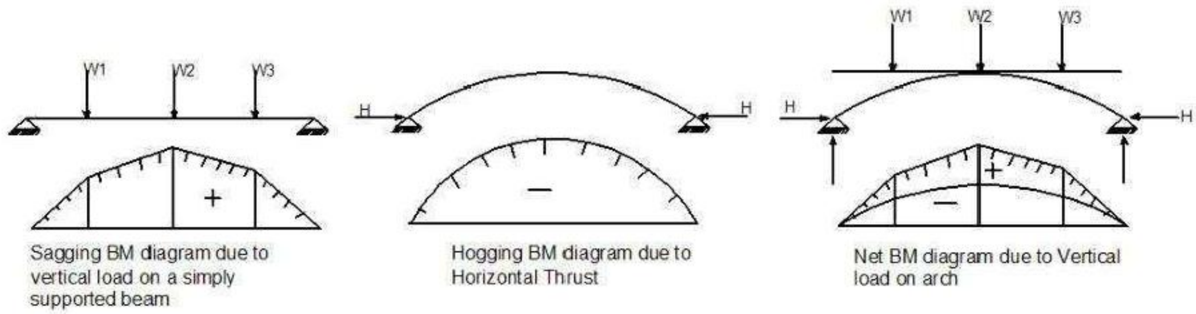
UNIT-I

Arches

An arch may be visualized as a curved beam in elevation with convexity upward

which is restrained at it ends from spreading outwards under the action of download vertical loads.

The inward horizontal reactions induced by the end restraints produce hogging moments in the arch whichwill counter act the static sagging moments set up by the vertical loads.



The consequent reduction in the net moments which is responsible for a significantly higher load carrying capacity of an arch as compared to the corresponding beam , presents the arch as an effective option in construction of bridges, buildings etc.,

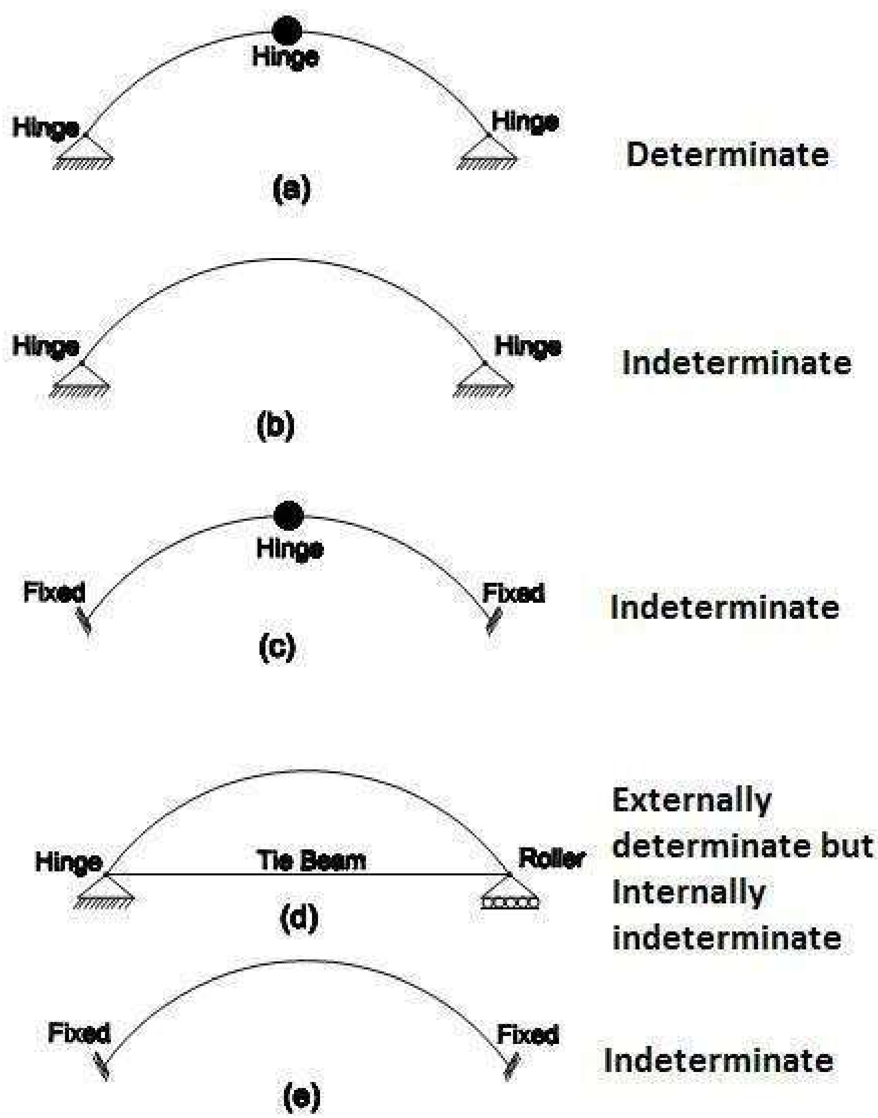
Since the transverse loading at any section normal to the axis of the girder is at an angle to the normal face, an arch is subjected to three restraining forces

1. Normal thrust
2. Radial shear and
3. Bending moment

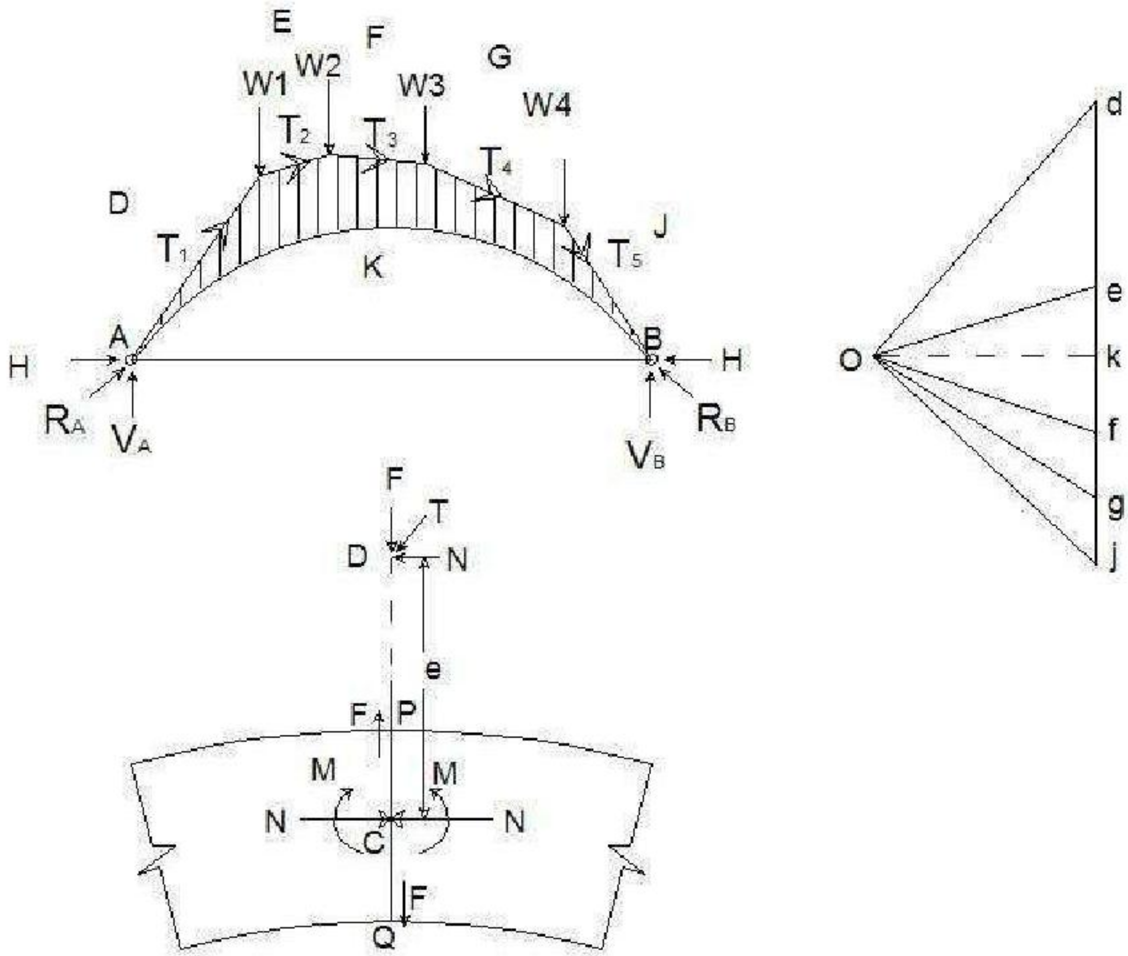
Thus the loads get transferred partly by axial compression and partly by flexure. In axial compression, at each cross section, the structure is subjected equal stresses. Reduction in bending moment results smaller and economical sections for an arch compared to a beam to transfer the same load.

Depending upon the number of hinges, arches may be divided into

a) Three hinged arch
b) Two hinged arch
c) Single hinge arch
d) Tied arch
e) Fixed arch



Linear (Theoretical) Arch and Line of Thrust:



Consider a system of jointed linkwork inverted about the support pointsA and B, with loads as shown in Fig. Under a given system of loading, each link will be in a state of compression. The magnitudes of thrusts T_1, T_2, T_3etc., can be known by the raysOd, Oe, etc., in. the force polygon drawn .Therefore a funicular polygon , which is encased in the corresponding arch with AB supports , is obtained with link work inverted.

This inverted linkwork or funicular polygon is known as "theoretical arch" or "linear arch" and also called as "line of thrust"

Actual arch: it is however not possible to construct the actual arch of shape of theoretical arch.The moving loads will change the shape of theoretical archand it cannot be made to change its shape to suit the varying load positions.Therefore in actual practice an arch is made

1) Parabolic
2) Circular and
3) Elliptical in shape

Considera cross-section PQ of the arch as shown in above. Let T be the resultant thrust acting through D along the linear arch. The thrust T is neither normal to the

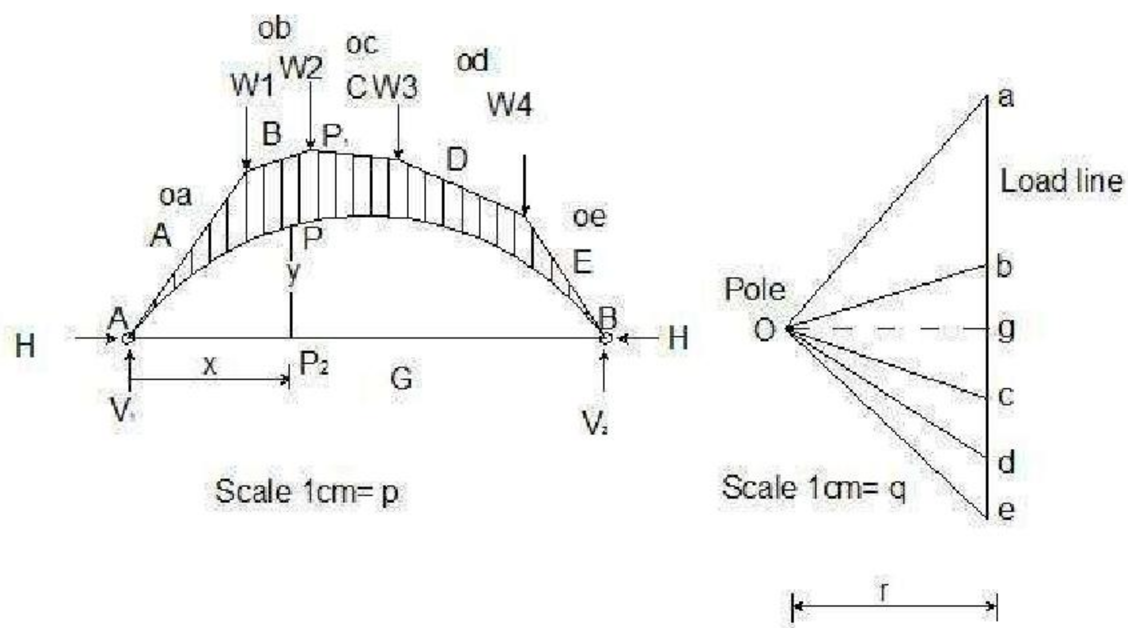
Therefore the resultant thrust T can be resolved into normal component (N)and tangentialcomponent (F)

The tangential component F willcause shear force at the section PQ.The Normal force N acts eccentrically (CD) to centre of cross section.
Thus the action of N at D can be transformed into two forces as

(i) A normal thrust N at C.
(ii) A bending moment M=N x e at C.

Where e-eccentricity CD
Hence the arch is subjected to straining actions
i) shear force also called as radial shear(F)
ii) Bendingmoment(M)
and iii) normal thrust(N)

Eddy's theorem:



Eddy's theorem states that "The bending moment at any section of an arch is proportional to the vertical intercept between the linear arch (or theoretical arch) and the centre line of the actual arch."

Consider any point on arch at a distance x from left support and y from horizontal Let P_1, P_2 section passing through P.

H- Horizontal reaction

Funicular polygon represents the B.M diagram due to the external loads. P_1, P_2 is the vertical interception of B.M diagram at X

Let arch is drawn to a scale 1cm=pmetre, and load diagram is plotted to a scale 1Cm =q kilometre.

If the distance of pole from load line is r cm.

Scale of B.M diagram= pqr KN-m

But theoretically, B.M at

P $M=V_1x-W_1(x-a)-Hy$

$=\mu_x-Hy$

Where μx usual B.M at section due to loading on s.sbeam .from fig, $\mu x = P_1P_2 = P_1P_2$ X scale of B.M diagram.$=P_1P_2(pqr)$

$Hy=PP_2=PP_2$ X scale of B.M diagram.$=PP_2(pqr)$

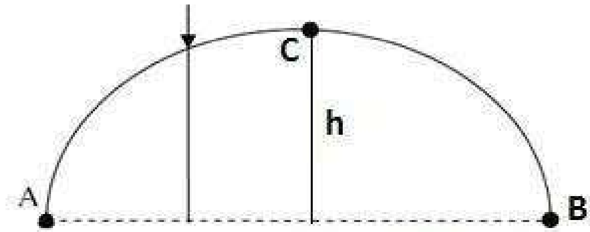
Therefore, $M_P=\mu_x-Hy=P_1P_2(pqr)-PP_2(pqr)=PP_1(pqr)$

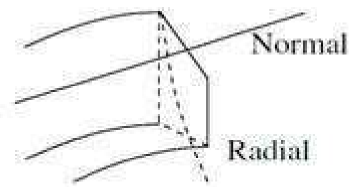
Hence the ordinate (PP_1) between linear arch and actual arch is proportional to the B.M This proves Eddy's theorem.

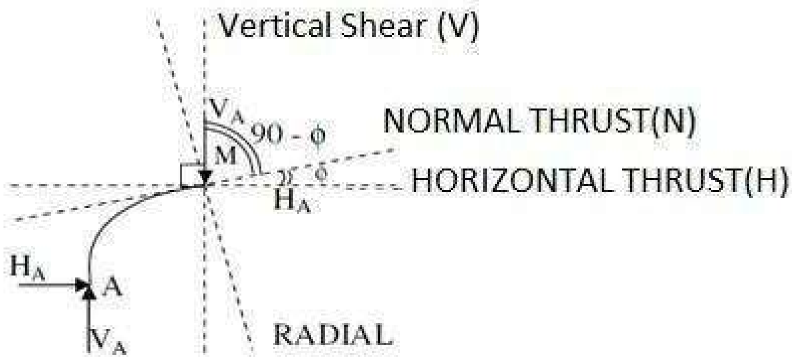
Three hinged arch:

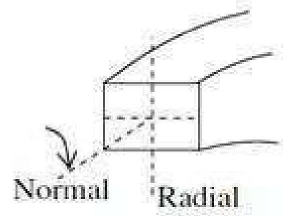
A three hinged arch is a statically determinate structure. It consists of two hinges at each abutment support called as springing and third hinge at crown.It has four reaction components.V_A&H at left hinge and V_B&H at right hinge. H being same with three available equations from static equilibrium and one additional equation i.e B.M at hinge crown is zero.Thus the value of H can be obtained easily.

Normal Thrust and Radial Shear:









Let reactions at A are H and V_A

Let reactions at B are H and V_B

Since BM at crown is zero, $M_C = V_A \times \frac{L}{2} - Hh = 0$

Let $\mu_c = V_A \times \frac{L}{2} \vee V_B \times \frac{L}{2}$

$$H = \frac{\mu_c}{h}$$

Similarly V_A and V_B can be obtained.

After obtaining V_A, V_B and H, radial shear (F) and Normal Thrust (N) can be obtained as follows

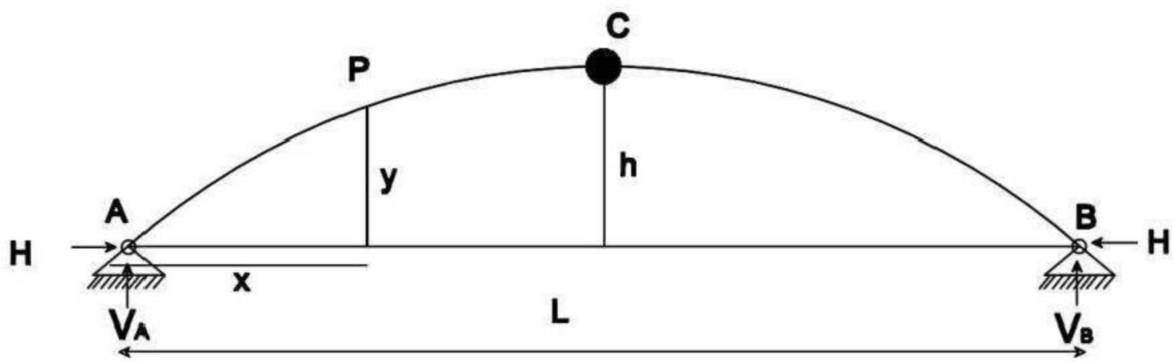
Resolving along the section,

Radial Shear (F) = HSin ϕ – VCos ϕ

Resolving normal to the section

Normal Thrust (N) = H Cos ϕ + VSin ϕ

Three hinged parabolic arch:



Equation of parabolic with respect to its span and rise can be obtained as follows.

Let y=kx(L-x) ---(1)

Where k- constant

At x=L/2, y=h central rise

Substituting in equation (1)

$$h=\frac{KL}{2}\left(L-\frac{L}{2}\right)$$

$$h=\frac{kl^2}{4}$$

Therefore,

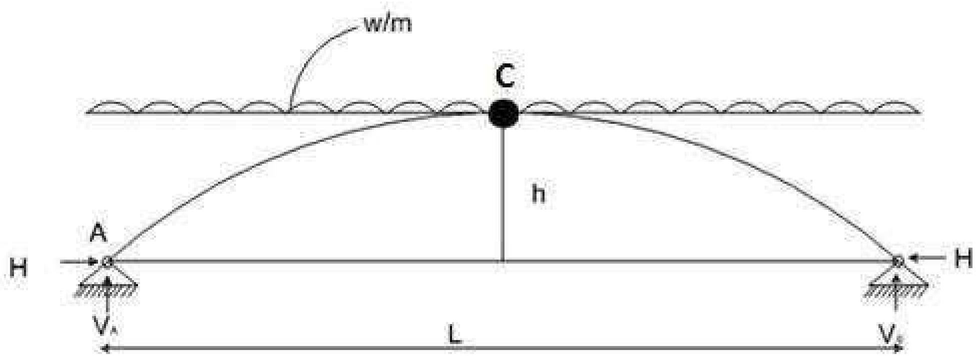
$$k=\frac{4h}{l^2}$$

Then the equation of parabola becomes

$$y=\frac{4hx(L-x)}{l^2}$$

According to Eddy's theorem, the vertical interception between the linear arch and the center line of the actual arch gives the bending moment at a section. When the arch is subjected to udl throughout, the Bending moment diagram would be an arch, since arch is also a parabolic arch. Hence parabolic arch will not have bending moment due to udl. It will be subjected to pure compression only.

1. A three hinged Parabolic arch of span L and rise h carries u.d.l w/m .runthroughout. Show that the horizontal thrust at each support is $\frac{WL^2}{8H}$ and also the B.M at any point of arch is zero.



$$\text{Total load} = WL$$

$$V_A = V_B = \frac{WL}{2}$$

Taking moment of all the forces about hinge A,

$$V_B L - \frac{WL^2}{2} = 0$$

Taking moment of forces left about C ,

$$H_A \times h = \frac{WL}{2} \times \frac{L}{2}$$

$$H_A = \frac{WL^2}{8H}$$

$$H_A = H_B = \frac{wL^2}{8h}$$

B.M at any section x-x

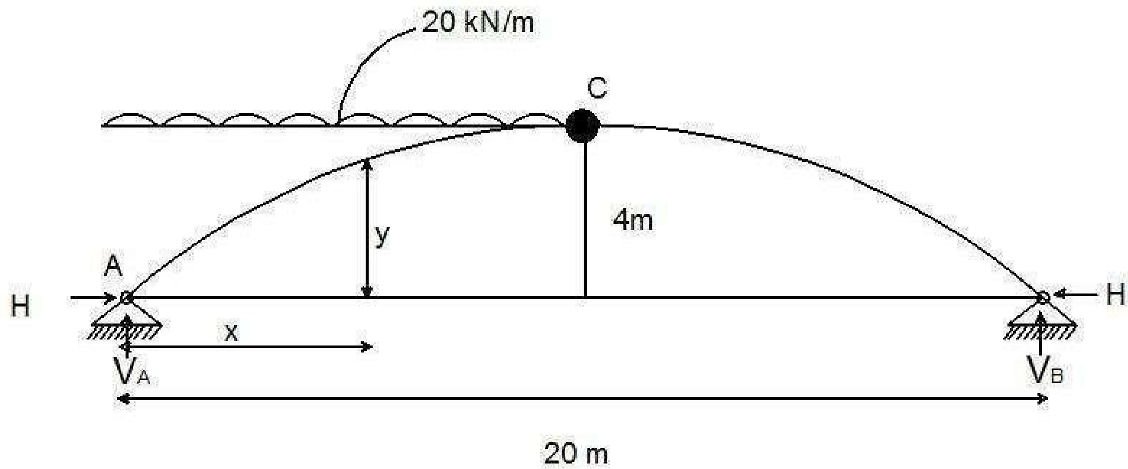
$$M_x = V_A x - \frac{wx^2}{2} - Hy$$

$$M_x = \frac{wL}{2}x - \frac{wx^2}{2} - \frac{wL^2}{8h} \times \frac{4hx(L-x)}{L^2}$$

$$M_x = \frac{wL}{2}x - \frac{wx^2}{2} - \frac{wLx}{2} + \frac{wx^2}{2}$$

$= 0$

2. A three hinged parabolic arch of span 20m and rise 4m carries audl of 20kN/m run on the left of the span. Find the maximum B.M for the arch.



Taking moments about A

$20V_B=(20X10)5$

$V_B=50$ kN

$V_A=150$ kN,

Taking moments about C towards right $Hx4=50x10$

$H=125kN$

At any distance of x from A,

$$y=\frac{4hx(L-x)}{L^2}=\frac{4\times 4(L-x)}{400}$$

$$y=\frac{x(20-x)}{25}$$

To obtain max B.M, $\frac{dM_x}{dx}=0$

$$M_x=V_A x-\frac{20x^2}{2}-Hy$$

$$M_x=150x-10x^2-125\times\frac{x(20-x)}{25}$$

$$M_x=50x-5x^2$$

$$\frac{dM_x}{dx}=50-10x=0$$

Hence x=5m

$$M_{max} = 50\times 5 - 5\times 5^2 = +125 \text{ kN-m}$$

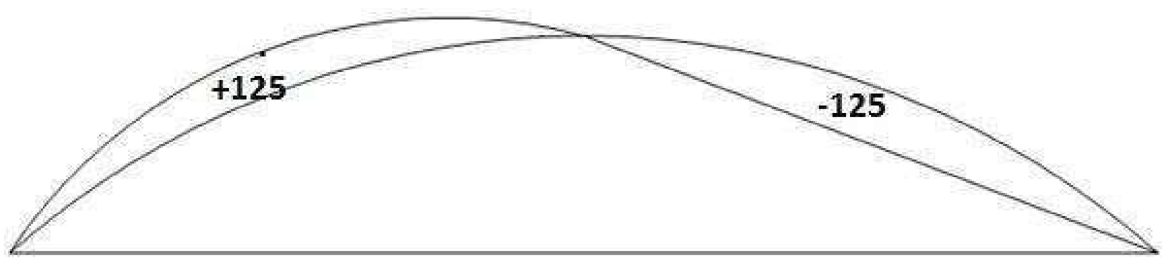
Similarly at a distance x from B,

$M_x = 5x^2 - 50x$

$$\frac{dM_x}{dx} = 10x - 50 = 0$$

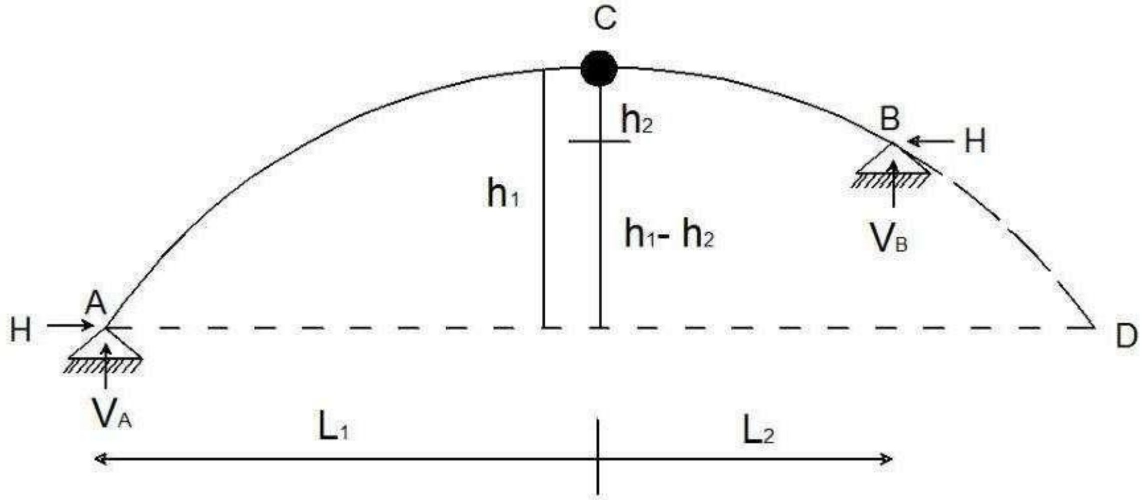
Hence x=5m

$M_x = -125 \ kN-m$



Bending Moment Diagram

Parabolic arch at different levels of height:



$$L_1 = \frac{L\sqrt{\sqrt{h1}}}{\sqrt{h1}+\sqrt{h2}} \;; \qquad L_2 = \frac{L\sqrt{\sqrt{h2}}}{\sqrt{h1}+\sqrt{h2}}$$

Let ACB is extended upto D

where L is the horizontal length between AD=

$L=2L_1$ h_1- height of C above A,

h_2- height of c

aboveB.

d=difference of

levels $d=h_1-h_2$.

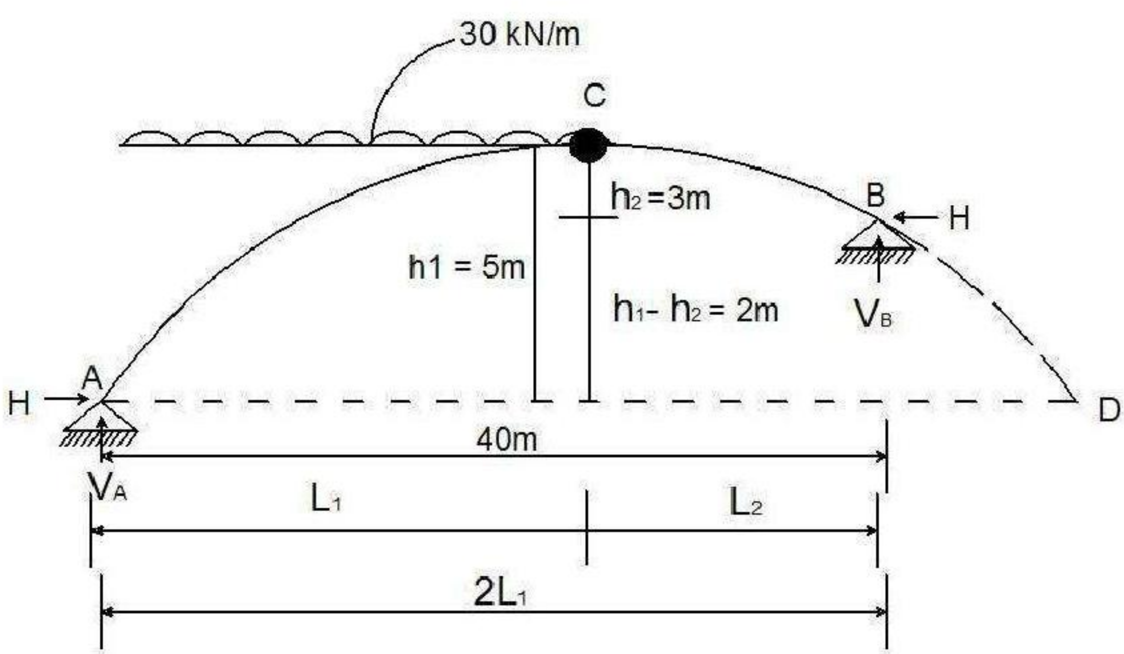
L_1= horizontal lengths betweenAC, L_2= horizontal lengths betweenCB.

L=span of arch AB.

Ordinate at any point above AD line

$$y= \frac{4\,h_2\,x\,(L-x)}{L^2}$$

3. A three hinged parabolic arch having suppourts at different levels as shown in the fig.carries a udl of 30kN/m over the span left of the crown. Determine the horizontal thurst developed.also find the B.M, normal thrust and radial shear at a section 15m from left suppourts.



h_2=3m,h_1=5
m L_1
+L_2=40m

$L_1 = \frac{L\sqrt{\sqrt{h1}}}{\sqrt{h1}+\sqrt{h2}}$; $L_2 = \frac{L\sqrt{\sqrt{h2}}}{\sqrt{h1}+\sqrt{h2}}$

By substituting the values of L, h_1& $h_2$in the above formulas we

get, L_1= 22.55m, L_2=17.45m

$$y=\frac{4\,h_2\,x\,(L-x)}{L^2}=\frac{20(Lx-x^2)}{L^2}$$

When x=15m , L=40m, h=3m

y=2.94m

Taking moments about C towards

right, V_BX17.45=H x 3

Therefore H=5.82V_B-----(1)

Taking moments about A,

M_A=V_Bx40+Hx2-

((30x2.55^2)/2)=0 V_A= 528 kN,

V_B = 147.7 kN,

Substitute the $V_B$in equation (1) we

get H=859.6KN

when x=15m,y=4.44m.

M_{15}=V_Ax15-Hx4.44-((30x15^2)/2)

=740.4kN-m

$$\frac{dy}{dx}=\tan\theta=\frac{20}{45.1\,X\,45.1}(45.1-2\,x)$$

When

X=15m θ

=8.44^0

V_{15} = 528.8-(30X15)

= 78.8kN.

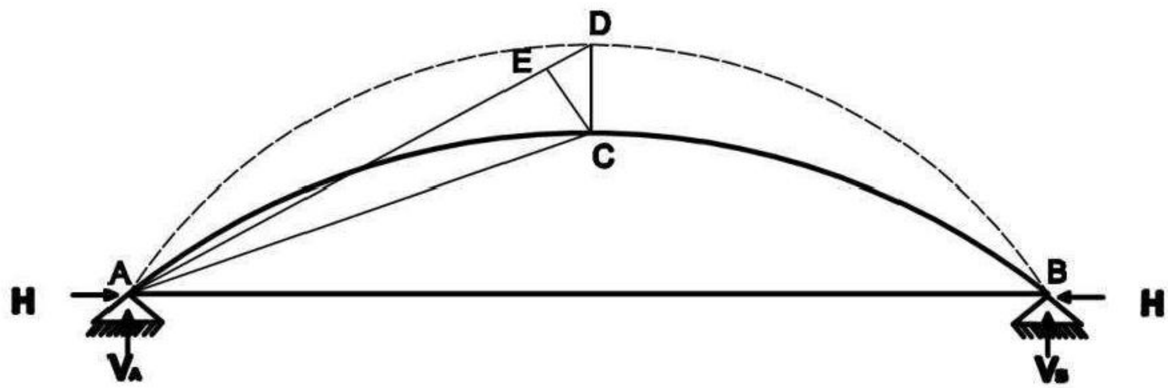
$=8.44^0$,V=78.8kN. N =861.86kN

Radial shear $F=H\sin\theta-V\cos\theta$

substitute H=859.6kN, θ =8.44^0,V=78.8kN.

F=48.31kN.

Effect of change in temperature on three hinged arch:



Consider a three hinged arch ACB of span L and centaral rise h. Let t is the rise of temperature α is the co-efficient of expansion.

Let length of arch increases with rise of temperature. Since end hinges A&B cannot under go any displacement, the crown hinge c of arch will rise upwards from C to D.

Therefore increase in arch = arcAD-arc AC

Length of chord AD= length of chord AC

(1+αt)

=(AC+ACαt)

Increse in length of arch =AD-AC

=(AC+ACαt)-AC

=ACαt --- (1)

Through C draw perpendicular CE on AD

Cosidering AC=AE

ED=ACαt-----(2)

CD=EDsecθ=ACαtsecθ

$<ADC=<ACM$

$CD = AC\alpha t\sec\theta = AC\alpha t \frac{AC}{CM} = AC^2\alpha t/CM$----(3)

$AC^2 = AM^2+CM^2$

$= (l/2)^2+h^2$

$= (l^2+4h^2)/4$--- (4)

Substitute (4)in (3)

$CD=((l^2+4h^2)/4)\frac{\alpha t}{h}$

$CD =(l^2+4h^2)\frac{\alpha t}{4h}$

There fore $dh=(l^2+4h^2)\frac{\alpha t}{4h}$

Effect of temperature rise on horizontal thrust (H):

Let h- ride of arch

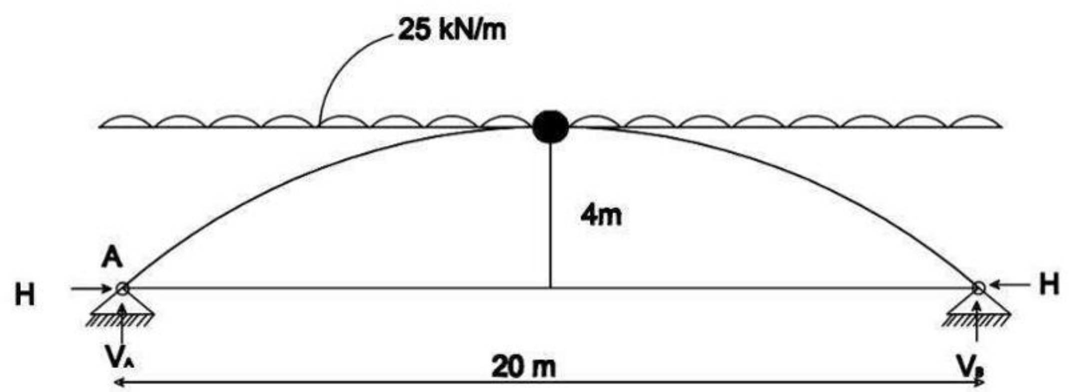
dh- increse in rise of arch or in h

dH-decrease of horizontal thrust due to increase in

temperature H- horizontal thrust

$\frac{-dH}{H} = \frac{dh}{h}$ or $dH = \frac{-dh}{h}(H)$

4. A three hinged arch of span 20m and rise 4m carries udl of 25kN/m.find the horizontal thrust for the arch. If now the arch is subjected to a rise of temperature of 40^0C,find what change in horizontal thrust will occur. Take $\alpha=12\times10^{-6}$per^{0}C.



Before rise in temperature

Taking moments about crown

c,

$$\frac{WL}{2}10=Hx4+\frac{w}{2}10^2$$

Therefore H=312.5kN.

Increase in arch, $dh=(l^2+4h^2)\frac{\alpha t}{4h}$

$$=(20^2+4x4^2)\frac{12\,x\,10-6}{4\,X\,4}X\,40$$

dh=0.01392m

decrease in $H=dh=\frac{-dh}{h}(H)$

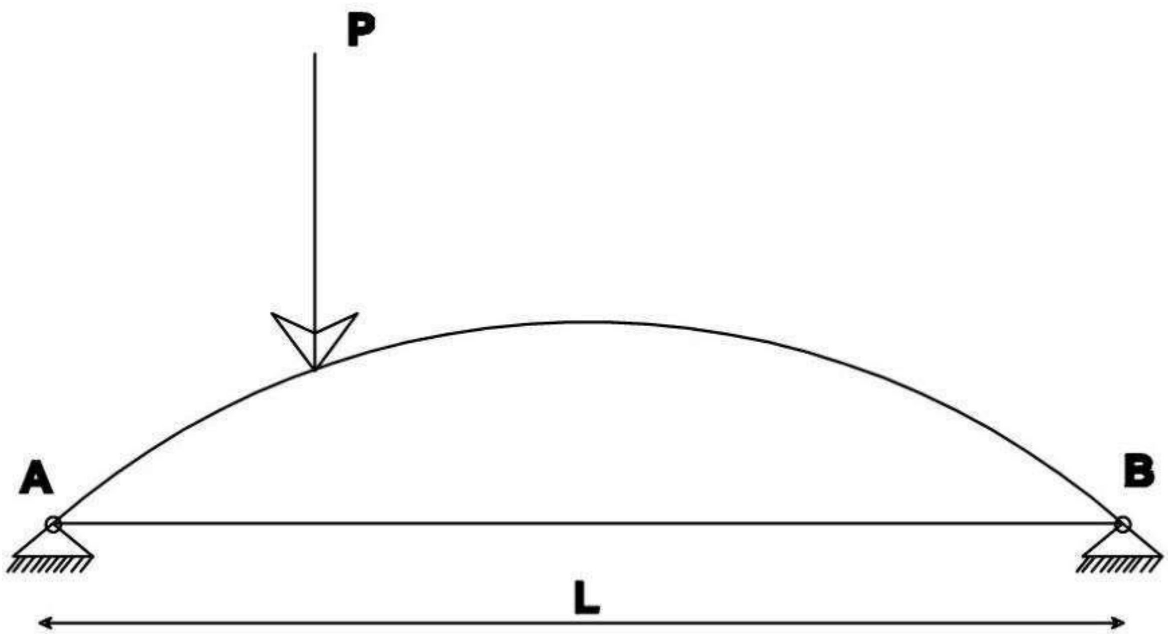
$$=0\frac{-0.01392}{4}X\,312.5$$

There fore dH=-1.0875 kN

Two Hinged Arches

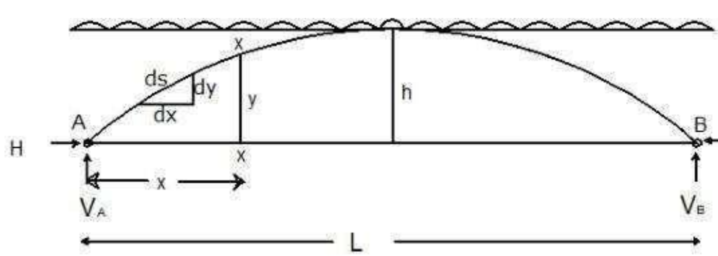
Two-hinged arch is the statically indeterminate structure to degree one. A typical two-hinged arch has four unknown reactions, but there are only three equations of equilibrium available. Hence, the degree of static indeterminacy is one for two hinged arch.

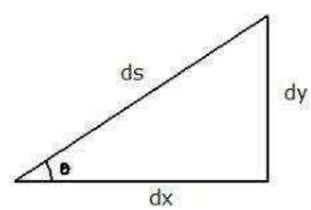
Thought three hinged arch have a hinge at the crown, it cannot move efficient than two hinged arch, but it makes the calculation simple. Two hinged arches are more practicable Two hinged arches are parabolic or circular. In two hinged arches horizontal thrust may be taken as redundant force.



A typical two hinged arch is shown below

Determination of 'H' using First Theorem of "Castigliano" :





Let H – Redundant Force at

B But B is actually hinged

Bending Moment at X-X

$M_x = \mu_x - Hy$

Where μ_x - bending Moment for Simply Supported

beam We know that Strain Energy stored in arch,

$$U = \int_A^B \frac{M_x^2}{2EI}$$

Since B is hinged , there is no horizontal displacement at B

$$\frac{\partial U}{\partial H} = 0$$

But

$$U = \int_A^B \frac{(\mu \text{¿¿} x - Hy)^2}{2EI} ds \text{¿}$$

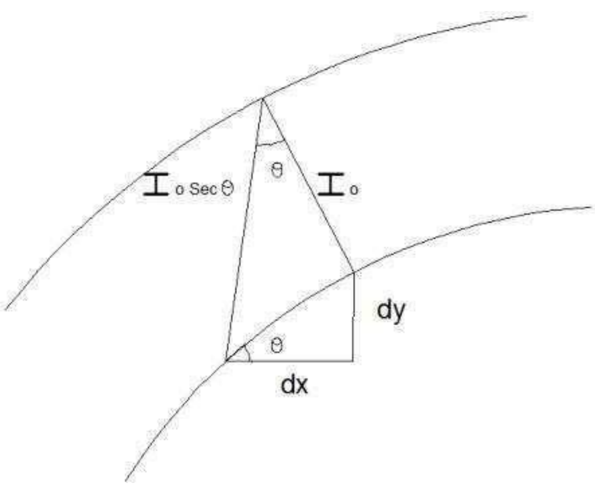
$$\frac{\partial U}{\partial H} = \int_A^B \frac{2(\mu \text{¿¿} x - Hy) \times (-y)}{2EI} ds = 0 \text{¿}$$

$$\frac{\partial U}{\partial H} = -\int_A^B \frac{\mu_x y}{EI} ds + \int_A^B \frac{H y^2}{EI} = 0$$

$$H = \frac{\int_A^B \frac{\mu_x y}{EI} ds}{\int_A^B \frac{y^2}{EI} ds}$$

But ds = $\sqrt{(dx^2 \text{¿} + dy^2) \text{¿}}$

For Simplicity, I varies as I= $I_0 sec\theta$



Where $I_0 = Moment\ of\ Inertiaat\ crown$

Also $ds = dx\ sec\theta$

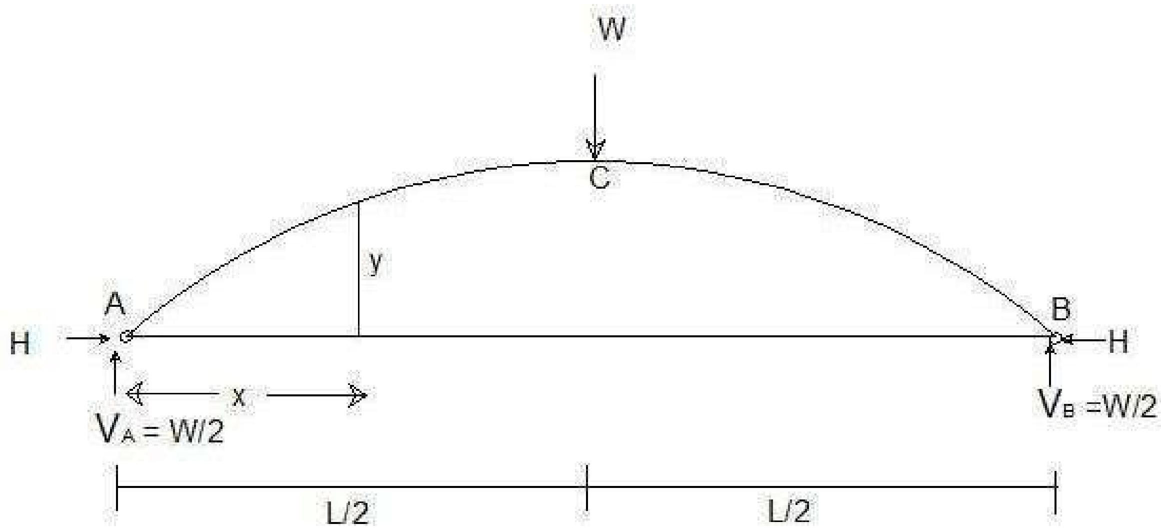
Substituting in equation ‘H’

$$H=\frac{\int_0^l \frac{(\mu \text{¿¿}\, x\; y)\, dx\sec\theta}{E\,I_0\; \sec\theta}}{\int_0^l \frac{y^2\, dx\sec\theta}{E\,I_0 \sec\theta}}\text{¿}$$

$$H=\frac{\int_0^l \mu_x\, y\, dx}{\int_0^l y^2 dx}$$

Analysis of Two hinged Parabolic Arches:

Ex: A two hinged parabolic arch of span L and rise 'h' carries a concentrated load W at the crown. Determine the expression for horizontal thrust H developed at Springing.



$$V_A = V_B = \frac{W}{2} \quad \text{(due to symmetry)}$$

Take a Section X-X

$$\mu_x = \frac{W}{2}x$$

$$y = \frac{4hx(L-x)}{l^2}$$

$$\int_0^l \mu_x y\,dx = \int \frac{W}{2}x \times \frac{4hx(L-x)}{l^2}$$

$$\int_0^l \mu_x y\,dx = \frac{2Wh}{l^2} \times 2\int_0^{l/2} ¿¿)\ \text{dx}$$

$$= \frac{4wh}{l^2} ¿]\ \text{over } 0 - L/2$$

$$= \frac{5Whl^2}{48}$$

Denominato
r

$$\int_0^l y^2\,dx = \int_0^l \left[\frac{4hx(L-x)}{l^2}\right]^2 dx$$

$$¿ \frac{16h^2}{l^4} \times 2\int_0^{l/2} [l^2 x^2 ¿ -2lx^3 + x^4]\,dx\ ¿$$

$$¿ \frac{32h^2}{l^4}\left[l^2\frac{x^3}{3} - 2l\frac{x^4}{4} + \frac{x^5}{5}\right]_0^{l/2}$$

$$¿\frac{32h^2}{l^4}\left[\frac{l^5}{2}-\frac{l^5}{32}+l^5\right]$$

$$¿\frac{8}{15}h^2 l$$

Therefore

$$\text{Horizontal Thrust (H)} = \frac{\frac{5\,Whl^2}{48}}{\frac{8}{15}h^2 l} = \frac{25}{128}\frac{Wl}{h}$$

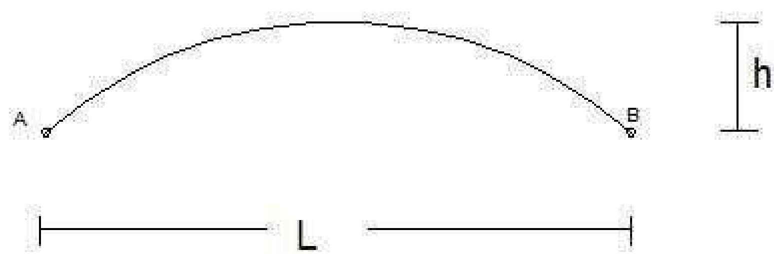
Effect of Temperature on Two hinged Arch:

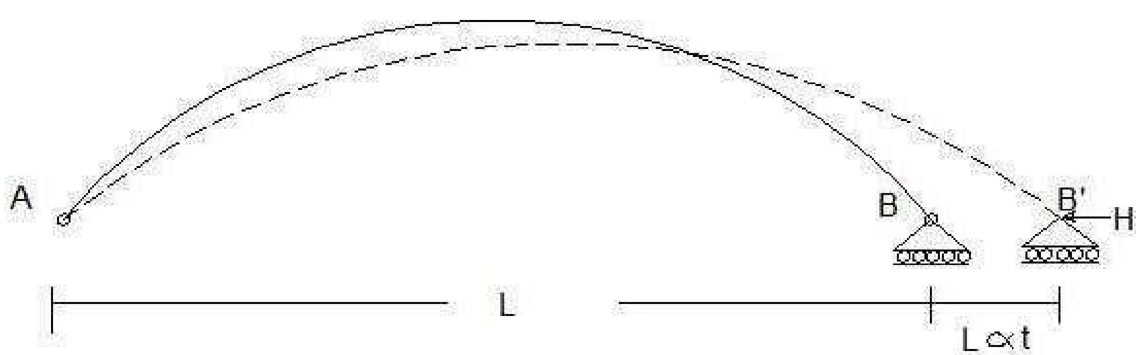
Consider two hinged arch subjected to rise in temperature by ‘t’ degrees. If B hinge is replaced by a roller as shown in figure.

Let α = *Coefficient of thermal expansion*

H = Horizontal Thrust required to bring back B to B which infers that H is the horizontal Thrust developed in the two hinged arch using the theorem of Castigliano.

Change in Displacement = $\frac{dU}{dH} = L\alpha t$





In this case M = -Hy

$$\frac{dM}{dH} = -y$$

$$\frac{dU}{dH} = L\alpha t$$

$$\int \frac{d}{dH}\frac{\left(M^2\right)}{2EI} ds = L\alpha t$$

$$\int \frac{M}{EI}\frac{dM}{dH} ds = L\alpha t$$

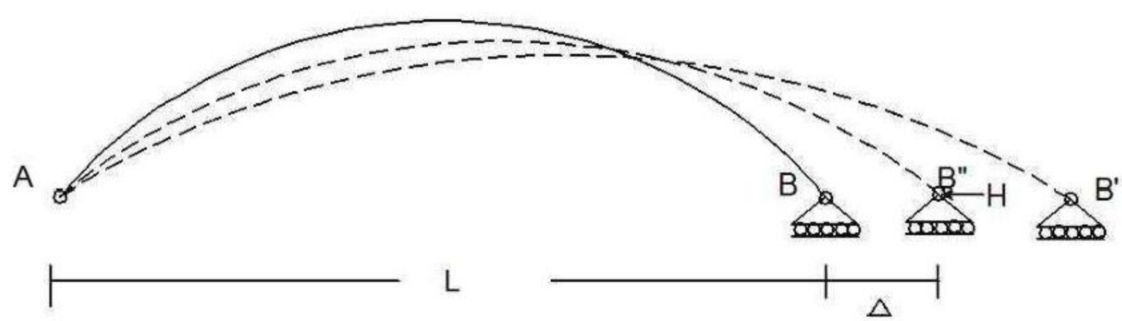
$$\int (-Hy)(-y)\frac{ds}{EI} = L\alpha t$$

$$H = \frac{L\alpha t}{\int y^2 \frac{ds}{EI}}$$

The above expression is Horizontal Thrust due to temperature change only.
If it is confined with loading and settlement of support then total horizontal thrust

$$H = \frac{\int \frac{\mu_x y}{EI} ds + L\alpha t - \Delta}{\int y^2 \frac{ds}{EI}}$$

Effect of yielding of Supports:



Let B hinge is replaced by roller
Let due to loads, the support moves to point B
Let $\mu - Moment\ due\ ¿\ simply\ supported\ case$

$y - Moment\ due\ ¿\ horizontal\ force$

BB = $\int \frac{\mu_x y}{EI} ds$

If H- Horizontal force at springing level
M –Moment due to H = Hy

$¿\int \frac{\mu_x y}{EI} ds$

$$¿\int \frac{(Hy)\,y}{EI} ds$$

$$¿\int \frac{H\,y^2}{EI} ds$$

Since the support is yielding by Δ from B to B

$$\int \frac{H\,y^2}{EI} ds = B\,B - \Delta$$

$$H = \frac{\int \frac{\mu_x\,y}{EI} ds - \Delta}{\int y^2 \frac{ds}{EI}}$$

If the support is elastic and if yields by k due to unit horizontal force at support then $\Delta = kH$

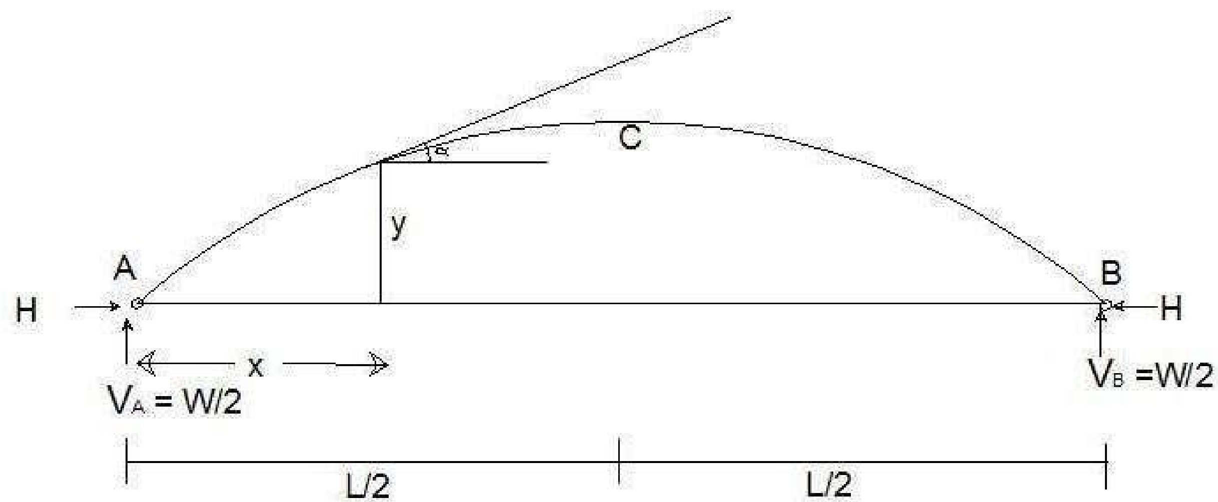
$$H \int \frac{y^2}{EI} ds = B\,B - \Delta$$

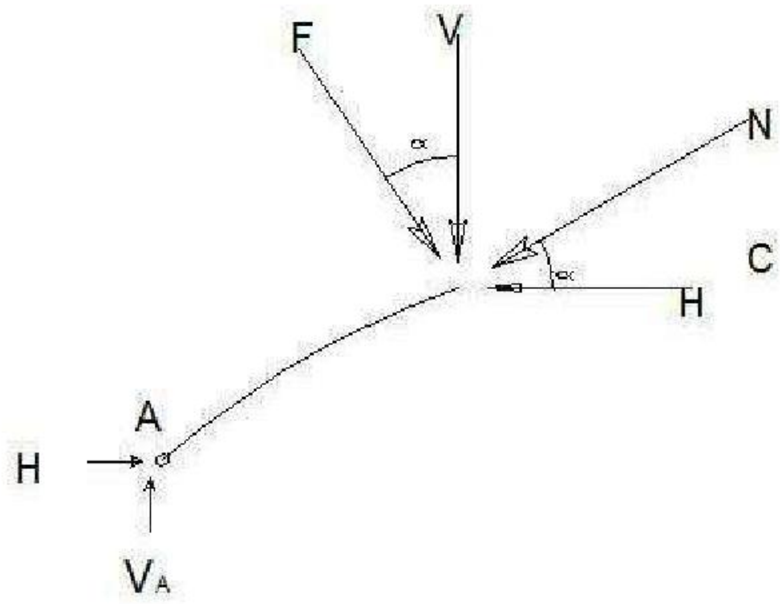
$$H \int \frac{y^2}{EI} ds = \int \frac{\mu_x\,y}{EI} ds - kH$$

$$H\left(\int \frac{y^2}{EI} ds + k\right) = \int \frac{\mu_x\,y}{EI} ds$$

$$\mathrm{H} = \frac{\int \frac{\mu_x\,y}{EI} ds}{\int \frac{y^2}{EI} ds + k}$$

Effect of Shortening of Rib:





The cross section of the arch is subjected to normal thrust also. The arch being made up of elastic material , counters shortening of the rib. This shortening reduces the horizontal thrust developed.

Expression for horizontal thrust due to rib shortening:

From figure N = VSinα + HCos α

M = $\mu - Hy$

Where V = Beam Shear

Starain Energy U = $\int \frac{M^2}{2EI} ds + \int \frac{N^2}{2EA} ds$

Where A –Area of Crossection at section X-X

If the support is unyielding, the horizontal displacement of arch is Zero.

$\frac{dU}{dH} = 0$

$$\int \frac{M}{EI}\frac{dM}{dH} ds + \int \frac{N}{EA}\frac{dN}{dH} ds = 0$$

$$\text{But } \frac{dM}{dH} = \frac{-y \wedge dN}{dH} = cos\alpha$$

$$\int \frac{(\mu - Hy)(-y)}{EI} ds + \int \frac{(VSin\alpha + HCos\alpha)(\cos\alpha)}{EA} ds = 0$$

$$-\int \frac{\mu y}{EI} ds + H\int \frac{y^2}{EI} ds + \int \frac{VSin\alpha\, Cos\alpha}{EA} ds + H\int \frac{\cos^2\alpha}{EA} ds = 0$$

$$H = \frac{\int \frac{\mu y}{EI} ds - \int \frac{VSin\alpha Cos\alpha}{EA} ds}{\int \frac{y^2}{EI} ds + \int \frac{\cos^2\alpha}{EA} ds}$$

Neglecting effect of shear

$$H = \frac{\int \frac{\mu y}{EI} ds}{\int \frac{y^2}{EI} ds + \int \frac{\cos^2\alpha}{EA} ds}$$

Considering the term $\int \frac{\cos^2\alpha}{EA} ds$ at crown and springings,it has some definite value.

Usually Crossection areat at crown is small and at springing is large

Hence $\frac{A}{Cos\alpha}$ = Constant = A_m $ASec\alpha$

$$\int \frac{\cos^2\alpha}{EA} ds \; becomes \int \frac{Cos\alpha}{E\,A_m}$$

$$\int \frac{\cos^2\alpha}{EA} ds = \int_0^L \frac{1}{E\,A_m} dx = \frac{L}{E\,A_m}$$

Substitute in Expression H

$$H = \frac{\int \frac{\mu y}{EI} ds}{\int \frac{y^2}{EI} ds + \frac{L}{E\,A_m}}$$

$$H = \frac{\int \frac{\mu_x y}{EI} ds - \Delta}{\int y^2 \frac{ds}{} - \frac{L}{}}$$

$EI \quad E\, A_m$

Where $\Delta = kH$

UNIT-II

MOMENT DISTRIBUTION METHOD

Distribution and carryover of moments – Stiffness and carry over factors – Analysis of continuous beams – Plane rigid frames with and without sway – Neylor"s simplification.

Hardy Cross (1885-1959)

- ✦ Moment Distribution is an **iterative method** of solving an **indeterminate Structure**.
- ✦ Moment distribution method was first introduced by **Hardy Cross in 1932.**
- ✦ Moment distribution is ***suitable for analysis of all types of indeterminate beams and rigid frames***.
- ✦ It is also called a **'relaxation method'** and it consists of successive ***approximations using a series of cycles***, ***each converging towards final result***.
- ✦ It is comparatively ***easier than slope deflection method***. It involves solving number of simultaneous equations with several unknowns, but in this method does not involve any simultaneous equations.
- ✦ It is **very easily** remembered and **extremely useful for checking computer output** of **highly indeterminate structures**.
- ✦ It is ***widely used*** in the analysis of ***all types of indeterminate beams and rigid frames***.
- ✦ The moment-distribution method was very popular among engineers.
- ✦ It is ***very simple*** and is being used even today for ***preliminary analysis of small structures***.
- ✦ The ***primary concept*** used in this methods are,
  - Fixed End Moments
  - Relative **or** Beam Stiffness **or** Stiffness factor
  - Distribution factor
  - Carry over moment or Carry over factor

Basic Concepts

- ✦ In moment-distribution method, counterclockwise beam end moments are taken as positive.
- ✦ The counterclockwise beam end moments produce clockwise moments on the joint.
- ✦ ***Note the sign convention:***

Anti-clockwise is positive (+)

Clockwise is negative (-)

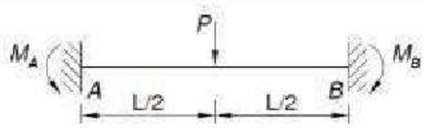
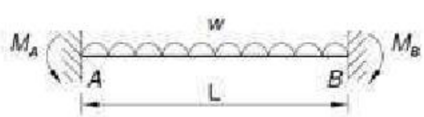
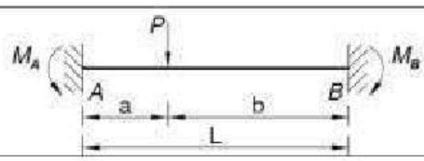
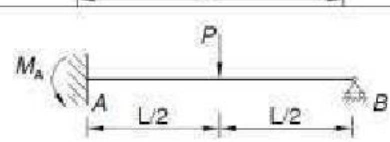
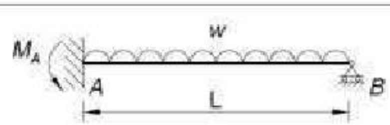
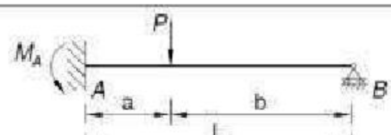
Assumptions in moment distribution method

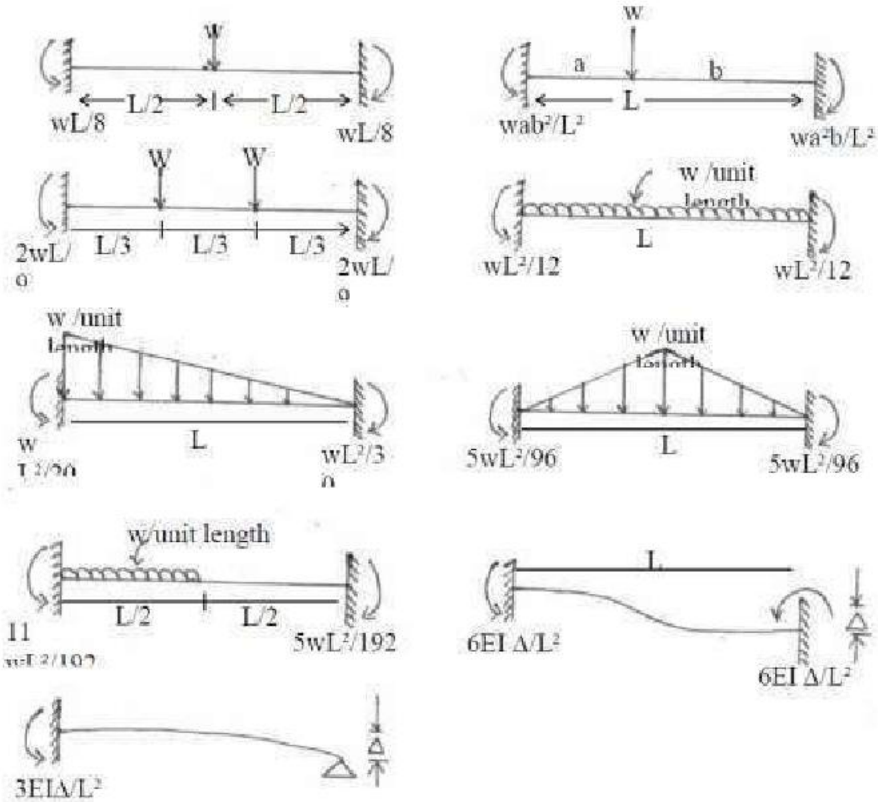
- ***All the members*** of the structures are ***assumed to be fixed*** and fixed end moments due to external loads are obtained.
- All the hinged joints are released by applying an equal and opposite moment.
- The joints are allowed to deflect (rotate) one after the other by releasing them successively.
- The unbalanced moment at the joint is shared by the members connected at the joint when it is released.
- The unbalanced moment at a joint is distributed in to the two spans with their distribution factor.

- Hardy cross method makes use of the ability of various structural members at a joint to sustain moments in proportional to their relative stiffness.

Fixed End Moments

- All members of a given frame are initially assumed fixed at both ends.
- The loads acting on these fixed beams produce fixed end moments at the ends.
- FEM are the moments exerted by the supports on the beam ends.
- These (non-existent) moments keep the rotations at the ends of each member zero.

M_A	Configuration	M_B
$+\frac{PL}{8}$		$-\frac{PL}{8}$
$+\frac{wL^2}{12}$		$-\frac{wL^2}{12}$
$+\frac{Pab^2}{L^2}$		$-\frac{Pa^2b}{L^2}$
$+\frac{3PL}{16}$		-
$+\frac{wL^2}{8}$		-
$+\frac{Pab(2L-a)}{2L^2}$		-



Relative or Beam Stiffness or Stiffness factor

- When a structural member of uniform section is subjected to a moment at one end, then the moment required so as to rotate that end to produce unit slope is called the ***stiffness of the member***.
- Stiffness is the member of ***force required to produce unit deflection***.
- It is also the moment required to produce unit rotation at a specified joint in a beam or a structure. It can be extended to denote the torque needed to produce unit twist.
- It is the moment required to rotate the end while acting on it through a unit rotation, without translation of the far end being
    - ✓ Beam is hinged or simply supported at both ends

      $k = 3\ EI / L$
    - ✓ Beam is hinged or simply supported at one end and fixed at other end

      $k = 4\ EI / L$
    - ✓ Stiffness of members in continuous beams and rigid frames
        - ➢ ***Stiffness of all intermediate members*** $k = 4\ EI / L$
        - ➢ ***Stiffness of edge members***,
            - ❖ If edge support is fixed $k = 4\ EI / L$
            - ❖ If edge support is hinged or roller $k = 3\ EI / L$
- Where, E = Young“s modulus of the beam material

  I = Moment of inertia of the beam

  L = Beam“s span length

Distribution factor

- When several members meet at a joint and a moment is applied at the joint to produce rotation without translation of the members, the moment is distributed among all the members meeting at that joint proportionate to their stiffness.
- Distribution factor = Relative stiffness / Sum of relative stiffness at the joint
- If there is 3 members,

  Distribution factors = $k_1 / (k_1+k_2+k_3)$, $k_2 / (k_1+k_2+k_3)$, $k_3 / (k_1+k_2+k_3)$

Carry over moment

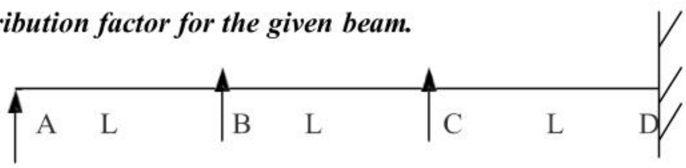
- Carry over moment: It is defined as the moment induced at the fixed end of the beam by the action of a moment applied at the other end, which is hinged.
- Carry over moment is the same nature of the applied moment.

Carry over factor (C.O):

- A moment applied at the hinged end B "carries over" to the fixed end „A", a moment equal to half the amount of applied moment and of the same rotational sense. C.O =0.5

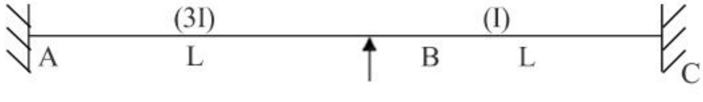
Problem:

1. ***Find the distribution factor for the given beam.***



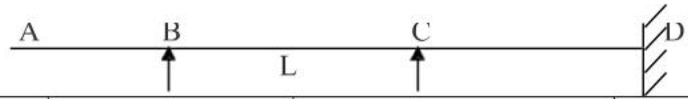
Joint	Member	Relative stiffness	Sum of Relative stiffness	Distribution factor
A	AB	4EI / L	4EI / L	(4EI / L) / (4EI / L) = 1
B	BA	3EI /L	3EI /L + 4EI / L = 7EI / L	(3EI / L) / (7EI / L)= 3/7
	BC	4EI / L		(4EI / L) / (7EI / L) = 4/7
C	CB	4EI / L	4EI / L + 4EI / L =8EI / L	(4EI / L) / (8EI / L) =4/8
	CD	4EI / L		(4EI / L) / (8EI / L)= 4/8
D	DC	4EI / L	4EI / L	(4EI / L)/ (4EI / L) = 1

2. ***Find the distribution factor for the given beam.***



Joint	Member	Relative stiffness	Sum of Relative stiffness	Distribution factor
A	AB	4E (3I) / L	12EI / L	(12EI / L) / (12EI / L) = 1
B	BA	4E(3I) /L	12EI /L + 4EI / L = 16EI / L	(12EI / L) / (16EI / L)= 3/4
	BC	4EI / L		(4EI / L) / (16EI / L) = 1/4
C	CB	4EI / L	4EI / L	(4EI / L) / (4EI / L) =1

3. Find the distribution factor for the given beam.



Joint	Member	Relative stiffness	Sum of Relative	Distribution factor
B	BA	0 (no support)	3EI / L	0
	BC	3EI / L		(3EI / L) / (3EI / L) =1
C	CB	3EI / L	3EI /L + 4EI / L	(3EI / L) / (7EI / L)= 3 / 7
	CD	4EI / L	= 7EI / L	(4EI / L) / (7EI / L) = 4 / 7
D	DC	4EI / L	4EI / L	(4EI / L) / (4EI / L) =1

Flexural Rigidity of Beams:

- ✦ The product of young"s modulus (E) and moment of inertia (I) is called Flexural Rigidity (EI) of Beams. The unit is N.mm^2.

Constant strength beam:

- ✦ If the flexural Rigidity (EI) is constant over the uniform section, it is called Constant strength beam.

Sway:

- ✦ Sway is the lateral movement of joints in a portal frame due to the unsymmetrical dimensions, loads, moments of inertia, end conditions, etc.

What are the situations where in sway will occur in portal frames?

- ✦ Eccentric or unsymmetrical loading
- ✦ Unsymmetrical geometry
- ✦ Different end conditions of the columns
- ✦ Non-uniform section of the members
- ✦ Unsymmetrical settlement of supports
- ✦ A combination of the above

What are symmetric and antisymmetric quantities in structural behaviour?

- ✦ When a symmetrical structure is loaded with symmetrical loading, the bending moment and deflected shape will be symmetrical about the same axis.
- ✦ Bending moment and deflection are symmetrical quantities

Steps involved in Moment Distribution Method:

1. Calculate fixed end moments due to applied loads following the same sign convention and procedure, which was adopted in the slope-deflection method.
2. Calculate relative stiffness.
3. Determine the distribution factors for various members framing into a particular joint.

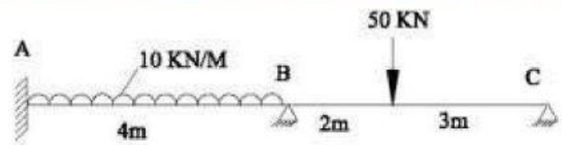
4. Distribute the net fixed end moments at the joints to various members by multiplying the net moment by their respective distribution factors in the first cycle.
5. In the second and subsequent cycles, carry-over moments from the far ends of the same member (carry-over moment will be half of the distributed moment).
6. Consider this carry-over moment as a fixed end moment and determine the balancing moment. This procedure is repeated from second cycle onwards till convergence

Advantages of Fixed Ends or Fixed Supports

1. Slope at the ends is zero.
2. Fixed beams are stiffer, stronger and more stable than SSB.
3. In case of fixed beams, fixed end moments will reduce the BM in each section.
4. The maximum defection is reduced.

Problem:

1. Analyse the frame given in figure by moment distribution method and draw the B.M.D & S.F.D



Step: 1 - Fixed end moment

$M^F_{AB} = -WL^2/12 = -10\times4^2/12 = -13.33$ KNM

$M^F_{BA} = WL^2/12 = 10\times4^2/12 = -13.33$ KNM

$M^F_{BC} = -Wab^2/L^2 = -50\times2\times3^2/5^2 = -36$ KNM

$M^F_{CB} = Wa^2b/L^2 = 50\times2^2\times3/5^2 = 24$ KNM

Step: 2 - Stiffness

$K_{AB} = K_{BA} = 4EI/L = EI$

$K_{BC} = K_{CB} = 3EI/L = 0.6EI$

Step: 3 - Distribution factor

Joint B

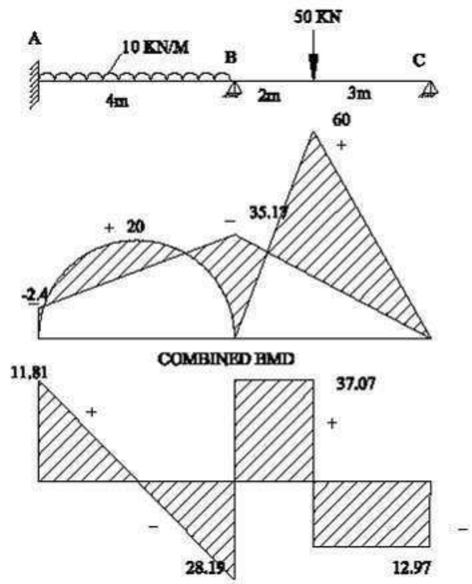
$D^F_{BA} = K_{BA}/(K_{BA}+K_{BC}) = 0.63$

$D^F_{BC} = K_{BC}/(K_{BA}+K_{BC}) = 0.37$

Step: 4 - Moment distribution

MEMBER	AB	B		CB
		BA	BC	
DF	0	0.67	0.33	0
FEM	-13.33	+13.33	-36	+24
BALANCING	0	0	0	-24
CF	0	0	-12	0
M	-13.33	+13.33	-48	0
BALANCING	0	21.84	12.83	0
CF	10.92	0	0	0
M-FINAL	**-2.4**	**35.17**	**-35.17**	**0**

Step: 5 - Reactions



SFD

Span AB:

R_A = 11.81 KN

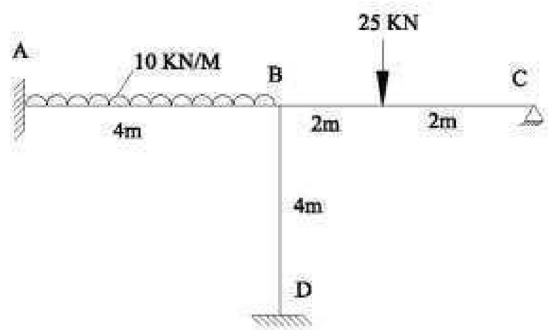
R_{B1} = 28.19 KN

Span BC:

R_{B2} = 37.03 KN

R_C = 12.97 KN

2. Analyse the frame given in figure by moment distribution method and draw the B.M.D&S.F.D



Step: 1 - Fixed end moment

M^F_{AB} = $-WL^2/12$ = $-10\times4^2/12$ = -13.33 KNM

M^F_{BA} = $WL^2/12$ = $10\times4^2/12$ = -13.33 KNM

M^F_{BC} = $-WL/8$ = $-25\times4/8$ = -12.5 KNM

M^F_{CB} = $WL/8$ = $25\times4/8$ = 12.5 KNM

M^F_{BD} = 0

M^F_{DB} = 0

Step: 2 - Stiffness

K_{AB} = K_{BA} = $4EI/L$ = EI

K_{BC} = K_{CB} = $3EI/L$ = 0.75EI

K_{BD} = K_{DB} = $4EI/L$ = EI

Step: 3 - Distribution factor

Joint B

D^F_{BA} = $K_{BA}/(K_{BA}+K_{BC}+K_{BD})$ = 0.36

D^F_{BC} = $K_{BC}/(K_{BA}+K_{BC}+K_{BD})$ = 0.28

D^F_{BD} = $K_{BD}/(K_{BA}+K_{BC}+K_{BD})$ = 0.36

Step: 4 - Moment distribution

MEMBER	AB	B			DB	CB
		BA	BC	BD		
DF	0	0.36	0.28	0.36	0	
FEM	-13.33	+13.33	-12.5	0	0	+12.5
CF	0	0	-6.25	0	0	-12.5
M(initial)	-13.33	+13.33	-18.75	0	0	0
BALANCING	0	+1.95	1.52	1.95	0	0
MF	0.98	0	0	0	0.98	0
M-FINAL	**-12.35**	**15.28**	**-17.23**	**1.95**	**0.98**	**0**

Step: 5 - Find reactions:

Span AB:

R_A = 19.27 KN

R_{B1} = 20.73 KN

Span BC:

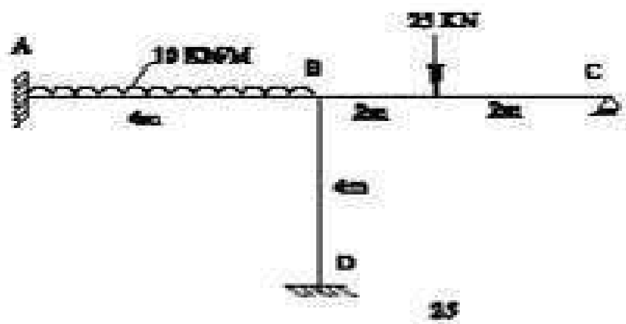
R_{B2} = 16.32 KN

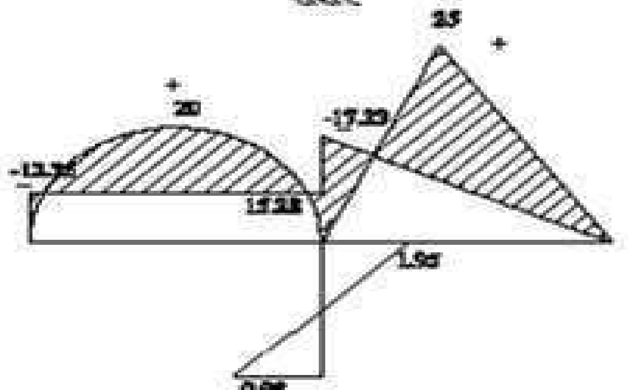
R_C = 8.68 KN

Span BD:

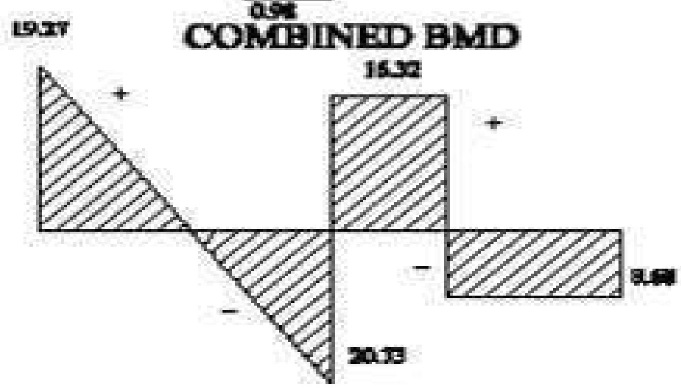
R_{B3} = -0.73 KN

R_D = 0.73 KN





COMBINED BMD

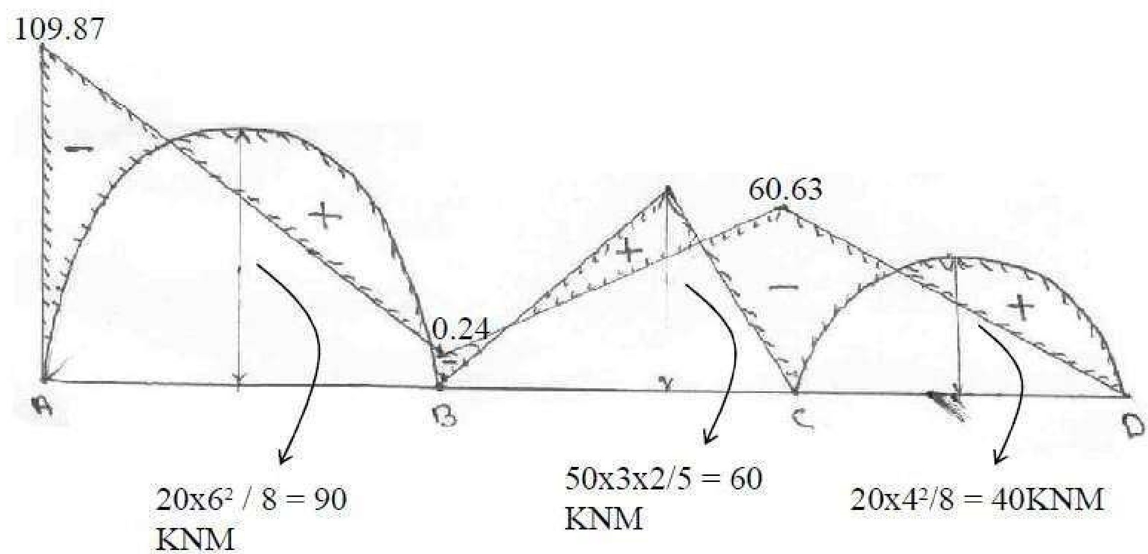


SFD

Step: 3 - Moment Distribution

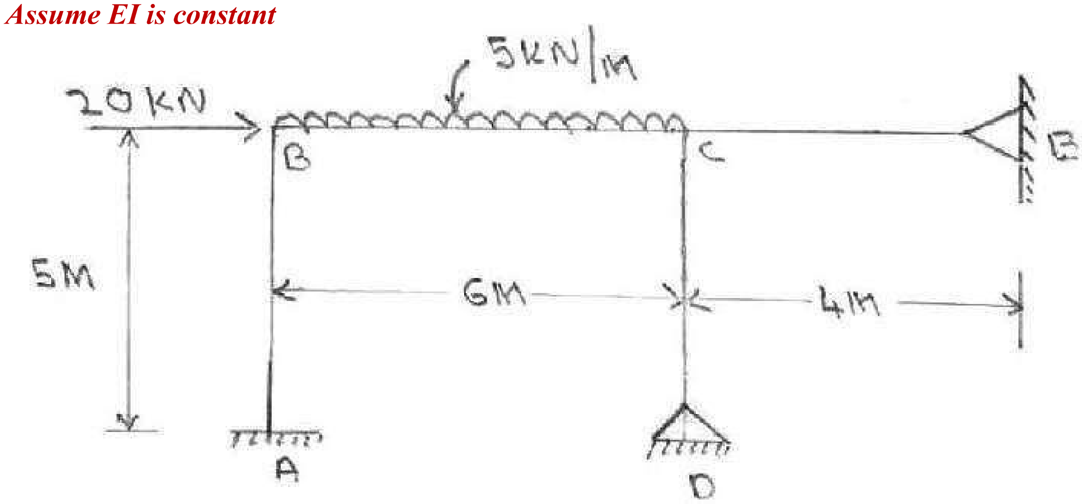
Jt	A	B		C		D	
Member	AB	BA	BC	CB	CD	DC	
D.F		0.46	0.54	0.51	0.49		
FEM	-100	+20	+33.6	+93.6	-26.67	+26.67	
Release jt. 'D'						-26.67	
CO					-13.34		
Initial moments	-100	+20	+33.6	+93.6	-40.01	0	
Balance		-24.66	-28.94	-27.33	-26.26		
C.O	-12.33		-13.67	--14.47			
Balance		+6.29	+7.38	+7.38	+7.09		
C.O	+3.15		+3.69	+3.69			
Balance		-1.7	-1.99	-1.88	-1.81		
C.O	-0.85		-0.94	-1			
Balance		+0.43	+0.51	+0.51	+0.49		
C.O	+0.22		+0.26	+0.26			
Balance		-0.12	-0.14	-0.13	-0.13		
C.O	-0.06						
Final moments	**-109.87**	**+0.24**	**-0.24**	**+60.63**	**-60.63**		

Step: 4 – BMD



6. Analysis the frame shown in figure by moment distribution method and draw BMD.

Assume EI is constant



Step: 1 - Fixed end moments

$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = M_{FCE}$ $M_{FEC} = 0$

$M_{FBC} = -\dfrac{5 \times 6^2}{12} = -15\,\text{kNm}$

$M_{FCB} = +15\,\text{kNm}$

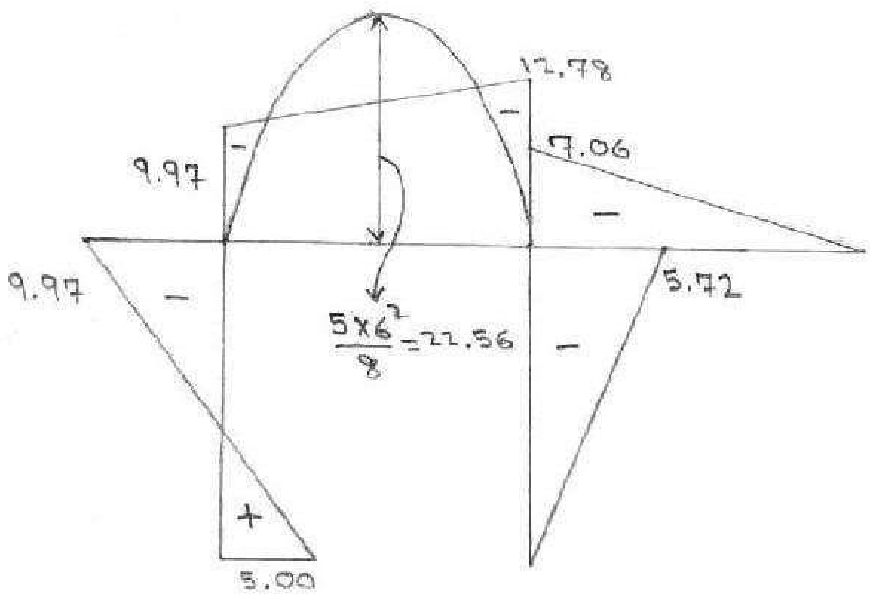
Step:2 - Distribution factor

Jt.	Member	Relative stiffness (K)	ΣK	DF = $\dfrac{K}{\sum K}$
B	BA	I/5	$\dfrac{11}{30}I$	0.55
	BC	I/6		0.45
C	CB	I/6 = 0.17 I		0.33
	CD	$\dfrac{3}{4}$I/5 = 0.15 I	0.51 I	0.3
	CE	$\dfrac{3}{4} \times \dfrac{I}{4} = 0.19\,I$		0.37

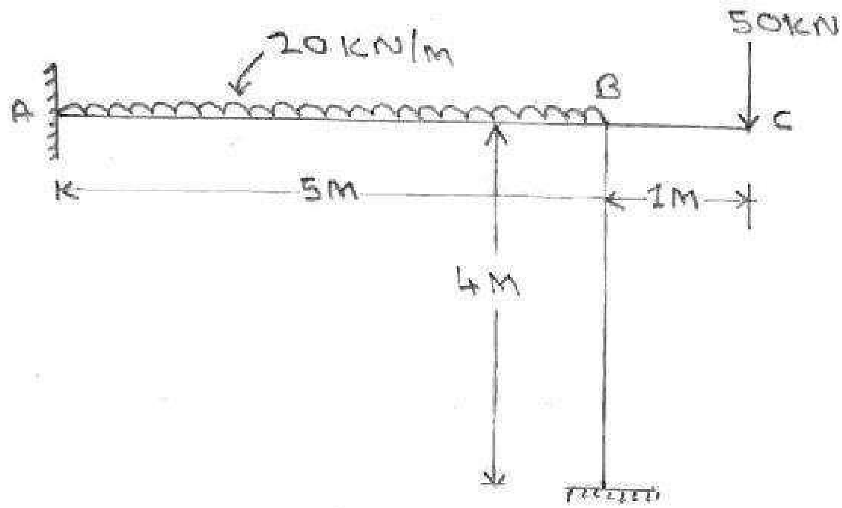
Step: 3 - Moment Distribution

Jt	A	B		C			D	E
Member	AB	BA	BC	CB	CD	CE	DC	EC
D.F	0	0.55	0.45	0.33	0.3	0.37	1	1
FEM	0	0	-15	+15	0	0	0	0
Balance		8.25	6.75	-4.95	-4.5	-5.55		
C.O	4.13		-2.48	3.38				
Balance		1.36	1.12	-1.12	-1.01	-1.25		
C.O	0.68		0.56	0.56				
Balance		0.31	0.25	-0.18	-0.17	-0.21		
C.O	0.16		-0.09	0.13				
Balance		0.05	0.04	-0.04	-0.04	-0.05		
C.O	0.03							
Final moments	**5**	**9.97**	**-9.97**	**12.78**	**-5.72**	**-7.06**	**0**	**0**

Step: 4 – BMD



8. Analyze the frame shown in figure by moment distribution method and draw BMD and SFD



Step: 1 - Fixed end moments

$$M_{FAB} = -\frac{20x5^2}{12} = -41.67 \text{ KNM}$$

$M_{FBA} = +41.67$ KNM

$M_{FBD} = M_{FDB} = 0$

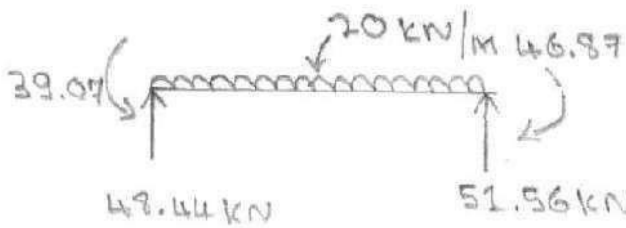
$M_{FBC} = -50 \times 1 = -50$ KNM

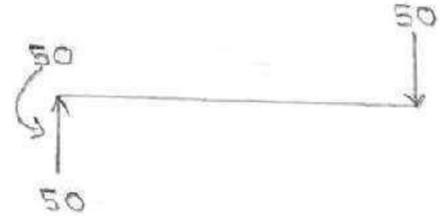
Step:2 - Distribution factor

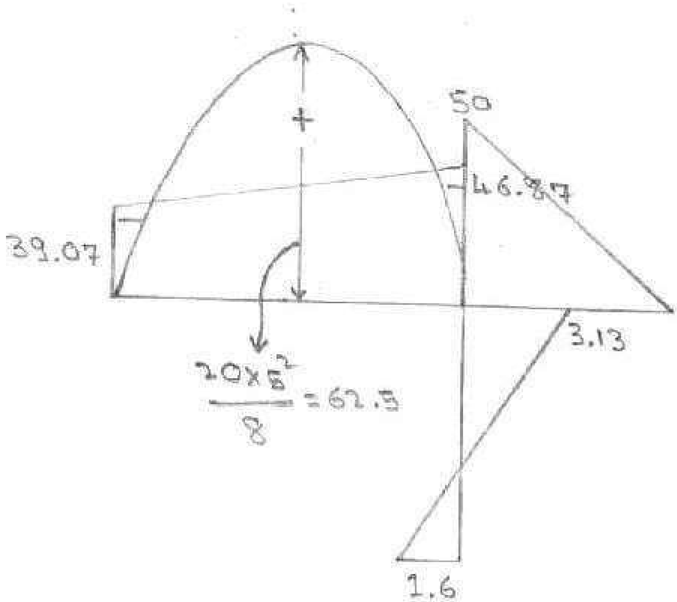
Jt.	Member	K	ΣK	$DF = \frac{K}{\sum K}$
B	BA	2I/5 = 0.4I		0.62
	BC	0	0.65I	0
	BD	I/4 = 0.25 I		0.38

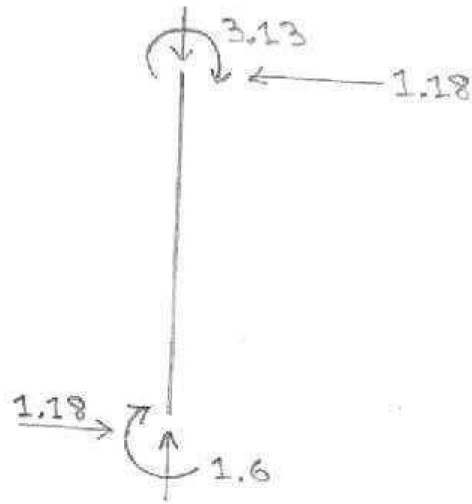
Step: 3 - Moment Distribution

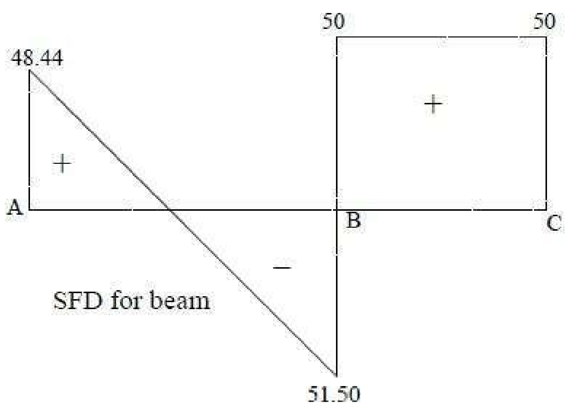
Jt	A	B			D
Member	AB	BA	BC	BD	DB
D.F	01	0.62	0	0.38	0
FEM	-41.67	41.67	-50	0	0
Balance		5.2	0	3.13	
C.O	2.6				1.6
Final moments	**-39.07**	**46.87**	**-50**	**3.13**	**1.6**

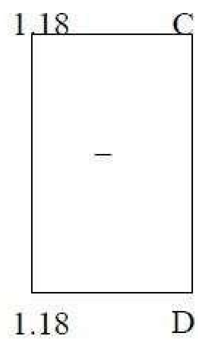








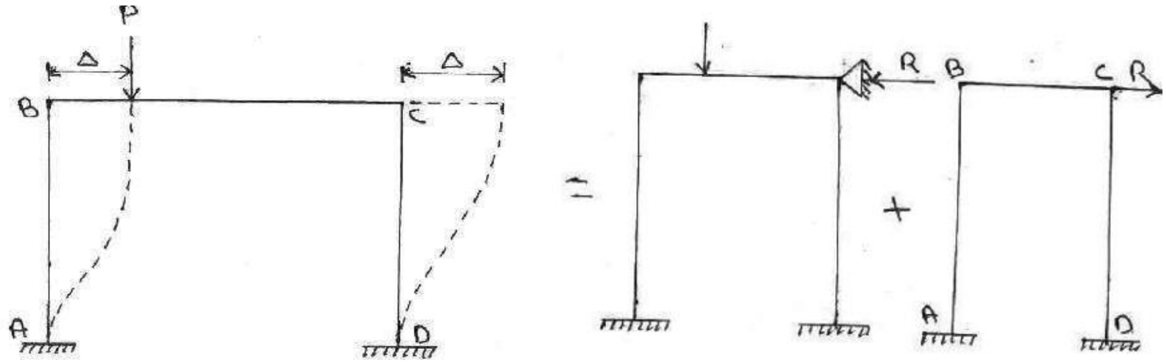




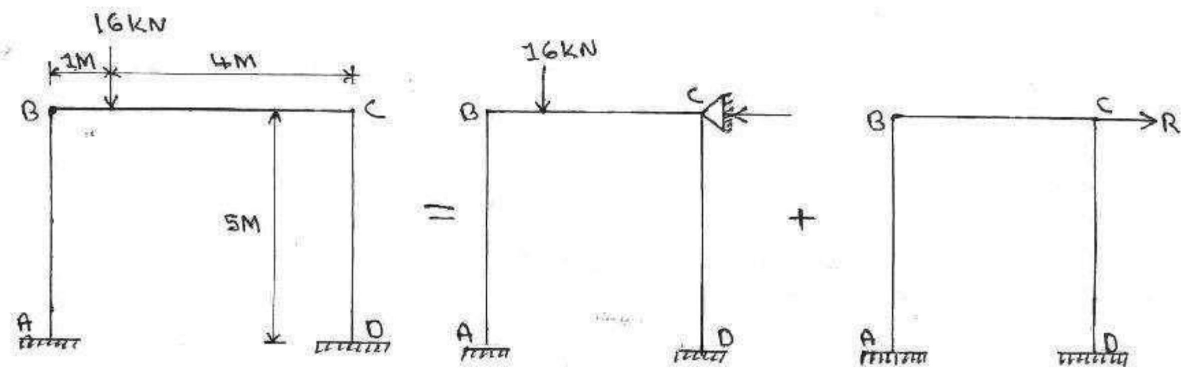
Transvers shear force diagram for

Moment distribution method for frames with side sway:

- Frames that are non symmetrical with reference to material property or geometry (different lengths and I values of column) or support condition or subjected to nonsymmetrical loading have a tendency to side sway.



__9.__ Analyze the frame shown in figure by moment distribution method. Assume EI is constant.



Non Sway Analysis:

- First consider the frame held from side sway as shown in figure.

Step: 1 - Fixed end moments

$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

$$M_{FBC} = -\frac{16\times1\times4^2}{5^2} = -10.24 \text{ kNm}$$

$$M_{FCB} = \frac{16\times1^2\times4}{5^2} = 2.56 \text{ kNm}$$

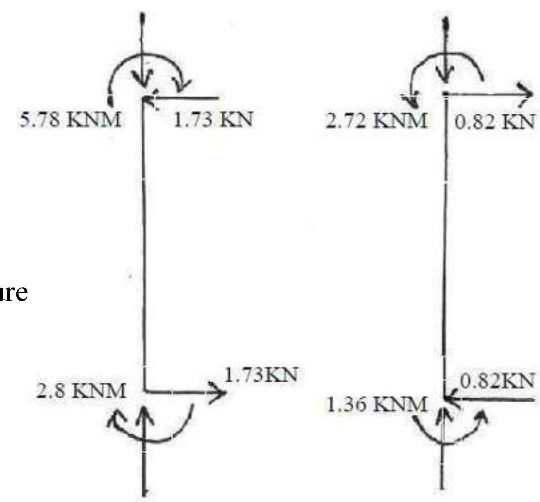
Step:2 - Distribution factor

Jt.	Member	Relative stiffness K	ΣK	$DF = \frac{K}{\sum K}$
B	BA BC	I/5 = 0.2 I I/5 = 0.2 I	0.4 I	0.5 0.5
C	CB CD	I/5 = 0.2I I/5 = 0.2I	0.4I	0.5 0.5

Step: 3 - Moment Distribution

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F	0	0.5	0.5	0.5	0.5	0
FEM	0		-10.24	2.56	0	0
Balance		5.12	5.12	-1.28	-1.28	
CO	2.56		-0.64	2.56		-0.64
Balance		0.32	0.32	-0.08	-0.08	
CO	0.16		-0.64	0.16		-0.64
Balance		0.32	0.32	-0.08	-0.08	
C.O	0.16		-0.04	0.16		-0.04
Balance		0.02	0.02	-0.08	-0.08	
C.O	0.01					-0.04
Final moments	**2.89**	**5.78**	**-5.78**	**2.72**	**-2.72**	**-1.36**

FBD of columns:



By seeing of the FBD of columns

$R = 1.73 - 0.82$

(Using $\Sigma F_x = 0$ for entire frame) = 0.91 KN ($\leftarrow$)

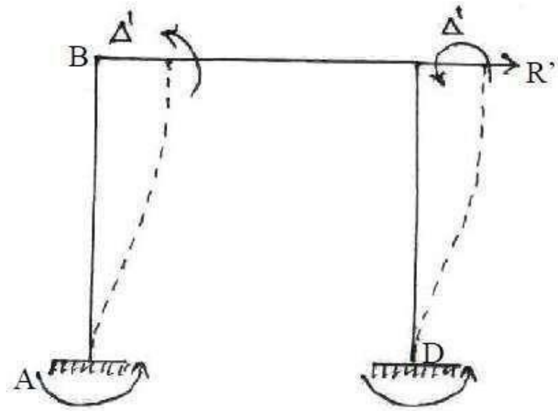
Now apply

R = 0.91 kN acting opposite as shown in figure

for the sway analysis.

ii) Sway analysis:

- For this we will assume a force R' is applied at C causing the frame to deflect Δ" as shown in figure



Since both ends are fixed, columns are of same length & I and assuming joints B & C are temporarily restrained from rotating and resulting fixed end moment are

$$M'_{BA} = M'_{AB} = M'_{CD} = M'_{DC} = \frac{6EI\Delta'}{l^2}$$

Assume $M^1_{BA} = -100$ kNm

$\therefore\ M'_{AB} = M'_{CD} = M'_{DC} = -100$ kNm

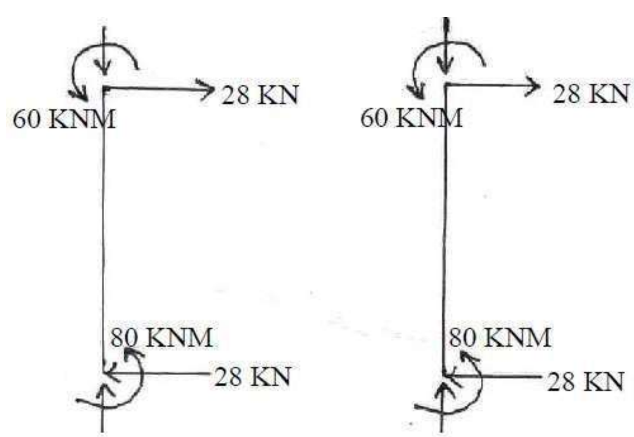
Moment distribution table for sway analysis:

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F	01	0.5	0.5	0.5	0.5	0
FEM	−100	−100	0	0	−100	−100
Balance		50	50	50	50	
CO	25		25	25		25
Balance		−12.5	−12.5	−12.5	−12.5	
CO	-6.25		−6.25	-6.25		−6.25
Balance		3.125	3.125	3.125	3.125	
C.O	1.56		1.56	1.56		1.56
Balance		−0.78	−0.78	−0.78	−0.78	
C.O	-0.39		−0.39	−0.39		0.39
Balance		0.195	0.195	0.195	0.195	
C.O	0.1					0.1
Final moments	**- 80**	**- 60**	**60**	**60**	**– 60**	**– 80**

FBD of columns:

Using $\sum F_x = 0$ for the entire frame

$R' = 28 + 28 = 56$ KN $(\rightarrow)$



✦ Hence R" = 56KN creates the sway moments shown in above moment distribution table. Corresponding moments caused by R = 0.91KN can be determined by proportion.

✦ Thus final moments are calculated by adding non sway moments and sway moments calculated for R = 0.91KN, as shown below

$$M_{AB} = 2.89 + \frac{0.91}{56}\ (-80) = 1.59\ \text{kNm}$$

$$M_{BA} = 5.78 + \frac{0.91}{56}\ (-60) = 4.81\ \text{kNm}$$

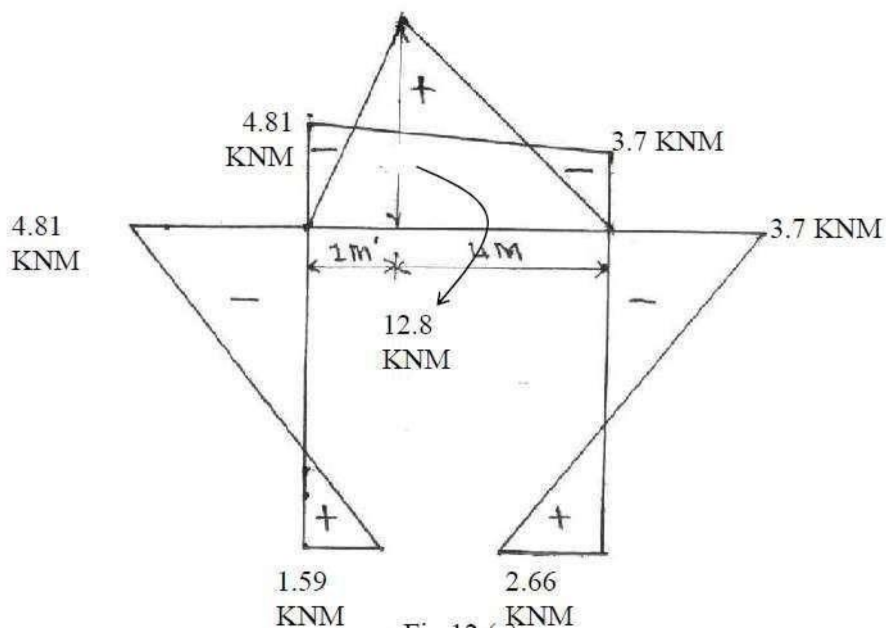
$$M_{BC} = -5.78 + \frac{0.91}{56}(60) = -4.81\ \text{kNm}$$

$$M_{CB} = 2.72 + \frac{0.91}{56}(60) = 3.7\ \text{kNm}$$

$$M_{CD} = -2.72 + \frac{0.91}{56}(-60) = -3.7\ \text{kNm}$$

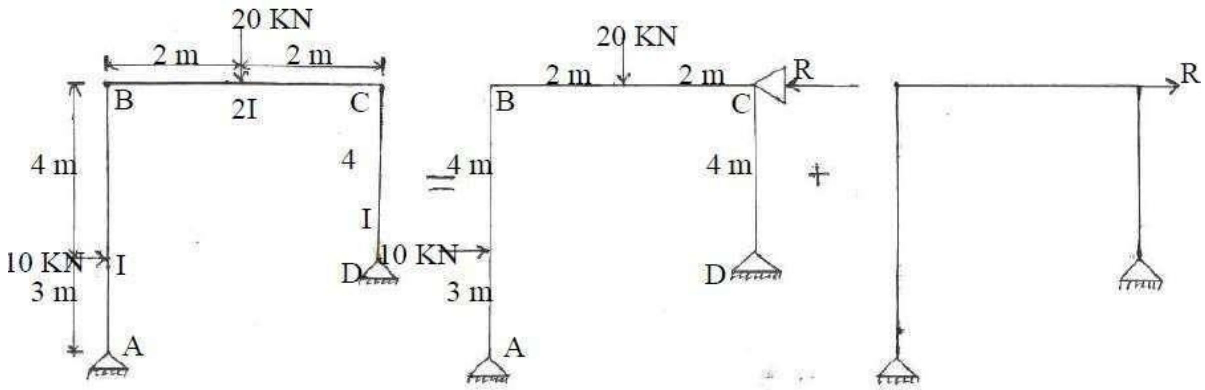
$$M_{DC} = -1.36 + \frac{0.91}{56}\ (-80) = -2.66\ \text{kNm}$$

BMD:



Moment distribution method for frames with side sway:

1. Analysis the rigid frame shown in figure by moment distribution method and draw BMD



i) Non Sway Analysis:

First consider the frame held from side sway as shown in figure 2

FEM calculation:

$$M_{FAB} = -\frac{10\times3\times4^2}{7^2} = -9.8\,KNM$$

$$M_{FBA} = +\frac{10\times3^2\times4}{7^2} = 7.3\,KNM$$

$$M_{FBC} = -\frac{20\times4}{8} = -10\ KNM$$

$$M_{FCB} = +10KNM$$

$$M_{FCD} = M_{FDC} = 0$$

Distribution Factor:

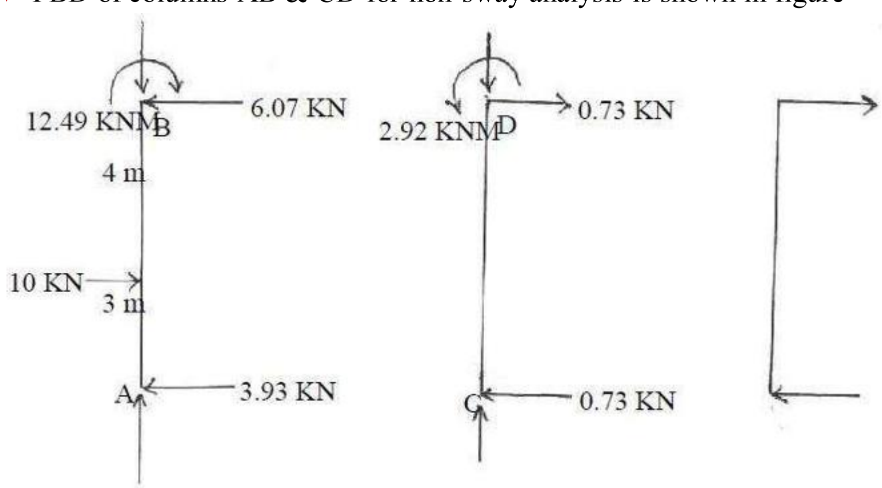
Joint	Member	Relative stiffness k	Σk	$DF = \frac{K}{\sum K}$
B	BA	$\frac{3}{4}\times\frac{I}{7} = 0.11I$	0.61 I	0.18
	BC	2I/4 = 0.5I		0.82
C	CB	2I/4 = 0.5I		0.72
	CD	$\frac{3}{4}\times\frac{I}{4} = 0.19\ I$	0.69 I	0.28

Moment distribution for non sway analysis:

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F	1	0.18	0.82	0.72	0.28	1
FEM	-9.8	7.3	-10	10	0	0
Release jt. 'D'	+9.8					
CO		4.9				
Initial moments	0	12.2	-10	10	0	0
Balance		-0.4	-1.8	-7.2	-2.8	
CO			-3.6	-0.9		
Balance		0.65	2.95	0.65	0.25	
C.O			0.33	1.48		
Balance		-0.06	-0.27	-1.07	-0.41	
C.O			-0.54	-0.14		
Balance		0.1	0.44	0.1	0.04	
Final moments	**0**	**12.49**	**-12.49**	**2.92**	**-2.92**	**0**

FBD of columns:

- FBD of columns AB & CD for non-sway analysis is shown in figure



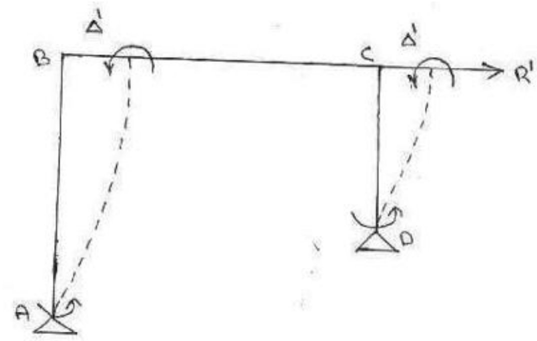
Applying $\Sigma F_x = 0$ for frame as a Whole, $R = 10 - 3.93 - 0.73$

$= 5.34$ kN ($\leftarrow$)

- Now apply R = 5.34KN acting opposite as shown in figure for sway analysis

ii) Sway analysis:

- For this we will assume a force R' is applied at C causing the frame to deflect Δ' as shown in figure.



Since ends A & D are hinged and columns AB & CD are of different lengths

$M'_{BA} = -3EI\Delta'/L^2_1$

$M'_{CD} = -3EI\Delta'/L^2_2$

$$\therefore \frac{M^1_{BA}}{M^1_{CD}} = \frac{3EI\,\Delta'/l_1^2}{3EI\,\Delta^1/l_2^2} = \frac{l_2^2}{l_1^2} = \frac{4^2}{7^2} = \frac{16}{49}$$

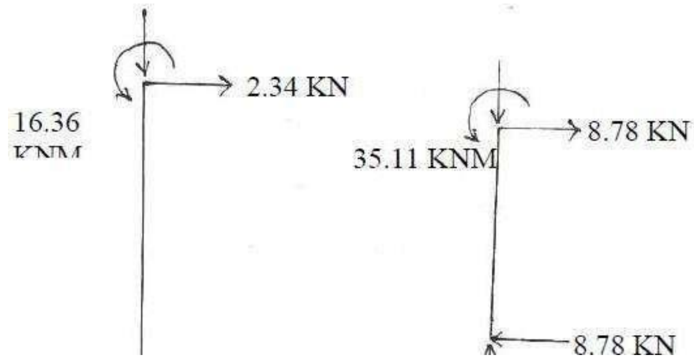
Assume $M^1_{BA} = -16\,\text{kNm}\ \&\ M'_{AB} = 0$

$\&\ M^1_{CD} = -49\,\text{kNm}\ \&\ M'_{DC} = 0$

Moment distribution table for sway analysis:

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F	1	0.18	0.82	0.72	0.28	1
FEM	0	-16	0	0	-49	0
Balance		2.88	13.12	35.28	13.72	
CO			17.64	6.56		
Balance		-3.18	-14.46	-4.72	-1.84	
CO			-2.36	-7.23		
Balance		0.42	1.94	5.21	2.02	
C.O			2.61	0.97		
Balance		-0.47	-2.14	-0.7	-0.27	
C.O			0.35	-1.07		
Balance		0.06	0.29	0.77	0.3	
C.O			0.39	0.15		
Balance		-0.07	-0.32	-0.11	-0.04	
Final moments	**0**	**-16.36**	**16.36**	**35.11**	**-35.11**	**0**

FBD of columns AB & CD for sway analysis moments is shown in fig.



Using $\Sigma f_x = 0$ for the entire frame
$R' = 11.12$ kN ($\rightarrow$)

Hence $R' = 11.12$ kN creates the sway moments shown in the above moment distribution table. Corresponding moments caused by $R = 5.34$ kN can be determined by proportion.

Thus final moments are calculated by adding non-sway moments and sway moments determined for R = 5.34 KN as shown below.

$M_{AB} = 0$

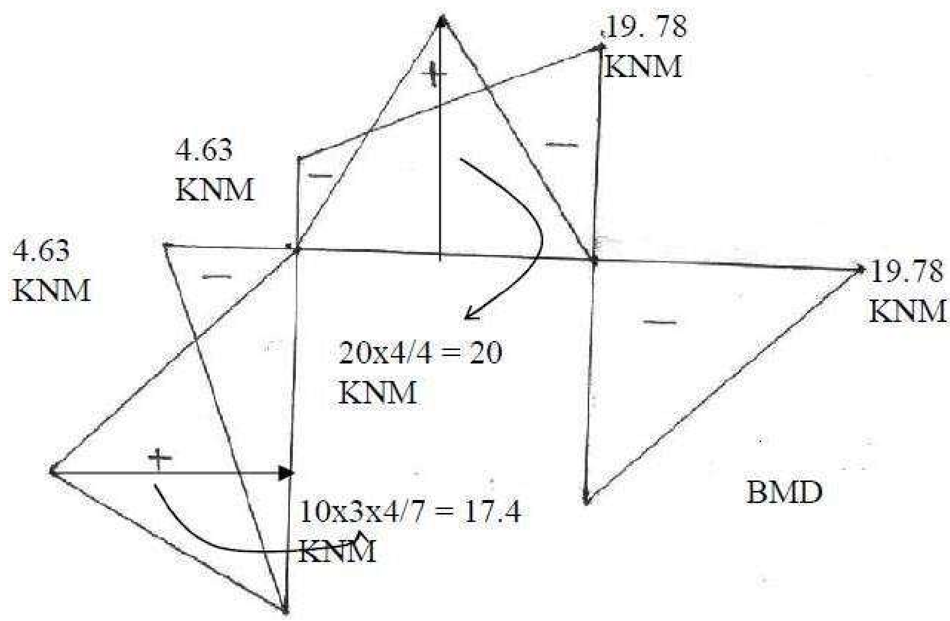
$M_{BA} = 12.49 + \frac{5.34}{11.12}(-16.36) = 4.63$ kNm

$M_{BC} = -12.49 + \frac{5.34}{11.12}(16.36) = -4.63$ KNM

$M_{CB} = 2.92 + \frac{5.34}{11.12}(35.11) = 19.78$ kNm

$M_{CD} = -2.92 + \frac{5.34}{11.12}(-35.11) = -19.78$ KNM

$M_{DC} = 0$



UNIT-III
Kanis Method

This method was developed by Dr. Gasper Kani of Germany in 1947. This method offers an iterative scheme for applying slope deflection method. We shall now see the application of Kani's method for different cases.

I. Beams with no translation of joints:

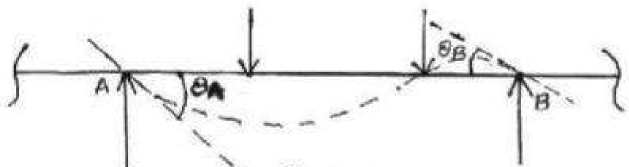


Fig.1 (a)

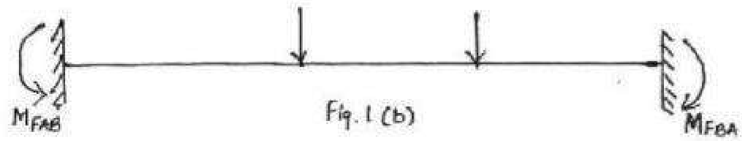


Fig.1 (b)

Fig.1 (c)

Fig.1 (d)

Let AB represent a beam in a frame, or a continuous structure under transverse loading, as show in fig. 1 (a) let the M_{AB} & M_{BA} be the end moment at ends A & B respectively.

Sign convention used will be: clockwise moment +ve and anticlockwise moment –ve.

The end moments in member AB may be thought of as moments developed due to a superposition of the following three components of deformation.

1. The member 'AB' is regarded as completely fixed. (Fig. 1 b). The fixed end moments for this condition are written as M_{FAB} & M_{FBA}, at ends A & B respectively.
2. The end A only is rotated through an angle θ_A by a moment $2M'_{AB}$ inducing a moment M'_{AB} at fixed end B.
3. Next rotating the end B only through an angle θ_B by moment $2M'_{BA}$ while keeping end 'A' as fixed. This induces a moment M'_{BA} at end A.

Thus the final moment M_{AB} & M_{BA} can be expressed as super position of three moments

$$\left.\begin{aligned} M_{AB} &= M_{FAB} + 2M'_{AB} + M'_{BA} \\ M_{BA} &= M_{FBA} + 2M'_{BA} + M'_{AB} \end{aligned}\right\} \qquad \ldots\ldots\ldots(1)$$

For member AB we refer end 'A' as near end and end 'B' as far end. Similarly when we refer to moment M_{BA}, B is referred as near end and end A as far end.

Hence above equations can be stated as follows. The moment at the near end of a member is the algebraic sum of (a) fixed end moment at near end. (b) Twice the rotation moment of the near end (c) rotation moment of the far end.

Rotation factors:

Fig. 2 shows a multistoried frame.

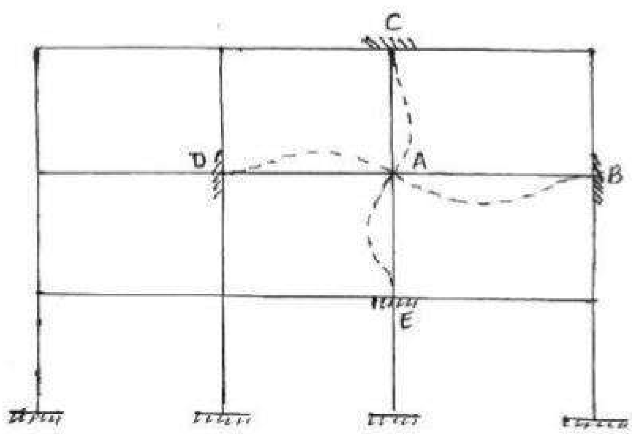


Fig. 2

Consider various members meeting at joint A. If no translations of joints occur, applying equation (1), for the end moments at A for the various members meeting at A are given by:

$$M_{AB} = M_{FAB} + 2M'_{AB} + M'_{BA}$$
$$M_{AC} = M_{FAC} + 2M'_{AC} + M'_{CA}$$
$$M_{AD} = M_{FAD} + 2M'_{AD} + M'_{DA}$$
$$M_{AE} = M_{FAE} + 2M'_{AE} + M'_{EA}$$

For equilibrium of joint A, $\Sigma M_A = 0$

$\therefore \Sigma M_{FAB} + 2\Sigma M'_{AB} + \Sigma M'_{BA} = 0$ ………………….(2)

where ,

ΣM_{FAB} =Algebraic sum of fixed end moments at A of all members meeting at A.

$\Sigma M'_{AB}$ = Algebraic sum of rotation moments at A of all member meeting at A.

$\Sigma M'_{BA}$ = Algebraic sum of rotation moments of far ends of the members meeting at A.

from equation (2)

$\Sigma M'_{AB} = -\frac{1}{2}\left[\sum M_{FAB} + \sum M'_{BA}\right]$ ……………… (3)

We know that $2M'_{AB} = \frac{4EI_{AB}}{L_{AB}}\theta_A = 4EK_{AB}\,\theta_A$

Where $K_{AB} = \frac{I_{AB}}{L_{AB}}$, relative stiffness of member AB

$M'_{AB} = 2E\,K_{AB}\,\theta_A$ …………….(4)

$\therefore \Sigma M'_{AB} = 2E\theta_A\,\Sigma K_{AB}$ …………….(5) (At rigid joint A all the members undergo same rotation θ_A)

Dividing Equation (4)/(5) gives

$$\frac{M'_{AB}}{\Sigma M'_{AB}} = \frac{K_{AB}}{\Sigma K_{AB}}$$

$\therefore M'_{AB} = \frac{K_{AB}}{\Sigma K_{AB}}\Sigma M'_{AB}$ ……………(5)

Substituting value of $\Sigma M'_{AB}$ from (3) in (5)

$$M'_{AB} = \left(-\frac{1}{2}\right)\frac{K_{AB}}{\Sigma K_{AB}}\left[\Sigma M_{FAB} + \Sigma M'_{BA}\right]$$
$$= U_{AB}\left[\Sigma M_{FAB} + \Sigma M'_{BA}\right] \text{………………(6)}$$

where $U_{AB} = -\frac{1}{2}\ \frac{K_{AB}}{\Sigma K_{AB}}$ is called as rotation factor for member AB at joint A.

Analysis Method:

In equation (6) ΣM_{FAB} is a known quantity. To start with the far end rotation moments M'_{BA} are not known and hence they may be taken as zero. By a similar approximation the rotation moments at other joints are also determined. With the approximate values of rotation moments computed, it is possible to again determine a more correct value of the rotation moment at A from member AB using equation (6).

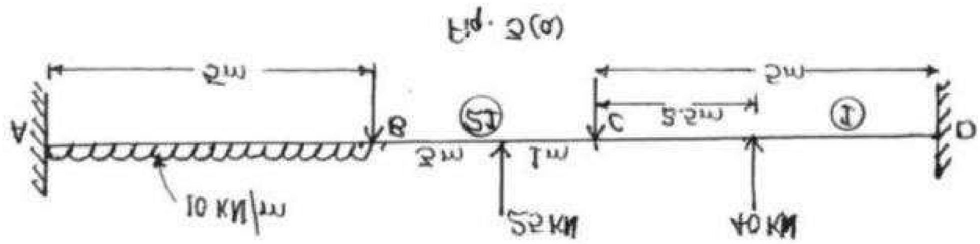
The process is carried out for sufficient number of cycles until the desired degree of accuracy is achieved.

The final end moments are calculated using equation (1).

Kani's method for beams without translation of joints, is illustrated in following

examples:

Ex: 1 Analyze the beam show in fig 3 (a) by Kani's method and draw bending



Solution:

a) Fixed end moments:

$$M_{FAB} = \frac{-10\times5^2}{12} = -20.83 \text{ kNm}$$

$$M_{FBA} = +20.83 \text{ kNm}$$

$$M_{FBC} = \frac{-25\times3\times1^2}{4^2} = -4.69 \text{ kNm}$$

$$M_{FCB} = \frac{25\times3^2\times1}{4^2} = 14.06 \text{ kNm}$$

$$M_{FCD} = \frac{-40\text{ x }5}{8} = -25 \text{ kNm}$$

$$M_{FDC} = 25 \text{ kNm}$$

b) Rotation Factors:

Jt.	Member	Relative stiffness (K)	ΣK	Rotation Factor

				$= -\frac{1}{2} \times \frac{K}{\sum K}$
B	BA	I/5 = 0.2I		- 0.14
	BC	2I/4 = 0.5I	0.7I	-0.36
C	CB	2I/4 = 0.5I		- 0.36
	CD	I/5 = 0.2I	0.7I	-0.14

c) Sum fixed end moment at joints:

$\Sigma M_{FB} = 20.83 - 4.69 = 16.14$ kNm

$\Sigma M_{FC} = 14.06 - 25 = -10.94$ kNm

The scheme for proceeding with method of rotation contribution is shown in figure 3 (b). The FEM, rotation factors and sum of fixed end moments are entered in appropriate places as shown in figure 3 (b).

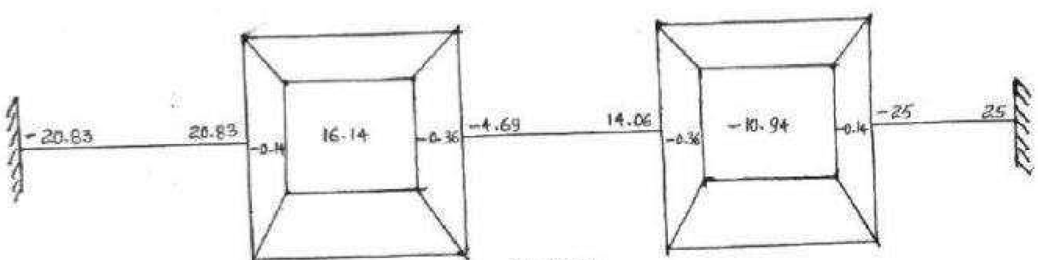


Fig. 3 (b)

d) Iteration Process:

Rotation contribution values at fixed ends A &D are zero. Rotation contributions at joints B & C are initially assumed as zero arbitrarily. These values will be improved in iteration cycles until desired degree of accuracy is achieved.

The calculations for two iteration cycles have been shown in following table. The remaining iteration cycle values for rotation contributions along with these two have been shown directly in figure 3 (c).

Jt	B		C	
Rotation Contribution	M'_{BA}	M'_{BC}	M'_{CB}	M'_{CD}
Iteration 1	-0.14 (16.14 + 0) = -2.26	-0.36 (16.14 + 0) = -5.81	-0.36 (-10.94 – 5.81)	-0.14 (-10.94 – 5.81) = 2.35

Iteration 2	-0.14 (16.14 + 6.03) = -3.1	-0.36 (16.14 + 6.03) = -7.98	= 6.03 -0.36 (-10.94 – 7.98) = 6.81	-0.14 (-10.94 – 7.98) = 2.65

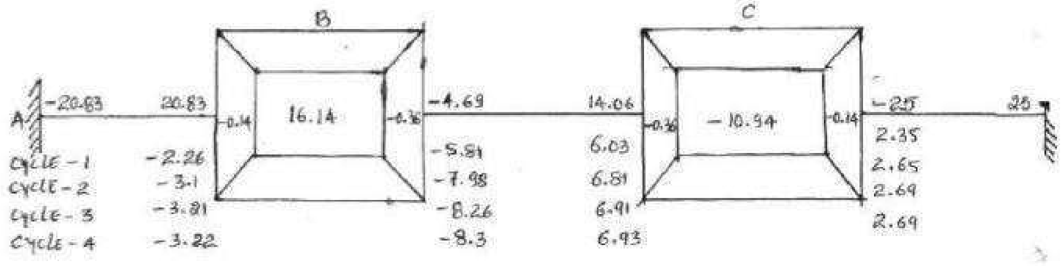


Fig.3(c)

Iterations are done up to four cycles yielding practically the same value of rotation contributions.

e) Final moments:

M_{AB} = - 20.83 + 0 – 3.22 = -24.05 kNm

M_{BA} = 20.83 + 2 × (-3.22) + 0 = 14.39 kNm

M_{BC} = –4.69 + 2 × (-8.3) + 6.93 = – 14.36 kNm

M_{CB} = 14.06 + (2 × 6.93) – 8.3 = 19.62 kNm

M_{CD} = –25 + (2 × 2.69) + 0 = -19.62 kNm

M_{DC} = 25 + 0 + 2.69 = 27.69 kNm

Bending moment diagram is shown in fig.3 (d)

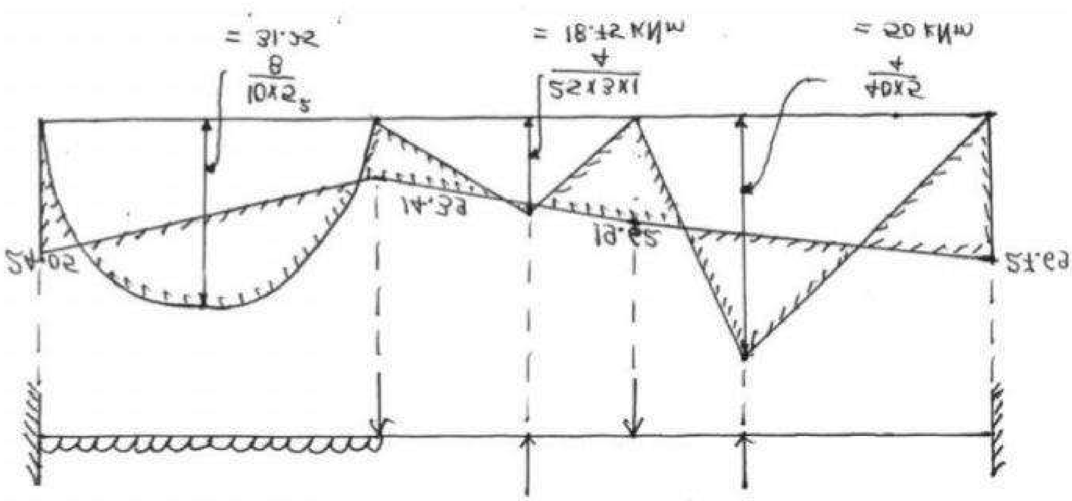


Fig.3 (d)

Ex 2: Analyze the continuous beam shown in fig. 4 (a)

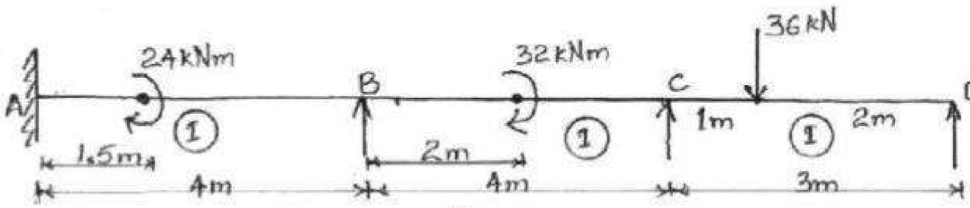


fig. 4 (a)

Solution:

a) Fixed end moments:

$$M_{FAB} = \frac{b(3a-1)}{l^2} M_o = \frac{2.5\ (3 \times 1.5 - 4)}{4^2} \times 24 = 1.88 \text{ kNm}$$

$$M_{FBA} = \frac{a(3b-1)}{1^2} M_o = \frac{1.5(3\times 2.5-4)}{4^2}\times 24 = 7.88 \text{ kNm}$$

$$M_{FBC} = \frac{M_o}{4} = \frac{32}{4} = 8 \text{ kNm}$$

$$M_{FCB} = \frac{M_o}{4} = 8 \text{ kNm}$$

$$M_{FCD} = -\frac{36\times 1\times 2^2}{3^2} = -16 \text{ kNm}$$

$$M_{FDC} = \frac{36\times 1^2\times 2}{3^2} = 8 \text{ kNm}$$

b) Modification in fixed end moments:

Actually end 'D' is a simply supported. Hence moment at D should be zero. To make moment at D as zero apply –8 kNm at D and perform the corresponding carry over to CD.

Modified $M_{FDC} = 8 - 8 = 0$

Modified $M_{FCD} = -16 + \frac{1}{2}(-8) = -20$ kNm

Now joint D will not enter the iteration process.

c) Rotation Factors:

Joint	**Member**	**Relative stiffness (K)**	**ΣK**	**Rotation Factor** $U = -\frac{1}{2} \times \frac{K}{\sum K}$
B	BA	I/4 = 0.25I	0.5 I	– 0.25
	BC	I/4 = 0.25I		–0.25
C	CB	I/4 = 0.25I		–0.25
	CD	$\frac{3}{4}\times\frac{I}{3} = 0.25I$	0.5I	–0.25

d) Sum of fixed end moments at joints:

$\Sigma M_{FB} = 7.88 + 8 = 15.88$ kNm

$\Sigma M_{FC} = 8 - 20 = -12$kNm

e) <u>Iteration process</u>

Joint	B		C	
Rotation Contribution	M'_{BA}	M'_{BC}	M'_{CB}	M'_{CD}
Rotation factor	–0.25	–0.25	–0.25	–0.25
Iteration 1 started at B assuming $M'_{CB} = 0$ & taking $M'_{AB} = 0'$ $M'_{DC} = 0.$	–0.25 × (15.88 + 0 + 0) = -3.97	–0.25 × (15.88 + 0 + 0) = -3.97	–0.25 × (-12 –3.97 + 0) = 3.97	–0.25 × (-12 –3.97 + 0) = 3.97
Iteration 2	–0.25 × (15.88 + 0 + 3.97) = –4.96	–0.25 × (15.88 + 0 + 3.97) = – 4.96	–0.25 × (-12 –4.96 + 0) = 4.24	–0.25 (–12 –4.96 + 0) = 4.24
Iteration 3	–0.25 (15.88 + 0 + 4.24) = –5.03	–0.25 (15.88 + 0 + 4.24) = –5.03	–0.25 × (-12 –5.03+ 0) = 4.26	–0.25 (–12 –5.03 + 0) = 4.26
Iteration 4	–0.25 (15.88 + 0 + 4.26) = –5.04	–0.25 (15.88 + 0 + 4.26) = –5.04	–0.25 × (-12 –5.03+ 0) = 4.26	–0.25 (–12 –5.03 + 0) = 4.26

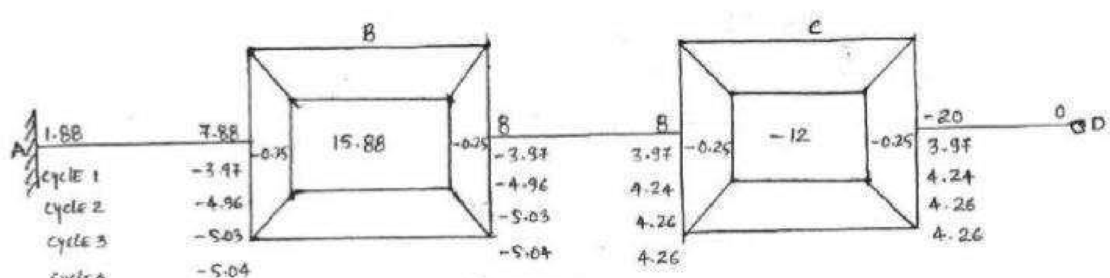


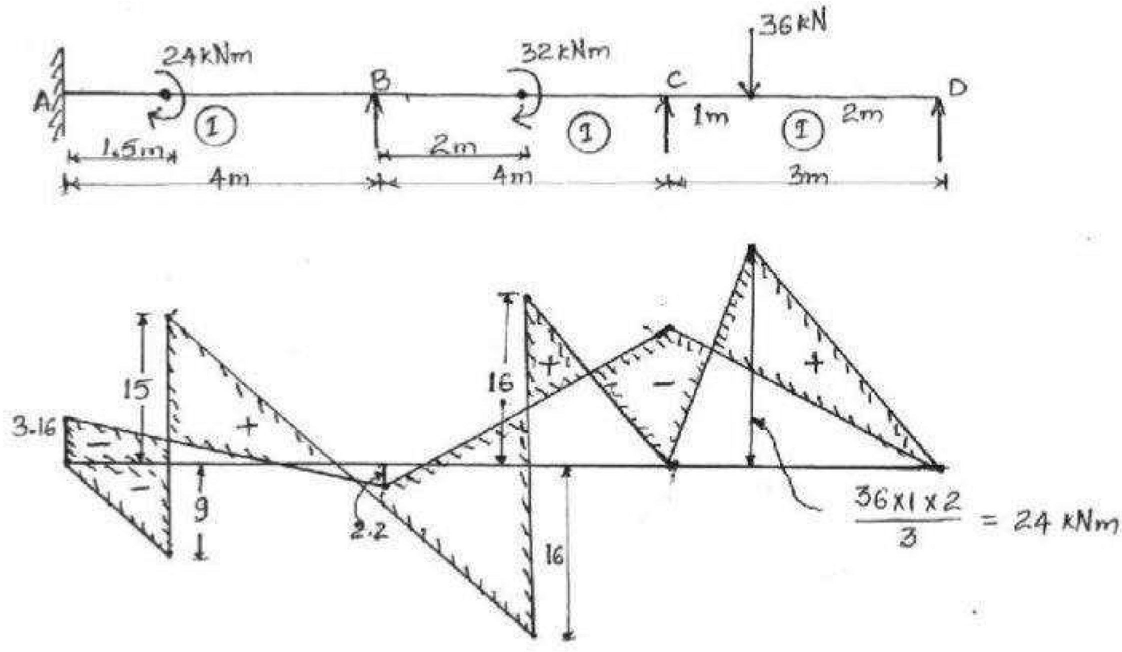
Fig. 4 (b)

Iteration process has been stopped after 4th cycle since rotation contribution values are becoming almost constant. Values of fixed end moments, sum of fixed end moments, rotation factors along with rotation contribution values after end of each cycle in appropriate places has been shown in fig. 4 (b).

f) Final moments

Member (ij)	**FEM M_{Fij} (kNm)**	$2M'_{ij}$ **(kNm)**	M^{1}_{ji} **(kNm)**	**Final moments** $\left(= M_{Fij} + 2M'_{ij} + M'_{ji}\right)$ **(kNm)**
AB	1.88	0	-5.04	-3.16
BA	7.88	2 (-5.04) = 10.08	0	-2.2
BC	8	2 (-504) = −10.08	4.26	+2.2
CB	8	2 × 4.26 = 8.52	-5.04	11.48
CD	-20	2 x 4.26 = 8.52	0	-11.48

BMD is shown below:



Ex 3: Analyze the continuous beam shown in fig. 5 (a) and draw BMD & SFD (VTU January 2005 exam)

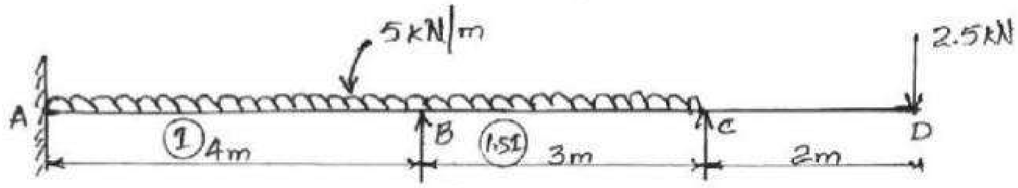


fig. 5 (a)

Solution:

a) Fixed end moments:

$$M_{FAB} = -\frac{5\times4^2}{12} = -6.67 \text{ kNm}$$

$$M_{FBA} = +6.67 \text{ kNm}$$

$$M_{FBC} = \frac{-5\times3^2}{12} = -3.75 \text{ kNm}$$

$$M_{FCB} = +3.75 \text{ kNm}$$

$$M_{CD} = -2.5 \text{ x } 2 = -5 \text{ kNm}$$

b) Modification in fixed end moments:

Since $M_{CD} = -5$ kNm; $M_{CB} = +5$kNm, for this add 1.25 kNm to M_{FCB} and do the corresponding carry over to M_{FBC}

∴ Now $M_{CB} = 5$ kNm

Modified $M_{FBC} = -3.75 + \frac{1}{2}(1.25) = -3.13$ kNm

Now joint C will not enter in the iteration process.

c) <u>Rotation factors:</u>

Jt.	Member	Relative stiffness (K)	ΣK	Rotation Factor $U = -\frac{1}{2} x \frac{K}{\sum K}$
B	BA	I/4 = 0.25I		− 0.2
	BC	$\frac{3}{4} \times \frac{1.5I}{3} = 0.375I$	0.625I	−0.3
C	CB	1.5I/3 = 0.5I		−0.5
	CD	0	0.5I	0

d) <u>Sum of fixed end moments at joints:</u>

$\Sigma M_{FB} = 6.67 - 3.13 = 3.54$ kNm

e) <u>Iteration Process</u>

Joint	B	
Rotation Contribution	M'_{BA} (kNm)	M'_{BC} (kNm)
Rotation factor	− 0.2	−0.3
Iteration 1 started at B taking $M'_{AB} = 0$ & $M'_{CB} = 0$	$-0.2 \times (3.54 + 0 + 0) = -0.71$	$-0.3 \times (3.54 + 0 + 0) = -1.06$

Since 'B' is the only joint needing rotation correction, the iteration process will stop after first iteration. Value of FEMs, sum of FEM at joint, rotation factors along with rotation contribution values in appropriate places is shown in fig. 5 (b)

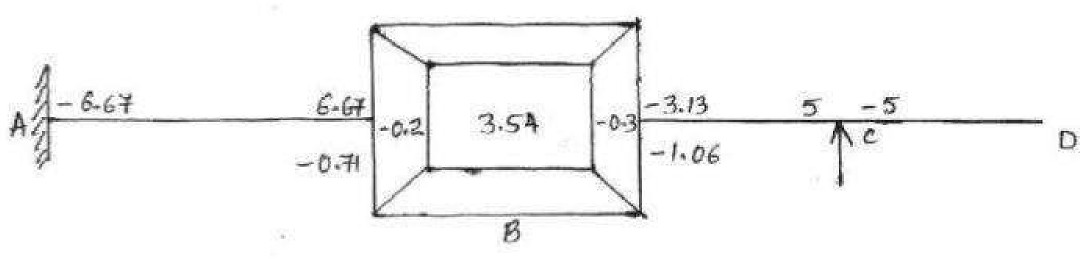
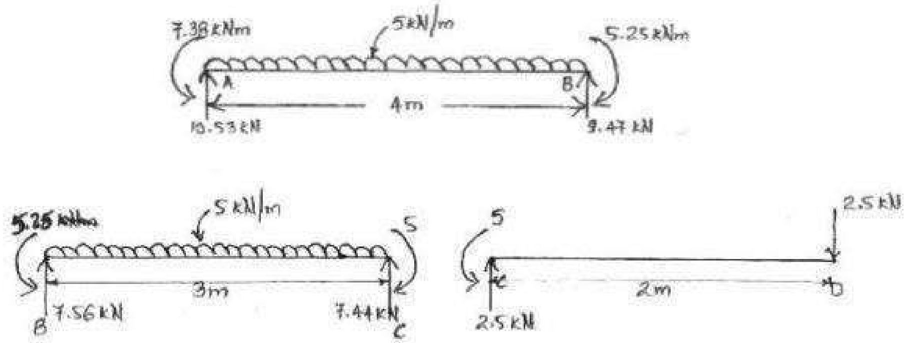


Fig.5(b)

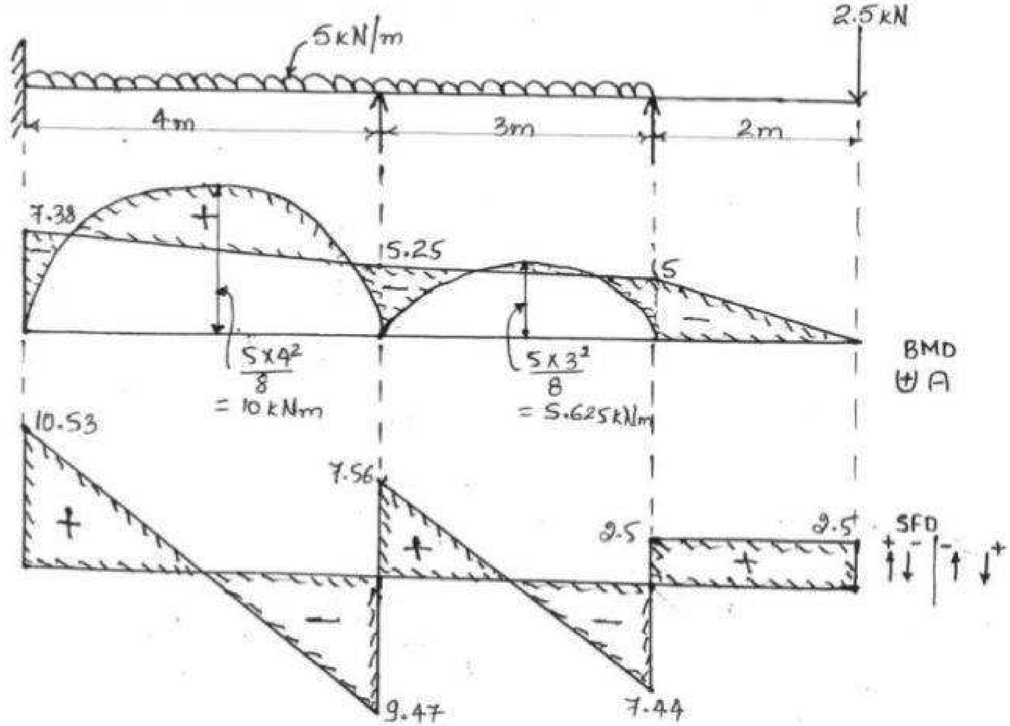
(f) Final moments:

Member (ij)	FEM M_{Fij} (kNm)	$2M'_{ij}$ (kNm)	M'_{ji} (kNm)	Final moments $\left(= M_{Fij} + 2M'_{ij} + M'_{ji}\right)$ (kNm)
AB	– 6.67	0	–0.71	–7.38
BA	6.67	2 × (-0.71) =	0	5.25
BC	–3.13	2 × (–1.06)	0	–5.25
CB				+ 5
CD	-	-	-	– 5
DC				0

FBD of each span along with reaction values which have been calculated from statics are shown below:



BMD and SFD are shown below



II. Kani's method for members with translatory joints:

Fig. 6 shows a member AB in a frame which has undergone lateral displacement at A & B so that the relative displacement is $\Delta = \Delta_B - \Delta_A$

If ends A & B are restrained from rotation FEM corresponding to this displacement are

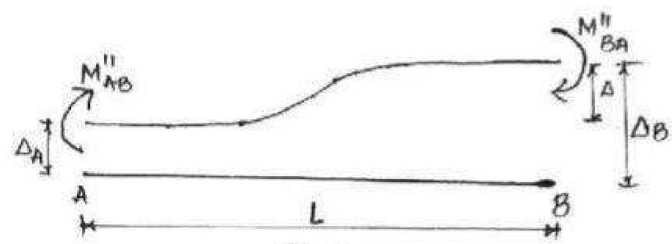


Fig. 6

$$M''_{AB} = M''_{BA} = \frac{6EI\,\Delta}{L^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

When translation of joints occurs along with rotations the true end moments are given by

$M_{AB} = M_{FAB} + 2M'_{AB} + M'_{BA} + M''_{AB}$

$M_{BA} = M_{FBA} + 2M'_{BA} + M'_{AB} + M''_{BA}$

If 'A' happens to be a joint where two or more members meet then from equilibrium of joint we have

$$\Sigma M_{AB} = 0$$

$$\Sigma M_{FAB} + 2\Sigma M'_{AB} + \Sigma M'_{BA} + \Sigma M''_{AB} = 0$$

$$\therefore \Sigma M'_{AB} = -\frac{1}{2}\left(\Sigma M_{FAB} + \Sigma M'_{BA} + \Sigma M''_{AB}\right) \ldots\ldots\ldots\ldots\ldots (8)$$

we know from equation (5)

$$M'_{AB} = \frac{K_{AB}}{\Sigma K_{AB}} \Sigma M'_{AB}$$

$$\therefore \text{Using equation (8) } M'_{AB} = -\frac{1}{2}\frac{K_{AB}}{\Sigma K_{AB}}\left(\Sigma M_{FAB} + \Sigma M'_{BA} + \Sigma M''_{AB}\right)$$

$$= U_{AB}\left(\Sigma M_{FA} + \Sigma M'_{BA} + \Sigma M''_{AB}\right)$$

Similarly

$$M'_{BA} = U_{BA}\left(\Sigma M_{FB} + \Sigma M'_{AB} + \Sigma M''_{BA}\right) \ldots\ldots\ldots\ldots\ldots (9)$$

Using the above relationships rotation contributions can be determined by iterative procedure. If lateral displacements are known the displacement moments can be determined from equation (7). If lateral displacements are unknown then additional equations have to be developed for analyzing the member.

Ex 4: In a continuous beam shown in fig. 7 (a). The support 'B' sinks by 10mm. Determine the moments by Kani's method & draw BMD.

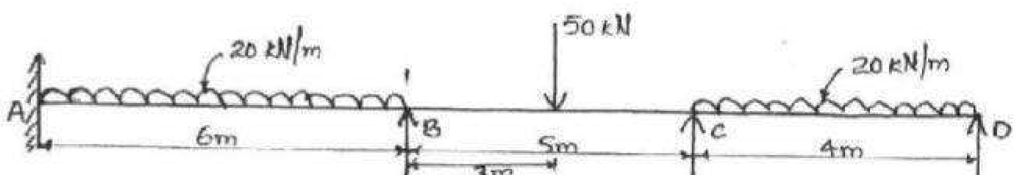


Fig. 7 (a)

Take I = 1.2×10^{-4} mm^4 & E = 2×10^5 N/mm²

Solution:

(a) Calculation of FEM:

$$M_{FAB} = -\frac{20\times 6^2}{12} - \frac{6\times 2\times 10^5 \times 1.2\times 10^{-4}\times 10^{12}\times 10}{(6000)^2\times 10^6}$$

$= -60 - 40$

$= -100$ kNm

M_{FBA} = +60 –40= 20 kNm

$$M_{FBC} = -\frac{50\times3\times2^2}{5^2} + \frac{6\times 2\times10^5 \times1.2\times10^{-4}\times10^{12}\times10}{(5000)^2\times10^6}$$

$= -24 + 57.6$

$= 33.6$ kNm

$$M_{FCB} = +\frac{50\times3^2\times2}{5^2} + \frac{6\times 2\times10^5 \times1.2\times10^{-4}\times10^{12}\times10}{(5000)^2\times10^6}$$

$= 36 + 57.6$

$= 93.6$ kNm

C & D are at same level

$$\therefore M_{FCD} = -\frac{20\times4^2}{12} = -26.67 \text{ kNm}$$

M_{FDC} = + 26.67 kNm

b) Modification in fixed end moments:

Since end 'D' is a simply supported, moment at 'D' is zero. To make moment at D as zero apply a moment of – 26.67 kNm at end D and perform the corresponding carry over to CD.

$\therefore$ Modified M_{FDC} = + 26.67 –26.67 = 0

Modified $M_{FCD} = -26.67 + \frac{1}{2}(-26.67)$

$= -40$ kNm

Other FEMs will be same as calculated earlier. Now joint 'D' will not enter the iteration process.

c) Rotation factors:

Joint	Member	Relative stiffness (K)	ΣK	Rotation Factor $U = -\frac{1}{2}\times\frac{K}{\sum K}$
B	BA	I/6 = 0.17 I	0.37 I	–0.23
	BC	I/5 = 0.2 I		–0.27
C	CB	I/5 = 0.2I	0.39I	–0.26
	CD	$\frac{3}{4}$ x I/4 = 0.19 I		–0.24

d) Sum of fixed end moments:

$$\Sigma M_{FB} = 20 + 33.6 = 53.6 \text{ kNm}$$
$$\Sigma M_{FC} = 93.6 - 40 = 53.6 \text{ kNm}$$

e) Iteration process:

Joint	B		C	
Rotation Contribution	M'_{BA} (kNm)	M'_{BC} (kNm)	M'_{CB} (kNm)	M'_{CD} (kNm)
Rotation factor	−0.23	−0.27	−0.26	−0.24
Iteration 1 (Started at B by taking $M'_{AB} = 0$ and assuming $M'_{CB} = 0$	−0.23 × (53.6 + 0 + 0) = −12.33	−0.27 x (53.6 + 0+0)= −14.47	−0.26×(53.6 – 14.47+ 0) = −10.17	−0.24 (53.6 – 14.47) 10.96= −9.39
Iteration 2	−0.23 (53.6 – 10.17) = −10.00	−0.27 (53.6 −10.17) = −11.73	−0.26 (53.6 −11.73) = −10.89	−0.24 (53.6 −11.73) =−10.05
Iteration 3	−0.23 (53.6−10.89) = −9.82	−0.27 (53.6 −10.89) =−11.53	−0.26 (53.6 −11.53) = −10.94	−0.24 (53.6 −11.53) =−10.10
Iteration 4	−0.23 (53.6 −10.94) = −9.81	−0.27 (53.6 −10.94) = −11.52	−0.26 (53.6 −11.52) = −10.94	−0.24 (53.6 −11.52) = −10.1

Iteration process has been stopped after fourth cycle since rotation contribution values are becoming almost constant. Values of FEMs, sum of fixed end moments, rotation factors along with rotation contribution values after end of each cycle in appropriate places has been shown in Fig. 7 (b).

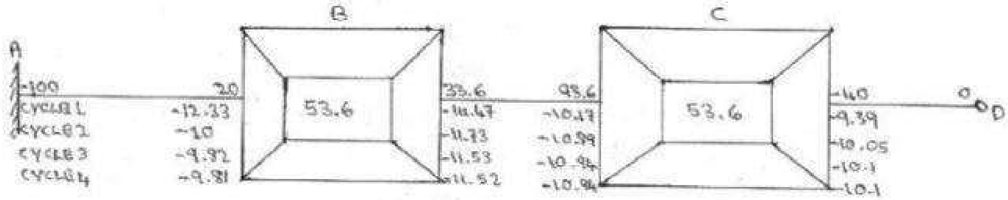
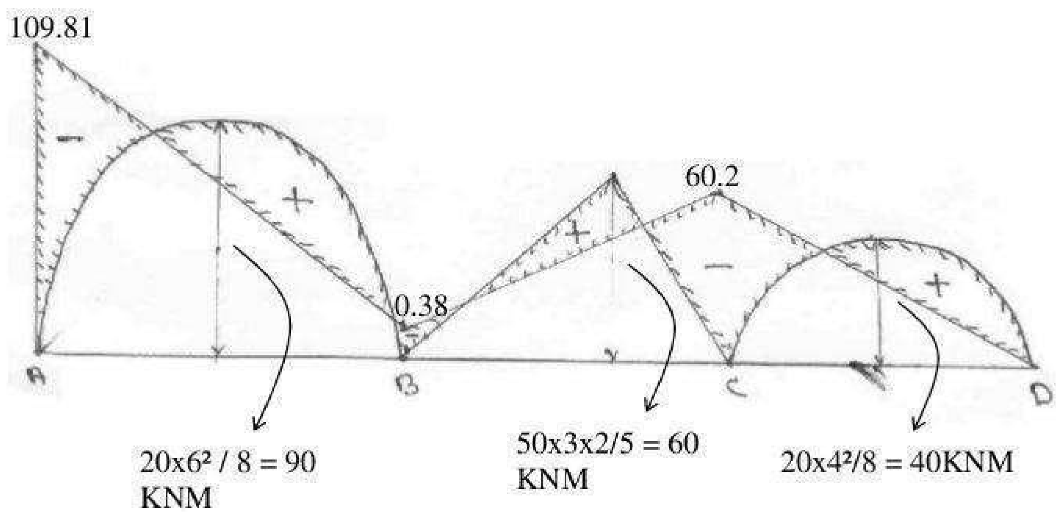


Fig:7(b)

f) Final moments:

Member (ij)	**FEM M_{Fij} (kNm)**	$2M'_{ij}$ **(kNm)**	M'_{ji} **(kNm)**	**Final moments** $\left(= M_{Fij} + 2M'_{ij} + M'_{ji}\right)$ **(kNm)**
AB	– 100	0	–9.81	–109.81
BA	20	2 × (-9.81)= -19.62	0	+0.38
BC	33.6	2 × (–11.52) = –23.04	–10.94	–0.38
CB	93.6	2 × (–10.94) = –21.88	–11.52	60.2
CD	–40	2 × (–10.1) = –20.2	0	–60.2
DC	0	0	0	0

g) BMD is shown below:



III) Analysis of frames with no translation of joints:

The frames, in which lateral translations are prevented, are analyzed in the same way as continuous beams. The lateral sway is prevented either due to symmetry of frame and loading or due to support conditions. The procedure is illustrated in following example.

Example-5. Analyze the frame shown in Figure 8 (a) by Kani's method. Draw BMD. (VTU Jan 2005 Exam).

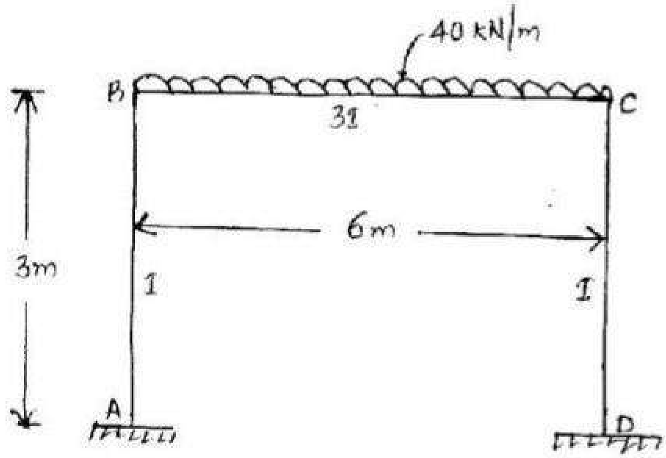


Fig-8(a)

Solution:

(a) Fixed end moments:

$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$

$$M_{FBC} = \frac{-40x6^2}{12} = -120kNm.$$

$M_{FCB} = +120kNm.$

(b) Rotation factors:

Joint	Member	Relative Stiffness (k)	Σk	Rotation factor = -½k/Σk
B	BC BA	3I/6 = 0.5I I/3 = 0.33I	0.83I	-0.3 -0.2
C	CB CD	3I/6 = 0.5I I/3 = 0.33I	0.83I	-0.3 -0.2

(c) Sum of FEM:

$\Sigma M_{FB} = -120+0 = -120$

$\Sigma M_{FC} = 120+0 = +120$

(d) Iteration process:

Joint	**B**		**C**	
Rotation Contribution	**M'_{BA}**	**M'_{BC}**	**M'_{CB}**	**M'_{CD}**
Rotation Factor	**-0.2**	**-0.3**	**-0.3**	**-0.2**
Iteration 1 stated with end B taking M'_{AB}=0 and assuming M'_{CB}=0	-0.2(-120+0) =24	-0.3(-120+0) =36	-0.2(120+36+0) = -46.8	-0.2(120+36+0) = -31.2
Iteration 2	-0.2(-120-46.8) =33.6	-0.3(-120-46.8) =50.04	-0.3(120+50.04) = -51.01	-0.2(120+50.04) = -34.01
Iteration 3	-0.2(-120-51.01) =34.2	-0.3(-120-51.01) =51.3	-0.3(120+51.3) = -51.39	-0.2 (120+51.3) = -34.26
Iteration 4	-0.2(-120-51.39) =34.28	-0.3(-120-51.39) =51.42	-0.3(120+51.42) = -51.43	-0.2 (120+51.42) = -34.28

The fixed end moments, sum of fixed and moments, rotation factors along with rotation contribution values at the end of each cycle in appropriate places is shown in figure 8(b).

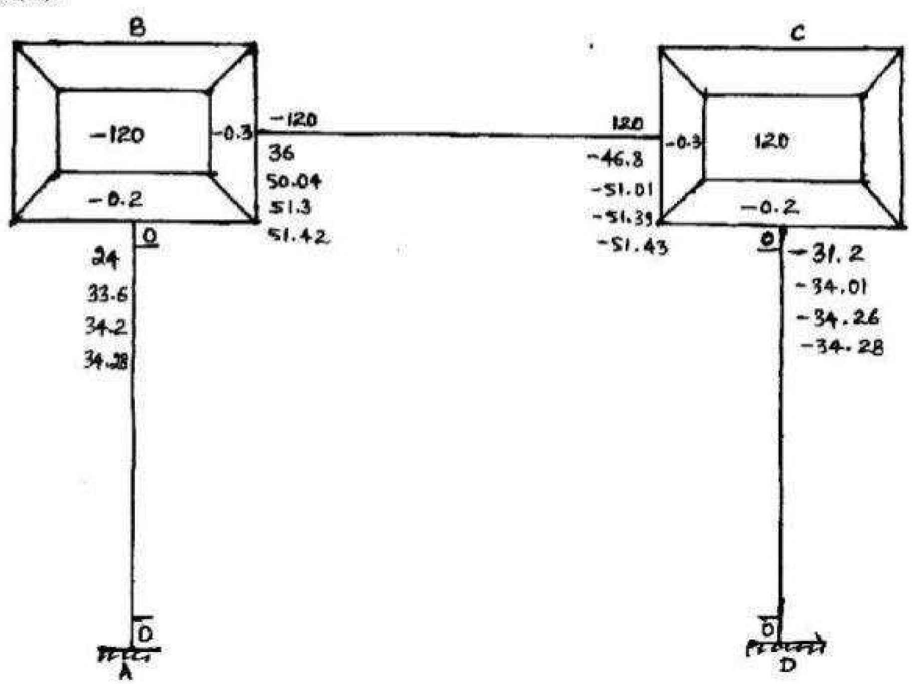


Fig-8(b)

(e) Final moments:

Member (ij)	M_{Fij}	$2M'_{ij}$ (kNm)	M'_{ji} (kNm)	(kNm) Final moment = $M_{Fij} + 2M'_{ij} + M'_{ji}$
AB	0	0	34.28	34.28
BA	0	2 x 34.28	0	68.56
BC	-120	2 x 51.42	-51.43	-68.59
CB	120	2 x (-51.43)	51.42	68.56
CD	0	2 x (-34.28)	0	-68.56
DC	0	0	-34.28	-34.28

BMD is shown below in figure-8 (c)

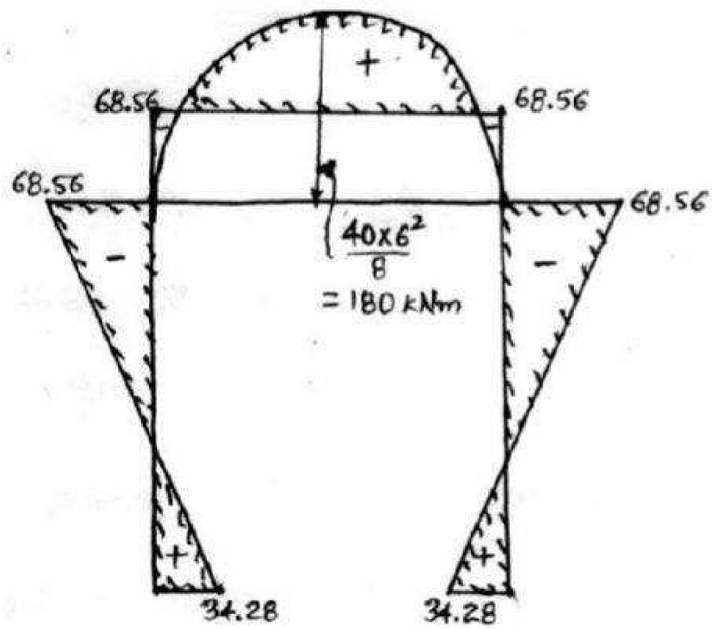


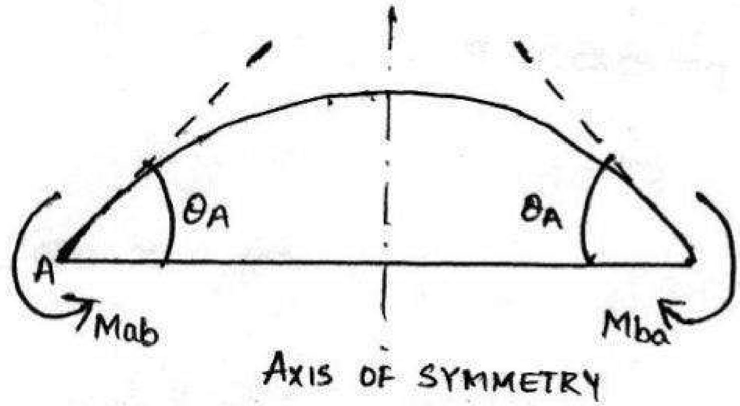
Fig-8 (c)

IV) Analysis of symmetrical frames under symmetrical loading:

Considerable calculation work can be saved if we make use of symmetry of frames and loading especially when analysis is done manually. Two cases of symmetry arise, namely, frames in which the axis of symmetry passes through the centerline of the beams and frames with the axis of symmetry passing through column line.

Case-1: (Axis of symmetry passes through center of beams):

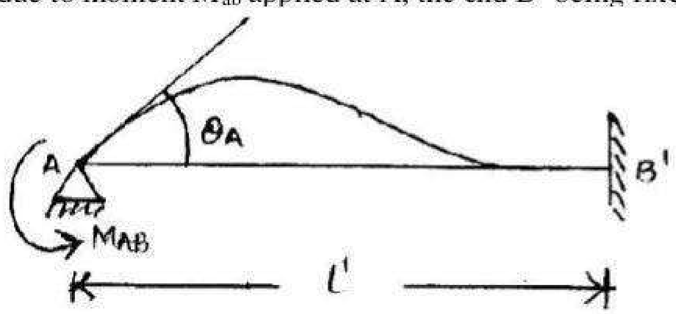
Let AB be a horizontal member of the frame through whose center, axis of symmetry passes. Let M_{ab} and M_{ba} be the end moments. Due to symmetry of deformation M_{ab} and M_{ba} are numerically equal but are opposite in their sense.



θ_A = Slope due to M_{ab} + slope due to M_{ba}

$$= \frac{M_{ab}l}{3EI} + \frac{M_{ba}l}{6EI} = \frac{M_{ab}l}{2EI}$$

Let this member be replaced by member AB' whose end A will undergo the rotation θ_A due to moment M_{ab} applied at A, the end B' being fixed.



$$\therefore \theta_A = \frac{M_{ab}l'}{4EI'}$$

Hence for equality of rotations between original member AB and the substitute member AB'

$$\frac{M_{ab}l}{2EI} = \frac{M_{ab}l'}{4EI'}$$

$$\frac{I}{l} = \frac{2I'}{l'}$$

$K = 2K'$

$\therefore K' = \frac{K}{2}$

Thus if K is the relative stiffness of original member AB, this member can be replaced by substitute member AB' having relative stiffness $\frac{K}{2}$. With this substitute member, the analysis need to be carried out for only, one half of the frame considering line of symmetry as fixed.

Example-6: Analyze the frame given in example-5 by using symmetry condition by Kani's method.

Solution:

Since symmetry axis passes through center of beam only one half of frame as shown in figure 9 (a) will be considered

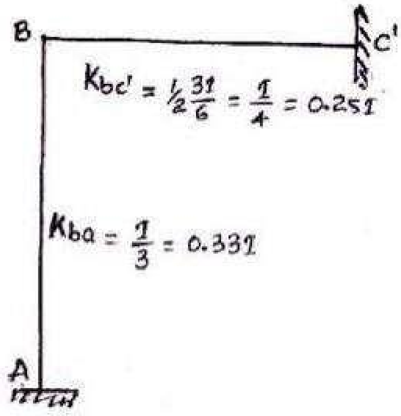


Fig-9(a)

∴ Rotation factors

$U_{BA} = -\frac{1}{2}$ x (0.33I / 0.33I + 0.25I) = -0.28

$U_{BC} = -\frac{1}{2}$ x (0.25I / 0.33I + 0.25I) = -0.22

The calculation of rotation contribution values is shown directly in figure-9(b)

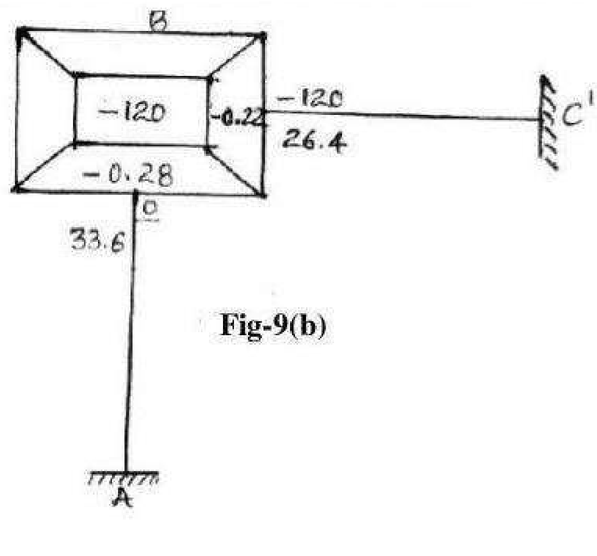


Fig-9(b)

Here we can see that rotation contributions are obtained in the first iteration only. The final moments for half the frame are shown in figure 9(c) and for full frame are shown in figure 9(d).

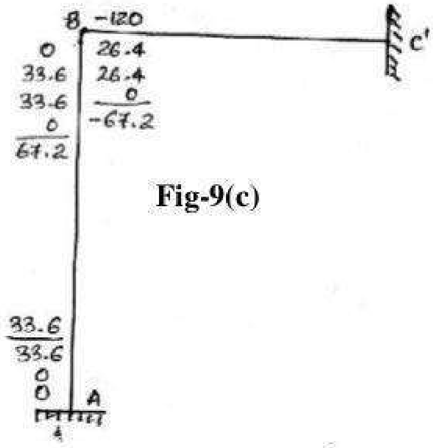


Fig-9(c)

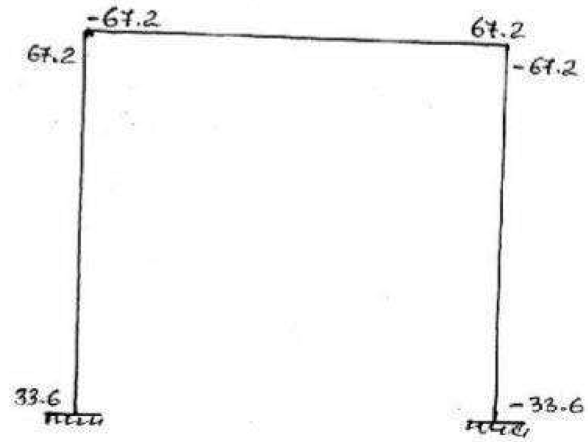


Fig-9(d)

Example-7: Analyze the frame shown in figure 10(a) by Kani's method.

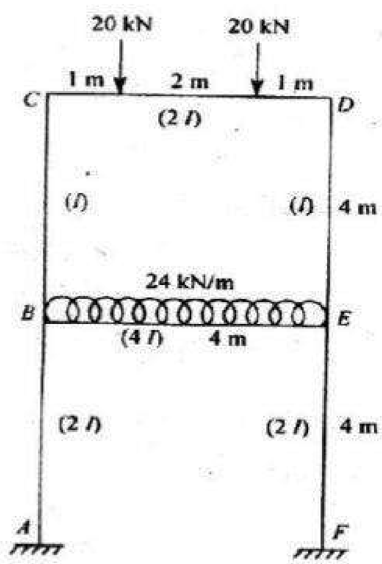


Fig-10(a)

Solution:

Analysis will be carried out taking the advantage of symmetry

(a) Fixed end moments:

$M_{Fcd} = -[(20 \times 1 \times 3^2 / 4^2) + (20 \times 3 \times 1^2 / 4^2)] = -15\text{kNm}$

$M_{Fbe} = -24 \times 4^2 / 12 = -32$ kNm.

The substitute frame is shown in figure 10(b)

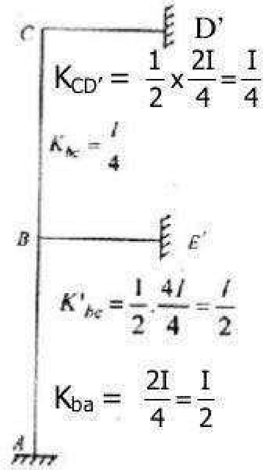


Fig-10(b)

(b) Rotation factors:

Joint	Member	Relative Stiffness K	Σk	Rotation factors $= -\frac{1}{2}\frac{K}{\Sigma K}$
B	BA	2I/4	5I/4	-1/5
	BE'	$\frac{1}{2} x \frac{4I}{4} = I/2$		-1/5
	BC	I/4		$-\frac{1}{10}$
C	CB	I/4	2I/4	-1/4
	CD'	$\frac{1}{2} x \frac{2I}{4} = \frac{I}{4}$		-1/4

Rotation contributions calculated by iteration process are directly shown in figure-10(c).

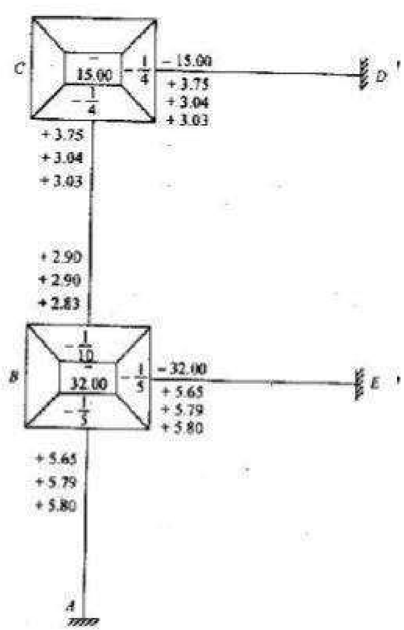


Fig-10(c)

The calculation of final moments for the substitute frame is shown in figure-10(d)

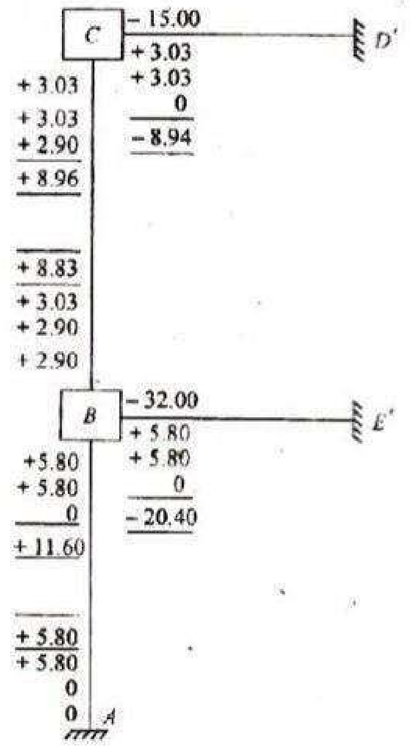


Fig-10(d)

Figure-10(e) shows final end moments for the entire frame.

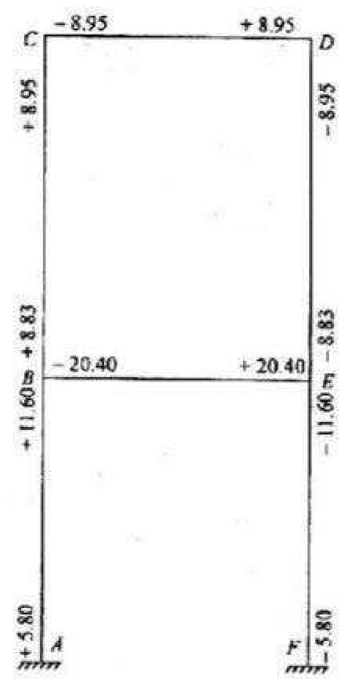


Fig-10(e)

Case 2: When the axis of symmetry passes through the column:

This case occurs when the number of bays is an even number. Due to symmetry of the loading and frame, the joints on the axis of symmetry will not rotate. Hence it is sufficient if half the frame is analyzed. The following example illustrates the procedure.

Example-8: Analyze the frame shown in figure-11(a) by Kani's method, taking advantage of symmetry and loading.

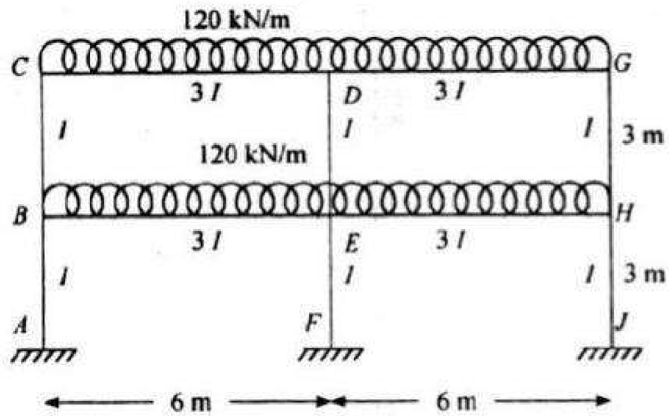


Fig-11(a)

Solution:

Only half frame as shown in figure-11(b) will be considered for the analysis.

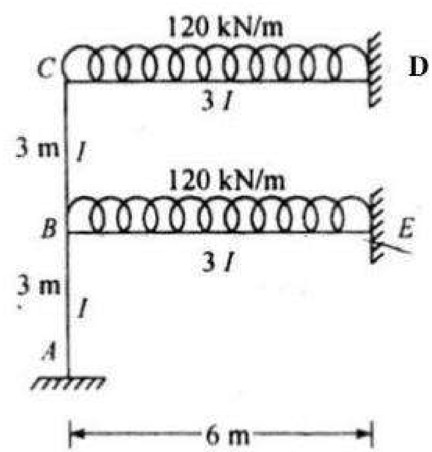


Fig-11(b)

(a) Fixed end moments:

$$M_{FBE} = -\frac{120 \times 6^2}{12} = -360 kNm$$

$$M_{FCD} = -\frac{120 \times 6^2}{12} = -360 kNm$$

(b) Rotation factors:

Joint	Member	Relative Stiffness k	Σk	Rotation factors = $-\frac{1}{2}\frac{K}{\Sigma K}$
B	BA	I/3	7I/6	-1/7
	BE	3I/6 = I/2		-3/14
	BC	I/3		-1/7
C	CB	I/3	5I/6	-1/5
	CD	3I/6 = I/2		-3/10

(c) Iteration process:

The iteration process for calculation of rotation contribution values at C & B was carried up to four cycles and values for each cycle are shown in figure-11(c).

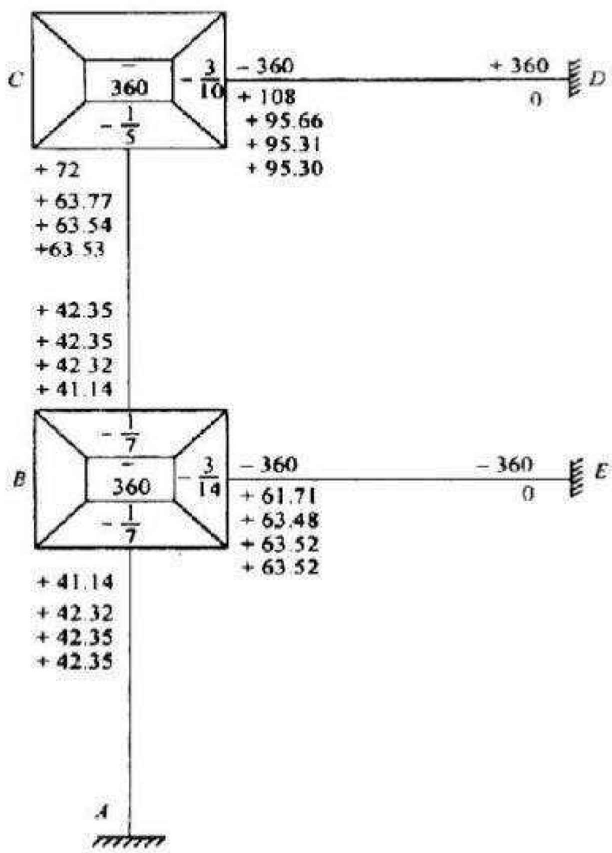


Fig-11(c)

Final moments calculations for half the frame are shown in figure-11(d) and final end moments of all the members of the frame are shown in figure-11(e).

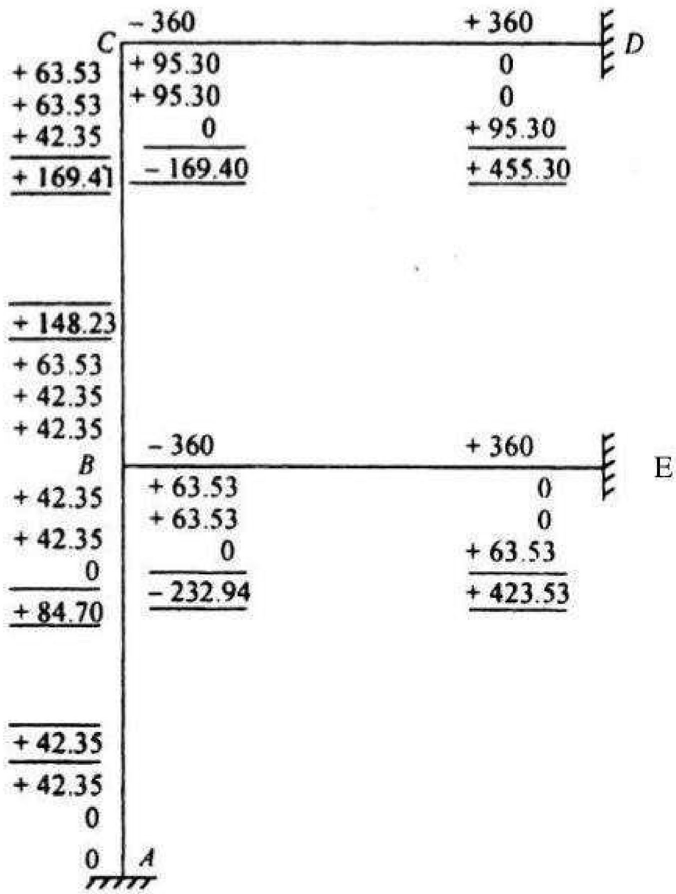


Fig-11(d)

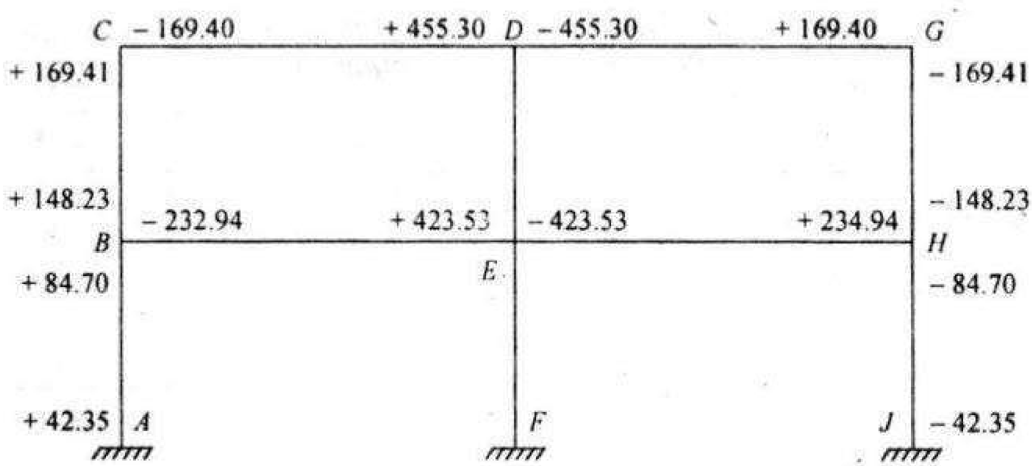


Fig-11(e)

UNIT-IV

FLEXIBILITY METHODS

This method is also known as force method or compatibility method. In this method the degree of static indeterminacy of the structure is determined and the redundants are identified. A coordinate is assigned to each redundant. Thus,P1, P2 Pn are the redundants at the coordinates 1,2, n.If all the redundants are removed , the resulting structure known as released structure, is statically determinate. This released structure is also known as basic determinate structure.

From the principle of super position the net displacement at any point in statically indeterminate structure is some of the displacements in the basic structure due to the applied loads and the redundants. This is known as the compatibility condition and may be expressed by the equation;

$\Delta_1 = \Delta_1 L + \Delta_1 R$ Where $\Delta 1$ - - - - Δn = Displ. At Co-ord.at 1,2 - -n

$\Delta_2 = \Delta_2 L + \Delta_2 R$ $\Delta_1 L$ ---- $\Delta_n L$ = Displ.At Co-ord.at 1,2 --------- n

| | | Due to aplied loads

| | | $\Delta_1 R$ ----$\Delta_n R$ = Displ.At Co-ord.at 1,2 ---------n

$\Delta_n = \Delta_n L + \Delta_n R$ Due to Redudants

The above equations may be return as $[\Delta] = [\Delta_L] + [\Delta_R]$ -------- **(1)**

$\Delta_1 = \Delta_1 L + \delta_{11} P_1 + \delta_{12} P_2 +$ --------- $\delta_{1n} P_n$

$\Delta_2 = \Delta_2 L + \delta_{21} P_1 + \delta_{22} P_2 +$ --------- $\delta_{2n} P_n$

| | | | |

| | | | |---------------------------------**(2)**

$\Delta_n = \Delta_n L + \delta_{n1} P_1 + \delta_{n2} P_2 +$ --------- $\delta_{nn} P_n$

m$\Delta = [\Delta L] + [\delta] [P]$ ----------------------(3)

m$[P] = [\delta]^{-1} \{[\Delta] - [\Delta L]\}$------------------(4)

If the net displacements at the redundants are zero then

$\Delta 1, \Delta 2$ ------- $\Delta n = 0$,

Then m $[P] = - [\delta]^{-1} [\Delta L]$ --------------(5)

The redundants P1,P2,--------- Pn are Thus determined

DISPLACEMENT METHOD :

This method is also known as stiffness or equilibrium. In this method the degree of kinematic indeterminacy (D.O.F) of the structure is determined and the coordinate is assigned to each independent displacement component.

In general, The displacement components at the supports and joints are treated as independent displacement components. Let 1,2, -------- n be the coordinates assigned to these independent displacement components $\Delta 1, \Delta 2$--------Δn.

In the first instance **lock all the supports and** the joints to obtain the **restrained structure** in which no displacement is possible at the coordinates. Let P'1, P'2 , - - - - P'n be the forces required at the coordinates 1,2,--------- n in the restrained structure in which the displacements $\Delta 1, \Delta 2$--------Δn are zero. Next, Let the supports and joints be unlocked permitting displacements $\Delta 1, \Delta 2$ -------Δn at the coordinates. Let these displacements require forces in P1d, P2d,--------Pnd at coordinates 1,2,------- n respectively.

If P1, P2- - - - - - Pn are the external forces at the coordinates 1,2, ------- n, then the conditions of equilibrium of the structure may be expressed as:

$$
\begin{aligned}
P1 &= P'1 + P1d \\
P2 &= P'2 + P2d \text{ --(1)} \\
&\vdots \\
Pn &= P'n + Pnd
\end{aligned}
$$

or $$[P] = [P'] + [Pd] \text{ -------------------(2)}$$

$$
\begin{aligned}
P1 &= P'1 + K11\,\Delta 1 + K12,\ \Delta 2 + K13\Delta 3 + \text{-----------}\ K1n\,\Delta n \\
P2 &= P'2 + K21\,\Delta 1 + K22\,\Delta 2 + K23\Delta 3 + \text{------------}\ K2n\,\Delta n \\
&\vdots \qquad \text{---------------(3)} \\
Pn &= P'n + Kn1\,\Delta 1 + Kn2,\ \Delta 2 + Kn3\Delta 3 + \text{ - - - - - - - - -}Knn\,\Delta n \quad m
\end{aligned}
$$

i.e $$[P] = [P'] + [K][\Delta] \text{ ---------------------(4)}$$

$$m\ [\Delta] = [K]^{-1}\{[P] - [P']\} \text{ ---(5)}$$

Where P = External forces

P' = Locking forces

Pd=Forces due to displacements

If the external forces act only at the coordinates the terms P'1, P'2 ,-------- P'n vanish. i. e the Locking forces are zero,then

$$[\Delta] = [K]^{-1}[P] \text{ -------------------------(6)}$$

On the other hand if there are no external forces at the coordinates then [P]=0 then

$$[\Delta] = -[K]^{-1}[P'] \text{ ----------------------------(7)}$$

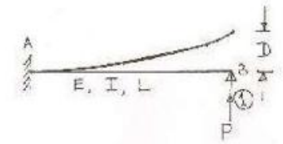
Thus the displacements can be found out. Knowing the displacements the forces are computed using slope deflection equations:

$$Mab = Mab + 2EI/L\,(2\theta a + \theta b + 3\delta/L)$$

$$Mba = Mba + 2EI/L\,(\theta a + 2\theta b + 3\delta/L)$$

Where Mab& Mba are the fixed end moments for the member AB due to external loading

FLEXIBILITY AND STIFFNESS MATRICES : **SINGLE CO-ORD.**



$D = \delta \times P$

$D = PL^3/3EI = \delta \times P$

m $\delta = L^3/3EI$

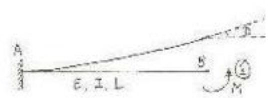
δ=Flexibility Coeff.

$P = K \times D$

$P = K \times PL^3/3EI$

$K = 3EI/L^3$

K=Stiffness Coeff.



$D = ML/EI$

$D = \delta \times M = ML/EI$

m $\delta = L/EI$

δ=Flexibility Coeff.

$M = K \times D = K \times ML/EI$

m $K = EI/L$

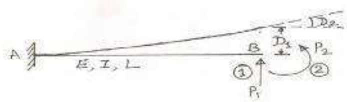
K=Stiffness Coeff.

$\delta \times K = 1$

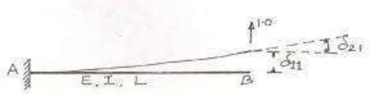
TWO CO-ORDINATE SYSTEM

$D_1 = \eth_{11}P_1 + \eth_{12} \times P_2$ & $D_2 = \eth_{21}P_1 + \eth_{22}P_2$

$$\text{m} \quad \begin{array}{c} D_1 \\ D_2 \end{array} \quad \begin{array}{cc} \eth_{11} & \eth_{12} \\ \eth_{21} & \eth_{22} \end{array} \quad \begin{array}{c} P_1 \\ P_2 \end{array}$$



$$[\eth] = \begin{bmatrix} L^3/3EI & L^2/2EI \\ L^2/2EI & L/EI \end{bmatrix}$$



Unit Force At Co-ord.(1)

$\eth_{11} = L^3/3EI$ $\quad$ $\eth_{21} = L^2/2EI$

Unit Force At Co-ord.(2)

$\eth_{12} = \eth_{21} = L^2/2EI$ $\quad$ $\eth_{22} = L/EI$

STIFFNESS MATRIX

Unit Displacement at (1)

$K_{11} = 12EI / L^3$

$K_{21} = -6EI / L^2$

Forces at Co-ord.(1) & (2)

Unit Displacement at (2)

$K_{12} = -6EI / L^2$

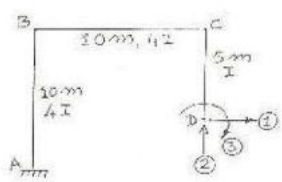
$K_{22} = 4EI / L$

Forces at Co-ord.(1) & (2)

$$P_1 = K_{11}D_1 + K_{12}D_2$$
$$P_2 = K_{21}D_1 + K_{22}D_2$$

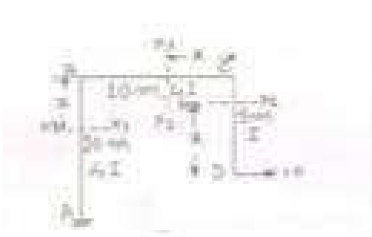
$$\therefore \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \qquad K = \begin{bmatrix} 12EI / L^3 & -6EI / L^2 \\ -6EI / L^2 & 4EI / L \end{bmatrix}$$

Develop the Flexibility and stiffness matrices for frame ABCD with reference to Coordinates shown

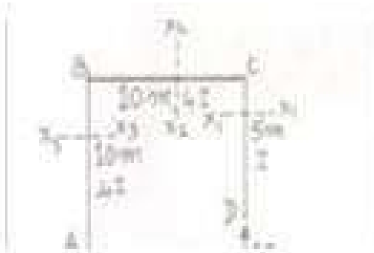


The Flexibility matrix can be developed by applying unit force successively at coordinates (1),(2) &(3) and evaluating the displacements at all the coordinates

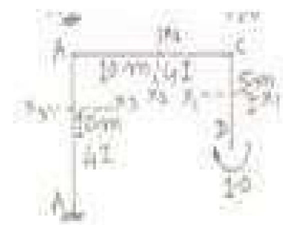
$\delta_{ij} = \int m_i m_j / EI \times ds$ $\quad\delta_{ij}$ =displacement at I due to unit load at j



Unit Load at (1)



Unit Load at (2)



Unit Load at (3)

Portion	DC	CB	BA
I	I	4I	4I
Origin	D	C	B
Limits	0 - 5	0 - 10	0 - 10
m_1	x	5	5 - x
m_2	0	x	10
m_3	- 1	-1	-1

$\delta_{11} = \int m_1.m_1 dx / EI = 125 / EI$

$\delta_{21} = \delta_{12} = \int m_1.m_2 dx / EI = 125 / 2EI$

$\delta_{31} = \delta_{13} = \int m_1.m_3 dx / EI = -25 / EI$

$\delta_{22} = \int m_2.m_2 dx / EI = 1000 / 3EI$

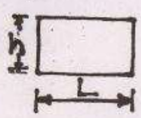
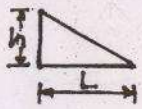
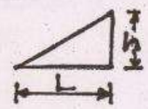
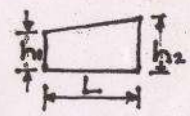
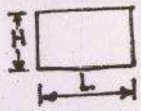
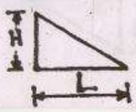
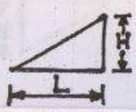
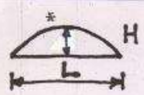
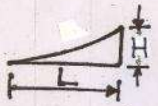
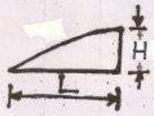
$\delta_{23} = \delta_{32} = \int m_2.m_3 dx / EI = -75 / 2EI$

$\delta_{33} = \int m_3.m_3 dx / EI = 10 / EI$

$$\therefore\ \boldsymbol{\delta} = 1 / 6EI \begin{bmatrix} 750 & 375 & -150 \\ 375 & 2000 & -225 \\ -150 & -225 & 60 \end{bmatrix}$$

INVERSING THE **FLEXIBILITY MATRIX [δ]**
THE **STIFENESS MATRIX [K]** CAN BE OBTAINED

DIAGRAM MULTIPLIERS.

m_j-Dia / M-Dia / m_i-Dia	 1	 2	 3	 4	 5
	hHL	$\frac{1}{2}hHL$	$\frac{1}{2}hHL$	$\frac{1}{2}hHL$	$\frac{HL}{2}(h_1+h_2)$
	$\frac{1}{2}hHL$	$\frac{1}{3}hHL$	$\frac{1}{6}hHL$	$\frac{hH}{6}(L+K_2)$	$\frac{HL}{6}(2h_1+h_2)$
	$\frac{1}{2}hHL$	$\frac{1}{6}hHL$	$\frac{1}{3}hHL$	$\frac{hH}{6}(L+K_1)$	$\frac{HL}{6}(h_1+2h_2)$
	$\frac{2}{3}hHL$	$\frac{1}{3}hHL$	$\frac{1}{3}hHL$	$\frac{Hh}{3L}(L^2+K_1K_2)$	$\frac{HL}{3}(h_1+h_2)$
	$\frac{1}{3}hHL$	$\frac{1}{12}hHL$	$\frac{1}{4}hHL$	$\frac{hH}{12}(L^2+K_1L+K_1^2)$	$\frac{HL}{12}(h_1+3h_2)$
	$\frac{2}{3}hHL$	$\frac{1}{4}hHL$	$\frac{5}{12}hHL$	$\frac{hH}{12}(5L^2-K_2L-K_2^2)$	$\frac{HL}{12}(3h_1+5h_2)$

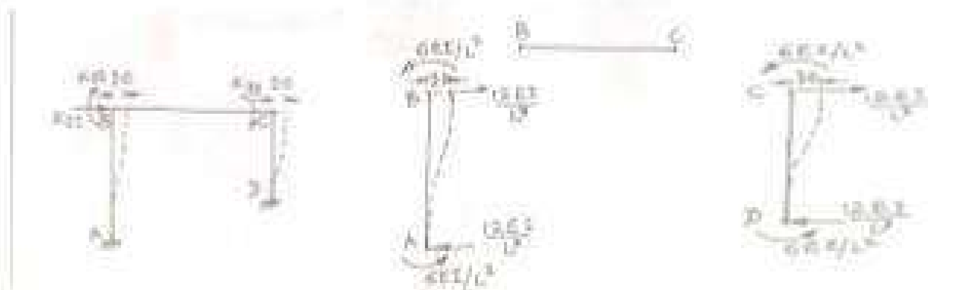
STIFFNESS MATRIX

Given Co-ordinates

$K_{11}=(12EI / L^3)_{AB} + (12EI / L^3)_{CD} = 0.144EI$

$K_{21}= -(6EI / L^2)_{AB} = -0.24EI$

$K_{31}= -(6EI / L^2)_{CD} = -0.24EI$



Unit Displacement at Co-ordinate (1)

Unit Dspl. at Co-ord.(2)

$K_{12}=K_{21}= -(6EI / L^2)_{AB} = -0.24EI$

$K_{22}=(4EI / L)_{AB} +(4EI / L)_{BC}= 3.2EI$

$K_{32}=(2EI / L)_{BC}= 0.82EI$

Unit Dspl. at Co-ord.(3)

$K_{13}=K_{31}= -(6EI / L^2)_{CD} = -0.24EI$

$K_{23}= K_{32}=(2EI / L)_{BC}= 0.82EI$

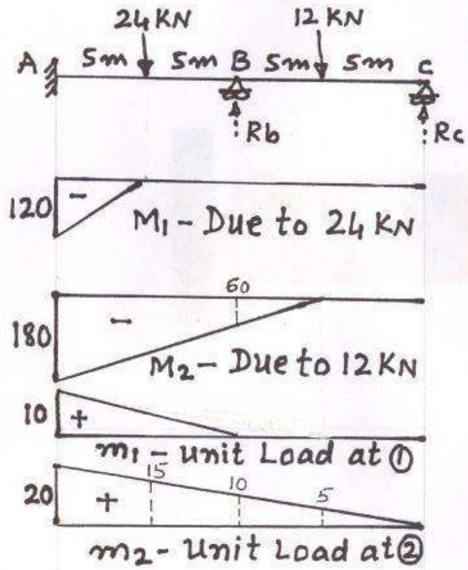
$K_{33}= (4EI / L)_{BC}+(4EI/L)_{CD}=2.4EI$

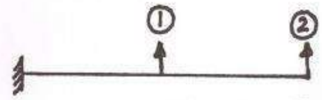
$$K=EI\begin{bmatrix} 0.144 & -0.24 & -0.24 \\ -0.24 & 3.2 & 0.8 \\ -0.24 & 0.8 & 2.4 \end{bmatrix}$$

Force Method (Flexibility or compatibility method)	**Displacement method** (Stiffness or equilibrium method)
1. Determine the degree of static indeterminacy (degree of redundancy), n	1. Determine the degree of kinematic indeterminacy , (degree of freedom), n
2. Choose the redundants.	2. Identify the independent displacement components
3. Assign coordinates 1, 2, ... ,n to the redundants	3. Assign coordinates 1, 2, ..., n to the independent displacement components.
4. Remove all the redundants to obtain to obtain the released structure.	4.Prevent all the independent displacement components to obtain the restrained structure.
5.Determine $[\Delta_L]$, the displacements at the coordinates due to applied loads acting on the released structure	5. Determine $[P^1]$ the forces at the coordinates in the restrained structure due to the loads other than those acting at the coordinates
6. Determine $[\Delta_R]$, the displacements at the coordinates due to the redundants acting on the released structure	6. Determine $[P_\Delta]$ the forces required at the coordinates in the unrestrained structure to cause the independent displacement components$[\Delta]$
7. Compute the net displacement at the coordinates $[\Delta] = [\Delta_L] + [\Delta_R]$	7. Compute the forces at the coordinates $[P] = [P^1] + [P_\Delta]$
8. Use the conditions of compatibility of displacement s to compute the redundants $[P] = [\delta]^{-1} \{[\Delta] - [\Delta_L]\}$	8. Use the conditions of equilibrium of forces to compute the displacements $[\Delta] = [k]^{-1} \{[P] - [P^1]\}$
9. Knowing the redundants, compute the internal member forces by using equations of statics.	9. Knowing the displacements, compute the internal member forces by using slope deflection equation.

LECTURE No .7

Example : 1





Ds = 2 , Rb, Rc Redundants $\frac{HL}{6}(2h_1+h_2)$

$$\Delta_{1L} = \frac{1}{EI}\left[-\frac{120\times5}{6}(2\times10+5)+\frac{10\times10}{6}\{2\times(-180)+(-60)\}\right]$$
$$= -9500/EI$$

$$\Delta_{2L} = \frac{1}{EI}\left[-\frac{120\times5}{6}(2\times20+15)+\frac{-180\times15}{6}(2\times20+5)\right]$$
$$= -25750/EI$$

$\frac{hHL}{3}$

$$\delta_{11} = \frac{1}{EI}\left[\frac{1}{3}\times10\times10\times10\right] = \frac{1000}{3EI}$$
$$\delta_{12} = \delta_{21} = \frac{1}{EI}\left[\frac{1}{6}\times10\times10(2\times20+10)\right] = \frac{2500}{EI}$$
$$\delta_{22} = \frac{1}{EI}\left[\frac{1}{3}\times20\times20\times20\right] = \frac{8000}{3EI}$$

$$\therefore [\delta] = \frac{1}{3EI}\begin{bmatrix}1000 & 2500\\ 2500 & 8000\end{bmatrix}$$

Using, $[P] = [\delta]^{-1}\{[\Delta]-[\Delta_L]\}$

$[\Delta] = 0$ Net displacements zero

$$\therefore \begin{bmatrix}P_1\\ P_2\end{bmatrix} = -3EI\begin{bmatrix}1000 & 2500\\ 2500 & 8000\end{bmatrix}^{-1}\begin{bmatrix}-9500/EI\\ -25750/EI\end{bmatrix} = \begin{bmatrix}19.93\\ 3.43\end{bmatrix}$$

ie $P_1 = R_b = 19.93$ KN & $P_2 = R_c = 3.43$ KN.

Draw B.M. & S.F. Diagram.

2 If supports B & C sinks down by units $\frac{200}{EI}$ & $\frac{100}{EI}$ units analyse the beam.

In this, net displacements, at 1 $\Delta_1 = -\frac{200}{EI}$ & at 2 $\Delta_2 = -\frac{100}{EI}$

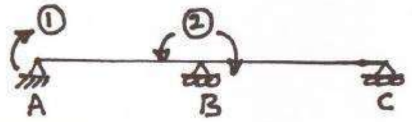
Compatibility Eqn is modified as –

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{3EI}\begin{bmatrix} 1000 & 2500 \\ 2500 & 8000 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -\frac{200}{EI} \\ -\frac{100}{EI} \end{bmatrix} - \begin{bmatrix} -\frac{9500}{EI} \\ -\frac{25750}{EI} \end{bmatrix} \right\}$$

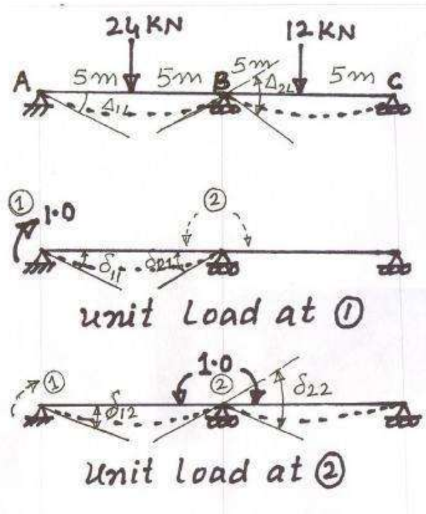
$$= 3EI \begin{bmatrix} 1000 & 2500 \\ 2500 & 8000 \end{bmatrix}^{-1} \begin{bmatrix} \frac{9300}{EI} \\ \frac{25650}{EI} \end{bmatrix} = \begin{bmatrix} 17.61 \\ 4.11 \end{bmatrix}$$

$\therefore$ $P_1 = R_b = 17.61$ KN & $P_2 = R_C = 4.11$ KN.

(3) Choose the support moments Ma, Mb as the redundants.



ie basic determinate structure will be two simply supported beams:



$$\Delta_{1L} = \frac{WL^2}{16EI} = \frac{24\times10^2}{16EI} = \frac{150}{EI}$$

$$\Delta_{2L} = \frac{24\times10^2}{16EI} + \frac{12\times10^2}{16EI} = \frac{225}{EI}$$

$$\delta_{11} = \frac{L}{3EI} = \frac{10}{3EI} \qquad \delta_{21} = \frac{L}{6EI} = \frac{10}{6EI} = \frac{5}{3EI}$$

$$\delta_{12} = \frac{L}{6EI} = \frac{5}{3EI} = \delta_{21}$$

$$\delta_{22} = \left(\frac{L}{3EI}\right)_{AB} + \left(\frac{L}{3EI}\right)_{BC}$$

$$= \frac{10}{3EI} + \frac{10}{3EI}$$

$$= \frac{20}{3EI}$$

$$\therefore [\delta] = \begin{bmatrix} \frac{10}{3EI} & \frac{5}{3EI} \\ \frac{5}{3EI} & \frac{20}{3EI} \end{bmatrix}$$

Using — $[P] = [\delta]^{-1} \left\{ [\Delta] - [\Delta_L] \right\}$ $\quad [\Delta] = 0$

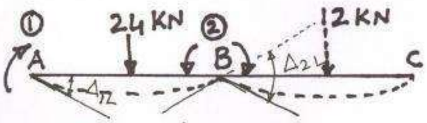
$$\therefore \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = - \begin{bmatrix} \frac{10}{3EI} & \frac{5}{3EI} \\ \frac{5}{3EI} & \frac{20}{3EI} \end{bmatrix}^{-1} \begin{bmatrix} \frac{150}{EI} \\ \frac{225}{EI} \end{bmatrix} = \begin{bmatrix} -32.10 \\ -25.70 \end{bmatrix}$$

$\therefore$ $P_1 = M_a = -32.10$ KNm $\quad \therefore$ $M_a = 32.10$ KNm (hogging)

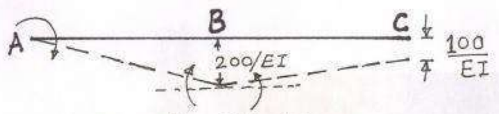
$P_2 = M_b = -25.70$ " $\quad \therefore$ $M_b = 25.70$ " (")

(Both M_a & M_b were assumed sagging)

(4) If sinking of supports B & C, which are $\frac{200}{EI}$ & $\frac{100}{EI}$, are considered:



Due to Loading



Due to Sinking

$$\Delta_{1L} = \frac{24 \times 10^2}{16EI} + \frac{200}{EI} \times \frac{1}{10} = \frac{170}{EI}$$

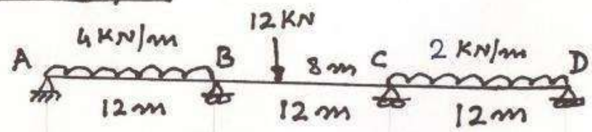
$$\Delta_{2L} = \frac{24 \times 10^2}{16EI} + \frac{12 \times 10^2}{16EI} - \frac{200}{EI} \times \frac{1}{10} - \left(\frac{200}{EI} - \frac{100}{EI} \right) \times \frac{1}{10} = \frac{195}{EI}$$

Considering net displ. $[\Delta] = 0$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = - \begin{bmatrix} \frac{10}{3EI} & \frac{5}{3EI} \\ \frac{5}{3EI} & \frac{20}{3EI} \end{bmatrix}^{-1} \begin{bmatrix} \frac{170}{EI} \\ \frac{195}{EI} \end{bmatrix} = \begin{bmatrix} -41.57 \\ -18.86 \end{bmatrix}$$

ie $P_1 = M_a = -41.57$ (hogging), $P_2 = M_b = -18.86$ (hogging)

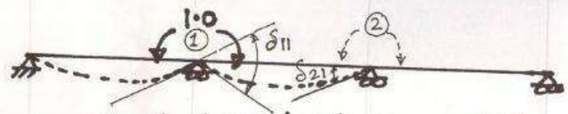
Example:



$D_s = 2$

EI – constant

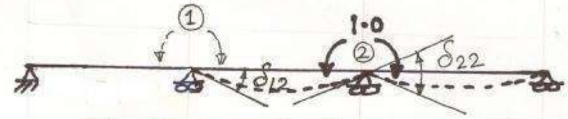
M_b & M_c redundants.



Unit Load at co-ord ①

$$\delta_{11} = \left(\frac{L}{3EI}\right)_{AB} + \left(\frac{L}{3EI}\right)_{BC} = \frac{12}{3EI} + \frac{12}{3EI} = \frac{8}{EI}$$

$$\delta_{21} = \left(\frac{L}{6EI}\right)_{BC} = \frac{12}{6EI} = \frac{2}{EI}$$

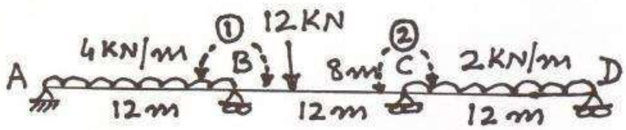


Unit Load at co-ord ②

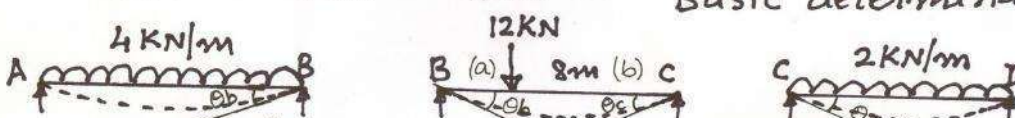
$$\delta_{12} = \left(\frac{L}{6EI}\right)_{BC} = \delta_{21} = \frac{2}{EI}$$

$$\delta_{22} = \left(\frac{L}{3EI}\right)_{BC} + \left(\frac{L}{3EI}\right)_{CD} = \frac{12}{3EI} + \frac{12}{3EI} = \frac{8}{EI}$$

$$\therefore [\delta] = \begin{bmatrix} \frac{8}{EI} & \frac{2}{EI} \\ \frac{2}{EI} & \frac{8}{EI} \end{bmatrix}$$



Since M_b & M_c are redundants, Basic determinate str. – AB, BC, & CD – Simply supported beams



$\theta_a = \theta_b = \dfrac{wl^3}{24EI}$ $\quad \theta_b = \dfrac{Pab(2l-a)}{6EIL}$ $\quad \theta_c = \dfrac{Pa(L^2-a^2)}{6EIL}$ $\quad \theta_c = \theta_d = \dfrac{wl^3}{24EI}$

Δ_{1L} = Displ. at co-ord. ① due to applied loads for determinate str

$$= (\theta_b)_{AB} + (\theta_b)_{BC} = \frac{4 \times 12^3}{24EI} + \frac{12 \times 4 \times 8\,(2 \times 12 - 4)}{6EI \times 12} = \frac{1184}{3EI}$$

Δ_{2L} = Displ. at co-ord. ② due to applied loads for determinate str.

$$= (\theta_c)_{BC} + (\theta_c)_{CD} = \frac{12 \times 4\,(12^2 - 4^2)}{6EI \times 12} + \frac{2 \times 12^3}{24EI} = \frac{688}{3EI}$$

Using compatibility condition.

$$[P] = [\delta]^{-1}\left\{[\Delta] - [\Delta_L]\right\} \quad \text{Net displ. } [\Delta] = 0$$

$$\therefore \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = - \begin{bmatrix} \frac{8}{EI} & \frac{2}{EI} \\ \frac{2}{EI} & \frac{8}{EI} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1184}{3EI} \\ \frac{688}{3EI} \end{bmatrix} = \begin{bmatrix} -45.0 \\ -17.4 \end{bmatrix}$$

ie $P_1 = M_b = -45.0$ KNm (hogging)

$P_2 = M_c = -17.4$ KNm (hogging)

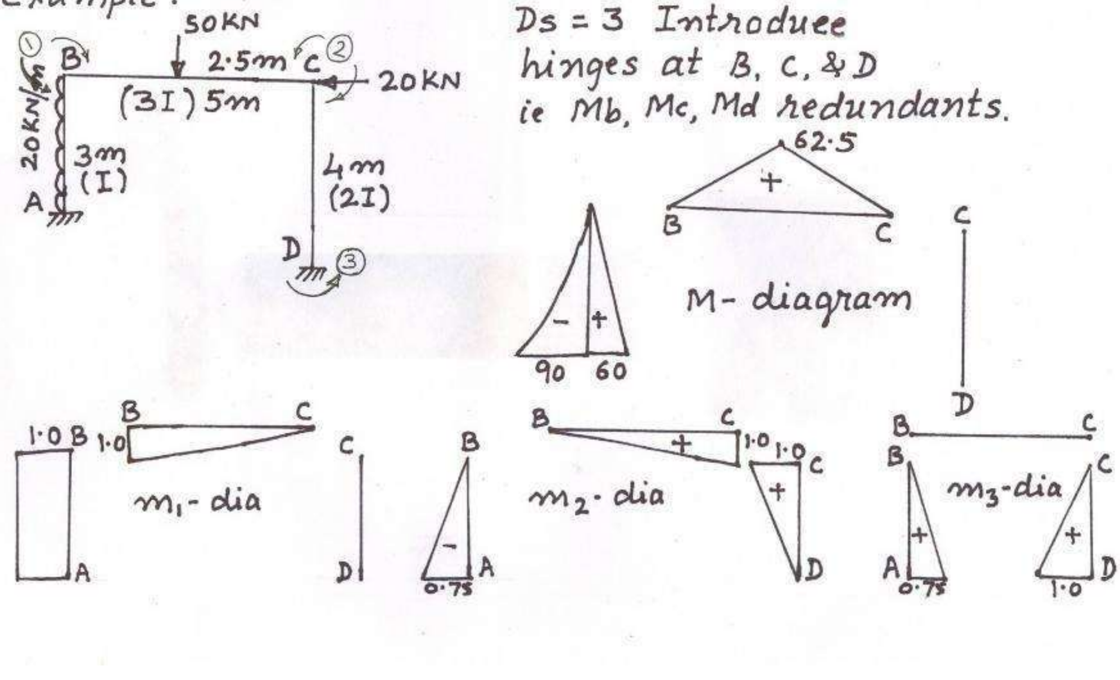
Draw B.M. & S.F. Diagrams.

The problem may also be solved by taking Rb & Rc redundant.

Example :

Ds = 3 Introduce hinges at B, C, & D

ie Mb, Mc, Md redundants.



FLEXIBILTY MATRIX:

$$\delta_{11} = \underset{(AB)}{\frac{1\times1\times3}{EI}} + \underset{(BC)}{\frac{1\times1\times5}{3\times3EI}} + 0 = \frac{3.56}{EI}$$

$$\delta_{22} = \underset{(AB)}{\frac{-0.75\times(-0.75)\times3}{3EI}} + \underset{(BC)}{\frac{1\times1\times5}{3\times3EI}} + \underset{(CD)}{\frac{1\times1\times4}{3\times2EI}} = \frac{1.785}{EI}$$

$$\delta_{33} = \frac{0.75\times0.75\times3}{3EI} + 0 + \frac{1\times1\times4}{3\times2EI} = \frac{1.223}{EI}$$

$$\delta_{12} = \delta_{21} = \frac{-0.75\times1\times3}{2\times EI} + \frac{1\times1\times5}{6\times3EI} + 0 = -\frac{0.847}{EI}$$

$$\delta_{23} = \delta_{32} = -\frac{0.75\times0.75\times3}{3\times EI} + 0 + \frac{1\times1\times4}{6\times2EI} = -\frac{0.23}{EI}$$

$$\delta_{31} = \delta_{13} = \frac{1\times0.75\times3}{2\times EI} + 0 + 0 = \frac{1.125}{EI}$$

$$\therefore [\delta] = \frac{1}{EI}\begin{bmatrix} 3.56 & -0.847 & 1.125 \\ -0.847 & 1.785 & -0.23 \\ 1.125 & -0.23 & 1.22 \end{bmatrix}$$

$$\Delta_{1L} = \frac{1\times(-90)\times3}{3\times EI} + \frac{1\times60\times3}{2\times EI} + \frac{1\times62.5(5+2.5)}{6\times3EI} + 0$$

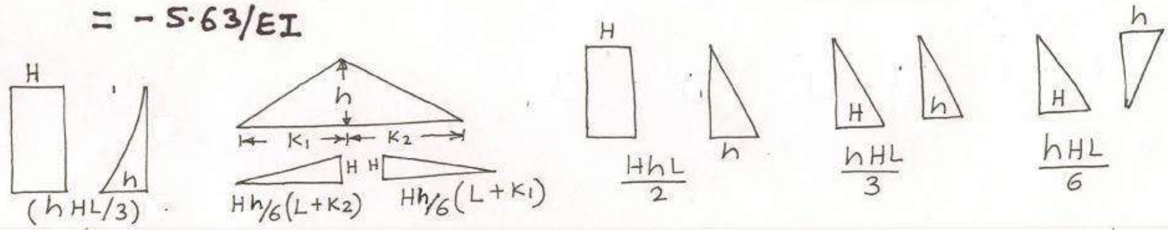
$$= 26.04/EI$$

$$\Delta_{2L} = \frac{-0.75(-90)\times3}{4\times EI} + \frac{-0.75\times60\times3}{3\times EI} + \frac{1\times62.5(5+2.5)}{6\times3EI} + 0$$

$$= 31.67/EI$$

$$\Delta_{3L} = \frac{0.75(-90)\times3}{4\times EI} + \frac{0.75\times60\times3}{3\times EI} + 0 + 0$$

$$= -5.63/EI$$



Using Compatibility condition,

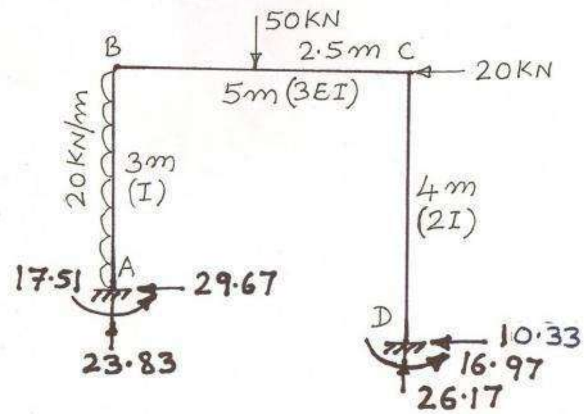
$$[P] = [\delta]^{-1}\{[\Delta]-[\Delta_L]\} \qquad [\Delta]=0$$

$$\therefore \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -EI \begin{bmatrix} 3.56 & -0.847 & 1.125 \\ -0.847 & 1.785 & -0.23 \\ 1.125 & -0.23 & 1.22 \end{bmatrix}^{-1} \frac{1}{EI} \begin{bmatrix} 26.04 \\ 31.67 \\ -5.63 \end{bmatrix} = \begin{bmatrix} -18.50 \\ -24.34 \\ +16.97 \end{bmatrix}$$

$\therefore$ $P_1 = M_b = 18.50$ (ten. out) $-ve$

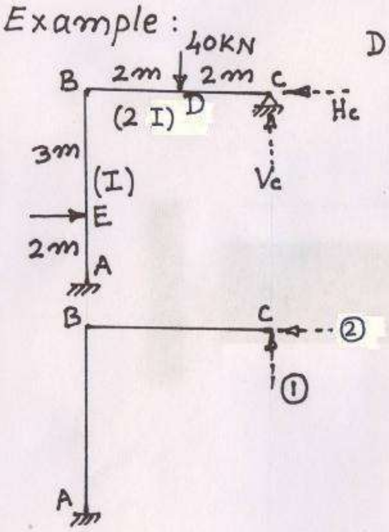
$P_2 = M_c = 24.34$ (- do -)

$P_3 = M_d = 16.97$ (ten. in) $+ve$



Example : $Ds = 2$ Remove hinge at C.

ie V_c & H_c redundants



S. No.	Seg-ment	Ori-gin	Limits	M	m_1	m_2	I
1	CD	C	0-2	0	x	0	2I
2	DB	C	2-4	$-40(x-2)$	x	0	2I
3	BE	B	0-3	-80	4	x	I
4	EA	B	3-5	$-80-50(x-3)$	4	x	I

$$\Delta_{1L} = \int_0^l \frac{M m_1}{EI}\,dx = -\frac{2133.33}{EI}$$

$$\Delta_{2L} = \int_0^l \frac{M m_2}{EI}\,dx = -\frac{1433.33}{EI}$$

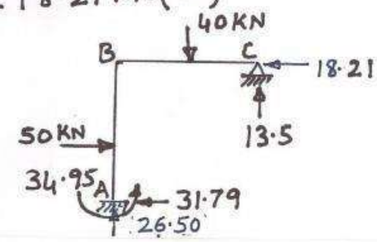
$$\delta_{11} = \int_0^l \frac{m_1\cdot m_1}{EI}\,dx = \frac{90.67}{EI} \qquad \delta_{12}=\delta_{21}=\int_0^l \frac{m_1 m_2}{EI}\,dx$$

$$\delta_{22} = \int_0^l \frac{m_2\cdot m_2}{EI}\,dx = \frac{41.67}{EI} \qquad\qquad = \frac{50}{EI}$$

Using compatibility condition, $[P] = [\delta]^{-1}\{[\Delta]-[\Delta_L]\}$

$$[\Delta]=0 \quad \therefore \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -\begin{pmatrix} 90.67/EI & 50/EI \\ 50/EI & 41.67/EI \end{pmatrix}^{-1} \begin{bmatrix} -2133.33/EI \\ -1433.33/EI \end{bmatrix} = \begin{bmatrix} 13.50 \\ 18.21 \end{bmatrix}$$

$\therefore \quad P_1 = V_c = 13.50\ KN\ (\uparrow) \qquad P_2 = H_c = 18.21\ KN\ (\leftarrow)$



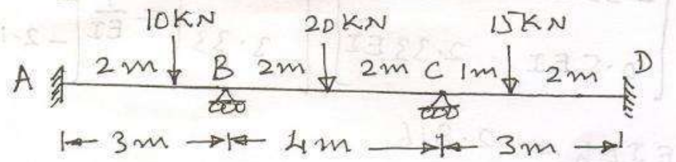
$M_b = 13.5\times4 - 40\times2 = -26.0$ (hogging)

$M_a = 13.5\times4 + 18.21\times5 - 40\times2 - 50\times2$

$= -34.95$ (hogging)

LECTURE No. 9

Ex-①



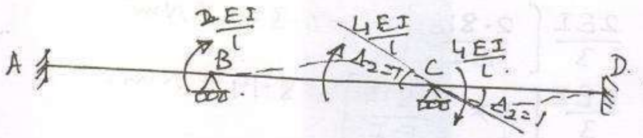
$D_K = 2$

θ_b, θ_c.

coord. ① & ②

co-ord

unit displ. at ①



unit displ. at ②

F.E.M: AB: $\overline{M}ab = -\dfrac{10\times2\times1^2}{3^2} = -2.22$ KNm

$\overline{M}ba = +\dfrac{10\times2^2\times1}{3^2} = 4.44$ "

BC: $\overline{M}bc = -\dfrac{20\times4}{8} = -10.00$ "

$\overline{M}cb = + \;-\;\;- = +10.00$ "

CD: $\overline{M}cd = -\dfrac{15\times2^2\times1}{3^2} = -6.67$ "

$\overline{M}dc = +\dfrac{15\times1^2\times2}{3^2} = 3.33$ ".

Locking forces: $P_1' = 4.44 - 10 = -5.56$

$P_2' = +10 - 6.67 = 3.33$

Stiffness Matrix:- $[K] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$

first colm: unit displ. at co-ord ①

$K_{11} = \left(\dfrac{4EI}{l}\right)_{AB} + \left(\dfrac{4EI}{l}\right)_{BC} = \dfrac{4E\times I}{3} + \dfrac{4EI}{4} = 2.33\,EI.$

$K_{21} = \left(\dfrac{2EI}{l}\right)_{BC} = \dfrac{2EI}{4} = 0.50\,EI.$

2nd colm: unit displ. at co-ord ②

$K_{12} = K_{21} = \left(\dfrac{2EI}{l}\right)_{BC} = 0.50\,EI.$

$K_{22} = \left(\dfrac{4EI}{l}\right)_{BC} + \left(\dfrac{4EI}{l}\right)_{CD} = \dfrac{4EI}{4} + \dfrac{4EI}{3} = 2.33\,EI$

$\therefore$ No ext. loads at the co-ord. $P_1 = P_2 = 0$.

$[\Delta] = [K]^{-1}[\{P\} - \{P'\}] \qquad [P] = 0.$

$\therefore [\Delta] = -[K]^{-1}[P'].$

$$\therefore \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = - \begin{bmatrix} 2.33\,EI & 0.5\,EI \\ 0.5\,EI & 2.33\,EI \end{bmatrix}^{-1} \begin{bmatrix} -5.56 \\ 3.33 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 2.816 \\ -2.032 \end{bmatrix}.$$

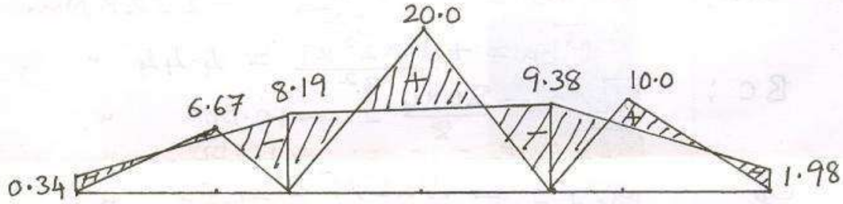
$$\therefore \quad EI\Delta_1 = EI\theta_b = 2.816$$
$$EI\Delta_2 = EI\theta_c = -2.032$$

$\therefore$ $M_{ab} = -0.34$ KNm $M_{ba} = 8.19$ KNm

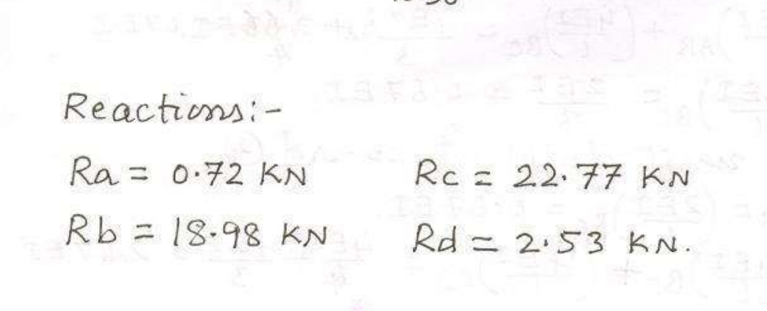
$M_{bc} = -8.19$ " $M_{cb} = 9.38$ "

$M_{cd} = -9.38$ " $M_{dc} = 1.98$ "

$$M_{ab} = -2.22 + \frac{2EI}{3}\left[\frac{2.816}{EI}\right] = -0.34 \text{ KNm}$$
$$M_{ba} = 4.44 + \frac{2EI}{3}\left[2 \times \frac{2.816}{EI}\right] = 8.19 \text{ KNm.}$$



BM Diagram



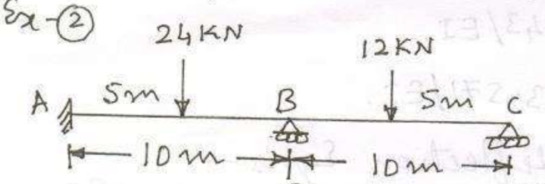
Reactions:-

$R_a = 0.72$ KN $R_c = 22.77$ KN

$R_b = 18.98$ KN $R_d = 2.53$ KN.

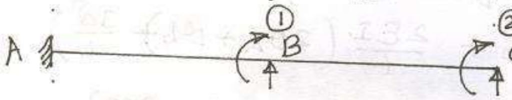
Supports B & C sinks down by $\frac{200}{EI}$ & $\frac{100}{EI}$ units.

Ex-(2)

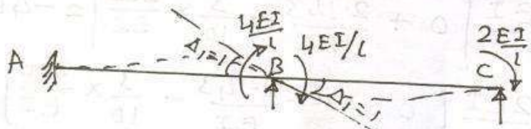


$DK = 2$

$\theta b, \theta c,$



co-ord. (1) & (2)

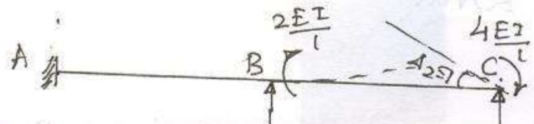


unit displ. at (1)

$K_{11} = \left(\frac{4EI}{l}\right)_{AB} + \left(\frac{4EI}{l}\right)_{BC}$

$= 0.8EI.$

$K_{21} = \left(\frac{2EI}{l}\right)_{BC} = 0.2EI.$



$K_{12} = K_{21} = 0.2EI$

unit displ. at (2)

$K_{22} = \left(\frac{4EI}{l}\right) = 0.4EI.$

F.E.Ms : Locking forces :

FEM = $(FEM)_{loading} + (FEM)_{sinking}$.

F.E.M. - due to sinking -

$\overline{M}_{ab} = -\frac{24\times10}{8} - \frac{6EI}{10^2}\times\frac{200}{EI}$

$= -30 - 12 = -42$ KNm

$\overline{M}_{ba} = +30 - 12 = +18$ "

$\overline{M}_{bc} = -\frac{12\times10}{8} + \frac{6EI}{10^2}\times\frac{100}{EI}$

$= -15 + 6 = -9.0$ KNm

$\overline{M}_{cb} = +15 + 6 = 21$ "

Locking force at (1) $P_1' = 18 - 9 = +9.0$

" (2) $P_2' = 21.0$ KNm.

$\therefore$ [Δ] No. Ext forces at co-ord (1) & (2) $\therefore P_1 = P_2 = 0$.

$\therefore \{\Delta\} = -[K]^{-1}\{P'\}$

$$\therefore \begin{bmatrix}\Delta_1 \\ \Delta_2\end{bmatrix} = -\begin{bmatrix}0.8EI & 0.2EI \\ 0.2EI & 0.4EI\end{bmatrix}^{-1}\begin{bmatrix}9.0 \\ 21.0\end{bmatrix} = \begin{bmatrix}2.143/EI \\ -53.571/EI\end{bmatrix}$$

$\therefore \Delta_1 = \theta_b = 2.143/EI.$

$\Delta_2 = \theta_c = -53.571/EI.$

$\therefore$ using slope deflection Eqns —

$$M_{ab} = \overline{M}_{ab} + \frac{2EI}{l}\left(2\theta_a + \theta_b - \frac{3\delta}{l}\right) \qquad \delta = -\frac{200}{EI}.$$

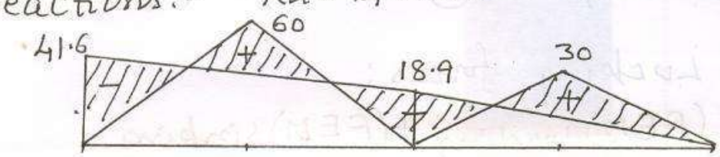
$$\therefore M_{ab} = -30 + \frac{2EI}{10}\left[0 + \frac{2.143}{EI} - \frac{3}{10} \times \frac{200}{EI}\right] = -41.6 \text{ KNm}$$

$$M_{ba} = +30 + \frac{2EI}{10}\left[0 + 2 \times \frac{2.143}{EI} - \frac{3}{10} \times \frac{200}{EI}\right] = 18.9 \text{ KNm}$$

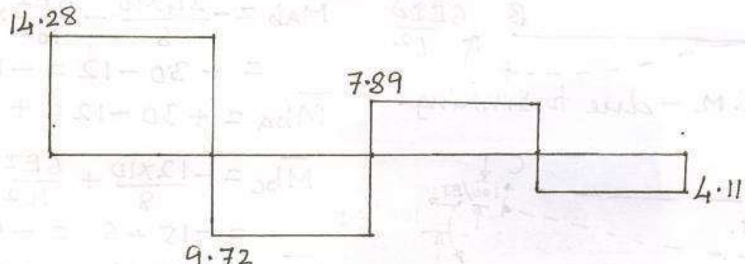
III^{ly} $M_{bc} = -18.9$ KNm

$M_{cb} = 0.$

Reactions:- $R_a = 14.28$ KN $\quad R_b = 17.61$ KN $\quad R_c = 4.11$ KN.

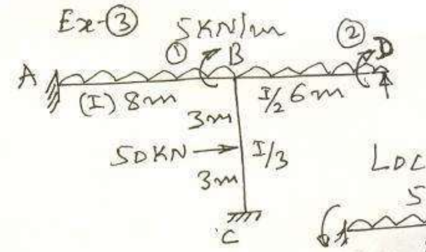


BMD



SFD

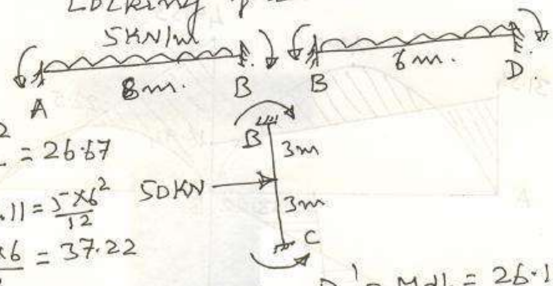
Ex-③



DK = 2.

θb, θd are the displ. co-ord. ①&②

Locking forces -



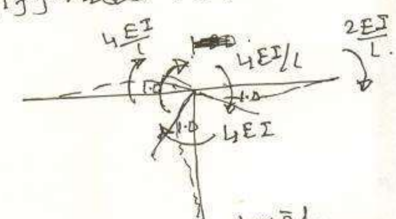
$\therefore -M_{ab} = M_{ba} = \frac{5\times 8^2}{12} = 26.67$

$-M_{bd} = M_{db} = 26.11 = \frac{5\times 6^2}{12}$

$M_{bc} = -M_{cb} = \frac{50\times 6}{8} = 37.22$

$\therefore P_1' = 26.67 - 26.11 + 37.22 = 38.06 \qquad P_2' = M_{db} = 26.11$

Stiffness Matrix:-



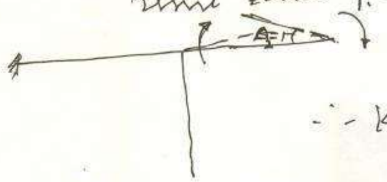
$\therefore K = \left(\frac{4EI}{l}\right)$ for AB, BD, BC.

$\therefore K_{11} = \frac{4E\times I}{8} + \frac{4EI/2}{6} + \frac{4EI/3}{6}$

$= 1.05EI$

$K_{21} = \left(\frac{2EI}{l}\right)_{BD} = \frac{2E\times I/3}{6} = 0.167EI$

unit displ. at ①



$K_{22} = \left(\frac{4EI}{l}\right)_{BD} = \frac{4E\times I/3}{6} = 0.33EI$

$K_{12} = K_{21} = 0.167EI.$

$\therefore K = EI\begin{bmatrix} 1.05 & 0.167 \\ 0.167 & 0.33 \end{bmatrix}.$

$\therefore$ Using Eql. Eqn -

$\{P\} = 0$ = Net forces.

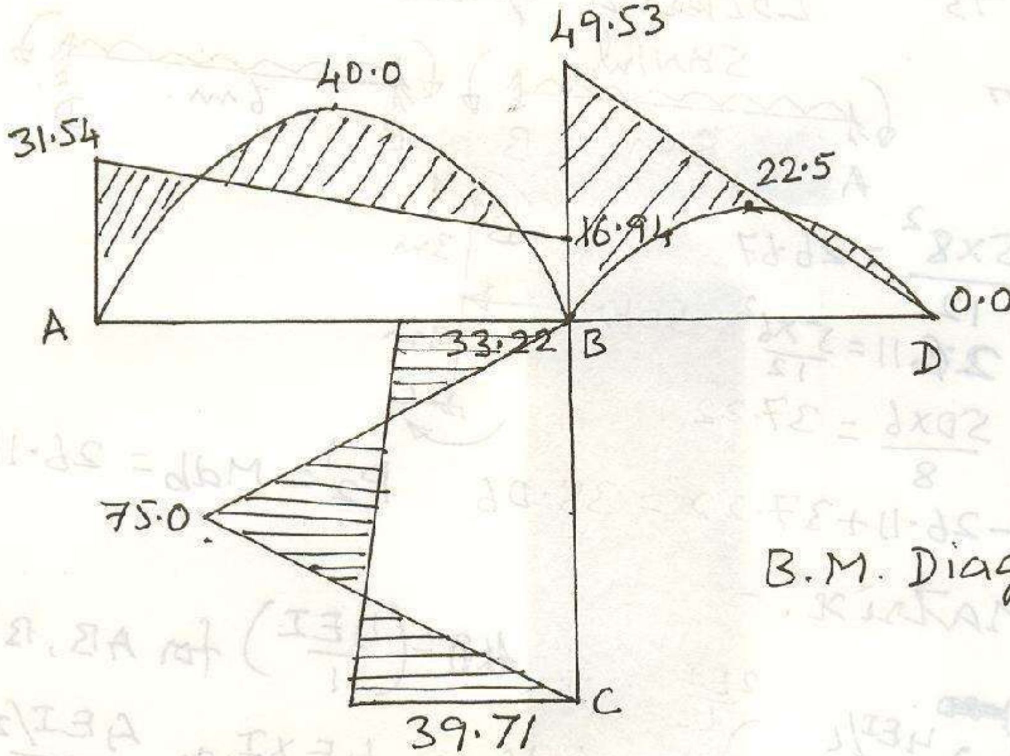
$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = +[K]^{-1}\left\{[P]-[P']\right\}$$

$$= -\frac{1}{EI}\begin{bmatrix} 1.05 & 0.167 \\ 0.167 & 0.33 \end{bmatrix}^{-1}\begin{bmatrix} 38.06 \\ 26.11 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} -19.47 \\ -101.55 \end{bmatrix}$$

$\therefore EI\Delta_1 = EI\theta_b = -19.47, \qquad EI\Delta_2 = EI\theta_d = -101.55$

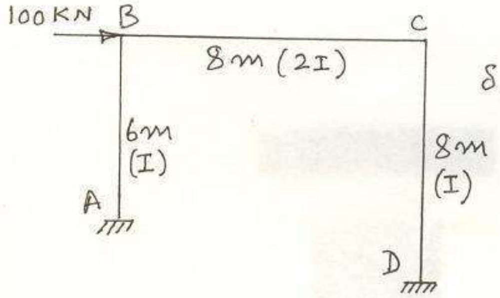
$M_{ab} = -31.54 \quad M_{bd} = -49.53 \quad M_{bc} = +33.22$ Draw B.M.D.

$M_{ba} = +16.94 \quad M_{db} = 0 \quad M_{cb} = -39.71$



B.M. Diagram.

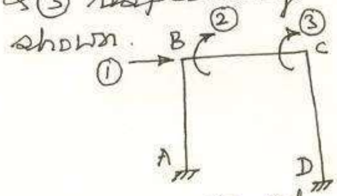
Ex-④



$D_K = 3$

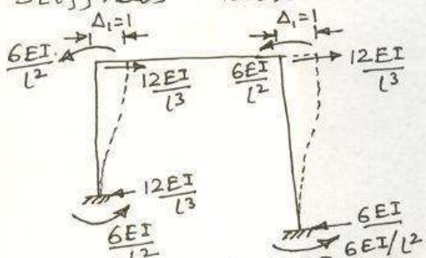
δ, θ_b, θ_c are the unknown displacements

Assign co-ord. ①, ② & ③ respectively as shown.



co-ord.

Stiffness Matrix:



unit displ. at ①

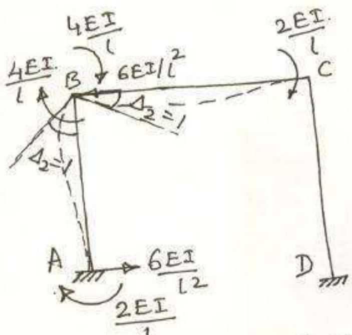
$$\therefore K_{11} = \left(\frac{12EI}{l^3}\right)_{AB} + \left(\frac{12EI}{l^3}\right)_{CD}$$

$$= \frac{12EI}{6^3} + \frac{12EI}{8^3}$$

$$= 0.08\,EI.$$

$$K_{21} = -\left(\frac{6EI}{l^2}\right)_{AB} = -0.167\,EI.$$

$$K_{31} = -\left(\frac{6EI}{l^2}\right)_{CD} = -0.094\,EI.$$



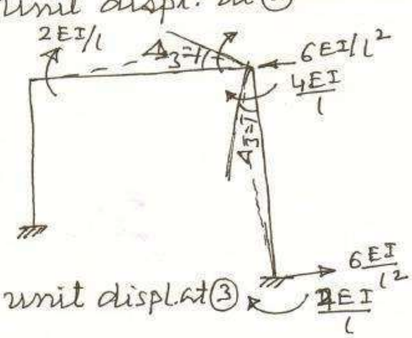
unit displ. at ②

$$\therefore K_{12} = -\left(\frac{6EI}{l^2}\right)_{AB} = K_{21} = -0.167\,EI.$$

$$K_{22} = \left(\frac{4EI}{l}\right)_{AB} + \left(\frac{4EI}{l}\right)_{BC}$$

$$= \frac{4EI}{6} + \left(\frac{4E\times 2I}{8}\right) = 1.667\,EI$$

$$K_{32} = \left(\frac{2EI}{l}\right)_{BC} = \frac{2E\times 2I}{8} = 0.5\,EI$$



unit displ. at ③

$$K_{13} = K_{31} = -0.094\,EI.$$

$$K_{23} = K_{32} = 0.5\,EI.$$

$$K_{33} = \left(\frac{4EI}{l}\right)_{BC} + \left(\frac{4EI}{l}\right)_{CD}$$

$$= \frac{4E\times 2I}{8} + \frac{4EI}{8}$$

$$= 1.5\,EI.$$

Locking forces — F.E.Ms. — are all zero.

$\therefore\ P_1' = P_2' = P_3' = 0$

Ext. forces — at the co-ordinates

$P_1 = 100\ KN. \quad P_2 = P_3 = 0.$

$\therefore\ \{\Delta\} = \{K\}^{-1}\{[P] - \{P'\}\}.$

$$\therefore \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 0.08 & -0.167 & -0.094 \\ -0.167 & 1.667 & 0.5 \\ -0.094 & 0.5 & 1.5 \end{bmatrix}^{-1}\left\{\begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$= \frac{1}{EI}\begin{bmatrix} 16.17 & 1.46 & 0.526 \\ 1.46 & 0.80 & -0.175 \\ 0.526 & -0.175 & 0.758 \end{bmatrix}\begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 1617 \\ 146 \\ 52.6 \end{bmatrix}$$

$\therefore\ EI\Delta_1 = EI\delta = 1617$

$EI\Delta_2 = EI\theta_b = 146$

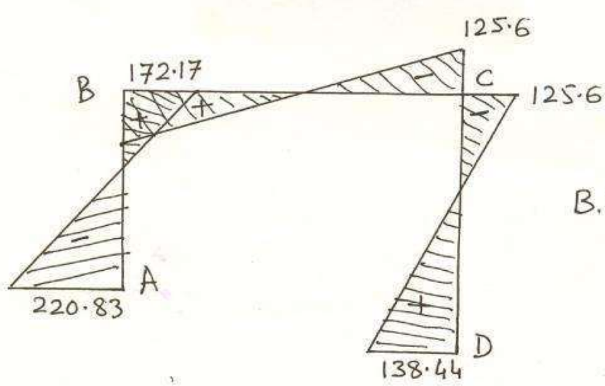
$EI\Delta_3 = EI\theta_c = 52.6$

From slope deflection Eqns :-

$M_{ab} = -220.83\ KNm \qquad M_{ba} = -172.17\ KNm$

$M_{bc} = +172.3\ KNm. \qquad M_{cb} = +125.6\ KNm.$

$M_{cd} = -125.3$ " $\qquad M_{dc} = -138.44$ ".



B.M. Diagram.